Control of linear systems subject to input constraints: a polynomial approach.*

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Abstract

A polynomial approach is pursued for locally stabilizing discrete-time linear systems subject to input constraints. Using the Youla-Kučera parametrization and geometric properties of polyhedra and ellipsoids, the problem of simultaneously deriving a stabilizing controller and the corresponding stability region is cast into standard convex optimization problems solved by linear, second-order cone and semidefinite programming. Key topics are touched on such as stabilization of MIMO plants or maximization of the size of the stability domain. Readily implementable algorithms are described. Their applicability is illustrated with several numerical examples.

Keywords

Linear Systems, Input Constraints, Polynomial Approach, Youla-Kučera Parametrization, Convex Programming.

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1 Introduction

The problem of control constraints appears in most practical control systems. Due to technological and safety reasons, the actuators cannot drive an unlimited energy to the controlled plant. This fact can be translated into bounds on control and state variables.

Control systems are often linearly designed. The modern theory of linear control provides efficient methods for computing control laws that guarantee stability and some performance requirements with respect to the linear closed-loop system. In general, this kind of design does not directly consider amplitude limitations on the control inputs. Then, the presence of input bounds can be source of parasitic equilibrium points and limit cycles, or can even lead the closed-loop system to an unstable behavior. In past years, this fact has motivated study of both analysis and design techniques taking control bounds into account, see [3, 11, 31] and references therein.

Control limitation may be handled implicitly, or a posteriori, through the so-called anti-windup strategies [19]. Alternatively, input constraints may be handled explicitly, or a priori, pursuing one of the two following approaches:

- Saturation avoidance methods – They consist in preventing input saturation. The closed-loop system therefore stays in a region of linear behavior. On the one hand, an interesting approach developed in the literature consists in determining a feedback control law that ensures positive invariance of a set included in a region of closed-loop linear behavior and including all admissible initial states. This positive invariant set is then considered as a linear local region of stability, see [15, 4] for comprehensive overviews. These methods may rely on the extended Farkas lemma [14], linear programming [32, 9], eigenstructure assignment [8, 9] or set-induced norms [29]. On the other hand, a more general convex programming approach can also be pursued [5]. It relies upon the Youla-Kučera parametrization of all stabilizing controllers [20, 36] and has been extended to handle $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance criteria using state-space arguments [30].
- Saturation allowance methods – They consist in letting the saturation occur. The closed-loop system is therefore non-linear. In this sense, significant results have lately emerged in the scope of global stabilization [28, 27] and semi-global stabilization [24]. They inherently require stability of the open-loop system. Relaxing this stability assumption, results addressing the local stabilization problem have also been obtained [31, 17, 16, 12, 13].

In this paper, we focus on a saturation avoidance method mixing some of the above mentioned techniques. On the one hand, we use the Youla-Kučera parametrization of all stabilizing controllers in the context of the polynomial approach to control systems [21, 22]. On the other hand, we use standard geometric properties of polyhedra and ellipsoids to come up with a convex programming formulation of the constrained stabilization problem, as in [5]. In order to provide practical design methods, we deliberately restrict ourselves to the scope of convex optimization, namely linear [34], second-order cone [25] or semidefinite programming [33]. Efficient polynomial-time interior-point methods have recently been developed for tackling all of these problems so that readily implementable constrained stabilization algorithms can be developed.

To the authors knowledge, our development is the first application of the polynomial approach to the control of linear systems subject to input constraints. Key points of our approach are enumerated below:

- We consider multivariable (MIMO) systems.
- We can guarantee that we describe the whole set of stabilizing controllers under input constraints.
- Stability is guaranteed by ensuring that the system state stays in the domain of linear closed-loop behavior.
- The region of admissible initial states can indifferently be a polyhedron or an ellipsoid.
- This region is not necessarily given as input data, i.e. we can simultaneously seek a stabilizing controller and the corresponding region of admissible initial states.
- The size of this region can be maximized.

The outline of the paper is as follows. In Section 2, we state the problems to be solved. We make the distinction between stabilization in a given region (Problem 1) and simultaneous derivation of the control law and the largest stability region (Problem 2). Section 3 imparts the necessary background material. Several technical results are exposed that will prove instrumental to the derivation in Section 4 of a control law that solves Problem 1. Section 5 is then devoted to the development of convex optimization algorithms for solving Problem 2. In Section 6, several examples from the control literature bear out the usefulness of our techniques. The paper winds up with some concluding remarks and directions for future research.

Notations: I_n is the $n \times n$ identity matrix, 1_n is the n-dimensional vector of all ones, A^T is the transpose of matrix A, $\|x\|$ is the Euclidean norm of vector x, vector inequality $x \geq y$ is meant componentwise, $A \succeq B$ means that matrix $A - B$ is positive semidefinite, $P(\mathtt{d})$ denotes a polynomial matrix in the indeterminate d and $\deg P$ stands for the highest degree of the polynomial entries of $P(\mathtt{d})$.

2 Problem Statement

Consider a multi-input multi-output (MIMO) observable discrete-time linear system

$$\begin{aligned} \xi_{k+1} &= F\xi_k + Gu_k \\ z_k &= H\xi_k \end{aligned} \qquad (1)$$

where $\xi_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $z_k \in \mathbb{R}^p$ stand for the state, input and output vectors, respectively. Input vector u_k is subject to hard constraints

$$-u^- \leq u_k \leq u^+ \tag{2}$$

where u^-, u^+ are given vectors with positive entries. Moreover, initial system state ξ_0 is supposed to belong to a given convex subset of the state-space. This subset can either be

- a polyhedron
$$\mathcal{P}_N = \{\xi : N\xi \leq \nu\} \tag{3}$$
where N is a matrix and ν is a vector, or

- an ellipsoid
$$\begin{array}{rcl}\mathcal{E}_\Pi & = & \{\xi : (\xi - \xi_c)^T \Pi (\xi - \xi_c) \leq 1\} \\ & = & \{\Pi^{-\frac{1}{2}}\eta + \xi_c : \|\eta\| \leq 1\}\end{array} \tag{4}$$
where Π is a positive semi-definite matrix and ξ_c is a vector.

Performing z-transform, linear system (1) can equivalently be written in a transfer function setting as

$$z(\mathtt{d}) = S(\mathtt{d})u(\mathtt{d}) + T(\mathtt{d})\xi_0 \tag{5}$$

where $\mathtt{d}$ stands for the usual backward shift operator [21] and rational matrices

$$\begin{array}{rcl}S(\mathtt{d}) & = & H(I_n - F\mathtt{d})^{-1}G\mathtt{d} \\ T(\mathtt{d}) & = & H(I_n - F\mathtt{d})^{-1}\end{array}$$

can be expressed using polynomial matrix fraction descriptions (MFDs, see [18] or [35]) as

$$\begin{array}{rcl}S(\mathtt{d}) & = & B(\mathtt{d})A^{-1}(\mathtt{d}) \\ & = & \bar{A}^{-1}(\mathtt{d})\bar{B}(\mathtt{d}) \\ T(\mathtt{d}) & = & \bar{A}^{-1}(\mathtt{d})\bar{C}(\mathtt{d}).\end{array} \tag{6}$$

System (5) should have no unstable hidden modes, so the matrix couples $A(\mathtt{d})$, $B(\mathtt{d})$ and $\bar{A}(\mathtt{d})$,$\bar{B}(\mathtt{d})$ in the above MFDs are assumed to be right and left coprime, respectively, for $|\mathtt{d}| \leq 1$. System (5) is to be driven by a dynamic controller

$$u(\mathtt{d}) = -K(\mathtt{d})z(\mathtt{d}) \tag{7}$$

where rational matrix $K(\mathtt{d})$ can be expressed using polynomial MFDs as

$$\begin{array}{rcl}K(\mathtt{d}) & = & Y(\mathtt{d})X^{-1}(\mathtt{d}) \\ & = & \bar{X}^{-1}(\mathtt{d})\bar{Y}(\mathtt{d}).\end{array} \tag{8}$$

Equivalently, dynamic controller (7) can be expressed in state-space setting as

$$\begin{array}{rcl}\eta_{k+1} & = & \bar{F}\eta_k + \bar{G}z_k \\ u_k & = & \bar{H}\eta_k + \bar{J}z_k\end{array} \tag{9}$$

where

$$K(\mathtt{d}) = \bar{H}(I_r - \bar{F}\mathtt{d})^{-1}\bar{G}\mathtt{d} + \bar{J}.$$

With these notations, two different problems are studied in this paper.

Problem 1 *Find dynamic controller (9) such that linear system (1) subject to input constraints (2) is stabilized when initialized within polyhedron (3) or ellipsoid (4).*

Problem 2 *Find dynamic controller (9) and the largest polyhedron (3) or ellipsoid (4) such that linear system (1) subject to input constraints (2) is stabilized when initialized within this set.*

Obviously, Problem 2 is tougher than Problem 1. On the one hand, in Section 4 it will be shown that Problem 1 can always be cast into a convex programming problem. On the other hand, Problem 2 appears to be intractable in its most general formulation, so that tractable approximation algorithms will be described in Section 5.

