Algebraic Approach to Robust Controller Design: A Geometric Interpretation *

Didier Henrion[1†] Michael Šebek[2,3] Sophie Tarbouriech[1]

[1] LAAS-CNRS
7 Avenue du Colonel Roche
31 077 Toulouse, Cedex 4, France

[2] Trnka Laboratory of Automatic Control
Faculty of Electrical Engineering
Czech University of Technology
Prague, Czech Republic

[3] Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Prague, Czech Republic

Key Words Linear Systems, Robust Control, Algebraic Approach, Youla-Kučera Parametrization, Bilinear Matrix Inequalities, Concave Minimization Problem.

Abstract The problem of robust controller design is addressed for a single-input single-output plant with a single uncertain parameter. Given one controller that stabilizes the nominal plant, the Youla-Kučera parametrization of all stabilizing controllers and quadratic forms over Hermite-Fujiwara matrices are used to provide clear and simple geometric answers to the following questions: Can the plant be robustly stabilized by a nominally stabilizing controller ? How can this robust controller be designed ? Thanks to recent results on bilinear matrix inequalities, this geometric interpretation allows to state the equivalence between robust controller design and the concave minimization problem.

1 Introduction

Since the pioneering work of Kharitonov, significant results have been achieved through the polynomial approach to linear systems robustness. In his monograph [1], Barmish presents a clear and comprehensive survey of existing techniques.

Given a nominally stable polynomial with a single uncertain parameter, the Eigenvalue Criterion of Białas (Theorem 4.7.6 in [1]) provides exact lower and upper uncertainty margins such that the polynomial is not destabilized. These bounds are easily computed as generalized eigenvalues of Hurwitz matrices. Control applications are obvious: one can readily conclude whether stability of the characteristic polynomial of a given uncertain closed-loop system is preserved over a given uncertainty bounding set.

Suppose now that we are given a SISO plant that has been stabilized by a controller using standard design methods. Moreover, imagine that the uncertainty on some plant parameters exceeds the bounds computed by the Eigenvalue Criterion, so that the resulting closed-loop system may be destabilized. Then an interesting question arises: Is there another controller, picked up from the whole set of nominally stabilizing controllers, that could robustly stabilize the uncertain plant ?

In this paper, we use the Youla-Kučera parametrization of all stabilizing controllers to give a geometric answer to this question and derive necessary and sufficient conditions of robust stabilizability. We show that our robust

*This work is part of the Barrande Project No. 97/005-97/026. It was also supported by the Grant Agency of the Czech Republic under contract No. 102/97/0861, by the Ministry of Education of the Czech Republic under contract No. VS97/034 and by the French Ministry of Education and Research under contract No. 10-INSA-96.

†Corresponding author. FAX +33 (0) 561 33 69 69. E-mail henrion@laas.fr

stability analysis problem can be interpreted as a special case of Bilinear Matrix Inequality feasibility problem [6] with indefinite quadratic terms. Thanks to recent works [5, 14], we offer clear geometric interpretations of the existence of a robust controller. It is finally shown that the task of designing a robust controller amounts to that of solving a concave minimization problem, a difficult global optimization problem – although many algorithms have been recently developed [10].

The paper is organized as follows. In Section 2, the Youla-Kučera parametrization of all stabilizing controllers is used to derive the expression of the characteristic polynomial of our closed-loop uncertain system. Dependence of polynomial coefficients on free design parameters is enlightened. Next to these preliminaries, we state the problem to be solved. In Section 3, we define the Hermite-Fujiwara matrix of a given polynomial and enumerate some useful properties. The Hermite-Fujiwara matrix of the closed-loop characteristic polynomial is then used in Section 4 to define quadratic forms over the set of design parameters. Using this feature, we establish our main result: a geometric necessary and sufficient condition for robust stabilizability. Relationship with BMI feasibility problems is then underlined and equivalence is stated between robust controller design and the concave minimization problem. A conceptual robust controller design algorithm is described and a simple illustrative example is finally proposed in Section 5. The conclusion features some general remarks about numerical aspects of linear system robust stabilization. Future research directions are finally mentioned.

Notation

$\mathbb{R}$	The field of real numbers.
x', X'	The transpose of vector x and matrix X.
$\langle x, y\rangle$	The inner product of two vectors, $\langle x, y\rangle = x'y$.
$\langle X, Y\rangle$	The inner product of two matrices, $\langle X, Y\rangle = \mathrm{trace} X'Y$.
$\|\|x\|\|$	The Euclidean norm of a vector, $\|\|x\|\| = \sqrt{\langle x, x\rangle}$.
$\mathcal{S}^n$	The unit sphere in $\mathbb{R}^n$, $\mathcal{S}^n = \{x \in \mathbb{R}^n : \|\|x\|\| = 1\}$.
$\otimes$	The Kronecker product.
I_n	The identity matrix of dimension n.
$\lambda_{\min}$	The minimal eigenvalue of a symmetric matrix, $\lambda_{\min} X = \min\langle x, Xx\rangle, x \in \mathcal{S}^n$.
$\lambda_{\max}$	The maximal eigenvalue of a symmetric matrix, $\lambda_{\max} X = \max\langle x, Xx\rangle, x \in \mathcal{S}^n$.
$\mathcal{RH}_\infty$	The set of proper rational functions that are analytic in the right half plane.
$\lceil n/2 \rceil$	$n/2$ when n is even and $(n+1)/2$ otherwise.

