Λ-Generalized Polynomials with Applications to Multivariable Decoupling

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Abstract. This work is intended to be a continuation of previous studies about a set of rational functions, known as generalized polynomials, which are basically defined as rational functions with poles in a symmetric region of the complex plane. In this paper, algebraic properties of Λ-generalized polynomials are examined, and via a suitable definition of a degree function it is proved that this set is a Euclidean ring. Actually, it is shown that widely used sets of rational functions like polynomials, proper rational and proper and stable rational functions can be regarded as particular cases of Λ-generalized polynomials. Some other results concerning division and coprimeness in the set of Λ-generalized polynomials, as well as canonical forms for Λ-generalized polynomial matrices are also presented.

An application of the concepts to the problem of decoupling with Λ-stability of linear square multivariable systems is also presented. It is proved that a linear system is decouplable with Λ-stability if and only if the so-defined Λ-stable interactor of the system is a diagonal matrix, or equivalently, if and only if the row and global zero structure of the system out of the region Λ coincide. The Λ-stable interactor and the row and global zero structure of the system out of Λ are defined via the Hermite column form and the Smith form of the system transfer function over the ring of Λ-generalized polynomials. If the system is decouplable with Λ-stability, a procedure to compute a decoupling state feedback is presented. It is also shown that the conditions for the row by row decoupling problem and decoupling with stability can be easily obtained as particular cases of our solution.

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1 Introduction

Algebraic concepts are fundamental in practically any branch of Engineering, and control theory is by no means the exception. The analysis and design of dynamic control systems relies to a deep extent on the notions and properties from algebraic objects. For instance, polynomials, proper rational functions, proper and stable rational functions, matrices with elements from these sets, and canonical forms related to these matrices are a key tool in system description, analysis and design. These rational functions can be regarded as particular cases of what is known as generalized polynomials, thus allowing a unified approach in the study of the algebraic properties of these objects.

Previous works on generalized polynomials are presented in [10], [11], [14], and [13]. In [10] it is shown that the set $\mathbb{R}_S$ of all transfer functions with denominators in a prescribed set S is a principal ideal domain, and system invariants under feedback and cascade control are characterized by the free $\mathbb{R}_S$-module generated by the columns of the system transfer function. Generalized polynomials are introduced in [11] as rational functions without poles in a certain region of the complex plane, and are used to describe the set of all transfer functions for particular signals using a set of admissible controllers. In [14] it is shown that the ring of proper and stable rational functions is a Euclidean domain, and the Euclidean division process in this ring is characterized. The algebra of proper and Ω-stable rational functions and matrices is examined in [13], where Ω is a symmetric region of the finite complex plane $\mathcal{C}$, containing $\mathcal{C}^+ := \{s \in \mathcal{C}, \mathrm{Re}(s) \geq 0\}$, and excluding at least one negative real number.

The aim of this work is to continue investigation on the algebraic properties of generalized polynomials and generalized polynomial matrices, and to present an application of these properties to the problem of decoupling with Λ-stability of linear square multivariable systems.

Regarding the study of generalized polynomials, there exist similarities between our work and the aforementioned references, since we are basically dealing with similar objects. However, we believe our approach is more general and comprehensive in the sense that comprises generalized polynomials as defined in previous works. In these references, generalized polynomials are limited to proper rational functions with poles in a region of the complex plane. With the inclusion of the point at infinity, and considering a region of the extended complex plane, a wider set of rational functions, like polynomials and nonproper rational functions, can also be considered as particular cases of Λ-generalized polynomials.

Besides the study of the algebraic properties of Λ-generalized polynomials and polynomial matrices, the main contribution of this paper is the solution of the decoupling problem with Λ-stability of linear square multivariable systems. It is proved that a linear system is decouplable with Λ-stability if and only if the so-defined Λ-stable interactor of the system is a diagonal matrix, or equivalently, if and only if the row and global zero structure of the system out of the region Λ coincide. The Λ-stable interactor and the row and global zero structure of the system out of Λ are defined via the Hermite column form and the Smith form of the system transfer function over the ring of Λ-generalized polynomials. If the system is decouplable with Λ-stability, a procedure to compute a decoupling state feedback is presented. It is also shown that the conditions for the row by row decoupling problem and decoupling with stability can be easily obtained as particular cases of our solution.