3 Preliminaries

First, we need to recall some well-known and widely used technical results of linear system control theory, namely the Youla-Kučera parametrization of all stabilizing controllers [20, 36], the extended Farkas lemma for inscribing a polyhedron into another polyhedron [14] and a lemma for inscribing an ellipsoid into a polyhedron [33].

Lemma 1 (Youla-Kučera Parametrization) *Let $\hat{X}(\mathtt{d})$, $\hat{Y}(\mathtt{d})$, $\hat{\bar{X}}(\mathtt{d})$, $\hat{\bar{Y}}(\mathtt{d})$ denote a particular solution to the double polynomial Bézout identity*

$$\begin{bmatrix} \bar{X}(\mathtt{d}) & \bar{Y}(\mathtt{d}) \\ -\bar{B}(\mathtt{d}) & \bar{A}(\mathtt{d}) \end{bmatrix} \begin{bmatrix} A(\mathtt{d}) & -Y(\mathtt{d}) \\ B(\mathtt{d}) & X(\mathtt{d}) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix} \tag{10}$$

where $A(\mathtt{d})$, $B(\mathtt{d})$ and $\bar{A}(\mathtt{d})$, $\bar{B}(\mathtt{d})$ are given coprime polynomial matrices. Then all linear controllers

$$\begin{aligned} u(\mathtt{d}) &= -Y(\mathtt{d})X^{-1}(\mathtt{d})z(\mathtt{d}) \\ &= -\bar{X}^{-1}(\mathtt{d})\bar{Y}(\mathtt{d})z(\mathtt{d}) \end{aligned}$$

that stabilize the linear system

$$\begin{aligned} z(\mathtt{d}) &= B(\mathtt{d})A^{-1}(\mathtt{d})u(\mathtt{d}) \\ &= \bar{A}^{-1}(\mathtt{d})\bar{B}(\mathtt{d})u(\mathtt{d}) \end{aligned}$$

are parametrized by

$$\begin{aligned} X(\mathtt{d}) &= \hat{X}(\mathtt{d}) + B(\mathtt{d})Q(\mathtt{d}) \\ Y(\mathtt{d}) &= \hat{Y}(\mathtt{d}) - A(\mathtt{d})Q(\mathtt{d}) \\ \bar{X}(\mathtt{d}) &= \hat{\bar{X}}(\mathtt{d}) + \bar{Q}(\mathtt{d})\bar{B}(\mathtt{d}) \\ \bar{Y}(\mathtt{d}) &= \hat{\bar{Y}}(\mathtt{d}) - \bar{Q}(\mathtt{d})\bar{A}(\mathtt{d}) \end{aligned} \tag{11}$$

where $Q(\mathtt{d})$, $\bar{Q}(\mathtt{d})$ are arbitrary stable rational matrices, or converging matrix power series in $\mathtt{d}$.

Lemma 2 (Extended Farkas Lemma) *Polyhedron*

$$\mathcal{P}_N = \{\xi : N\xi \leq \nu\}$$

is included within polyhedron

$$\mathcal{P}_M = \{\xi : M\xi \leq \mu\}$$

if and only if there exists a matrix P with non-negative entries such that

$$\begin{array}{rcl} PN & = & M \\ P\nu & \leq & \mu. \end{array}$$

Lemma 3 (Ellipsoid Inscribed Within Polyhedron) *Ellipsoid*

$$\begin{array}{rcl} \mathcal{E}_\Pi & = & \{\xi : (\xi - \xi_c)^T \Pi (\xi - \xi_c) \leq 1\} \\ & = & \{\Pi^{-\frac{1}{2}}\eta + \xi_c : \|\eta\| \leq 1\} \end{array}$$

is included within polyhedron

$$\mathcal{P}_M = \{\xi : M\xi \leq \mu\}$$

if and only if

$$\|\Pi^{-\frac{1}{2}}(M^i)^T\| + M^i\xi_c \leq \mu^i, \quad i = 1, 2, \ldots \tag{12}$$

where M^i and μ^i stand for the i-th row in matrix M and vector μ, respectively. Using a standard Schur complement argument [33], inequality (12) can equivalently be expanded into matrix form as follows

$$\left[\begin{array}{cc} (\mu^i - M^i\xi_c)\Pi & \star \\ M^i & \mu^i - M^i\xi_c \end{array}\right] \succeq 0, \quad i = 1, 2, \ldots \tag{13}$$

where a star denotes a symmetric block. When ellipsoid $\mathcal{E}_\Pi$ is centered about the origin, i.e. $\xi_c = 0$, the above matrix inequality becomes

$$\left[\begin{array}{cc} \Pi & \star \\ M^i & (\mu^i)^2 \end{array}\right] \succeq 0, \quad i = 1, 2, \ldots \tag{14}$$

4 Stabilization

In this section, we show that Problem 1 can be cast into standard convex optimization problems that can be solved with powerful and widely spread interior-point algorithms.

From equations (5) and (7), it follows that

$$\begin{array}{rcl} z(\mathtt{d}) & = & S(\mathtt{d})u(\mathtt{d}) + T(\mathtt{d})\xi_0 \\ & = & -S(\mathtt{d})K(\mathtt{d})z(\mathtt{d}) + T(\mathtt{d})\xi_0 \\ & = & [I_p + S(\mathtt{d})K(\mathtt{d})]^{-1}T(\mathtt{d})\xi_0. \end{array}$$

Thus

$$u(\mathtt{d}) = -K(\mathtt{d})[I_p + S(\mathtt{d})K(\mathtt{d})]^{-1}T(\mathtt{d})\xi_0. \tag{15}$$

Using MFDs (6), (8) and Lemma 1, we can write

$$\begin{aligned} Y(\mathtt{d})\bar{A}(\mathtt{d})[I_p + S(\mathtt{d})K(\mathtt{d})]X(\mathtt{d}) &= Y(\mathtt{d})[\bar{A}(\mathtt{d})X(\mathtt{d}) + \bar{B}(\mathtt{d})Y(\mathtt{d})] \\ &= Y(\mathtt{d}) \end{aligned}$$

so that

$$\begin{aligned} K(\mathtt{d}) &= Y(\mathtt{d})X^{-1}(\mathtt{d}) \\ &= Y(\mathtt{d})\bar{A}(\mathtt{d})[I_p + S(\mathtt{d})K(\mathtt{d})]. \end{aligned} \tag{16}$$

Upon reporting equation (16) into equation (15) and using parametrization (11), it holds

$$\begin{aligned} u(\mathtt{d}) &= -Y(\mathtt{d})\bar{A}(\mathtt{d})T(\mathtt{d})\xi_0 \\ &= -Y(\mathtt{d})\bar{C}(\mathtt{d})\xi_0 \\ &= [A(\mathtt{d})Q(\mathtt{d})\bar{C}(\mathtt{d}) - \hat{Y}(\mathtt{d})\bar{C}(\mathtt{d})]\xi_0 \end{aligned} \tag{17}$$

where $\hat{Y}(\mathtt{d})$ is a particular solution to Bézout equation (10) and $Q(\mathtt{d})$ is an arbitrary stable rational matrix of size $m \times p$.

Now let $u^j(\mathtt{d})$, $A^j(\mathtt{d})$, $Y^j(\mathtt{d})$ stand for the j-th row in vector $u(\mathtt{d})$ and matrices $A(\mathtt{d})$, $Y(\mathtt{d})$, respectively, so that

$$u^j(\mathtt{d}) = [A^j(\mathtt{d})Q(\mathtt{d})\bar{C}(\mathtt{d}) - \hat{Y}^j(\mathtt{d})\bar{C}(\mathtt{d})]\xi_0 \tag{18}$$

for $j = 1, 2, \ldots, m$. Then we use the Kronecker product [23] to write equation (18) as

$$u^j(\mathtt{d}) = [q(\mathtt{d})V^j(\mathtt{d}) + W^j(\mathtt{d})]\xi_0 \tag{19}$$

where $q^T(\mathtt{d})$ is column vector obtained by stacking together the columns of matrix $Q(\mathtt{d})$, and

$$\begin{aligned} V^j(\mathtt{d}) &= \bar{C}(\mathtt{d}) \otimes [A^j(\mathtt{d})]^T \\ W^j(\mathtt{d}) &= -\hat{Y}^j(\mathtt{d})\bar{C}(\mathtt{d}). \end{aligned} \tag{20}$$

Denote sequences $u^j(\mathtt{d})$, $q(\mathtt{d})$ as infinite power series

$$\begin{aligned} u^j(\mathtt{d}) &= u_0^j + u_1^j\mathtt{d} + u_2^j\mathtt{d}^2 + \cdots \\ q(\mathtt{d}) &= q_0 + q_1\mathtt{d} + q_2\mathtt{d}^2 + \cdots \end{aligned} \tag{21}$$

and polynomial matrices $V^j(\mathtt{d})$, $W^j(\mathtt{d})$ as

$$\begin{aligned} V^j(\mathtt{d}) &= V_0^j + V_1^j\mathtt{d} + V_2^j\mathtt{d}^2 + \cdots \\ W^j(\mathtt{d}) &= W_0^j + W_1^j\mathtt{d} + W_2^j\mathtt{d}^2 + \cdots \end{aligned} \tag{22}$$

Equating coefficients at like powers of the indeterminate $\mathtt{d}$ in relation (19) yields

$$u_i^j = \bar{M}_i^j(q)\xi_0 \tag{23}$$

where

$$\bar{M}_i^j(q) = W_i^j + \sum_{k=0,1,2\ldots} q_k V_{i-k}^j \tag{24}$$

is a row-vector parametrized by row-vectors q_k for $k = 0, 1, 2, \ldots$

Recall that input sequence $u(\mathtt{d})$ is subject to constraints (2). In view of relation (23), initial state vector ξ_0 is also constrained. This is captured by the following lemma.