2 Robust Control Design Problem

Using standard results on the parametrization of all stabilizing controllers, we first derive the expression of the closed-loop characteristic polynomial of a given uncertain SISO plant. Coefficients of the characteristic polynomial are shown to depend on free design parameters. Next to these preliminaries, we can state the problem to be solved.

2.1 Nominal plant

We consider a standard single-input single-output feedback linear system. The closed-loop transfer function is

$$T = \frac{P}{1 + PC} \tag{1}$$

where plant and controller transfer functions are proper and respectively read

$$P = \frac{b}{a}, \quad C = -\frac{y}{x}$$

for a, b and x, y two couples of relatively prime elements of $\mathcal{RH}_\infty$.

As described for instance in [13], the set of all controllers stabilizing plant P is generated by all solutions x, y over $\mathcal{RH}_\infty$ to Bézout equation

$$ax + by = 1. \tag{2}$$

All solutions to (2) read

$$\begin{array}{rcl} x &=& x_0 + bq \\ y &=& y_0 - aq \end{array} \tag{3}$$

where couple x_0, y_0 is a particular solution to (2) corresponding to a particular stabilizing controller

$$C_0 = -\frac{y_0}{x_0},$$

and q is a parameter that varies over $\mathcal{RH}_\infty$ while satisfying the non-restrictive constraint $x_0 + bq \neq 0$. Parametrization (3) is generally referred to as the Youla-Kučera parametrization or Q-parametrization.

Finally, rational functions involved in Bézout equation (2) and parametrization (3) can always be expressed as quotients of polynomials

$$\begin{array}{l} x = \frac{n_x}{d_c}, \quad y = \frac{n_y}{d_c}, \quad x_0 = \frac{n_{x_0}}{d_c}, \quad y_0 = \frac{n_{y_0}}{d_c}, \\ a = \frac{n_a}{d_p}, \quad b = \frac{n_b}{d_p}, \quad q = \frac{n_q}{d_q} \end{array} \tag{4}$$

for d_c, d_p and d_q stable polynomials of sufficiently high degree. Using equations (3) and (4), we derive the alternative expressions

$$P = \frac{n_b}{n_a}, \quad C_0 = -\frac{n_{y_0}}{n_{x_0}}$$

and

$$C = -\frac{d_q d_p n_{y_0} - d_c n_q n_a}{d_q d_p n_{x_0} + d_c n_q n_b} = -\frac{n_y}{n_x}. \tag{5}$$

2.2 Uncertain Plant. Characteristic Polynomial.

Now consider that some coefficients of the numerator and denominator of our plant are uncertain, viz.

$$P(r) = \frac{b + r\beta}{a + r\alpha}, \tag{6}$$

where parameter uncertainty r belongs to a real interval including the origin

$$r \in \mathfrak{I} = [r_{\min},\ r_{\max}], \quad r_{\min} \leq 0, \quad r_{\max} \geq 0. \tag{7}$$

Without loss of generality, we can assume that α and β belong to $\mathcal{RH}_\infty$ and read

$$\alpha = \frac{n_\alpha}{d_p}, \quad \beta = \frac{n_\beta}{d_p}, \tag{8}$$

where n_α, n_β are polynomials of degrees respectively less than or equal to those of n_a, n_b.

As a result, the transfer function of the closed-loop system is also uncertain:

$$T(r) = \frac{P(r)}{1 + P(r)C} = \frac{x(b + r\beta)}{x(a + r\alpha) + y(b + r\beta)}.$$

Using parametrization (3) and relations (4) and (8), straightforward calculations yield the equivalent polynomial expression

$$T(r) = \frac{(n_b + r n_\beta)(d_q d_p n_{x_0} + n_q d_c n_b)}{d_q d_p (n_{x_0} n_a + n_{y_0} n_b) + r\left[d_q d_p (n_{x_0} n_\alpha + n_{y_0} n_\beta) + n_q d_c (n_\alpha n_b - n_\beta n_a)\right]}.$$

The denominator of $T(r)$ is the characteristic polynomial of the closed-loop uncertain system. It can be written

$$d(r) = d_q d_0 + r(d_q d_1 + n_q d_2) \tag{9}$$

where

$$\begin{array}{rcl} d_0 & = & d_p(n_{x_0} n_a + n_{y_0} n_b) \\ d_1 & = & d_p(n_{x_0} n_\alpha + n_{y_0} n_\beta) \\ d_2 & = & d_c(n_\alpha n_b - n_\beta n_a). \end{array} \tag{10}$$

Closed-loop characteristic polynomial $d(r)$ therefore depends on known plant and controller data d_0, d_1, d_2, on parameter uncertainty r and most notably on *free design parameters* d_q, n_q. Recall that d_q is an arbitrary stable polynomial and n_q is an arbitrary polynomial of degree less than or equal to that of d_q.

2.3 Problem Statement

Suppose that there are no hidden modes in the closed-loop transfer function (1). Then stability of (1) and stability of the closed-loop characteristic polynomial $d(r)$ are equivalent, and the problem to be solved in this paper has two equivalent formulations.

Robust Control Design Problem (RCD Problem)

- *Given a controller C_0 that stabilizes the nominal plant $P(0)$ and given an uncertainty bounding set $\mathcal{I}$, find when possible a nominally stabilizing controller C that stabilizes the uncertain plant $P(r)$ for all $r \in \mathcal{I}$.*
- *Given polynomials d_0, d_1, d_2 and set $\mathcal{I}$, find polynomials d_q, n_q such that $d(r)$ is stable for all $r \in \mathcal{I}$.*

3 Hermite-Fujiwara matrix

In this section, we introduce the Hermite-Fujiwara matrix as a means of analyzing stability of a polynomial. Some relevant properties of this matrix are reviewed.

