Parabolic Set Simulation for Reachability Analysis of Linear Time Invariant Systems with Integral Quadratic Constraint

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February 26, 2019

Abstract

This work extends reachability analyses based on ellipsoidal techniques to Linear Time Invariant (LTI) systems subject to an integral quadratic constraint (IQC) between the past state and disturbance signals, interpreted as an input-output energetic constraint. To compute the reachable set, the LTI system is augmented with a state corresponding to the amount of energy still available before the constraint is violated. For a given parabolic set of initial states, the reachable set of the augmented system is overapproximated with a time-varying parabolic set. Parameters of this paraboloid are expressed as the solution of an Initial Value Problem (IVP) and the overapproximation relationship with the reachable set is proved. This paraboloid is actually supported by the reachable set on so-called touching trajectories. Finally, we describe a method to generate all the supporting paraboloids and prove that their intersection is an exact characterization of the reachable set. This work provides new practical means to compute overapproximation of reachable sets for a wide variety of systems such as delayed systems, rate limiters or energy-bounded linear systems.

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1 Introduction

We consider the reachability problem for Linear Time Invariant (LTI) with Integral Quadratic Constraints (IQC). Reachable set computation is an active field of research in control theory (see [3]). It has many applications such as state estimation (see [9]) or verification (see [2]) of dynamical systems.

IQC is a classical tool of robust control theory (see e.g. [16, 17]). It can model complex systems (infinite state dimension or non-linear dynamics) such as delays, rate limiters and uncertain systems (see [7, 15, 18] and [1]). Up to now, IQC have mainly been used to evaluate the stability of systems. Despite their modeling power, we still lack tools to manipulate such systems: computing their reachable set is challenging.

In this paper, we extend reachability analysis based on ellipsoidal techniques (see e.g. [4,12,13]) for LTI systems subject to an IQC. This IQC is a trajectory constraint (i.e. valid at any time) between past state-trajectory, input signals and unknown disturbance signals. To override dealing with constraints over the state-trajectories, we study the LTI system augmented with a state corresponding to the integral term in the IQC. For a given parabolic set of initial states, the reachable set of the augmented system is overapproximated by a time-varying parabolic set. Parameters of this paraboloid are expressed as the solution of an Initial Value Problem (IVP) partially described by a Riccati differential equation. This paraboloid is a tight overapproximation of the reachable set as it is supported by the reachable set on so-called *touching trajectories*. By studying touching trajectories that are close to violating the constraint, we find conditions to generate all the supporting time-varying parabolic sets. At a given time, the intersection of these supporting parabolic sets is an *exact* representation of the reachable set.

Related work Reachability analysis of LTI systems with ellipsoidal bounded inputs is studied in [4,12,13]. Such systems can model infinity norm bounded input-output LTI systems. The reachable set (which is convex and bounded; see [12]) can be overapproximated with time-varying ellipsoidal sets. Each ellipsoid is described by its parameters (center and radius) that are solution of an IVP. These parameters produce tight ellipsoids (i.e., ellipsoids touching the reachable set) which are external approximations of the reachable set. When multiple ellipsoids with different touching trajectories are considered, their intersection is a strictly smaller overapproximation of the reachable set. The accuracy of the overapproximation can be made arbitrarily small by adding more well chosen ellipsoids. The exact representation of the reachable set is possible by using a infinite set of ellipsoids.

An optimal control formulation of the reachable set problem is also possible [6, 14]. For some given cost function (usually linear in the case of hyperplane constraints), the maximal cost reached through the system flow for a given set of initial states defines a constraint over the reachable set: any state of the reachable set has lower cost. This optimization problem can be locally solved (see e.g. with the Pontryagin Maximum Principle -PMP-, see [5, 6, 14, 23]) leading to local description of the reachable set boundary. It also can be solved globally (using Hamilton-Jacobi-Bellman -HJB- viscosity subsolutions for example, see [22]) leading to global constraints over the reachable set. If the reachable set can be expressed as the intersection (possibly uncountable) of elements of the chosen function family, then the intersection of the resulting constraints gives an exact representation of the reachable set.

HJB and PMP based methods propagate the constraints along the flow of the dynamical system. Occupation measures and barrier certificates methods aim at finding constraints over the reachable tube of a dynamical system: [19] uses IQCs for verification purposes using barrier certificates where the positivity of the energetic state is ensured by using a nonnegative constant multiplier: [8,10] use an occupation measure approach where the integral constraint can be incorporated as a constraint over the moment of the trajectories. A hierarchy of semi-definite conditions are derived for polynomial dynamics. Then, off-the-shelf semi-definite programming solvers are used to solve the feasibility problem. Optimization-based methods do not usually take advantage of the model structure as they consider a large class of systems (convex, Lipschitz or polynomial dynamics for example).

The study of LTI systems with two norm bounded energy is closely related to the Linear Quadratic Regulator (LQR) problem. In the LQR problem, a quadratic integral is minimized at the terminal time. Optimal trajectories belong to a time-varying parabolic surface, whose quadratic coefficients are solution of a Riccati differential equation. [20] describes the reachable set of LTI systems with terminal IQC.

Contributions We study the reachable set computation of an LTI system with IQC. To the knowledge of the authors, this is the first paper to provide a set-based solution for reachable set computation for LTI systems with IQC.

- We extend the existing ellipsoidal method for reachability analysis of bounded-inputs LTI systems to reachability analysis of LTI systems with IQC. These parabolic constraints are defined by time-varying parameters which are solution of an IVP. Part of this IVP (the quadratic coefficient of the parabolic constraint) is a Riccati differential equation. The IVP convergence property is obtained thanks to the convergence property of the Riccati differential equation.

- These parabolic constraints are external approximations of the reachable set. The use of parabolic set is particularly suited to the system of interest: the approximation is tight in the sense that each constraint stays in contact with the boundary of the reachable set. We exhibit these touching trajectories. Under some conditions, the reachable set is exactly described by the intersection of well chosen time-varying paraboloid.

Plan The LTI system with temporal IQC and the reachability analysis problem are introduced (Section 1.1). Parabolic constraints and their associated parameter IVP are defined, their domain of definition is analyzed, the overapproximation property is formulated, as well as the touching trajectories (Section 2). A method to generate a set of time-varying parabolic constraints is described. The intersection of these paraboloids exactly describes the reachable set of the system (Section 3). An example of a stable system is described (Section 4).

1.1 Notation

Let $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ denote the set of real valued symmetric square matrices of size n. For $A \in \mathbb{S}^n$, we write $A \succ 0$ (resp. $A \prec 0$) when A is positive definite (resp. negative definite). We define the matrix norm $\|A\| = \sqrt{\mathrm{tr}(A^\top A)}$ for $A \in \mathbb{R}^{n\times m}$, where $\mathrm{tr}(B)$ is the trace of $B \in \mathbb{R}^{n\times n}$. Let a n-vector valued *signal* be a function that associates to a time instant in $[0, +\infty[$ a vector from $\mathbb{R}^n$. For a given interval $I \subseteq \mathbb{R}$, let $\mathbf{L}_2(I; \mathbb{R}^n)$ denote the Hilbert space of signals equipped with the norm: $\|\mathbf{u}\| = \sqrt{\int_{t\in I} \mathbf{u}^T(t)\mathbf{u}(t)dt} < \infty$. For a set $\Omega \subset \mathbb{R}^n$, let $\partial\Omega$ denote its boundary. Let $\mathscr{C}^1(I; \mathbb{R}^n)$ the set of functions from I to $\mathbb{R}^n$ which are continuous and differentiable with continuous derivative.