Lemma 4 *Input $u(\mathtt{d})$ satisfies constraints (2) if and only if*

$$\xi_0 \in \mathcal{P}_M = \{\xi : M(q)\xi \leq \mu\} \tag{25}$$

where

$$M(q) = \begin{bmatrix} \bar{M}_0(q) \\ -\bar{M}_0(q) \\ \bar{M}_1(q) \\ -\bar{M}_1(q) \\ \vdots \end{bmatrix} \quad \bar{M}_i(q) = \begin{bmatrix} \bar{M}_i^1(q) \\ \vdots \\ \bar{M}_i^m(q) \end{bmatrix} \quad \mu = \begin{bmatrix} u^+ \\ u^- \\ u^+ \\ u^- \\ \vdots \end{bmatrix}. \tag{26}$$

Recalling the statement of Problem 1, vector ξ_0 is assumed to belong either to polyhedron (3) or to ellipsoid (4). Using Lemmas 2, 3 and 4, we can now write necessary and sufficient conditions for this problem to be solved.

Theorem 1 *Problem 1 is solved*

- *in polyhedron (3) if and only if there exist P with non-negative entries and q such that*

$$\begin{array}{rcl} PN & = & M(q) \\ P\nu & \leq & \mu \end{array} \tag{27}$$

- *in ellipsoid (4) if and only if there exists q such that*

$$\|\Pi^{-\frac{1}{2}}(M^i(q))^T\| + M^i(q)\xi_c \leq \mu^i, \quad i = 1, 2, \ldots \tag{28}$$

  where $M^i(q)$ and μ^i stand for the i-th row in matrix $M(q)$ and vector μ, respectively.

A stabilizing controller is then retrieved from parameter q.

Several remarks are in order.

Remark 1 (Positive Invariance) As pointed out in the introduction, most of the existing methods to deal with local stabilization of constrained input systems hinge upon the concept of positive invariance: system trajectories, once initialized in some set, are guaranteed to stay within this set while converging to the origin. However, in this paper we do not rely on positive invariance of sets $\mathcal{P}_M$, $\mathcal{P}_N$ or $\mathcal{E}_\Pi$ to ensure stability under control constraints. On the one hand, inclusion equations (27) or (28) ensure that input u verifies the constraints and never saturates. On the other hand, stability of $Q(\mathtt{d})$ ensures stability of the closed-loop system.

Remark 2 (Maximum Positively Invariant Set) Even though our approch does not rely on positive invariance, we can still use the systematic method outlined in [11] in order to build the maximum positively invariant set for system (1) controlled by controller (9) subject to input constraints (2). Consider the extended state vector

$$\check{\xi}_k = \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix}$$

and the extended matrices

$$\check{F} = \begin{bmatrix} F + G\bar{J}H & G\bar{H} \\ \bar{G}H & \bar{F} \end{bmatrix} \quad \check{H} = \begin{bmatrix} \bar{J}H & \bar{H} \end{bmatrix}.$$

It follows that the dynamics of the closed-loop system (1) and (9) are captured by the equation

$$\check{\xi}_{k+1} = \check{F}\check{\xi}_k,$$

the system input being given by

$$u_k = \check{H}\check{\xi}_k.$$

Under the assumption that pair $(\check{F}, \check{H})$ is detectable, define the matrix

$$\check{M} = \begin{bmatrix} \check{H} \\ -\check{H} \\ \check{H}\check{F} \\ -\check{H}\check{F} \\ \check{H}\check{F}^2 \\ -\check{H}\check{F}^2 \\ \vdots \end{bmatrix}$$

The maximal set of admissible initial states $\check{\xi}_0$ such that the input constraints are satisfied is then the polyhedron

$$\check{P} = \left\{ \check{\xi} : \check{M}\check{\xi} \leq \mu \right\}.$$

By construction [11], this polyhedron is a positively invariant set for the closed-loop system (1) and (9). This means that for any initial condition $\check{\xi}_0$ chosen in $\check{P}$, the extended state vector $\check{\xi}_k$ remains in $\check{P}$ while converging to the origin and satisfying input constraints (2).

Remark 3 (Polynomial Youla-Kučera Parameter) In general, sequence $Q(\mathtt{d})$ can be expressed as an infinite stable power series. However, for practical purpose, we only consider polynomial, or finite-impulse-response $Q(\mathtt{d})$ in the sequel.

Remark 4 (Convex Programming Problem) When $Q(\mathtt{d})$ is a matrix polynomial, relations (27) become a finite-dimensional linear programming problem [34] that must be solved for a non-negative matrix P and vector coefficients q_k. Similarly, relations (28) become a finite-dimensional second-order cone programming problem [25] that must be solved for vector coefficients q_k. It follows from relation (17) that

$$\deg u \leq \max\{\deg A + \deg Q + \deg \bar{C}, \deg \hat{Y} + \deg \bar{C}\}.$$

In both convex programming problems, matrix $M(q)$ features $N = 2m(\deg u + 1)$ rows and n columns.

Remark 5 (Controller Order) As in Remark 4, we deduce from relation (11) that

$$\begin{array}{rcl} \deg X & \leq & \max\{\deg \hat{X}, \deg B + \deg Q\} \\ \deg Y & \leq & \max\{\deg \hat{Y}, \deg A + \deg Q\}. \end{array}$$

Consequently, the order of stabilizing controller (7) depends on the degree of Youla-Kučera parameter $Q(\mathtt{d})$. However, the dependence is not simple since cancellations may occur in equations (11). Moreover, any unimodular right factor shared by $X(\mathtt{d})$ and $Y(\mathtt{d})$ may artificially increase the degree of $X(\mathtt{d})$ and $Y(\mathtt{d})$ without affecting the controller order. In particular, if a static output feedback controller is expected, then it is sufficient but not necessary to restrict matrices $X(\mathtt{d})$, $Y(\mathtt{d})$ to be constant.

In view of the above remarks, we can now describe an algorithm for solving Problem 1.

Algorithm 1 (Stabilization)

Input: System (1) with input constraints (2). Polyhedron (3) or ellipsoid (4) of initial conditions that must be stabilized. Degree δ of Youla-Kučera parameter $Q(\mathtt{d})$.
Output: A stabilizing dynamic controller (9), if one exists.

Step 1: Compute polynomial matrices $A(\mathtt{d})$, $B(\mathtt{d})$, $\bar{A}(\mathtt{d})$, $\bar{B}(\mathtt{d})$, $\bar{C}(\mathtt{d})$ in MFDs (6).
Step 2: Solve Bézout equation (10) for polynomial matrices $\hat{X}(\mathtt{d})$, $\hat{Y}(\mathtt{d})$, $\hat{\bar{X}}(\mathtt{d})$, $\hat{\bar{Y}}(\mathtt{d})$.
Step 3: Assume polynomial matrix $Q(\mathtt{d})$ has degree δ. Build matrix $M(q)$ and vector μ using relations (20),(22),(24) and (26).
Step 4: If the stability region is polyhedron (3), solve linear programming problem (27) for a non-negative matrix P and a parameter q. If the stability region is ellipsoid (4), solve second-order cone programming problem (28) for a parameter q. If the problem is infeasible, conclude that there is no Youla-Kučera parameter of degree δ that solves the problem and exit.
Step 5: Report $Q(\mathtt{d})$ into relations (11) to build dynamic controller polynomial matrices $X(\mathtt{d})$, $Y(\mathtt{d})$. Derive the corresponding state-space realization (9).

Remark 6 (Increasing Controller Order) Clearly, if the above algorithm fails for some degree δ, the designer may try higher values of δ. Recalling Remark 5, this will most probably entail increasing the controller order. It is up to the designer to handle this trade-off, knowing that there is no straightforward method to find the minimum value of δ for which stability is ensured.