Definition 1 *Given a polynomial $p = p_0 + p_1 \mathsf{s} + \cdots + p_n \mathsf{s}^n$ where $p_i > 0$ for $i = 0, 1, \ldots, n$, the Hermite-Fujiwara matrix of p is defined as the symmetric matrix $H_n(p) = [h_{ij}]_{i,j=1,\ldots,m}$ where $m = \lceil n/2 \rceil$ and*

$$h_{ij} = \sum_{k=1}^{2i-1} (-1)^{k+2i-1} p_{n-k+1} p_{n+k-2i-2j+2}, \quad j \geq i.$$

For illustration, the Hermite-Fujiwara matrix is given below for the first values of n.

$$H_1 = p_1 p_0 \quad H_2 = p_2 p_1 \quad H_3 = \begin{bmatrix} p_3 p_2 & p_3 p_0 \\ p_3 p_0 & p_1 p_0 \end{bmatrix} \quad H_4 = \begin{bmatrix} p_4 p_3 & p_4 p_1 \\ p_4 p_1 & -p_3 p_0 + p_2 p_1 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} p_5 p_4 & p_5 p_2 & p_5 p_0 \\ p_5 p_2 & p_5 p_0 - p_4 p_1 + p_3 p_2 & p_3 p_0 \\ p_5 p_0 & p_3 p_0 & p_1 p_0 \end{bmatrix} \; H_6 = \begin{bmatrix} p_6 p_5 & p_6 p_3 & p_6 p_1 \\ p_6 p_3 & p_6 p_1 - p_5 p_2 + p_4 p_3 & -p_5 p_0 + p_4 p_1 \\ p_6 p_1 & -p_5 p_0 + p_4 p_1 & -p_3 p_0 + p_2 p_1 \end{bmatrix}.$$

Property 1 *Hermite-Fujiwara matrix $H_n(p)$ is positive definite if and only if polynomial p is Hurwitz stable.*

Proof See [11] for an early use of Hermite-Fujiwara matrices in linear system stability analysis. A more recent and more mathematical treatment can be found in [4].□

Remark 1 *In the sequel if n is greater than the degree of p, then $H_n(p)$ is the Hermite-Fujiwara matrix associated to p considered as a polynomial of degree n (that is, some leading coefficients p_i are zero).*

Definition 2 *Given two polynomials $p_0 = p_{0_{\text{even}}}(\mathsf{s}^2) + \mathsf{s} p_{0_{\text{odd}}}(\mathsf{s}^2)$ and $p_1 = p_{1_{\text{even}}}(\mathsf{s}^2) + \mathsf{s} p_{1_{\text{odd}}}(\mathsf{s}^2)$ of degree less than or equal to n, the compound Hermite-Fujiwara matrix of p_0 and p_1 is defined as*

$$H_n(p_0, p_1) = H_n(p_{0_{\text{even}}}(\mathsf{s}^2) + \mathsf{s} p_{1_{\text{odd}}}(\mathsf{s}^2)).$$

Remark 2 *For any polynomial p of degree less than or equal to n, $H_n(p, p) = H_n(p)$.*

Property 2 *Given a real number z and a polynomial p of degree less than or equal to n, we have*

$$H_n(zp) = z^2 H_n(p).$$

Proof From Definition 1, entries of the Hermite-Fujiwara matrix of a polynomial p are bilinear in the coefficients of p.□

Property 3 *Given two real numbers z_0, z_1 and two polynomials p_0, p_1 of degree less than or equal to n, we have*

$$H_n(z_0 p_0 + z_1 p_1) = z_0^2 H_n(p_0) + z_0 z_1 [H_n(p_0, p_1) + H_n(p_1, p_0)] + z_1^2 H_n(p_1).$$

Proof By simple inspection.□

For instance if $p_0 = p_{0_0} + p_{0_1}\mathbf{s} + p_{0_2}\mathbf{s}^2 + p_{0_3}\mathbf{s}^3$ and $p_1 = p_{1_0} + p_{1_1}\mathbf{s} + p_{1_2}\mathbf{s}^2 + p_{1_3}\mathbf{s}^3$, we have

$$H_3(z_0p_0 + z_1p_1) = \begin{bmatrix} (z_0p_{0_0} + z_1p_{1_0})(z_0p_{0_1} + z_1p_{1_1}) & (z_0p_{0_0} + z_1p_{1_0})(z_0p_{0_3} + z_1p_{1_3}) \\ (z_0p_{0_0} + z_1p_{1_0})(z_0p_{0_3} + z_1p_{1_3}) & (z_0p_{0_2} + z_1p_{1_2})(z_0p_{0_3} + z_1p_{1_3}) \end{bmatrix}$$

$$= z_0^2\begin{bmatrix} p_{0_0}p_{0_1} & p_{0_0}p_{0_3} \\ p_{0_0}p_{0_3} & p_{0_2}p_{0_3}\end{bmatrix} + z_0z_1\left(\begin{bmatrix} p_{0_0}p_{1_1} & p_{0_0}p_{1_3} \\ p_{0_0}p_{1_3} & p_{0_2}p_{1_3}\end{bmatrix} + \begin{bmatrix} p_{1_0}p_{0_1} & p_{1_0}p_{0_3} \\ p_{1_0}p_{0_3} & p_{1_2}p_{0_3}\end{bmatrix}\right) + z_1^2\begin{bmatrix} p_{1_0}p_{1_1} & p_{1_0}p_{1_3} \\ p_{1_0}p_{1_3} & p_{1_2}p_{1_3}\end{bmatrix}.$$

Generalization to the case of more than two polynomials also follows by inspection.