This work is organized as follows. In Section 2, Λ-generalized polynomials are introduced and their algebraic properties are studied. Λ-generalized polynomials will be basically defined as a set of rational functions with poles in a prescribed region of the extended complex plane. With a suitable definition of a degree function, it will be proved that this set is a Euclidean ring, and that widely used sets of rational functions like polynomials, proper rational and proper and stable rational functions can be regarded as particular cases of Λ-generalized polynomials. Conditions for division and coprimeness in the set of Λ-generalized polynomials in terms of zeros out of the region Λ are also presented. Section 3 deals with Λ-generalized polynomial matrices. Elementary operations over the ring of Λ-generalized polynomials are defined, and it is shown the relationship of these operations with unimodular matrices. Canonical forms of Λ-generalized matrices are introduced based on matrix equivalence and elementary operations, namely the Hermite form and the Smith form. An application of Λ-generalized polynomials and matrices is presented in Section 4, where we study the problem of decoupling with Λ-stability of linear multivariable systems. Finally, we end with some conclusions.

2 Λ-Generalized polynomials

In Section 2.1, Λ-generalized polynomials are introduced, and the algebraic properties of such polynomials are studied in Section 2.2.

2.1 Definition of Λ-generalized polynomials

Let $\mathbb{R}$ denote the field of real numbers. A polynomial in the variable s with coefficients in $\mathbb{R}$ is an expression of the form

$$a(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$$

where $a_i \in \mathbb{R}$, $i = 0, \ldots, n$. The set of such polynomials will be denoted by $\mathbb{R}[s]$.

As usual, if $a_n \neq 0$ then $a(s)$ has degree n, denoted deg $a(s)$, and if $a_n = 1$, $a(s)$ is said to be monic. A root of $a(s)$ is a complex number λ such that $a(\lambda) = 0$.

Consider now rational fractions of the form

$$f(s) = \frac{a(s)}{b(s)} \tag{2.1}$$

where $a(s)$ and $b(s)$ are coprime polynomials, and $b(s) \neq 0$. The set of rational functions over $\mathbb{R}$, denoted $\mathbb{R}(s)$, is the set of functions of the form (2.1).

Through this work, $\mathcal{C}$ will stand for the complex plane and $\mathcal{C}^*$ for the extended complex plane, i.e., $\mathcal{C}^* = \mathcal{C} \cup \{\infty\}$. Also, $\mathcal{C}_-$ will denote the open left half complex plane, and $\mathcal{C}_-^*$ its extended version, i.e., $\mathcal{C}_-^* = \mathcal{C}_- \cup \{\infty\}$.

Definition 2.1 Let $f(s) = a(s)/b(s)$ be a rational function. The point $s = w_1 \in \mathcal{C}^*$ is said to be a zero of $f(s)$ if $lim_{\, s \to w_1} f(s) = 0$, and the point $s = w_2 \in \mathcal{C}^*$ is said to be a pole of $f(s)$ if $lim_{\, s \to w_2} f(s) = \infty$.

According to this definition, any rational function $f(s) \in \mathbb{R}(s)$ has the same number of poles and zeros since the point at infinity is taken into account.

Definition 2.2 The rational function $f(s) = a(s)/b(s)$ is said to be proper if deg $b(s) \geq$ deg $a(s)$, and strictly proper if deg $b(s) >$ deg $a(s)$.

Definition 2.3 The rational function $f(s)$ is said to be proper and stable if $f(s)$ is proper, and its poles lie in the open left half complex plane $\mathcal{C}_-$.

Definition 2.4 Let Λ be a subset of $\mathcal{C}^*$ with the following properties:

- if $s_1 = \alpha + j\omega \in \Lambda$, then $s_2 = \alpha - j\omega \in \Lambda$,
- at least one point of the real axis belongs to Λ,

then Λ is called a symmetric subset of $\mathcal{C}^*$.