5 Largest Stability Regions

The control law designed in Theorem 1 is valid when the closed-loop system state is initialized in a given convex subset of the state-space. In this section, we assume that this region is not given beforehand. As stated in Problem 2, a natural objective is then to find the largest possible stability region for which a stabilizing control law can be designed. However, we shall see that this problem proves very intricate in the general case, so that we will be satisfied with an approximate, but tractable solution.

5.1 Polyhedral Regions

Assuming for simplicity that polyhedron $\mathcal{P}_M = \{\xi : M(q)\xi \leq \mu\}$ is bounded, there is unfortunately no closed formula for the volume of $\mathcal{P}_M$. However, given a fixed q, this volume may be computed by numerical algorithms featuring exponential complexity but that prove rather efficient in practice. The interested reader is referred to [7] for recent developments in this domain. Therefore, we can in principle construct a black box for computing the volume of $\mathcal{P}_M$ and feed this into a maximizer. This would be an example of an optimization problem with a very expensive function evaluation and numerical gradient computation. This possibility is not investigated in this paper.

We rather pursue a simpler and more pragmatic approach. Assuming we are given an arbitrary polyhedron $\mathcal{P}_N = \{\xi : N\xi \leq \nu\}$ as an initial guess, we try to find the largest scaling ε^{-1} such that Problem 1 is solved within homothetic polyhedron

$$\varepsilon^{-1}\mathcal{P}_N = \{\xi : N\xi \leq \varepsilon^{-1}\nu\} \subset \mathcal{P}_M = \{\xi : M(q)\xi \leq \mu\}.$$

Recalling relations (27), the resulting optimization problem turns out to be a mere linear programming problem. This is captured by the following algorithm.

Algorithm 2.1 (Largest Inscribed Homothetic Polyhedron)

Input: System (1) with input constraints (2). Initial guess polyhedron (3). Degree δ of Youla-Kučera parameter $Q(\mathtt{d})$.
Output: A stabilizing dynamic controller (9) and the corresponding polyhedron (25) of stabilizable initial conditions, if some exist.

Steps 1,2,3 and 5: See Algorithm 1.
Step 4: Solve the linear programming optimization problem

$$\begin{array}{ll} \min & \varepsilon \\ \text{s.t.} & PN = M(q) \\ & P\nu \leq \varepsilon\mu \end{array} \tag{29}$$

for a non-negative matrix P, a parameter q and a positive scalar ε. If the problem is infeasible, conclude that there is no Youla-Kučera parameter of degree δ that solves the problem and exit.

Remark 7 (Initial Guess Polyhedron) In order to find a suitable initial guess polyhedron $\mathcal{P}_N$, one can build matrix N and vector ν using the systematic method outlined in [11].

5.2 Ellipsoidal Regions

Because matrix inequality (13) is bilinear in matrices Π and $M(q)$, it is not suitable for simultaneously computing Π, ξ_c and controller parameter q. On the other hand, both Π

and $M(q)$ appear linearly in matrix inequality (14) when $\mathcal{E}_\Pi$ is centered about the origin. We will show that it is always possible to normalize input bounds, center ellipsoid $\mathcal{E}_\Pi$ and use formulation (14).

Lemma 5 *Assume first that F has no eigenvalue at 1. Then the dynamic equation with asymmetric input constraints*

$$\begin{array}{c} \xi_{k+1} = F\xi_k + Gu_k \\ -u^- \leq u_k \leq u^+ \end{array}$$

can equivalently be written as the dynamic equation with symmetric input constraints

$$\begin{array}{c} \tilde{\xi}_{k+1} = F\tilde{\xi}_k + G\tilde{u}_k \\ -\tilde{u}^+ \leq u_k \leq \tilde{u}^+ \end{array} \tag{30}$$

provided that $\tilde{\xi}_k = \xi_k - \xi_c$, $\xi_c = (I_n - F)^{-1}G\Phi$, $\Phi = (u^+ + u^-)/2$ and $\tilde{u} = (u^+ - u^-)/2$. Now if F has some eigenvalues at 1, just set $\tilde{\xi}_k = \xi_k$ and $\tilde{u} = \min(u^+, u^-)$ where the latter equality is meant componentwise.

The proof follows by using standard matrix manipulations. It is easy to show that vector ξ_c must verify $(I_n - F)\xi_c = G\Phi$. A sufficient condition for ξ_c to exist is that $I_n - F$ is non-singular, which holds if and only if F has no eigenvalue at 1.

Once system (1) is put into symmetric form (30), the inclusion of centered ellipsoid $\mathcal{E}_\Pi$ within polyhedron $\mathcal{P}_M$ holds as in Lemma 3 if and only if linear matrix inequality (14) holds. Recall that the lengths of the axes of ellipsoid $\mathcal{E}_\Pi$ are inversely proportional to the squared eigenvalues of Π. Therefore, in order to maximize the size of ellipsoid $\mathcal{E}_\Pi$ we can minimize the trace or the maximum eigenvalue of Π. This gives rise to the following algorithm.

Algorithm 2.2 (Largest Inscribed Ellipsoid)

Input: System (1) with input constraints (2). Degree δ of Youla-Kučera parameter $Q(\mathtt{d})$.
Output: A stabilizing dynamic controller (9) and the corresponding polyhedron (25) of stabilizable initial conditions, if some exist.

Step 0: If $u^+ \neq u^-$, perform input centering as in Lemma 5 and work on equivalent system (30).
Steps 1,2,3 and 5: See Algorithm 1.
Step 4: Solve either the semidefinite optimization problem

$$\begin{array}{ll} \min & \mathrm{trace}\Pi \\ \text{s.t.} & \begin{bmatrix} \Pi & \star \\ M^i(q) & (\mu^i)^2 \end{bmatrix} \succeq 0, \quad i = 1, 2, \ldots, N \end{array} \tag{31}$$

or the semidefinite optimization problem

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda I_n \succeq \Pi \\ & \begin{bmatrix} \Pi & \star \\ M^i(q) & (\mu^i)^2 \end{bmatrix} \succeq 0, \quad i = 1, 2, \ldots, N \end{array} \tag{32}$$

for a semidefinite positive matrix Π and a parameter q. If the problem is infeasible, conclude that there is no Youla-Kučera parameter of degree δ that solves the problem and exit.

Remark 8 (No Initial Guess) One significant advantage of Algorithm 2.2 over Algorithm 2.1 is that no initial guess polyhedron $\mathcal{P}_N$ is needed to start the optimization process.

6 Numerical Examples

We will now illustrate the ideas exposed in the paper on several numerical examples found in the control literature. In the sequel, all the computations involving polynomials and polynomial matrices were readily performed with the Polynomial Toolbox Version 2.0 for MATLAB [26]. All the linear, second-order cone and semidefinite programming problems were implemented with the user-friendly interface LMITOOL 2.0 for MATLAB [10] and solved with the interior-point methods of the SDPHA package 3.0 [6]. Vertices of polytope $\mathcal{P}_M$ were very efficiently computed with the software QHULL [1]. Finally, closed-loop simulations were performed on Simulink 2.0 for MATLAB.

6.1 First Example and Algorithm 1

As a simple first illustrative example, we consider the SISO discrete-time linear system studied in [32]

$$\xi_{k+1} = \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix} \xi_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

when only the first component of the state vector is available for feedback, i.e.,

$$z_k = [1 \quad 0]\xi_k.$$

System input is constrained to

$$-7 \leq u_k \leq 7$$

and initial state vector ξ_0 is assumed to belong to convex polyhedron $\mathcal{P}_N = \{\xi : N\xi \leq \nu\}$ where

$$N = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ -1.5 & 2 \\ 1.5 & -2 \end{bmatrix} \quad \nu = \begin{bmatrix} 5 \\ 5 \\ 10 \\ 10 \end{bmatrix}.$$

We aim at solving Problem 1, i.e. finding a stabilizing feedback controller for the above system.