Property 4 *Given a real vector* $z = [z_0 \quad z_1 \, \cdots \, z_k]'$ *and* $k+1$ *polynomials* $p_0, p_1, \ldots, p_k$ *of degree less than or equal to* n*, we have*

$$\begin{aligned} H_n(\sum_{i=0}^{k} z_i p_i) &= [z' \otimes I_{\lceil n/2 \rceil}] \begin{bmatrix} H_n(p_0) & H_n(p_0,p_1) & \cdots & H_n(p_0,p_k) \\ H_n(p_1,p_0) & H_n(p_1) & & \vdots \\ \vdots & & \ddots & \\ H_n(p_k,p_0) & \cdots & & H_n(p_k) \end{bmatrix} [z \otimes I_{\lceil n/2 \rceil}] \\ &= \sum_{i=0}^{k}\sum_{j=0}^{k} z_i z_j H_n(p_i,p_j). \end{aligned} \tag{11}$$

4 Solution to the RCD Problem

In this section, we first build quadratic forms over the set of design parameters by invoking properties of the Hermite-Fujiwara matrix of the closed-loop characteristic polynomial defined in Section 2. Then we can state our main result: geometric necessary and sufficient conditions for robust stabilizability. Connection with BMI feasibility is emphasized and shown to provide nice geometric interpretations to the RCD problem. An equivalent concave minimization problem is described and an RCD algorithm is finally proposed and illustrated.

4.1 Robust Stability Condition

Let n and l respectively denote degrees of characteristic polynomial d and design polynomial d_q. Moreover, let

$$\begin{aligned} d_q &= z_0 + z_1\mathbf{s} + \cdots + z_l\mathbf{s}^l \\ n_q &= z_{l+1} + z_{l+2}\mathbf{s} + \cdots + z_{2l+1}\mathbf{s}^l. \end{aligned} \tag{12}$$

We note $k = 2l+1, m = \lceil n/2 \rceil$ and collect all design parameters into a vector

$$z = [z_0 \quad z_1 \, \cdots \, z_k]' \in \mathbb{R}^{k+1}.$$

Finally, for $i = 0, 1, \ldots, l$ we define the uncertain polynomials

$$\begin{aligned} p_i(r) &= \mathbf{s}^i d_0 + r\mathbf{s}^i d_1 \\ p_{i+l+1}(r) &= r\mathbf{s}^i d_2 \end{aligned} \tag{13}$$

where d_0, d_1, d_2 were given in (10). With this notation, characteristic polynomial (9) becomes

$$d(r) = \sum_{i=0}^{k} z_i p_i(r).$$

According to Property 4, the Hermite-Fujiwara matrix of $d(r)$ reads

$$\begin{aligned} M(z,r) = H_n(d(r)) &= [z' \otimes I_m]G(r)[z \otimes I_m] \\ &= \sum_{i=0}^{k}\sum_{j=0}^{k} z_i z_j G_{ij}(r) \in \mathbb{R}^{m \times m} \end{aligned} \tag{14}$$

where

$$G(r) = \begin{bmatrix} G_{00}(r) & \cdots & G_{0k}(r) \\ \vdots & \ddots & \vdots \\ G_{k0}(r) & \cdots & G_{kk}(r) \end{bmatrix} \in \mathbb{R}^{m(k+1)\times m(k+1)} \tag{15}$$

and

$$G_{ij}(r) = H_n(p_i(r), p_j(r)). \tag{16}$$

Note the separation in relation (14) between design vector z and matrix $G(r)$. A simple inspection reveals that $G(r)$ is a quadratic polynomial matrix in the indeterminate r, i.e. $G(r) = G_0 + G_1 r + G_2 r^2$. Moreover, matrix coefficients of $G(r)$ only depend on known plant and controller data.

From Property 1, $d(r)$ is stable if and only if vector z is such that

$$M(z, r) > 0 \quad \forall r \in \mathcal{I} \tag{17}$$

or equivalently

$$\langle v, M(z,r)v\rangle > 0 \quad \forall v \in \mathcal{S}^m \quad \forall r \in \mathcal{I}.$$

Recalling equation (14), this condition can also be written

$$\langle z, F(v,r)z\rangle > 0 \quad \forall v \in \mathcal{S}^m \quad \forall r \in \mathcal{I}$$

where

$$F(v,r) = \begin{bmatrix} \langle v, G_{00}(r)v\rangle & \cdots & \langle v, G_{0k}(r)v\rangle \\ \vdots & \ddots & \vdots \\ \langle v, G_{k0}(r)v\rangle & \cdots & \langle v, G_{kk}(r)v\rangle \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}. \tag{18}$$

In association with $F(v,r)$, we define the set

$$\mathcal{K} = \{z \in \mathbb{R}^{k+1} : \langle z, F(v,r)z\rangle > 0, \quad \forall v \in \mathcal{S}^m, \quad \forall r \in \mathcal{I}\}.$$

Note that $\mathcal{K}$ is the union of lines through the origin since $z \in \mathcal{K}$ implies that $\gamma z \in \mathcal{K}$ for any scalar γ. With these notations, our main result is as follows.