1.2 System

For a given input signal $\mathbf{u} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^p)$, given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_u \in \mathbb{R}^{n\times p}$, and a given terminal time $t > 0$, we study the trajectories

$\mathbf{x} \in \mathbf{L}_2([0,t];\mathbb{R}^n)$ of the LTI system:

$$\begin{cases} \dot{\mathbf{x}}(\tau) = A\mathbf{x}(\tau) + B\mathbf{w}(\tau) + B_u\mathbf{u}(\tau) \quad \text{with } \tau \in [0,t] \\ \mathbf{x}(0) = x_0 \end{cases} \tag{1}$$

where $\mathbf{w} \in \mathbf{L}_2([0,t];\mathbb{R}^m)$ is an unknown disturbance that satisfies:

$$x_{q0} + \int_0^\tau \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix}^\top M \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix} ds \geq 0 \text{ for all } \tau \in [0,t] \tag{2}$$

for given initial conditions $(x_0, x_{q0}) \in \mathbb{R}^n \times \mathbb{R}^+$, and given matrix

$$M = \begin{bmatrix} M_x & M_{xu} & M_{xw} \\ M_{xu}^\top & M_u & M_{uw} \\ M_{xw}^\top & M_{uw}^\top & M_w \end{bmatrix} \in \mathbb{S}^{n+m+p} \tag{3}$$

with $M_w \prec 0$.

In this work, the constraint (2) is expressed as a constraint over a state $\mathbf{x_q} \in \mathbf{L}_2([0,t];\mathbb{R})$ defined for $s \in [0,t]$ by:

$$\mathbf{x_q}(\tau) = x_{q0} + \int_0^\tau \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix}^\top M \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix} ds, \tag{4}$$

then

$$\mathbf{x_q}(\tau) \geq 0 \text{ for all } \tau \in [0,t]. \tag{5}$$

The constrained dynamical system $\mathscr{S}(\mathcal{X}_0, t)$ is then defined for a given set of initial states $\mathcal{X}_0 \subset \mathbb{R}^n \times \mathbb{R}$ and a terminal time $t > 0$:

$$(\mathbf{x}, \mathbf{x_q}) \in \mathscr{S}(\mathcal{X}_0, t) \Leftrightarrow \begin{cases} \mathbf{x} \text{ is solution of (1)} \\ \text{and } \mathbf{x_q} \text{ is solution of (4)} \\ \text{with } (x_0, x_{q0}) \in \mathcal{X}_0 \\ \mathbf{x_q} \text{ satisfies (5)} \end{cases} \tag{6}$$

Let the reachable set be defined by:

$$\mathcal{R}(\mathcal{X}_0, t) = \left\{ (\mathbf{x}(t), \mathbf{x_q}(t)) | (\mathbf{x}, \mathbf{x_q}) \in \mathscr{S}(\mathcal{X}_0, t) \right\}. \tag{7}$$

Then, $\mathcal{R}(\mathcal{X}_0, t) \subseteq \mathcal{X}_+$ where $\mathcal{X}_+$ is the subset of the state-space where the constraint $x_q \geq 0$ is satisfied:

$$\mathcal{X}_+ = \mathbb{R}^n \times \mathbb{R}^+.$$

1.3 Paraboloids

We overapproximate the reachable set $\mathcal{R}(\mathcal{X}_0, t)$ of $\mathscr{S}(\mathcal{X}_0, t)$ with *paraboloids*:

Definition 1 (Paraboloid)**.** *Given $(E, f, g) \in \mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R}$, define the value function:*

$$\begin{aligned} h: \; \mathbb{R}^n \times \mathbb{R} \; &\to \mathbb{R} \\ (x, x_q) \; &\mapsto x^\top E x - 2f^\top x + g + x_q, \end{aligned}$$

and the paraboloid:

$$\mathcal{P}(E, f, g) = \left\{ (x, x_q) \in \mathbb{R}^{n+1} \middle| h(x, x_q) \leq 0 \right\}.$$

Let $\mathbb{P} = \{\mathcal{P}(E, f, g) | E \in \mathbb{S}^n, f \in \mathbb{R}^n, g \in \mathbb{R}\}$ be the set of paraboloids.

Definition 2 (Scaled Paraboloid)**.** *For $\mathcal{P} \in \mathbb{P}$ with parameters (E, f, g) and a scaling factor $\gamma > 0$, let $\gamma\mathcal{P} \in \mathbb{P}$ be the scaled paraboloid defined by parameters $(\gamma E, \gamma f, \gamma g)$.*

Scaled paraboloids satisfy the following:

Property 1. *Given $\mathcal{P} \in \mathbb{P}$ and $\gamma \geq 1$, it holds $\mathcal{P} \cap \mathcal{X}_+ \subseteq \gamma\mathcal{P} \cap \mathcal{X}_+$.*

Proof: Let h and h' (resp.) the value functions of $(E, f, g) = \mathcal{P}$ and $\gamma\mathcal{P}$ (resp.) evaluated at $(x, x_q) \in \mathcal{P}$. Since $(x, x_q) \in \mathcal{P}$, $h \leq 0$, i.e. $x^\top E x - 2f^\top x + g \leq -x_q$. Then, $h' = \gamma(x^\top E x - 2f^\top x + g) + x_q \leq -(\gamma - 1)x_q$. Since $(x, x_q) \in \mathcal{X}_+$ and since $\gamma - 1 \geq 0$, we have $(\gamma - 1)x_q \geq 0$ i.e. $h' \leq 0$ meaning that $(x, x_q) \in \gamma\mathcal{P} \cap \mathcal{X}_+$. $\square$

For P a time-dependent subset of $\mathcal{X}_+$ that associates to a time t of an interval $I \subset \mathbb{R}^+$ a subset of $P(t)$ of $\mathcal{X}_+$, we define a *touching trajectory*:

Definition 3 (Touching Trajectory)**.** *A trajectory $\mathbf{X}^*$ solution of (1,4) is a touching trajectory of P when $\mathbf{X}^*(t)$ belongs to the surface of $P(t)$ at any time $t \in I$, i.e. $\mathbf{X}^*(t) \in \partial P(t)$.*

1.4 Problem Statement

We are now ready to state the problems which are respectively solved in Theorem 1 (in Section 2) and Theorem 2 (in Section 3), the main results of our paper.

Problem 1. *Find an overapproximation of the reachable set $\mathcal{R}(\mathcal{P}_0, t)$ at any $t > 0$ for a given paraboloid of initial conditions $\mathcal{P}_0 \in \mathbb{P}$.*

Problem 2. *Find an exact characterization of the reachable set $\mathcal{R}(\mathcal{P}_0, t)$ at any $t > 0$ for a given paraboloid of initial conditions $\mathcal{P}_0 \in \mathbb{P}$.*

2 Overapproximation with Paraboloids

In this section, Problem 1 is solved using time-varying paraboloids $P : I \to \mathbb{P}$ where I is the interval of definition of P. Parameters $(\mathbf{E}(\cdot), \mathbf{f}(\cdot), \mathbf{g}(\cdot))$ of $P(\cdot)$ are solution of a Riccati differential equation that guarantees an overapproximation relationship with the reachable set, i.e. $\mathcal{R}(\mathcal{P}_0, t) \subseteq P(t)$ for any $t \in I$. Existence and domain of definition I of P are expressed. We prove that the overapproximations P are *tight* since there are so-called *touching trajectories* of $\mathcal{R}(\mathcal{P}_0, t)$ that both belong to the surface of $P(t)$ and to the surface of $\mathcal{R}(\mathcal{P}_0, t)$ for $t \in I$. Finally, the method is presented for a simple toy example.