First, it is easily found that

$$\begin{aligned} A(\mathtt{d}) &= 0.8621 - 1.7241\mathtt{d} + \mathtt{d}^2 \\ B(\mathtt{d}) &= 0.4310\mathtt{d}^2 \\ \bar{C}(\mathtt{d}) &= [0.8621 - 1.0345\mathtt{d} \quad 0.4310\mathtt{d}^2] \end{aligned}$$

in MFDs (6). A particular solution to Bézout equation (10) is then

$$\begin{aligned}\hat{X}(\mathtt{d}) &= 1.1600 + 2.3200\mathtt{d}\\ \hat{Y}(\mathtt{d}) &= 6.5888 - 5.3824\mathtt{d}.\end{aligned}$$

Recalling Remark 3, we assume that $Q(\mathtt{d})$ is a polynomial of degree δ. Using Theorem 1, we find that the lowest degree for which linear programming problem (27) turns out to be feasible is $\delta = 3$. Matrices of the linear programming problem read

$$P = \begin{bmatrix} 0 & 0.4667 & 0.4667 & 0\\ 0.4667 & 0 & 0 & 0.4667\\ 0.3076 & 0.1060 & 0 & 0.4932\\ 0 & 0.2015 & 0.5462 & 0.0530\\ 0.0192 & 0 & 0 & 0.1338\\ 0 & 0.0192 & 0.1338 & 0\\ 0.4116 & 0 & 0 & 0.4942\\ 0 & 0.4116 & 0.4942 & 0\\ 0.1891 & 0 & 0 & 0\\ 0 & 0.1891 & 0 & 0\\ 0 & 0 & 0.2742 & 0\\ 0 & 0 & 0 & 0.2742\\ 0 & 0.2986 & 0.5247 & 0\\ 0.2986 & 0 & 0 & 0.5247 \end{bmatrix} \quad M(q) = \begin{bmatrix} -1.1667 & 0\\ 1.1667 & 0\\ 0.9414 & -0.5833\\ -0.9414 & 0.5833\\ 0.2199 & -0.2293\\ -0.2199 & 0.2293\\ 1.1529 & -0.1653\\ -1.1529 & 0.1653\\ 0.1891 & 0.3781\\ -0.1891 & -0.3781\\ -0.4112 & 0.5483\\ 0.4112 & -0.5483\\ -1.0857 & 0.4524\\ 1.0857 & -0.4524 \end{bmatrix}$$

for polynomial

$$Q(\mathtt{d}) = 6.0731 + 5.2855\mathtt{d} + 3.0815\mathtt{d}^2 + 1.0495\mathtt{d}^3$$

and a vector μ whose entries are all equal to 7. Reporting $Q(\mathtt{d})$ into relations (11), a stabilizing output controller is given by polynomials

$$\begin{aligned}X(\mathtt{d}) &= 1.1600 + 2.3200\mathtt{d} + 2.6177\mathtt{d}^2 + 2.2783\mathtt{d}^3 + 1.3282\mathtt{d}^4 + 0.4524\mathtt{d}^5\\ Y(\mathtt{d}) &= 1.3533 + 0.5320\mathtt{d} + 0.3834\mathtt{d}^2 - 0.8773\mathtt{d}^3 - 1.2721\mathtt{d}^4 - 1.0495\mathtt{d}^5.\end{aligned}$$

The resulting closed-loop system is guaranteed to be stable for any initial condition chosen within polyhedron $\mathcal{P}_N$. The system input is represented in Figure 1 when the system state is initialized to

$$\xi_0 = \begin{bmatrix} 6\\ -0.5 \end{bmatrix} \in \mathcal{P}_N.$$

For comparison, Figure 2 shows the requested polyhedron of stabilizable initial conditions $\mathcal{P}_N$ together with the actual polyhedron of stabilizable initial conditions $\mathcal{P}_M$ corresponding to the controller found above. In contrast to the results achieved in [32], our approach does not require that polyhedron $\mathcal{P}_N$ be positively invariant. Indeed, system stability is ensured by the mere inclusion of polyhedron $\mathcal{P}_N$ into polyhedron $\mathcal{P}_M$. Another significant advantage of our approach with respect to that proposed in [32] is that we can design dynamic output feedback controllers. On the other hand, we may come up with controllers of relatively high order.

6.2 Second Example

We now illustrate on a very simple numerical example that the choice of parameter $Q(\mathtt{d})$ may influence the size of the polyhedron of stabilizable initial conditions.

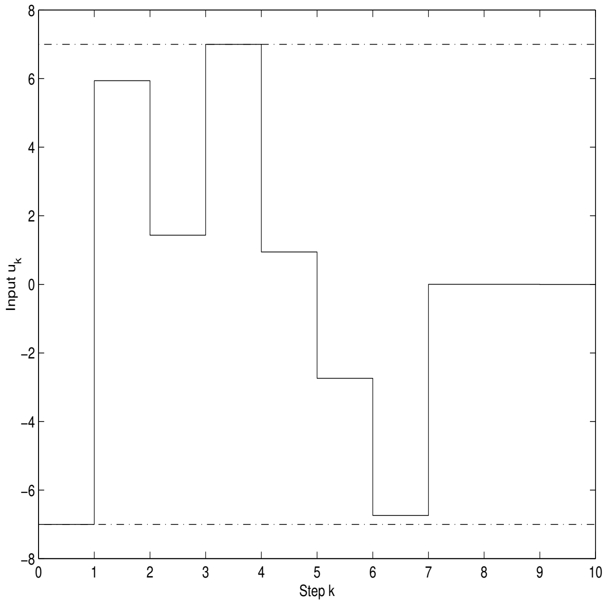

Figure 1: System input.

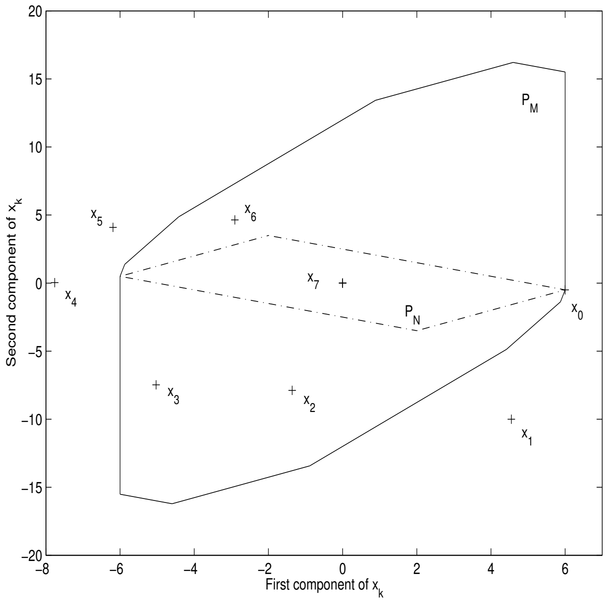

Figure 2: Polyhedra $\mathcal{P}_M$ and $\mathcal{P}_N$ with system state ξ_k.

Consider the unstable discrete-time system

$$\begin{aligned} \xi_{k+1} &= 2\xi_k + u_k \\ z_k &= \xi_k \end{aligned}$$

where system input is constrained to

$$-1 \leq u_k \leq 1.$$

Assume that the polyhedron of admissible initial conditions is given by

$$\mathcal{P}_N = \{\xi : -1/\alpha \leq \xi \leq 1/\alpha\}$$

where α is a positive scalar.

The system polynomials are readily computed as $A(\mathtt{d}) = 1 - 2\mathtt{d}$, $B(\mathtt{d}) = 1$ and $\bar{C}(\mathtt{d}) = 1$. A particular solution to Bézout equation (10) is $\hat{X}(\mathtt{d}) = 0$ and $\hat{Y}(\mathtt{d}) = 1$.

Then, we successively build linear programming problem (27) for increasing values of δ, the degree of polynomial $Q(\mathtt{d})$. We can easily find parameters Q_i such that the threshold α is minimized and hence the size of polyhedron $\mathcal{P}_N$ is maximized. In Table 1, we reported values of the Q_is and the minimum achievable α for increasing values of δ. When degree δ tends to infinity, $Q(\mathtt{d})$ tends to a converging infinite sequence and threshold α tends to a finite value, namely 1. This is not surprising since it is well-known that in general stabilization of an open-loop unstable system with a bounded control can be achieved only locally.

δ	α	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5
–	2	0	0	0	0	0	0
0	4/3	2/3	0	0	0	0	0
1	8/7	6/7	4/7	0	0	0	0
2	16/15	14/15	4/15	8/15	0	0	0
3	32/31	30/31	28/31	24/31	16/31	0	0
4	64/63	62/63	60/63	56/63	48/63	32/63	0
5	128/127	126/127	124/127	120/127	112/127	96/127	64/127

Table 1 – Threshold α and parameters Q_i for increasing values of degree δ.

The example clearly illustrates the fact that the size of the domain of stabilizable initial conditions can be enlarged using the degrees of freedom inherent to the approach.