Theorem 1 *The RCD problem is feasible if and only if set $\mathcal{K}$ is not empty. Moreover, to any vector z in $\mathcal{K}$ corresponds a robust controller solution to the RCD problem.*

Proof The RCD problem has a solution if and only if characteristic polynomial $d(r)$ is stable for any uncertainty $r \in \mathcal{I}$. As shown above, stability of $d(r)$ is equivalent to positive definiteness of Hermite-Fujiwara matrix $M(z,r)$, which is in turn equivalent to the existence of a design parameter vector z in set $\mathcal{K}$.□

In the sequel, we expose several equivalent formulations for Theorem 1 that are believed to give clear geometric insights into the RCD problem.

4.2 BMI Formulation

According to equation (14), positive definite inequality (17) can also be written

$$\sum_{i=0}^{k}\sum_{j=0}^{k} z_i z_j G_{ij}(r) > 0 \quad \forall r \in \mathcal{I}. \tag{19}$$

Since matrices $G_{ij}(r)$ are symmetric, the above inequality can be interpreted as a special form of a parametrized Bilinear Matrix Inequality (BMI) with indefinite quadratic terms $z_i^2 G_{ii}(r)$ – see §4 in [3]. In [6], Goh *et al.* show that many of the problems considered in the robust control literature can be formulated as BMIs. In view of inequality (19), this is also true for our RCD problem.

4.3 Geometric Interpretation

Following [5], we define the *matrix field of values*

$$\mathcal{F} = \{F(v,r) = \begin{bmatrix} \langle v, G_{00}(r)v\rangle & \cdots & \langle v, G_{0k}(r)v\rangle \\ \vdots & \ddots & \vdots \\ \langle v, G_{k0}(r)v\rangle & \cdots & \langle v, G_{kk}(r)v\rangle \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}, \quad \forall v \in \mathcal{S}^m, \quad \forall r \in \mathcal{I}\}$$

as an extension of the concept of a field of values – see Chapter 1 in [9]. The problem of finding a robust stabilizing controller may be converted into a matrix field of values problem as follows.

Corollary 1 *The RCD problem is feasible if and only if there exists a symmetric matrix $Z \in \mathbb{R}^{(k+1)\times(k+1)}$ such that*

$$\begin{aligned} &\langle Z, F\rangle > 0 \quad \forall F \in \mathcal{F} \\ &\mathrm{rank} Z = 1. \end{aligned} \tag{20}$$

Proof Suppose that the RCD problem has a solution. From Theorem 1, we are then given a vector $z \in \mathcal{K}$. From the definition of $\mathcal{K}$, $Z = zz'$ clearly solves (20). Conversely, suppose that there exists a rank one symmetric matrix Z such that (20) holds. Let σ_1 be the only non-zero singular value of Z and u_1 the corresponding right singular vector, so that $\sigma_1 u_1 u_1' = Z$. Then $z = \sqrt{\sigma_1} u_1$ belongs to $\mathcal{K}$ and the RCD problem has a solution.□

Remark 3 *Corollary 1 has the following geometric interpretation [14]. In order to solve the RCD problem, one has to find a hyperplane in $\mathbb{R}^{(k+1)\times(k+1)}$ that strictly separates the origin from matrix field of values $\mathcal{F}$. Moreover, the symmetric matrix Z that defines the perpendicular to this hyperplane must be of rank one. The existence of such a matrix ensures that a vector z belongs to $\mathcal{K}$.*

Another nice geometric interpretation of the RCD problem is as follows. First, let $F_s(v,r)$ stand for the symmetric part of $F(v,r)$, viz.

$$F_s(v,r) = \frac{1}{2}[F(v,r) + F'(v,r)]. \tag{21}$$

Let ρ be a real number such that

$$\rho^{-1} > \max\{0, \lambda_{\max} F_s(v,r)\} \quad \forall v \in \mathcal{S}^m \quad \forall r \in \mathcal{I}. \tag{22}$$

Note that (22) ensures that the symmetric matrix

$$P(v,r) = I_{k+1} - \rho F_s(v,r) \tag{23}$$

is positive definite for all $v \in \mathcal{S}^m$ and $r \in \mathcal{I}$. In association with $P(v,r)$, we define the set

$$\mathcal{C} = \{z \in \mathbb{R}^{k+1} : \langle z, P(v,r)z\rangle \leq 1, \quad \forall v \in \mathcal{S}^m, \quad \forall r \in \mathcal{I}\}.$$

Since $P(v,r)$ is always positive definite, $\mathcal{C}$ is the intersection of an infinite number of centered ellipsoids. Hence $\mathcal{C}$ is convex, centered around the origin and its radius is the maximal norm of any element belonging to $\mathcal{C}$. The next result mimics Lemma 1 in [14] and provides a second geometric interpretation to RCD problem feasibility.