Parameters of P are expressed as the solutions of an initial value problem. For given $E_0 \in \mathbb{S}^n$, let $\mathbf{E}$ be the solution of the following Riccati differential equation with initial condition $\mathbf{E}(0) = E_0$:

$$\begin{aligned}\dot{\mathbf{E}}(t) = &- \mathbf{E}(t)A - A^\top \mathbf{E}(t) - M_x \\ &+ \left(B^\top \mathbf{E}(t) + M_{xw}^\top\right)^\top M_w^{-1} \left(B^\top \mathbf{E}(t) + M_{xw}^\top\right).\end{aligned} \tag{8}$$

Let $[0, T_E(E_0)[$ be the interval of definition of (8)'s solutions (existence, uniqueness, convergence properties and continuity of the solution are studied in [11]). Let $\mathbf{f}$ denote the solution of the following IVP with initial condition $\mathbf{f}(0) = f_0$:

$$\begin{aligned}\dot{\mathbf{f}}(t) = &- A^\top \mathbf{f}(t) + (M_{xu} + \mathbf{E}(t)B_u)\mathbf{u}(t) \\ &+ (\mathbf{E}(t)B + M_{xw})M_w^{-1}(B^\top \mathbf{f}(t) - M_{uw}^\top \mathbf{u}(t)).\end{aligned} \tag{9}$$

$\mathbf{f}$ satisfies a Linear Varying Parameters differential equation with a continuous input signal. On $[0, T_E(E_0)[$, solution $\mathbf{f}$ to (9) exists, it is unique and continuous. By continuity of $\mathbf{f}$ and $\mathbf{u}$ over $[0, T_E(E_0)[$, $\mathbf{g}$ is defined on $[0, T_E(E_0)[$. For $t \in [0, T_E(E_0)[$, let:

$$\mathbf{g}(t) = g_0 + \int_0^t \begin{bmatrix} \mathbf{f}(\tau) \\ \mathbf{u}(\tau) \end{bmatrix}^\top G \begin{bmatrix} \mathbf{f}(\tau) \\ \mathbf{u}(\tau) \end{bmatrix} d\tau \tag{10}$$

where $G = \begin{bmatrix} BM_w^{-1}B^\top & B_u - BM_w^{-1}M_{uw}^\top \\ (B_u - BM_w^{-1}M_{uw}^\top)^\top & M_u - M_{uw}M_w^{-1}M_{uw}^\top \end{bmatrix}$.

Definition 4 (Time-Varying Paraboloid)**.** *For an initial paraboloid $\mathcal{P}_0 \in \mathbb{P}$, let the time-varying paraboloid*

$$\begin{aligned} P \colon I &\to \mathbb{P} \\ t &\mapsto \mathcal{P}(\mathbf{E}(t), \mathbf{f}(t), \mathbf{g}(t)) \end{aligned}$$

be defined by the time-varying coefficients $(\mathbf{E}, \mathbf{f}, \mathbf{g})$ *solutions of (8,9,10) with initial condition* $\mathcal{P}(E_0, f_0, g_0) = \mathcal{P}_0$. *Let* $P = \mathcal{T}(\mathcal{P}_0)$ *be the function that associates to initial paraboloid the time-varying paraboloid. Let* $T_P(P) = T_E(E_0)$ *and* $\mathcal{I}(P) = [0, T_P(P)[$ *be the interval of definition of* P.

For $P = \mathcal{T}(\mathcal{P}_0)$, let $h(t, \cdot)$ the value function of $P(t)$ at $t \in \mathcal{I}(P)$. For $X_t = (x_t, x_{q,t}) \in \mathbb{R}^{n+1}$, $w_t \in \mathbb{R}^m$, let $h_X(t, w_t) = h(t, \mathbf{X}(t))$ be the value function along the trajectory $\mathbf{X} = (\mathbf{x}, \mathbf{x_q})$ solution of (1,4) generated by $\mathbf{w}$ such that $\mathbf{w}(t) = w_t$ and $\mathbf{X}(t) = X_t$.

Property 2. *The maximum time derivative of the value function* $h(t, \mathbf{X}(t))$ *along the trajectories* $\mathbf{X}$ *for a disturbance* w_t *at* t *exists and it is equal to zero.*

Proof: The time derivative of h_X is the quadratic function:

$$\dot{h}_X = \begin{bmatrix} x_t \\ \mathbf{u}(t) \\ w_t \end{bmatrix}^\top H(t) \begin{bmatrix} x_t \\ \mathbf{u}(t) \\ w_t \end{bmatrix} \tag{11}$$

where H is obtained using (1,4,8-10). $H(t)$ is a function of $\mathbf{E}(t)$, $\mathbf{f}(t)$ and $\mathbf{g}(t)$. The quadratic coefficient in w_t is $\begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}^\top H(t) \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} = M_w$. Since $M_w \prec 0$, the supremum of $w_t \mapsto \dot{h}_X(t, w_t)$ exists and is reached for $w_t = \mathbf{w}^*(t) = \arg\max_{w_t \in \mathbb{R}^m} \dot{h}_X(t, w_t)$ with:

$$\mathbf{w}^* = -M_w^{-1} \left(B^\top (\mathbf{E}\mathbf{x} - \mathbf{f}) + \begin{bmatrix} M_{xw} \\ M_{uw} \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right). \tag{12}$$

Using (8,9,10) in (11), we get $\max_{w_t \in \mathbb{R}^m} \dot{h}_X = 0$. □

We can now state one of our main results:

Theorem 1 (Solution to Problem 1)**.** *Let* $P = \mathcal{T}(\mathcal{P}_0)$ *for a set of initial states* $\mathcal{P}_0$. *The reachable set* $\mathcal{R}(\mathcal{P}_0, t)$ *of* $\mathscr{S}(\mathcal{P}_0, t)$, $t > 0$, *is overapproximated by* $P(t)$, *i.e.:*

$$\forall t \in \mathcal{I}(P), \mathcal{R}(\mathcal{P}_0, t) \subseteq P(t) \cap \mathcal{X}_+.$$

Proof: Using Property 2, by integration of $\dot{h}_X$, if $h_X(0) \leq 0$ then $\forall t \in \mathcal{I}(P), h_X(t) \leq 0$, i.e.:

$$\mathbf{X}(0) \in P(0) \Rightarrow \mathbf{X}(t) \in P(t) \text{ for all } t \in \mathcal{I}(P).$$

The constraint (5) ensures that $\mathbf{X}(t) \in \mathcal{X}_+$. □

Property 3. *Let $\mathbf{X}^*$ be a trajectory generated by $\mathbf{w}^*$ defined in (12) such that initial condition satisfies $\mathbf{X}^*(0) \in \partial\mathcal{P}_0$. At any time $t \in \mathcal{I}(P)$, $\mathbf{X}^*$ satisfies $\mathbf{X}^*(t) \in \partial P(t)$.*

Proof: $\mathbf{X}^*$ is the trajectory generated by the optimal disturbance $\mathbf{w}^*$. Using Property 2, $\dot{h}_{X^*} = 0$. Since $h_{0,\mathbf{X}^*(0)} = 0$, by integration, $h_t(\mathbf{X}^*(t)) = 0$. $\square$

Trajectories generated by $\mathbf{w}^*$ defined in (12) stay in contact with the surface of their time-varying paraboloids. Touching trajectories of P do not necessarily belong to $\mathscr{S}(\mathcal{P}_0, t)$, $t \in \mathcal{I}(P)$, as the energetic constraint might be locally violated.

Remark 1. *Property 2 and (8,9,10) can be derived solving the following optimal control problem (for $t > 0$):*

$$\begin{aligned} \max_{\mathbf{w} \in \mathbf{L}_2([0,t];\mathbb{R}^m)} \quad & \int_0^t \begin{bmatrix} \mathbf{x}(\tau) \\ \mathbf{u}(\tau) \\ \mathbf{w}(\tau) \end{bmatrix} M \begin{bmatrix} \mathbf{x}(\tau) \\ \mathbf{u}(\tau) \\ \mathbf{w}(\tau) \end{bmatrix} d\tau - x_{q,t} \\ s.t. \quad & \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{w} + B_u\mathbf{u} \\ & \mathbf{x}(t) = x_t \end{aligned}$$

for given $(x_t, x_{q,t}) \in \mathcal{X}_+$. This is a special instance of the LQR problem (see e.g. [20]).