6.3 Third Example and Algorithm 1

Consider the MIMO discrete-time system studied in [15]. Its state-space model reads

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} 1.7 & -3.3 \\ 1.3 & 0.3 \end{bmatrix} \xi_k + \begin{bmatrix} 3 & 2 \\ -2 & 2 \end{bmatrix} u_k \\ z_k &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi_k \end{aligned}$$

where the control vector is subject to the following constraints

$$-\begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \leq u_k \leq \begin{bmatrix} 1.0 \\ 1.4 \end{bmatrix}$$

and the initial state vector ξ_0 is assumed to belong to the polyhedron

$$\mathcal{P}_N = \{\xi : \underbrace{\begin{bmatrix} -0.076 & 0.536 \\ -0.544 & 0.384 \\ 0.076 & -0.536 \\ 0.544 & -0.384 \end{bmatrix}}_{N} \xi \leq \underbrace{\begin{bmatrix} 1.0 \\ 1.4 \\ 1.5 \\ 1.5 \end{bmatrix}}_{\nu} \}.$$

We aim at solving Problem 1, i.e. determining a state-feedback compensator such that the closed-loop system is stabilized for any initial condition chosen in $\mathcal{P}_N$. Following Algorithm 1, the polynomial matrices in MFD (6) read

$$\begin{aligned} A(\mathtt{d}) &= \begin{bmatrix} -0.0667 + \mathtt{d} & -0.2667 \\ 0.6937 & -0.3500 + \mathtt{d} \end{bmatrix} & B(\mathtt{d}) &= \begin{bmatrix} 1.188\mathtt{d} & -1.500\mathtt{d} \\ 1.521\mathtt{d} & -0.1667\mathtt{d} \end{bmatrix} \\ \bar{A}(\mathtt{d}) &= \begin{bmatrix} -0.0625 + \mathtt{d} & -0.6875 \\ 0.2708 & -0.3542 + \mathtt{d} \end{bmatrix} & \bar{B}(\mathtt{d}) &= \begin{bmatrix} 1.1875\mathtt{d} & -1.5000\mathtt{d} \\ 1.5208\mathtt{d} & -0.1667\mathtt{d} \end{bmatrix}. \end{aligned}$$

Particular solutions to the double Bézout identity (10) are given by

$$\begin{aligned} \hat{X}(\mathtt{d}) &= \begin{bmatrix} -1.7000 & 3.3000 \\ -1.3000 & -0.3000 \end{bmatrix} & \hat{Y}(\mathtt{d}) &= \begin{bmatrix} 0.8000 & 0.4800 \\ -0.5000 & 2.5800 \end{bmatrix} \\ \hat{\bar{X}}(\mathtt{d}) &= \begin{bmatrix} -1.6800 & 1.2800 \\ -3.3300 & -0.3200 \end{bmatrix} & \hat{\bar{Y}}(\mathtt{d}) &= \begin{bmatrix} 0.8000 & 0.4800 \\ -0.5000 & 2.5800 \end{bmatrix}. \end{aligned}$$

For a first degree ($\delta = 1$) parameter matrix

$$Q(\mathtt{d}) = \begin{bmatrix} -0.5017 + 0.1654\mathtt{d} & 0.7760 + 0.3677\mathtt{d} \\ -0.7751 - 0.4976\mathtt{d} & -0.5365 + 0.2719\mathtt{d} \end{bmatrix}$$

linear programming problem (27) admits a non-negative solution

$$P = \begin{bmatrix} 0.9818 & 0.0008 & 0.0013 & 0.0086 \\ 0.0008 & 0.0120 & 0.9813 & 0.0042 \\ 0.3491 & 0.0006 & 0.0021 & 0.4308 \\ 0.0324 & 0.4591 & 0.3794 & 0.0289 \\ 0.0211 & 0.0268 & 0.3962 & 0.1384 \\ 0.4435 & 0.2315 & 0.0684 & 0.1198 \\ 0.0034 & 0.9746 & 0.0189 & 0.0008 \\ 0.0194 & 0.0014 & 0.0039 & 0.9752 \\ 0.8933 & 0.0850 & 0.0286 & 0.0742 \\ 0.0105 & 0.0255 & 0.8752 & 0.0364 \\ 0.6855 & 0.0291 & 0.0227 & 0.3142 \\ 0.0081 & 0.2936 & 0.6709 & 0.0085 \end{bmatrix}$$

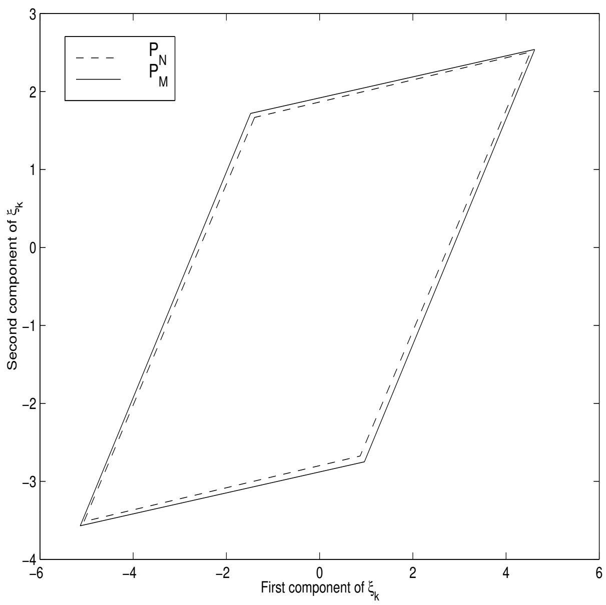

Figure 3: Polyhedra $\mathcal{P}_N$ and $\mathcal{P}_M$ obtained with Algorithm 1.

leading to polyhedron

$$\mathcal{P}_M = \{\xi : \underbrace{\begin{bmatrix} -0.0703 & 0.5226 \\ -0.5285 & 0.3656 \\ 0.0703 & -0.5226 \\ 0.5285 & -0.3656 \\ 0.2076 & 0.0208 \\ -0.0716 & 0.4676 \\ -0.2076 & -0.0208 \\ 0.0716 & -0.4676 \\ 0.0892 & -0.2439 \\ 0.1047 & 0.2458 \\ -0.0892 & 0.2439 \\ -0.1047 & -0.2458 \end{bmatrix}}_{M(q)} \xi \leq \underbrace{\begin{bmatrix} 1.0 \\ 1.4 \\ 1.5 \\ 1.5 \\ 1.0 \\ 1.4 \\ 1.5 \\ 1.5 \\ 1.0 \\ 1.4 \\ 1.5 \\ 1.5 \end{bmatrix}}_{\mu} \}.$$

Polyhedra $\mathcal{P}_N$ and $\mathcal{P}_M$ are represented in Figure 3. One can check that $\mathcal{P}_N \subset \mathcal{P}_M$.

Reporting matrix $Q(\mathtt{d})$ into relations (11), polynomial matrices of the controller right MFD are given by

$$\begin{aligned} X(\mathtt{d}) &= \begin{bmatrix} -1.7000 + 0.5668\mathtt{d} + 0.9427\mathtt{d}^2 & 3.3000 + 1.7262\mathtt{d} + 0.0287\mathtt{d}^2 \\ -1.300 - 0.6338\mathtt{d} + 0.3345\mathtt{d}^2 & -0.3000 + 1.2696\mathtt{d} + 0.5139\mathtt{d}^2 \end{bmatrix} \\ Y(\mathtt{d}) &= \begin{bmatrix} 0.5599 + 0.3801\mathtt{d} - 0.1654\mathtt{d}^2 & 0.3887 - 0.6790\mathtt{d} - 0.3677\mathtt{d}^2 \\ -0.4232 + 0.4862\mathtt{d} + 0.4976\mathtt{d}^2 & 1.8539 + 0.3766\mathtt{d} - 0.2719\mathtt{d}^2 \end{bmatrix}. \end{aligned}$$

One can check that this is a fourth order controller. The input sequence (17) is a polynomial vector of degree two given by

$$u(\mathtt{d}) = \begin{bmatrix} -0.0703 + 0.2076\mathtt{d} + 0.0892\mathtt{d}^2 & 0.5226 + 0.0208\mathtt{d} - 0.2439\mathtt{d}^2 \\ -0.5285 - 0.0716\mathtt{d} + 0.1047\mathtt{d}^2 & 0.3656 + 0.4676\mathtt{d} + 0.2458\mathtt{d}^2 \end{bmatrix} \xi_0$$

which means that any initial condition ξ_0 chosen in $\mathcal{P}_M$ is steered to the origin in three steps at most. For comparison, the same performance was achieved in [15] with a complex variable regulating scheme.

6.4 Fourth Example and Algorithms 2.1 and 2.2

As a fourth illustrative example, we consider the MIMO discrete-time linear system studied in [32]. Its state-space model reads

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix} \xi_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ z_k &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi_k. \end{aligned}$$

Note that this system was studied in the first example when only the first component of the state vector was available for feedback. System input is constrained to

$$-7 \leq u_k \leq 7.$$

Suppose that we aim at solving Problem 2, i.e. both a stabilizing controller and the largest region of stabilizable initial conditions are sought.