Corollary 2 *The RCD problem is feasible if and only if the radius of $\mathcal{C}$ is strictly greater than one.*

Proof First suppose that the radius of $\mathcal{C}$ is strictly greater than one and therefore that there is a vector z such that $\|z\| > 1$ and $\langle z, P(v,r)z\rangle \leq 1$ for all $v \in \mathcal{S}^m$ and $r \in \mathcal{I}$. From equation (21) and (23), this is equivalent to $\|z\| - \rho\langle z, F(v,r)z\rangle \leq 1$ from which it follows that $\rho\langle z, F(v,r)z\rangle \geq \|z\| - 1 > 0$ for all $v \in \mathcal{S}^m$ and $r \in \mathcal{I}$. Since ρ is positive, z belongs to $\mathcal{K}$. According to Theorem 1, z is solution to the RCD problem. Conversely, suppose that the RCD problem has a solution and therefore that there exists a vector z in the set $\mathcal{K}$. Since $\mathcal{K}$ is the union of lines through the origin, we may assume without loss of generality that $z \in \mathcal{S}^{k+1}$. Therefore $\langle z, P(v,r)z\rangle = \|z\| - \rho\langle z, F(v,r)z\rangle = 1 - \rho\langle z, F(v,r)z\rangle < 1$ for all $v \in \mathcal{S}^k$ and $r \in \mathcal{I}$. Consequently, the radius of the set $\mathcal{C}$ is strictly greater than $\|z\| = 1$.□

4.4 Concave Minimization Problem

The geometric condition described in Corollary 2 holds if and only if the solution $\hat{z}$ to the optimization problem

$$\begin{array}{ll} \max & \|z\| \\ \text{s.t.} & z \in \mathcal{C} \end{array} \tag{24}$$

is such that $\|\hat{z}\| > 1$. (24) is a non-linear optimization problem where a convex function is to be maximized over a convex set. Besides its simplicity, it is a difficult non-convex global optimization problem usually called a *concave minimization problem* – see Chapter I, §2 in [10]. It is well known that problem (24) is multiextremal, and therefore that $\|z\|$ may possess (possibly very many) local maxima with respect to $\mathcal{C}$. Nevertheless, several algorithms that solve (24) are described in the literature. For more details, the reader is referred to [10].

Remark 4 *It may not be necessary to solve (24) for its global optimum $\hat{z}$. We can stop as soon as a vector z has been found such that $\|z\| > 1$.*

4.5 Conceptual Algorithm

Following along the same line as [14], we finally present a conceptual algorithm to solve the RCD problem. In (24), vector z is constrained to the intersection of an infinite number of ellipsoids. However, this infinite dimensional problem can be solved by considering a sequence of finite dimensional problems.

RCD Algorithm

Input *Plant and controller transfer functions P, C_0 and the uncertainty bounding set $\mathfrak{I}$.*

Output *If the RCD problem is feasible, the algorithm returns the transfer function C of a robust controller. Otherwise, the algorithm states that the RCD problem is infeasible.*

Step 1 *Using relations (4) and (10), compute polynomials d_0, d_1, d_2. Select l to be the degree of design polynomials n_q and d_q. Let n be the degree of characteristic polynomial (9), and let $k = 2l + 1$, $m = \lceil n/2 \rceil$. Build quadratic polynomial matrix $G(r)$ using Definition 2 and equations (13), (15) and (16). Finally, choose arbitrary initial guess $v_1 \in \mathcal{S}^m$, $r_1 \in \mathfrak{I}$.*

Step i = 2,3,...

1. *Using equations (18) and (21), compute matrix $F_s(v_{i-1}, r_{i-1}) \in \mathbb{R}^{(k+1)\times(k+1)}$. Select a real number ρ_{i-1} such that $\rho_{i-1}^{-1} > \max\{0, \lambda_{\max} F_s(v_{i-1}, r_{i-1})\}$, and let $P_{i-1} = I_{k+1} - \rho_{i-1} F_s(v_{i-1}, r_{i-1})$.*
2. *Solve for a globally maximizing vector ζ_i the problem*

$$\begin{array}{ll} \max & \|\zeta\| \\ \text{s.t.} & \langle \zeta, P_j \zeta \rangle \leq 1 \quad j = 1, \ldots, i-1. \end{array} \tag{25}$$

3. *If $\|\zeta_i\| \leq 1$, stop, the RCD problem does not have a solution.*
4. *Otherwise, build matrix $M(\zeta_i, r)$ as in (14) and solve for a globally minimizing pair v_i, r_i the problem*

$$\begin{array}{rcll} \lambda_i & = & \min & \langle v, M(\zeta_i, r) v \rangle \\ & & \text{s.t.} & v \in \mathcal{S}^{k+1} \quad r \in \mathfrak{I}. \end{array} \tag{26}$$

5. *If $\lambda_i > 0$, stop, $z = \zeta_i$ is a solution to the RCD problem. Design polynomials n_q, d_q are given by equations (12). Finally, numerator n_y and denominator n_x of the controller transfer function are retrieved from equation (5).*
6. *Otherwise, go to Step i+1.*

Remark 5 *It can be shown [14] that this algorithm will stop in a finite number of steps if the RCD problem has a solution. Otherwise the RCD problem is infeasible. This may be regarded as an important limitation since if the RCD problem has no solution, there is no guarantee for the algorithm to stop in Step 3. Further investigation is currently underway to improve the above algorithm so that convergence be guaranteed. Constraint dropping scheme [7] may prove useful for this purpose.*

Remark 6 *Existence of a robustly stabilizing controller does not depend upon the choice of nominal controller C_0. Hence if one starts from another nominally stabilizing controller, one just gets another robustly stabilizing controller.*

Remark 7 *If the algorithm stops in Step 3, then there is no robustly stabilizing controller parametrized by polynomials n_q and d_q of prespecified degree l. However, there could be a solution of a higher degree.*

Remark 8 *The solution v_i, r_i to problem (26) is such that v_i is the eigenvector of $M(\zeta_i, r_i)$ associated to the eigenvalue*

$$\begin{array}{rcl} \lambda_i & = & \min \quad \lambda_{\min} M(\zeta_i, r) \\ & & \text{s.t.} \quad r \in \mathcal{I}. \end{array}$$

The above eigenvalue can be found for instance by a one-dimensional search over real interval $\mathcal{I}$.