Remark 2 (Representation of paraboloids)**.** *In [20], the time-varying value function is a quadratic function defined by its quadratic coefficient $\mathbf{S}$, its center $\mathbf{x_c}$ and its value at the center ρ. $\mathbf{S}$, $\mathbf{x_c}$ and ρ satisfied an IVP. In this formulation, the center $\mathbf{x_c}$ can diverge when the determinant of $\mathbf{S}$ vanishes. However, the corresponding time-varying value function is time-continuous and can be extended continuously. In this paper, we choose to work with variables $\mathbf{E}$, $\mathbf{f}$ and $\mathbf{g}$ (see Definition 1) to avoid this issue.*

Example 1. *Let $A = -1$, $B = 1$, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $B_u = 0$ and $\mathbf{u} : [0, \infty[\mapsto 0$. Solutions of ODE (8) (that is $\dot{\mathbf{E}} = -\frac{1}{2}\mathbf{E}^2 + 2\mathbf{E} - 1$) diverge for $E_0 < E^-$ (see Figure 1) where $E^- < E^+$ are the roots of the equation $-\frac{1}{2}E^2 + 2E - 1 = 0$ for $E \in \mathbb{R}$, $E^- = 2 - \sqrt{2}$ and $E^+ = 2 + \sqrt{2}$. Figure 2 shows the trajectory of the paraboloid for E_0 in the stable region $E_0 > E^-$ while Figure 3 shows the trajectories of the paraboloid for E_0 in the unstable region $E_0 < E^-$.*

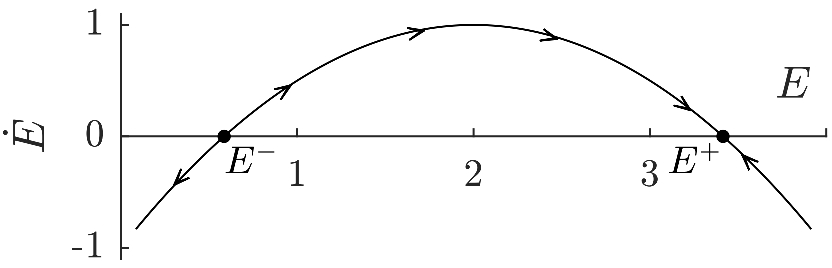


Figure 1: Convergence analysis of (8)'s solutions for Example 1

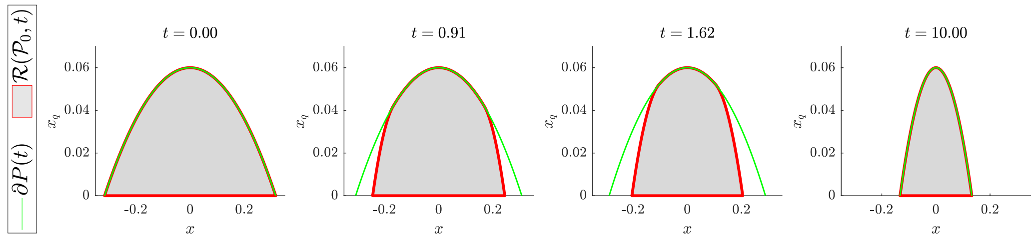


Figure 2: Time-varying paraboloid overapproximates the reachable set at different time instants t in $\{0.00, 0.91, 1.62, 10.00\}$ for an initial maximum energetic level of $x_{q,0} = 0.06$. The solution of (8) converges to a constant value when $t \to +\infty$. The shaded regions are the reachable set $\mathcal{R}(\mathcal{P}_0, t)$, the thin green lines are the boundary of the overapproximation $P(t)$ of Theorem 1.

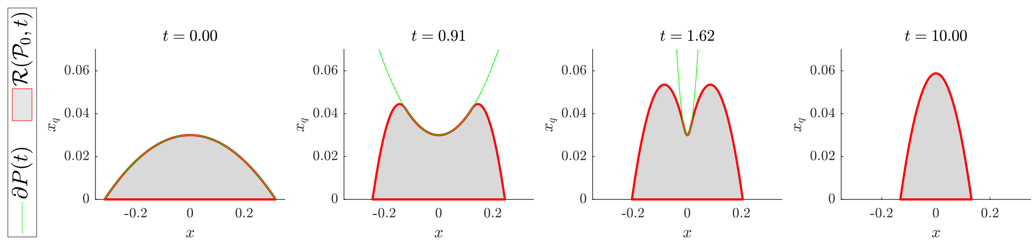


Figure 3: Time-varying paraboloid overapproximates the reachable set at different time instants t in $\{0.00, 0.91, 1.62, 10.00\}$ for an initial maximum energetic level of $x_{q,0} = 0.03$. The solution of (8) has a finite escape time and diverge at $t = 1.68$. The shaded regions are the reachable set $\mathcal{R}(\mathcal{P}_0, t)$, the thin green lines are the boundary of the overapproximation $P(t)$ of Theorem 1.

3 Exact Reachable Set Computation

In this section, a set of time-varying paraboloids is defined. At a given time, the intersection of the paraboloids gives better overapproximations of the reachable set. With additional assumptions, the reachable set is exactly characterized. This approach relies on the use of Property 1 and preliminary results showing that for any state of the overapproximation, there exists a trajectory in $\mathscr{S}(\mathcal{P}_0, t)$, $t > 0$, leading to this state.

Let Π, a set of time-varying paraboloids (Definition 4), be defined by:

$$\Pi = \left\{ \mathcal{T}(\gamma \mathcal{P}_0) \middle| \gamma \geq 1, \exists (x, x_q) \in \mathcal{P}_0^{\epsilon_q}, \dot{\mathbf{x}}_{\mathbf{q}}^{*,\gamma}(0) \geq 0 \right\} \tag{13}$$

where $\mathcal{P}_0^{\epsilon_q}$ is a small set of states near $\partial \mathcal{P}_0$ in the half-plane $x_q \leq 0$:

$$\mathcal{P}_0^{\epsilon_q} = \left\{ (x, x_q) \middle| \begin{array}{r} -x^\top E_0 x + 2 f_0^\top x - g_0 \in [-\epsilon_q, 0] \\ x_q \in [-\epsilon_q, 0] \end{array} \right\}$$

where $(E_0, f_0, g_0) = \mathcal{P}_0$, $\epsilon_q > 0$, and $(\mathbf{x}^{*,\gamma}, \mathbf{x}_{\mathbf{q}}^{*,\gamma})$ the touching trajectory of $\gamma \mathcal{P}_0$ s.t. $(\mathbf{x}^{*,\gamma}, \mathbf{x}_{\mathbf{q}}^{*,\gamma})(0) = (x, x_q)$. Π is defined such that each touching trajectory of rising energy belongs to a time-varying paraboloid P of Π.

Direct computation gives $\dot{\mathbf{x}}_{\mathbf{q}}^{*,\gamma}(0) = \gamma^2 a + \gamma b + c$ where $a < 0$ (consequence of $M_w \prec 0$) and b and c in $\mathbb{R}$. If there is a $(x, x_q) \in \mathcal{P}_0^{\epsilon_q}$ such that $\dot{\mathbf{x}}_{\mathbf{q}}^{*,1}(0) > 0$, since $\mathcal{P}_0^{\epsilon_q}$ is bounded and since $a < 0$, there exists a $\overline{\gamma} > 1$ such that $\forall (x, x_q) \in \mathcal{P}_0^{\epsilon_q}, \dot{\mathbf{x}}_{\mathbf{q}}^{*,\overline{\gamma}}(0) \leq 0$. Therefore, the set of scalings $\gamma \geq 1$ such that $\dot{\mathbf{x}}_{\mathbf{q}}^{*,\gamma}(0) \geq 0$ is $[1, \overline{\gamma}]$. Π is at most a bounded set of time-varying paraboloids.

Let $\mathcal{I}(\Pi) \subseteq \mathbb{R}^+$ be the set of time instant $t \in \mathcal{I}(\Pi)$ where there exists a $P \in \Pi$ that is defined at t (i.e. $t \in \mathcal{I}(P)$). Since for each $P \in \Pi$, 0 belongs to the interval $\mathcal{I}(P)$, we have:

$$\mathcal{I}(\Pi) = \left[0, \sup_{P \in \Pi} \{ T_P(P) \} \right[\tag{14}$$

For $t \in \mathcal{I}(\Pi)$, let

$$\Pi^\cap(t) = \bigcap_{P \in \Pi \text{ s.t. } t \in \mathcal{I}(P)} P(t) \tag{15}$$

the intersection of all the defined time-varying paraboloids P of Π at time t (see Figure 4).