First, we illustrate Algorithm 2.1 and seek the largest homothetic polyhedron of stabilizable initial conditions. As in [32] we select

$$\mathcal{P}_N = \{\xi : \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ -1.5 & 2 \\ 1.5 & -2 \end{bmatrix}}_{N} \xi \leq \underbrace{\begin{bmatrix} 5 \\ 5 \\ 10 \\ 10 \end{bmatrix}}_{\nu} \}$$

as an initial guess polyhedron. Open-loop polynomial matrices in relations (6) read

$$\begin{aligned} A(\mathtt{d}) &= 0.8621 - 1.7241\mathtt{d} + \mathtt{d}^2 & B(\mathtt{d}) &= \begin{bmatrix} 0.4310\mathtt{d}^2 \\ 0.8621\mathtt{d} - 0.6897\mathtt{d}^2 \end{bmatrix} \\ \bar{A}(\mathtt{d}) &= \begin{bmatrix} -1.0345 + \mathtt{d} & 0.4310\mathtt{d} \\ -0.3448 & -0.6897 + \mathtt{d} \end{bmatrix} & \bar{B}(\mathtt{d}) &= \begin{bmatrix} 0.4310\mathtt{d} \\ -0.6897\mathtt{d} \end{bmatrix}. \end{aligned}$$

Particular solutions to the double Bézout identity (10) are given by

$$\begin{aligned} \hat{X}(\mathtt{d}) &= \begin{bmatrix} -0.8000 - 0.4400\mathtt{d} & -0.5000 - \mathtt{d} \\ 0.4000 + 0.7040\mathtt{d} & -1.2000 + 1.6000\mathtt{d} \end{bmatrix} \\ \hat{Y}(\mathtt{d}) &= \begin{bmatrix} 0.0960 + 1.0208\mathtt{d} & -2.8400 + 2.3200\mathtt{d} \end{bmatrix} \\ \hat{\bar{X}}(\mathtt{d}) &= 1.1600 \\ \hat{\bar{Y}}(\mathtt{d}) &= \begin{bmatrix} 1.0208 & 2.3200 \end{bmatrix}. \end{aligned}$$

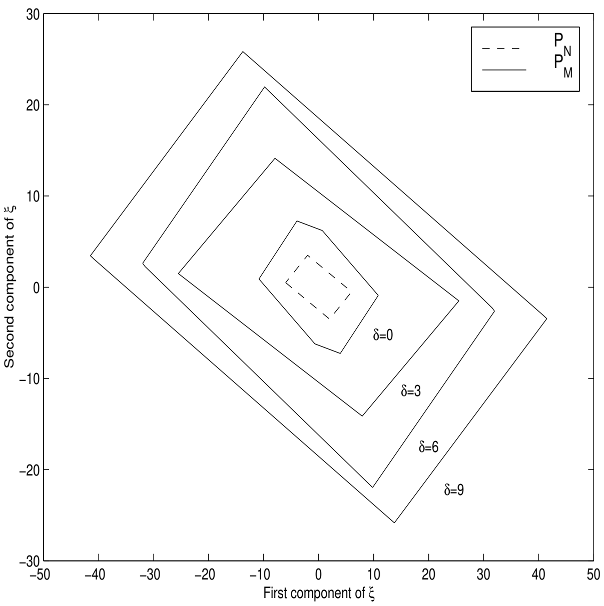

Figure 4: Stability polyhedra $\mathcal{P}_M$ for increasing values of δ obtained with Algorithm 2.1.

Linear programming optimization problem (29) is solved for increasing values of δ, the degree of Youla-Kučera polynomial matrix $Q(\mathtt{d})$. The corresponding stability polyhedra $\mathcal{P}_M$ are represented in Figure 4. As can be seen, we considerably enlarged the set of stabilizable initial conditions found in [32]. Of course, this was done at the price of a higher order controller.

Second, we illustrate Algorithm 2.2 and seek the largest ellipsoid of stabilizable initial conditions. We carry out the semidefinite programming optimization problem of Algorithm 2.2 for increasing values of δ. Note that input constraints are symmetric so that there is no need to perform input centering Step 0. The corresponding stability polyhedra $\mathcal{P}_M$ are represented in Figure 5. Here also, we note a significant improvement over the results achieved in [32]. The closed-loop system is then simulated for $\delta = 9$ and three initial conditions ξ_0. As can be seen in Figure 6, input constraints are always satisfied.

Finally, in order to illustrate the point raised in Remark 5, we try to minimize the degree of polynomial matrices $X(\mathtt{d})$ and $Y(\mathtt{d})$ in equations (11), hoping that the order of the resulting controller be minimized as well. We obtained a first-order controller whose right MFD polynomial matrices are given by

$$\begin{aligned} X(\mathtt{d}) &= \begin{bmatrix} -0.8000 - 0.4400\mathtt{d} - 0.1510\mathtt{d}^2 & -0.5000 - \mathtt{d} - 0.5925\mathtt{d}^2 \\ 0.4000 + 0.4019\mathtt{d} + 0.2416\mathtt{d}^2 & -1.200 + 0.4149\mathtt{d} + 0.9481\mathtt{d}^2 \end{bmatrix} \\ Y(\mathtt{d}) &= \begin{bmatrix} 0.3981 + 0.4167\mathtt{d} + 0.3504\mathtt{d}^2 & -1.6549 - 0.0510\mathtt{d} + 1.3747\mathtt{d}^2 \end{bmatrix}. \end{aligned}$$

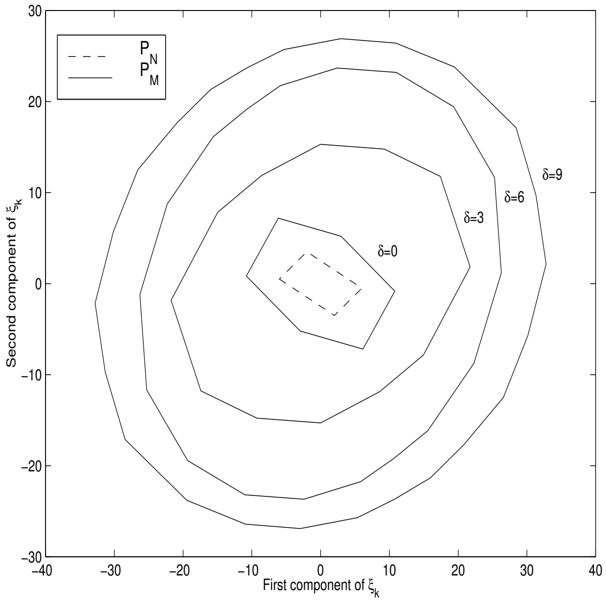

Figure 5: Stability polyhedra $\mathcal{P}_M$ for increasing values of δ obtained with Algorithm 2.2.

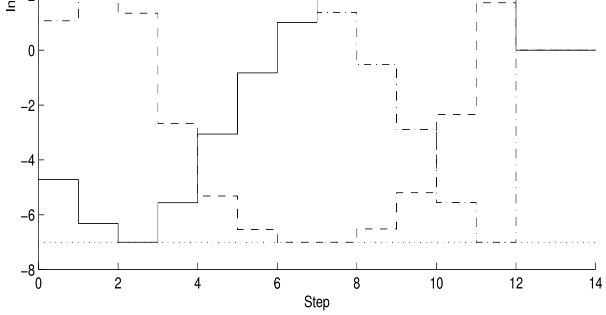

Figure 6: Input profile for $\delta = 9$ and various initial conditions.

6.5 Fifth Example and Algorithms 1 and 2.2

Consider the MIMO discrete-time linear system studied in [2]. Its state-space model reads

$$\begin{aligned}\xi_{k+1} &= \begin{bmatrix} 1.4 & -0.3 \\ -1.2 & 0.7 \end{bmatrix} \xi_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_k \\ z_k &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi_k\end{aligned}$$

where input constraints are as follows

$$-0.5 \leq u_k \leq 5.$$

First, we suppose as in [2] that the initial state vector ξ_0 belongs to the polyhedron

$$\mathcal{P}_N = \{\xi : \underbrace{\begin{bmatrix} -1.0 & 0.2 \\ 0.5 & -1.1 \\ 1.0 & -0.2 \\ -0.5 & 1.1 \end{bmatrix}}_{N} \xi \leq \underbrace{\begin{bmatrix} 5 \\ 2 \\ 0.5 \\ 10 \end{bmatrix}}_{\nu}.$$

We aim at solving Problem 1, i.e. finding a controller that stabilizes any initial condition picked up from $\mathcal{P}_N$. Following Algorithm 1, we get

$$\begin{aligned}A(\mathtt{d}) &= 1.613 - 3.387\mathtt{d} + \mathtt{d}^2 & B(\mathtt{d}) &= \begin{bmatrix} 1.613\mathtt{d} - 0.6452\mathtt{d}^2 \\ -1.613\mathtt{d} + 0.3226\mathtt{d}^2 \end{bmatrix} \\ \bar{A}(\mathtt{d}) &= \begin{bmatrix} -1.129 + \mathtt{d} & -0.4839 \\ -1.935 & -2.258 + \mathtt{d} \end{bmatrix} & \bar{B}(\mathtt{d}) &= \begin{bmatrix} -0.6452\mathtt{d} \\ 0.3226\mathtt{d} \end{bmatrix}\end{aligned}$$

in MFDs (6). Particular solutions to the double Bézout identity (10) are given by

$$\begin{aligned}\hat{X}(\mathtt{d}) &= \begin{bmatrix} -1.4000 + 0.4000\mathtt{d} & 0.3000 - 0.4400\mathtt{d} \\ 1.2000 - 0.2000\mathtt{d} & -0.7000 + 0.2200\mathtt{d} \end{bmatrix} \\ \hat{Y}(\mathtt{d}) &= \begin{bmatrix} -2.7200 + 0.6200\mathtt{d} & 1.0700 - 0.6820\mathtt{d} \end{bmatrix} \\ \hat{\bar{X}}(\mathtt{d}) &= 0.5200 + 0.6654\mathtt{d} + 0.4801\mathtt{d}^2 + 0.2420\mathtt{d}^3 + 0.0665\mathtt{d}^4 \\ \hat{\bar{Y}}(\mathtt{d}) &= \begin{bmatrix} 0.5136 + 0.6220\mathtt{d} + 0.5032\mathtt{d}^2 + 0.2813\mathtt{d}^3 + 0.0861\mathtt{d}^4 \\ -0.1230 - 0.1408\mathtt{d} - 0.1275\mathtt{d}^2 - 0.0894\mathtt{d}^3 - 0.0339\mathtt{d}^4 \end{bmatrix}.\end{aligned}$$