Remark 9 *Problem (25) is referred to as a* global minimization problem of a concave function subject to separable quadratic constraints. *Several algorithms exist to solve (25) – see Chapter XI, §2.4 in [10]. However, it may not be necessary to solve (25) for the global maximum at each step. As pointed out in Remark 4, one can stop as soon as a point ζ_i such that $\|\zeta_i\| > 1$ has been found. As underlined in [14], this may be detrimental to the overall speed of convergence of the algorithm but may avoid spending a lot of time in solving (25) globally.*

5 Illustration

We consider Example 4.2.3 in [1] where robust stability of a simple first order plant described by

$$P(r) = \frac{1}{\mathtt{s} - r}$$

and uncertainty bounds

$$r \in \mathcal{I} = [-2, 2]$$

was studied. As pointed out in [1], a feedback compensator that stabilizes the nominal plant $P(0)$ is

$$C_0 = 1.$$

The resulting characteristic polynomial is $\mathtt{s} + 1 - r$, so when $r \geq 1$ the closed-loop system becomes unstable. As a result, we can conclude that C_0 is *not* a robustly stabilizing controller. Our task is then to find another nominally stabilizing controller C such that the closed-loop system is robustly stable for all $r \in \mathcal{I}$.

Using the notations of Section 2, we choose $d_p = 1 + \mathtt{s}, d_c = 1$ and $n_{x_0} = 1, n_{y_0} = 1, n_a = \mathtt{s}, n_b = 1, n_\alpha = -1, n_\beta = 0$. Closed-loop characteristic polynomial components are thus $d_0 = \mathtt{s}^2 + 2\mathtt{s} + 1, d_1 = -\mathtt{s} - 1$ and $d_2 = 1$. With the notations of §4, we select $l = 1$, hence $n = 3$, $k = 3$ and $m = 2$. Equations (13), (15) and (16) yield

$$G(r) = \left[\begin{array}{cc|cc|cc|cc} . & . & 1 & 1 & . & . & . & . \\ . & 2 & 1 & 1 & . & . & . & . \\ \hline . & . & 2 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}\right] - \left[\begin{array}{cc|cc|cc|cc} . & . & . & 1 & . & . & . & . \\ . & 3 & 1 & 2 & . & 2 & . & 1 \\ \hline . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \hline . & . & . & 1 & . & . & . & . \\ . & 2 & 1 & 1 & . & . & . & . \\ \hline . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}\right] r + \left[\begin{array}{cc|cc|cc|cc} . & . & . & . & . & . & . & . \\ . & 1 & . & 1 & . & . & . & 1 \\ \hline . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & . \\ . & 1 & . & 1 & . & . & . & 1 \\ \hline . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}\right] r^2$$

where a dot denotes a zero entry. With our implementation of the algorithm, the RCD problem is found to be feasible and we obtain the following design polynomials

$$\begin{array}{rcl} d_q & = & 1.8471 + 0.1100\mathtt{s} \\ n_q & = & -1.3730 - 1.3547\mathtt{s}. \end{array}$$

Equation (5) provides the corresponding second order controller

$$C = -\frac{1.8471 + 3.3571\mathtt{s} + 1.4647\mathtt{s}^2}{0.5011 + 0.6294\mathtt{s} + 0.1100\mathtt{s}^2}.$$

One can check that the closed-loop characteristic polynomial (9) is then

$$d(r) = 1.8741 + 3.8582\mathtt{s} + 2.0940\mathtt{s}^2 + 0.1100\mathtt{s}^3 - r(0.5011 + 0.6294\mathtt{s} + 0.1100\mathtt{s}^2).$$

Using the Eigenvalue Criterion of Białas, we obtain the uncertainty bounding set

$$\mathcal{I}_{\max} = (-\infty, 3.7400]$$

such that stability of $d(r)$ is preserved for any $r \in \mathcal{I}_{\max}$. The inclusion $\mathcal{I} \subset \mathcal{I}_{\max}$ thus confirms that C is a *robust* stabilizing controller for $P(r)$ and uncertainty bounding set $\mathcal{I}$.

6 Conclusion

In this paper we considered the problem of robust controller design for an uncertain SISO plant. We used the parametrization of all stabilizing controllers and reduced Hermite matrices to state our main contribution: a necessary and sufficient condition for robust stabilizability. This result is aimed at providing simple and clear geometric interpretations of the existence of a robust controller. Relationships with BMI feasibility and the concave minimization problem were enlightened and a robust controller design algorithm was finally proposed.

We must underline the fact that the design of a robust controller as described in this paper can be a very difficult numerical task. The reasons behind this are three-fold.

- First, we consider *real* and *highly structured* parametric uncertainties. It is now well understood that the more structured the uncertainty, the more difficult the problem. Moreover, problems involving real uncertainty are generally far more difficult than problems involving purely complex uncertainty. For a discussion on this topic, see the survey paper [15].