We now prove that when some assumption about boundedness of (8)'s solutions (Assumption 1) and touching trajectories behavior around the null energetic surface (Assumption 2) holds, then $\mathcal{R}(\mathcal{P}_0, t) = \Pi^\cap(t) \cap \mathcal{X}_+$, for given $t \in \mathcal{I}(\Pi)$ (Theorem 2, Section 3.5). To achieve that:

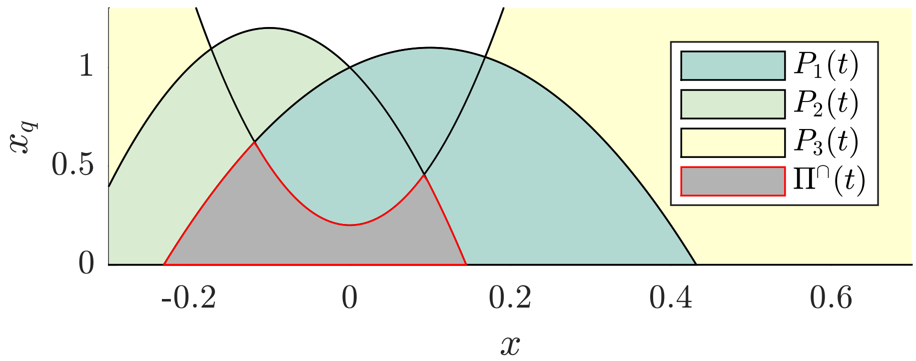


Figure 4: For $t \geq 0$, $P_i \in \Pi$, $i = 1, 2, 3$. Light color shaded area are time-varying constraints of Π at t. Grey color shaded are is their intersection $\Pi^\cap(t)$.

- we prove the overapproximation relationship $\mathcal{R}(\mathcal{P}_0, t) \subseteq \Pi^\cap(t)$ (Section 3.1);
- we prove that any state $(x, x_q) \in \Pi^\cap(t)$ is reachable from a state $(x, x'_q) \in \partial\Pi^\cap(t)$ (Section 3.2);
- for a state $X_t \in \partial\Pi^\cap(t)$, we find a touching trajectory $\mathbf{X}^*$ of $\Pi^\cap$ such that $\mathbf{X}^*(t) = X_t$ (Section 3.3);
- these touching trajectories $(\mathbf{x}^*, \mathbf{x_q}^*)$ of $\Pi^\cap$ satisfy the state constraint $\mathbf{x_q}(\cdot) \geq 0$ over $[0, t]$ (Section 3.4);
- finally, we conclude that any $X_t \in \Pi^\cap(t)$ is reachable from $\mathcal{P}_0$, thus $\mathcal{R}(\mathcal{P}_0, t) = \Pi^\cap(t) \cap \mathcal{X}_+$ (Section 3.5).

3.1 Overapproximation Relationship

If $\mathcal{Y} \subseteq \mathcal{Z}$ subsets of $\mathbb{R}^{n+1}$, then $\mathcal{R}(\mathcal{Y}, t) \subseteq \mathcal{R}(\mathcal{Z}, t)$. This result is stated in Property 4 for the specific case of scaled paraboloids (see Definition 2).

Property 4. *For a set of initial states $\mathcal{P}_0 \in \mathbb{P}$ and a scaling factor $\gamma \geq 1$, let $P_\gamma = \mathcal{T}(\gamma\mathcal{P}_0)$. For any trajectory $\mathbf{X} \in \mathscr{S}(\mathcal{P}_0, t)$, it holds $\mathbf{X}(t) \in P_\gamma(t)$ for all $t \in \mathcal{I}(P_\gamma)$.*

Proof: Using Property 1 and Theorem 1, $\mathbf{X}(t) \in P(t)$ for any $t \in \mathcal{I}(P)$. □
As each time-varying paraboloid is an overapproximation of the reachable set, the intersection of these paraboloids is as well an overapproximation.

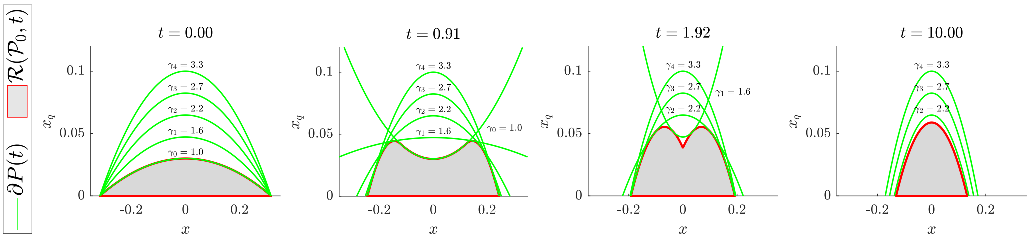


Figure 5: Time-varying paraboloids overapproximate the reachable set at different time instants t in $\{0.00, 0.91, 1.62, 10.00\}$ for different initial conditions $P_i(0) = \gamma_i \mathcal{P}_0$, where scaling factors γ_i are respectively equal to 1.0, 1.6, 2.2, 2.7 and 3.3 for $i = 0, \dots, 4$. The shaded regions are the reachable set $\mathcal{R}(\mathcal{P}_0, t)$, the thin green lines are the boundary of the overapproximation $P(t)$ of Theorem 1.

Property 5. *$\mathcal{R}(\mathcal{P}_0, t) \subseteq \Pi^\cap(t)$ for any $t \in \mathcal{I}(\Pi)$.*

Proof: This is a direct consequence of Property 4. $\square$

Example 2 (Continued from Example 1)**.** *In the case where the solution of* (8) *does not converge (i.e. $E_0 < \mathbf{E}^-$), Figure 5 shows several paraboloid trajectories with different initial energetic levels (i.e. different initial scaling). Since all the scalings γ_i are greater than 1, $\mathcal{P}_0 \cap \mathcal{X}_+ \subset \gamma_i \mathcal{P}_0 \cap \mathcal{X}_+$. Therefore, each time-varying paraboloid is a valid constraint that bounds $\mathcal{R}(\mathcal{P}_0, t)$, $t \in \mathcal{I}(\Pi)$ (Theorem 1). Therefore, $\mathcal{R}(\mathcal{P}_0, t) \subseteq P^\cap(t) = P_0(t) \cap P_1(t) \cap \dots \cap P_4(t)$ where $P_i = \mathcal{T}(\gamma_i \mathcal{P}_0)$, and γ_i are resp. equal to 1, 1.6, 2.2, 2.7 and 3.3 for $i = 0, \dots, 4$. In this case, the overapproximation $P^\cap(t)$ is strictly included in $P_0(t)$.*

Observations in Example 2 motivate the use of multiple time-varying paraboloids to get better overapproximation of the reachable set $\mathcal{R}(\mathcal{P}_0)$.

3.2 Past trajectory for states in the interior of $\Pi^\cap(t)$

Property 6. *For a trajectory $(\mathbf{x}, \mathbf{x_q}) \in \mathscr{S}(\mathcal{P}_0, T)$, $T > 0$, $\mathbf{x}$ is time-continuous.*

Proof: Let $f : t, x \mapsto Ax + B\mathbf{w}(t) + B_u \mathbf{u}(t)$. Since for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, $f(., x)$ is locally measurable over $\mathbb{R}^+$, $f(t, .)$ is Lipschitz over $\mathbb{R}^n$, (1) has a unique solution $\mathbf{x}$ (see [21], Theorem 1.1) that is time-continuous. $\square$

$\mathbf{x}$ trajectories are time-continuous, however, this in not necessarily true for $\mathbf{x_q}$ trajectories: $\mathbf{x_q}$ might have steps due to sudden release of the energy through the disturbance $\mathbf{w}$. We use the following property to prove that the state $(x, \alpha x_q)$ is reachable from the state (x, x_q) for any given $\alpha \in [0, 1]$.