Then linear programming problem (27) is solved for $\delta = 3$, the degree of Youla-Kučera polynomial matrix $Q(\mathtt{d})$. The corresponding stability polyhedron $\mathcal{P}_M$ is represented in Figure 7. Polynomial matrices of the controller left MFD are given by

$$\begin{aligned}\bar{X}(\mathtt{d}) &= 0.6200 + 0.6654\mathtt{d} + 0.4801\mathtt{d}^2 + 0.2420\mathtt{d}^3 + 0.0665\mathtt{d}^4 \\ \bar{Y}(\mathtt{d}) &= \begin{bmatrix} 0.5136 + 0.6220\mathtt{d} + 0.5032\mathtt{d}^2 + 0.2813\mathtt{d}^3 + 0.0861\mathtt{d}^4 \\ -0.1230 - 0.1408\mathtt{d} - 0.1275\mathtt{d}^2 - 0.0894\mathtt{d}^3 - 0.0339\mathtt{d}^4 \end{bmatrix}.\end{aligned}$$

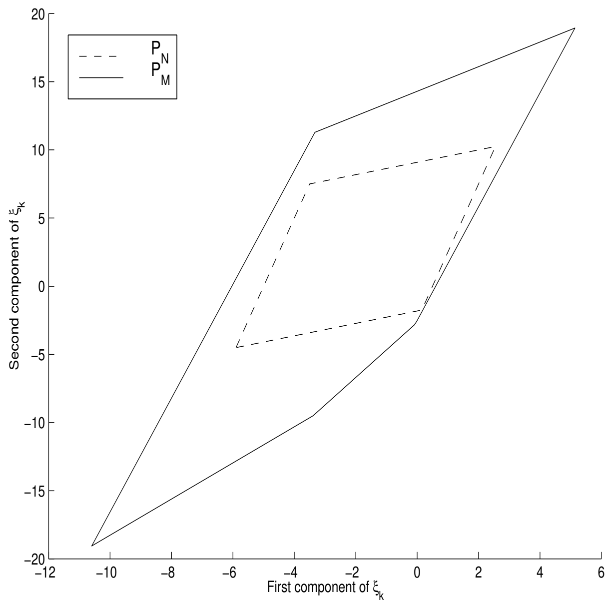

Figure 7: Stability polyhedra $\mathcal{P}_N$ and $\mathcal{P}_M$.

Then, we try to solve Problem 2 with Algorithm 2.2. We assume that polyhedron $\mathcal{P}_N$ is not given anymore and that the stability region must be found together with the stabilizing controller. Note that input constraints are not symmetric so that input centering Step 0 must be carried out. Equivalent linear system (30) reads

$$\tilde{\xi}_{k+1} = \begin{bmatrix} -1.4 & -0.3 \\ -1.2 & 0.7 \end{bmatrix} \tilde{\xi}_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tilde{u}_k$$

with symmetric input constraints

$$-2.75 \leq \tilde{u}_k \leq 2.75$$

and centering vector

$$\xi_c = \begin{bmatrix} -2.8133 \\ 3.7500 \end{bmatrix}.$$

Polynomial matrices in MFDs (6) are as follows

$$\begin{aligned} A(\mathtt{d}) &= 1.613 - 3.387\mathtt{d} + \mathtt{d}^2 & B(\mathtt{d}) &= \begin{bmatrix} 4.435\mathtt{d} - 1.774\mathtt{d}^2 \\ -4.435\mathtt{d} + 0.8871\mathtt{d}^2 \end{bmatrix} \\ \bar{A}(\mathtt{d}) &= \begin{bmatrix} -1.129 + \mathtt{d} & -0.4839 \\ -1.935 & -2.258 + \mathtt{d} \end{bmatrix} & \bar{B}(\mathtt{d}) &= \begin{bmatrix} -1.774\mathtt{d} \\ 0.8871\mathtt{d} \end{bmatrix}. \end{aligned}$$

Particular solutions to the double Bézout identity (10) are given by

$$\begin{aligned} \hat{X}(\mathtt{d}) &= \begin{bmatrix} -1.4000 + 0.4000\mathtt{d} & 0.3000 - 0.4400\mathtt{d} \\ 1.2000 - 0.2000\mathtt{d} & -0.7000 + 0.2200\mathtt{d} \end{bmatrix} \\ \hat{Y}(\mathtt{d}) &= \begin{bmatrix} -0.9891 + 0.2255\mathtt{d} & 0.3891 - 0.2480\mathtt{d} \end{bmatrix} \\ \hat{\bar{X}}(\mathtt{d}) &= 0.6200 + 0.8022\mathtt{d} + 0.8004\mathtt{d}^2 + 0.6838\mathtt{d}^3 + 0.4399\mathtt{d}^4 \\ \hat{\bar{Y}}(\mathtt{d}) &= \begin{bmatrix} 0.1411 + 0.1911\mathtt{d} + 0.2086\mathtt{d}^2 + 0.2144\mathtt{d}^3 + 0.2158\mathtt{d}^4 \\ -0.0406 - 0.0552\mathtt{d} - 0.0606\mathtt{d}^2 - 0.0629\mathtt{d}^3 - 0.0643\mathtt{d}^4 \end{bmatrix}^T. \end{aligned}$$

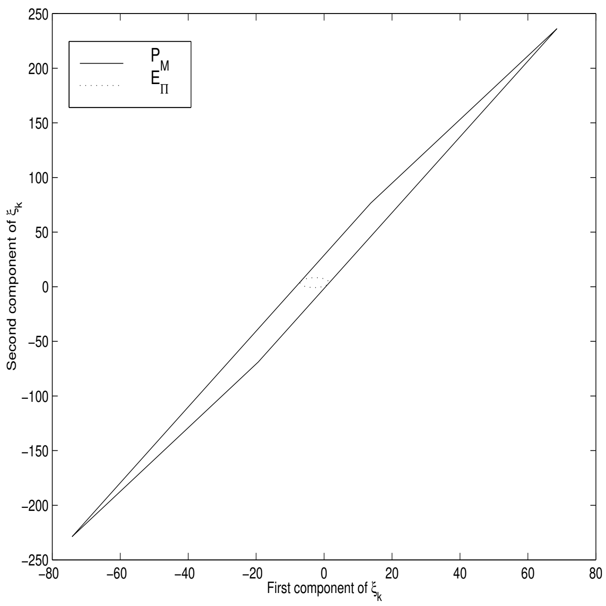

Figure 8: Stability ellipsoid $\mathcal{E}_\Pi$ with polyhedron $\mathcal{P}_M$.

Then, semidefinite programming problem (31) is solved for $\delta = 3$. We get

$$\Pi = 10^{-4} \cdot \begin{bmatrix} 555.2 & -19.95 \\ -19.95 & 491.6 \end{bmatrix}.$$

Ellipsoid $\mathcal{E}_\Pi = \{\xi : (\xi - \xi_c)^T \Pi (\xi - \xi_c) \leq 1\}$ and inscribed polyhedron $\mathcal{P}_M = \{\xi : M(q)(\xi - \xi_c) \leq \mu\}$ are represented in Figure 8. As we can see, polyhedron $\mathcal{P}_M$ is very elongated in one particular direction. Upon increasing degree δ we can show that $\mathcal{P}_M$ features an infinite direction and that the domain of stabilizable initial conditions is not bounded in this case.

7 Conclusion

We have provided a simple solution to the difficult problem of locally stabilizing a SISO discrete-time linear system subject to hard input constraints. We have proposed practical and easily implementable algorithms for computing a stabilizing control law and the largest associated stability region for linear systems subject to input constraints. The strength of our technique is that it only resorts to convex programming, in contrast to other local stabilization methods relying upon non-convex bilinear problems [31, 17, 13]. On the downside, the designer must handle the trade-off between increasing the compensator order and enlarging the region of admissible initial states. Our technique can possibly result in relatively high order compensators.

Our work hints at several directions for future research. It can be extended to

- polyhedral or ellipsoidal constraints on system state, input or output

- uncertain systems, following the techniques described in [35]
- continuous-time systems
- performance requirements around the origin and trade-off between performance and the size of the stability domain, as in [16].

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