- Second, we solve a robust *design* problem. In the literature, most achievements involving real parametric uncertainty pertain to the scope of robust stability analysis, see [1].

- Third, we obtain constructive *necessary and sufficient* conditions for the existence of a robust controller. Usually, robust stabilizability conditions are only sufficient. In particular, we do not restrict ourselves to any subset of the left half plane when studying stability. To illustrate this point, note for instance that the robust controller design method recently described in [12] handles stability regions that are polytopes in the space of characteristic polynomial coefficients. As a result, the design algorithm rely upon linear programming and can easily and efficiently be implemented. Clearly, the stability condition is only sufficient since it is well known that the stability region of a Hurwitz polynomial in the space of its coefficients is a highly non-convex set that can only poorly and conservatively be approximated by a polytope. Nevertheless, the solution proposed in [12] constitutes an interesting tradeoff between computational load and stability requirements. On the other hand, our algorithm may be rather considered as an ultimate design tool in the sense that we can definitively know whether a robustly stabilizing controller exists or not, but this may be done at the expense of computational complexity.

Some aspects of our work need further investigation. Potential directions of research are enumerated below.

- We may use Bézoutian matrices to reduce problem dimensions by half. Indeed, in [4], positive definitiveness of the Hermite-Fujiwara matrix is shown to be equivalent to positive definitiveness of a reduced Bézoutian matrix.

- The specific structure of the concave minimization problem (convex criterion, convex feasible set) together with the outer approximation algorithm proposed in [7] may be exploited to derive a more powerful and tractable design algorithm.

- There are several LMI relaxation schemes that may prove useful for approximating the radius of $\mathcal{C}$ in Corollary 2. Lower and upper bounds on the global optima to problems (24) and (25) can be derived that readily provide necessary or sufficient conditions for robust stabilizability. Such bounds can efficiently be computed using widely spread interior-point algorithms [2]. A survey of existing relaxation methods and guaranteed bounds on suboptimal solutions can be found in [8].

- By considering determinants of polynomial matrices, our results should be generalized to the multi-input multi-output case.

- Finally, an extension to more general types of uncertainty is required to accommodate a wider range of practical control problems.

Acknowledgements

The first author would like to thank Prof. Jean-Baptiste Hirriart-Urruty (Université Paul Sabatier, Toulouse, France) and Prof. Stephen Boyd (Stanford University, California) for helpful comments on optimization aspects and Prof. Vladimír Kučera (Institute of Information Theory and Automation, Prague, Czech Republic) for several valuable discussions on linear system robust stabilization.

References

[1] B. R. Barmish, New Tools for Robustness of Linear Systems, Macmillan, 1994.

[2] S. Boyd, L. El Ghaoui, E. Féron and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics, Vol. 15, 1994.

[3] S. Boyd and L. Vandenberghe, Semidefinite Programming Relaxations of Non-convex Problems in Control and Combinatorial Optimization, in: Communications, Computation, Control and Signal Processing – a Tribute to Thomas Kailath, Kluwer, 1997.

[4] P. A. Fuhrmann and B. N. Datta, On Bézoutians, Vandermonde matrices and the Liénard-Chipart stability criterion, Linear Algebra and its Applications, Vol. 120, 1989, pp. 23–37.

[5] K. C. Goh, L. Turan, M. G. Safonov, G. P. Papavassilopoulos and J. H. Ly, Biaffine Matrix Inequality Properties and Computational Methods, Proceedings of the American Control Conference, Baltimore, Maryland, 1994, pp. 850–855.

[6] K. C. Goh, M. G. Safonov and J. H. Ly, Robust Synthesis via Bilinear Matrix Inequalities, International Journal of Robust and Nonlinear Control, Vol. 6, No. 9/10, 1996, pp. 1079–1095.

[7] C. Gonzaga and E. Polak, On Constraint Dropping Schemes and Optimality Functions for a Class of Outer Approximation Algorithms, SIAM Journal on Control and Optimization, Vol. 17, 1979, pp. 477–493.

[8] D. Henrion, S. Tarbouriech, D. Arzelier and M. Šebek, LMI Approximations for the Radius of the Intersection of Ellipsoids, submitted for publication, 1998.

[9] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.

[10] R. Horst and H. Tuy, Global Optimization: Deterministic Approaches. Third Revised and Enlarged Edition, Springer, 1996.

[11] E. I. Jury and B. D. O. Anderson, Some Remarks on Simplified Stability Criteria for Continuous Linear Systems, IEEE Transactions on Automatic Control, Vol. 17, No. 2, 1972, pp. 371–372.

[12] F. J. Kraus and V. Kučera, Robust Assignment of Polynomial Matrix Polytopes, Proceedings of the European Control Conference, Brussels, Belgium, 1997.

[13] V. Kučera, Diophantine Equations in Control - A Survey, Automatica, Vol. 29, No. 6, 1993, pp. 1361–1375.

[14] M. G. Safonov and G. P. Papavassilopoulos, The Diameter of an Intersection of Ellipsoids and BMI Robust Synthesis, Proceedings of the IFAC Symposium on Robust Control Design, Rio de Janeiro, Brazil, 1994, pp. 313–317.

[15] P. M. Young, M. P. Newlin and J. C. Doyle, μ Analysis with Real Parametric Uncertainty, Proceedings of the IEEE Conference on Decision and Control, Brighton, England, 1991, pp. 1251–1256.