Property 7. *For $t \geq 0$, if $(x, x_q) \in \mathcal{R}(\mathcal{P}_0, t)$ then $(x, \alpha x_q) \in \mathcal{R}(\mathcal{P}_0, t)$ for all $\alpha \in [0, 1]$.*

Proof: For $\epsilon > 0$, let $\mathbf{w} \in \mathbf{L}_2([0, t+\epsilon]; \mathbb{R}^m)$, s.t. $\mathbf{w}^\top(s) M_w \mathbf{w}(s) = -(1-\alpha)\mathbf{x_q}(t)\frac{1}{\epsilon}$ when $s \in [t, t+\epsilon]$. Then $\int_t^{t+\epsilon} \mathbf{w}^\top(s) M_w \mathbf{w}(s) ds \to -(1-\alpha)\mathbf{x_q}(t)$ when $\epsilon \to 0$. Using Cauchy-Schwartz inequality: $\left| \int_t^{t+\epsilon} (-M_w)^{\frac{1}{2}} \mathbf{w}(s) ds \right| \leq \sqrt{\int_t^{t+\epsilon} 1 ds} \sqrt{\int_t^{t+\epsilon} -\mathbf{w}^T(s) M_w \mathbf{w}(s) ds}$ and the time-continuity of $\mathbf{x}$ (from Property 6), the quantity $\int_t^{t+\epsilon} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ 0 \end{bmatrix}^\top M \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{u}(s) \\ \mathbf{w}(s) \end{bmatrix} ds \to 0$ when $\epsilon \to 0$. By integration, $\mathbf{x}_q(t+\epsilon) \to \alpha \mathbf{x_q}(t)$ when $\epsilon \to 0$. Since $\mathbf{x}$ is continuous (Property 6), $\mathbf{x}(t+\epsilon) \to \mathbf{x}(t)$ when $\epsilon \to 0$. By continuity of $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{w}$ over $[t, t+\epsilon]$, $\mathbf{x_q}$ is continuous over $[t, t+\epsilon]$. Then, there exists a $t' \in [t, t+\epsilon]$ such that $\mathbf{x_q}(\tau) \geq \alpha \mathbf{x_q}(t) \geq 0$ for all $\tau \in [t, t']$ and $\mathbf{x_q}(t') \to \alpha \mathbf{x_q}(t)$ when $\epsilon \to 0$. Therefore, the constraint $\mathbf{x_q}(\cdot) \geq 0$ is satisfied over $[t, t']$ and the trajectory $(\mathbf{x}, \mathbf{x_q})$ is a valid trajectory of $\mathscr{S}(\mathcal{P}_0, t')$ for all $t \leq t'$. $\square$

3.3 Past trajectory for states in $\partial\Pi^\cap(t)$

The value function $\tilde{h}$ of a time-varying paraboloid $\tilde{P}$ can be approximated at the first order along a touching trajectory $\mathbf{X}^*$ of another time-varying paraboloid P when initial conditions $P(0)$ and $\tilde{P}(0)$ are close.

Property 8. *For any $\lambda, \tilde{\lambda} \in [1, \overline{\lambda}]$, any $t \in \mathcal{I}(P) \cap \mathcal{I}(\tilde{P})$:*

$$\left| \tilde{h}_t(\mathbf{X}^*(t)) - (\lambda - \tilde{\lambda})\lambda^{-1} \mathbf{x}^*_\mathbf{q}(0) \right| \leq N(\lambda - \tilde{\lambda})^2$$

where $P = \mathcal{T}(\lambda \mathcal{P}_0)$, $\tilde{P} = \mathcal{T}(\tilde{\lambda}\mathcal{P}_0)$, $\tilde{h}_t$ is the value function of $\tilde{P}(t)$, $\mathbf{X}^* = (\mathbf{x}^*, \mathbf{x}^*_\mathbf{q})$ is a touching trajectory of P and $N > 0$ a scalar.

Proof: Let $\mathbf{n} = (\mathbf{E} - \tilde{\mathbf{E}})\mathbf{x}^* - (\mathbf{f} - \tilde{\mathbf{f}})$. Using (1,12,8-10), $\mathbf{n}$ satisfies the linear time varying differential equation: $\dot{\mathbf{n}} = (-A^\top + M_{xw} M_w^{-1} B^\top + \tilde{\mathbf{E}} B M_w^{-1} B^\top)\mathbf{n}$. Since t belongs to $\mathcal{I}(\tilde{P})$ and by time-continuity of $\tilde{\mathbf{E}}(\cdot)$ over $[0, t]$, there is a scalar $K > 0$ that bounds $\|\tilde{\mathbf{E}}(\cdot)\|$ over $[0, t]$. Then, there exists $L > 0$ upper bound of $\|-A^\top + M_{xw} M_w^{-1} B^\top + \tilde{\mathbf{E}}(\cdot) B M_w^{-1} B^\top\|$ over $[0, t]$. Using the Grönwall inequality, it holds $\|\mathbf{n}(\tau)\| \leq e^{L\tau}\|\mathbf{n}(0)\|$ for $\tau \in [0,t]$. Since $\tilde{\lambda}^{-1}\tilde{P}(0) = \lambda^{-1}P(0) = \mathcal{P}_0$, it holds $\mathbf{n}(0) = (\lambda - \tilde{\lambda})n_0$ with $n_0 = E_0\mathbf{x}^*(0) - f_0$. Therefore, $\|\mathbf{n}(\tau)\| \leq \left|\lambda - \tilde{\lambda}\right| e^{L\tau}\|n_0\|$. Along the touching trajectory $\mathbf{X}^* = (\mathbf{x}^*, \mathbf{x}^*_{\mathbf{q}})$ of P and by using (8-10), $\dot{\tilde{h}}_t$ is equal to : $\dot{\tilde{h}}_t(\mathbf{X}^*(t)) = \mathbf{n}^\top(t)BM_w^{-1}B^\top\mathbf{n}(t)$. By integration, we have $\left|\tilde{h}_t(\mathbf{X}^*(t)) - \tilde{h}_0(\mathbf{X}^*(0))\right| \leq N(\lambda - \tilde{\lambda})^2$, where $R = \left|n_0BM_w^{-1}B^\top n_0\right|(2L)^{-1}e^{2LT}$ a finite constant (since $M_w \prec 0$). Since $\mathbf{X}^*$ is a touching trajectory of P: $\lambda(\mathbf{x}^{*\top}(0)E_0\mathbf{x}^*(0) - 2f_0^\top\mathbf{x}^*(0) + g_0) + \mathbf{x}^*_{\mathbf{q}}(0) = 0$. Direct computation gives: $\tilde{h}_0(\mathbf{X}^*(0)) = (\lambda - \tilde{\lambda})\lambda^{-1}\mathbf{x}^*_{\mathbf{q}}(0)$. Thus, $\left|\tilde{h}_t(\mathbf{X}^*(t)) - (\lambda - \tilde{\lambda})\lambda^{-1}\mathbf{x}^*_{\mathbf{q}}(0)\right| \leq N(\lambda - \tilde{\lambda})^2$. $\square$

When Property 8 holds, if $N(\lambda - \tilde{\lambda})^2 \leq (\lambda - \tilde{\lambda})\lambda^{-1}\mathbf{x}^*_{\mathbf{q}}(0)$, then the sign of $\tilde{h}_t(\mathbf{X}^*(t))$ is equal the sign of $(\lambda - \tilde{\lambda})\lambda^{-1}\mathbf{x}^*_{\mathbf{q}}(0)$. Since $\tilde{h}_t(\mathbf{X}^*(t)) > 0 \Rightarrow \mathbf{X}^*(t) \notin \Pi^\cap(t)$, Property 8 provides a way to identify states that do not belongs to $\Pi^\cap(t)$.

Property 9. *Let $\mathbf{X}^*$ a touching trajectory of $P = \mathcal{T}(\lambda\mathcal{P}_0)$ (where $\lambda \in [1, \overline{\lambda}]$ given) and $t \in \mathcal{I}(P)$ given. If there is a $\tilde{\lambda} \in [1, \overline{\lambda}]$, s.t. $t \in \mathcal{I}(\tilde{P})$ (where $\tilde{P} = \mathcal{T}(\tilde{\lambda}\mathcal{P}_0)$) and $\left|\lambda - \tilde{\lambda}\right| \leq \left|N^{-1}\mathbf{x}^*_{\mathbf{q}}(0)\right|$, then*

$$(\lambda - \tilde{\lambda})\lambda^{-1}\mathbf{x}^*_{\mathbf{q}}(0) > 0 \Rightarrow \tilde{h}_t(\mathbf{X}^*(t)) > 0$$

where $\tilde{h}$ is the value function of $\tilde{P}$ and $N > 0$ a scalar.

Proof: This is a direct consequence of Property 8 and of the property: $(|a - b| \leq c) \wedge (c < |b|) \Rightarrow (ab > 0)$ for $a, b, c \in \mathbb{R}$. $\square$

The existence of $\tilde{\lambda}$ in Property 9 is conditioned by t belonging to $\mathcal{I}(\tilde{\lambda}\mathcal{P}_0)$. In this work, to ensure the existence of such $\tilde{\lambda}$ at a given time $t \in \mathcal{I}(\Pi)$, we enforce the boundedness of $\|\mathbf{E}(\cdot)\|$ on $[0,T]$.

Assumption 1. *There is a scalar $K > 0$, such that for any $(\mathbf{E}, \mathbf{f}, r) = P \in \Pi^\cap$, $\|\mathbf{E}(\cdot)\|$ is bounded by K on $[0,T]$.*

The domain of definition of $P \in \Pi$ is only defined by the domain of definition of its parameter $\mathbf{E}$ (see Section 2). Thus, when Assumption 1 holds, we have $[0,T] \subset \mathcal{I}(P)$ and therefore $[0,T] \subset \mathcal{I}(\Pi)$. Property 9 can then be restated when Assumption 1 holds:

Property 10. *Let $P_\lambda \in \Pi$, $t \in [0,T]$, $\lambda \in [1, \overline{\lambda}]$, for $X_t \in \partial P_\lambda(t)$ if $X_t \in \Pi^\cap(t)$ then the touching trajectory $\mathbf{X}^*$ of P such that $\mathbf{X}^*(t) = X_t$ is a touching trajectory of $\Pi^\cap$.*

Proof: Since Assumption 1 holds in Property 9, the constant N can be chosen independently from $\tilde{\mathbf{E}}$ and $\mathbf{E}$ (i.e. from P and $\tilde{P}$). Let $(\mathbf{x}^*, \mathbf{x_q}^*) = \mathbf{X}^*$. Lets assume that either $(\mathbf{x_q}^*(0) < 0) \wedge (\lambda < \overline{\lambda})$ or $(\mathbf{x_q}(0) > 0) \wedge (\lambda > 1)$. For both cases, we can choose a $\tilde{\lambda} = \lambda - \eta$, with η s.t. $\eta \mathbf{x_q}^*(0) > 0$, $|\eta| < \left|N^{-1}\mathbf{x_q}^*(0)\right|$ and $\tilde{\lambda} \in [1, \overline{\lambda}]$. Since $\tilde{\lambda} \in [1, \overline{\lambda}]$, $\tilde{P} \in \Pi$. Then Property 9 shows that $X_t \notin \tilde{P}(t)$, i.e. $X_t \notin \Pi^{\cap}(t)$! Therefore, either $(\mathbf{x_q}^*(0) < 0) \wedge (\lambda = \overline{\lambda})$ or $(\mathbf{x_q}(0) > 0) \wedge (\lambda = 1)$ or $(\mathbf{x_q}^*(0) = 0)$ and $\mathbf{X}^*(0) \in \partial P(0)$. Similar computation than in proof of Property 1 gives $\mathbf{X}(0) \in \lambda' \mathcal{P}_0$ for any $\lambda' \in [1, \overline{\lambda}]$. Therefore, $\mathbf{X}(0)$ belongs to the intersection which is $\Pi^{\cap}(0)$. Finally, thanks to Property 5, $\mathbf{X}$ is a touching trajectory of $\Pi^{\cap}$. $\square$

Since $\Pi^{\cap}(t)$ is an intersection of closed sets, $\Pi^{\cap}(t)$ is closed as well. In the general case, for an infinite intersection $\mathcal{Y}^{\cap} = \bigcap_{i \in \mathbb{N}} Y_i$ of closed sets Y_i, $i \in \mathbb{N}$, any boundary point $y \in \partial \mathcal{Y}^{\cap}$ does not necessarily belongs to the boundary of any Y_i, $i \in \mathbb{N}$ (e.g. $\bigcap_{\epsilon \in]1,2]} [-\epsilon, \epsilon] = [-1, 1]$, but there is no $\epsilon \in]1, 2]$ such that $1 \in \partial[-\epsilon, \epsilon]$).

Property 11. *For any $X_t \in \partial \Pi^{\cap}(t)$, there is a $P \in \Pi$ such that $X_t \in \partial P(t)$.*

Proof: Let $Q(t, x, \lambda) = -x^\top \mathbf{E}_\lambda(t) x + 2\mathbf{f}_\lambda^\top(t) x - \mathbf{g}_\lambda(t)$ where $(\mathbf{E}_\lambda, \mathbf{f}_\lambda, \mathbf{g}_\lambda) = P_\lambda = \mathcal{T}(\lambda \mathcal{P}_0)$ with $\lambda \in [1, \overline{\lambda}]$. By continuity of (8,9,10) solutions, and since Assumption 1 holds, $Q(t, x, \cdot)$ is continuous over the closed interval $[1, \overline{\lambda}]$. To this respect, for any $x \in \mathbb{R}^n$, the minimum of $x_q = Q(t, x, \cdot)$ exists and is reached for a $\lambda^* \in [1, \overline{\lambda}]$. Therefore, for any $(x, x_q) \in \partial \Pi^{\cap}(t)$, there is $P \in \Pi$ s.t. $(x, x_q) \in \partial P(t)$. $\square$

Lemma 1 shows that any state $X_t \in \partial \Pi^{\cap}(t)$ (with $t \in \mathcal{I}(\Pi)$ given) is the terminal state of a touching trajectory $\mathbf{X}$ of $\Pi^{\cap}$ with initial state $X_0 \in \partial \Pi^{\cap}(0)$.

Lemma 1. *If Assumption 1 holds, any state $X_t \in \partial \Pi^{\cap}(t)$ has a past touching trajectory $\mathbf{X}$ of $\Pi^{\cap}$.*

Proof: This is a direct consequence of Assumption 1, Property 10 and Property 11. $\square$

3.4 Past trajectory constraint $\mathbf{x_q}(\cdot) \geq 0$

Touching trajectories of $P \in \Pi$ with initial state in $\Pi^{\cap}(0)$ are also touching trajectories of $\Pi^{\cap}$ (Lemma 1). We enforce the touching trajectories $(\mathbf{x}^*, \mathbf{x_q}^*)$

of $\Pi^\cap$ to satisfy the constraint $\mathbf{x_q^*}(\cdot) \geq 0$ by assuming that no touching trajectory has a rising $\mathbf{x_q^*}$ state close to the the null plan $x_q = 0$.

Assumption 2 (Falling touching trajectories)**.** *Any touching trajectory* $(\mathbf{x}^*, \mathbf{x_q}^*)$ *of* $\Pi^\cap$ *have a strictly decreasing energetic state on the null energetic surface:*

$$\mathbf{x_q}^*(t) \in [-\epsilon_q, 0] \Rightarrow \dot{\mathbf{x}}_\mathbf{q}^*(t) < 0.$$

This assumption was found reasonable for several stable IQC systems study. We use the following intermediate result:

Property 12. *Consider a function* $f : [0,1] \to \mathbb{R}$ *continuous and differentiable over* $[0,1]$ *with a continuous derivative* f' *over* $[0,1]$ *such that* f *satisfy* $f(0) \leq 0$ *and* $\forall x \in [0,1], (f(x) \in [-\epsilon, 0]) \Rightarrow f'(x) < 0$*, for a* $\epsilon > 0$*. Then* $\forall x \in [0,1], f(x) < 0$*.*

Proof: Let $x \in [0,1]$ such that $f(x) \in [\epsilon, 0]$. Since $f'(x) < 0$ and f' is continuous, there exists a neighborhood $[x, x+\eta]$, $\eta > 0$, such that $f'(\cdot) < 0$ over $[x, x+\eta]$. Therefore, by integration $f(\cdot) \leq 0$ over $[x, x+\eta]$. $f(x) \in [-\epsilon, 0] \Rightarrow \exists \eta > 0, \forall y \in [x, x+\eta], f(y) \leq 0$. Since f is continuous and $f(0) < 0$, any function f non negative would violate this statement (direct consequence of the intermediate value theorem). □

3.5 Exact characterization of the reachable set

Exact characterization of $\mathcal{R}(\mathcal{P}_0, t)$ by $\Pi^\cap(t)$ for $t \in \mathcal{I}(\Pi)$ is guaranteed since ownership of touching trajectories is proven with Property 1, non-ownership is guaranteed locally by Property 8. Remaining trajectories of $\mathscr{S}(\mathcal{P}_0, t)$ can be constructed from Property 7 and touching trajectories of Π. Finally, all the trajectories satisfy the constraint since touching trajectories satisfy Assumption 2 and cannot violate the constraint $\mathbf{x_q}(\cdot) \geq 0$ temporarily.

Theorem 2 (Solution to Problem 2)**.** *If Assumption 1 and 2 hold, then* $\mathcal{R}(\mathcal{P}_0, t) = \Pi^\cap(t) \cap \mathcal{X}_+$ *for any* $t \in [0, T]$ *where* Π, $\Pi^\cap$ *and* $\mathcal{I}(\Pi)$ *are defined by (13,15,14).*

Proof: Thanks to Property 7, for $X_t \in \Pi^\cap(t)$, if the projection of X_t over $\partial\Pi^\cap(t)$ is reachable, then X_t is reachable. Since Assumption 1 holds, Lemma 1 shows that when $X_t \in \partial\Pi^\cap(t)$, there exists a touching trajectory $\mathbf{X} = (\mathbf{x}, \mathbf{x_q})$ of $\Pi^\cap$ s.t. $\mathbf{X}(t) = X_t$. By continuity of $\mathbf{E}$, of the optimal disturbance $\mathbf{w}^*$ of $\mathbf{X}$, of $\mathbf{u}$ and of $\mathbf{x}$, we have $\mathbf{x_q} \in \mathscr{C}^1([0,t]; \mathbb{R})$. Since Assumption 2 holds and $\mathbf{X}(\tau) \in \partial\Pi^\cap(\tau)$ for all $\tau \in [0, t]$, Property 12 can be applied to $\mathbf{x_q}$.

The existence of any $\tau \in [0,t]$ such that $\mathbf{x_q}(\tau) < 0$ would violate Property 12 since the terminal state satisfies $\mathbf{x_q}(t) \geq 0$! Therefore, $\mathbf{X} \in \mathscr{S}(\mathcal{P}_0, t)$ and $X_t \in \mathcal{R}(\mathcal{P}_0, t)$. These properties lead to $\Pi^{\cap}(t) \cap \mathcal{X}_+ \subseteq \mathcal{R}(\mathcal{P}_0, t)$. Finally, using Property 5, we have $\Pi^{\cap}(t) \cap \mathcal{X}_+ = \mathcal{R}(\mathcal{P}_0, t)$. $\square$

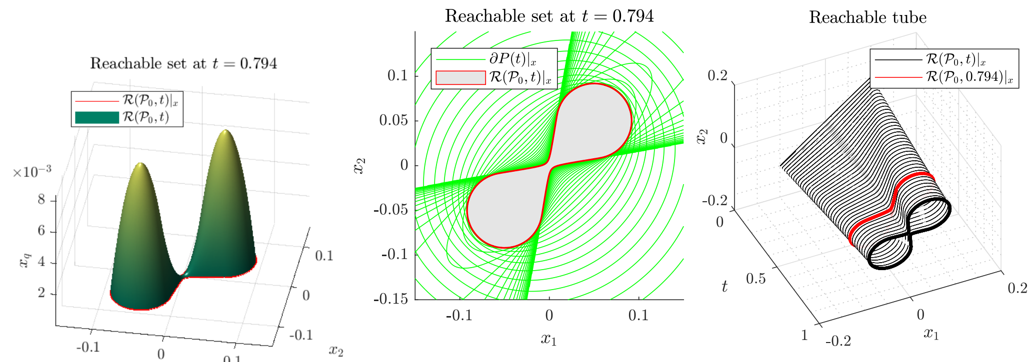
4 Example

We study the stable IQC system $\mathscr{S}(\mathcal{P}_0, t)$, defined in (6), at a given time t in $[0,1]$, for a parabolic set of initial states $\mathcal{P}_0 = \mathcal{P}(E_0, f_0, g_0)$, with $E_0 = \left[\begin{smallmatrix} a+b & a \\ a & a+b \end{smallmatrix}\right]$, $f_0 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $g_0 = 0.015$, $a = 10^{-2}$ and $b = 10^{-6}$, and for the following parameters $A = -I$, $B = I$, $B_u = 0$, $\mathbf{u} : \mathbb{R}^+ \mapsto 0$ and $M = \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2I \end{smallmatrix}\right]$ where $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

The reachable set $\mathcal{R}(\mathcal{P}_0, t)$ of $\mathscr{S}(\mathcal{P}_0, t)$, defined in (7), is computed using (13) and Theorem 2, for $t \in [0,1]$. Figures 6a and 6b show the reachable set $\mathcal{R}(\mathcal{P}_0, t)$ set at time $t = 0.794$ and its projection $\mathcal{R}(\mathcal{P}_0, t)|_x$ over the LTI state space (i.e. projection over (x_1, x_2) states). In Figure 6b, the constraints boundaries $\partial P(t)$ (for $P \in \Pi$, Π defined in Section 3) are touching the reachable set $\mathcal{R}(\mathcal{P}_0, t)$. The non-convexity of $\mathcal{R}(\mathcal{P}_0, t)$ arises from the non-positive solutions of the Riccati differential equation (8). Figure 6c represents the projection of the reachable tube $t \mapsto \mathcal{R}(\mathcal{P}_0, t)$ projected over the LTI dimension (x_1, x_2).

5 Conclusion

In this work, the reachability problem for an LTI system with energetic constraint is solved. The solution is a set-based method that relies on overapproximations with time-varying paraboloids. The paraboloid parameters are expressed as solutions of an IVP that involves a Riccati differential equation. We prove that with assumptions about touching trajectories of the reachable set and boundedness of Riccati differential equation's solutions, the intersection of a well-chosen set of paraboloids exactly describes the reachable set. Our method is tractable and has been used to exhibit the reachable set of a stable system. In some future works, weaker assumptions about the reachable set will be considered, the LTV case will be studied as well as the discrete time case. Most of the research around dynamical systems with integral constraints bring generic solutions, we hope that the linear case gives a better understanding about how such systems behave.



(a) Reachable set

(b) Reachable set of the LTI system

(c) Reachable tube of the LTI system

Figure 6: The green surface in (a) is the reachable set $\mathcal{R}(\mathcal{P}_0, t)$ at $t = 0.794$ of $\mathscr{S}(\mathcal{P}_0, t)$ computed using Theorem 2. Its projection over the LTI state space (x_1, x_2) (in solid red line) is shown in (b), each green line corresponds to one constraint $P \in \Pi$ computed with Theorem 1. (c) is the reachable tube $t \to \mathcal{R}(\mathcal{P}_0, t)$ of $\mathscr{S}(\mathcal{P}_0, t)$ projected over the LTI state space (x_1, x_2) for $t \in [0, 1]$. The red section corresponds to the time $t = 0.794$.

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