These restrictions on Λ are basically due to the fact that we are dealing with rational functions with real coefficients. As a special case, we can also define $\Lambda = \{\infty\}$.

The main concept of this section is the following.

Definition 2.5 Given a symmetric set $\Lambda \subset \mathcal{C}^*$, the set of Λ-generalized polynomials, denoted $\mathbb{R}_\Lambda(s)$, is defined as the set of rational functions whose poles are in Λ.

The reason for using the term "polynomial" in the previous definition is because the set $\mathbb{R}_\Lambda(s)$ shares much more properties (from the algebraic point of view) with the polynomials than it does with the rational functions, as we will see. Actually, it will be shown that polynomials are a particular case of Λ-generalized polynomials.

Example 2.1 In this example we specify some regions for Λ and find their corresponding set $\mathbb{R}_\Lambda(s)$.

a) When Λ is taken as the extended complex plane $\mathcal{C}^*$, then $\mathbb{R}_\Lambda(s)$ is the set of rational functions $\mathbb{R}(s)$.
b) If Λ is defined as the complex plane, $\Lambda = \mathcal{C}$, then $\mathbb{R}_\Lambda(s)$ is the set of proper rational functions. In this case $f(s) \in \mathbb{R}_\Lambda(s)$ has to have all its poles in finite positions within the complex plane as the proper rational functions do. The set of proper rational functions will be denoted by $\mathbb{R}_{\mathrm{p}}(s)$.
c) If $\Lambda = \{\infty\}$ then $\mathbb{R}_\Lambda(s)$ is the set of polynomials $\mathbb{R}[s]$. This can also be seen from the fact that any polynomial can be considered as a rational function $f(s) = a(s)/b(s)$, where $b(s) = 1$, i.e., a rational function with no finite poles.
d) If $\Lambda = \mathcal{C}^*_-$, then $\mathbb{R}_\Lambda(s)$ is the set of stable rational functions.
e) If $\Lambda = \mathcal{C}_-$, then $\mathbb{R}_\Lambda(s)$ is the set of proper and stable rational functions, hereafter denoted $\mathbb{R}_{\mathrm{ps}}(s)$.

2.2 Properties of Λ-generalized polynomials

In this section, we examine some of the algebraic properties of Λ-generalized polynomials.

With the usual definitions of addition and multiplication of rational functions, it can be seen that the set $\mathbb{R}_\Lambda(s)$ is closed under these operations, i.e., if $f(s)$, $g(s) \in \mathbb{R}_\Lambda(s)$, then

- $f(s) + g(s) \in \mathbb{R}_\Lambda(s)$, and
- $f(s)\,g(s) \in \mathbb{R}_\Lambda(s)$.

These properties, along with some others, convert $\mathbb{R}_\Lambda(s)$ to a ring. Furthermore, $\mathbb{R}_\Lambda(s)$ is a commutative ring with multiplicative unit 1 and additive unit 0.

It can also be seen that $\mathbb{R}_\Lambda(s)$ is an integral domain, since for any $f(s)$, $g(s) \in \mathbb{R}_\Lambda(s)$, we have that

- $f(s)g(s) = g(s)f(s)$, and
- $f(s)g(s) = 0$ implies that $f(s) = 0$, or $g(s) = 0$.

Definition 2.6 An element $f(s) \in \mathbb{R}_\Lambda(s)$ is said to be an invertible element (or a unit) of $\mathbb{R}_\Lambda(s)$ if there exists $g(s) \in \mathbb{R}_\Lambda(s)$ such that $f(s)g(s) = 1$.

The next step is to prove that $\mathbb{R}_\Lambda(s)$ is a Euclidean ring.

Definition 2.7 An entire ring R is said to be Euclidean if there exists a function $\gamma(\cdot)$ which associates a non-negative integer $\gamma(f)$ to any nonzero element f of R, such that the following properties are satisfied for $f,\ g \in R$, $g \neq 0$:

a) $\gamma(f) \leq \gamma(fg)$
b) there exist q and r in R such that
$$f = gq + r$$
and, either $r = 0$, or $\gamma(r) < \gamma(g)$.

Property b) is the so-called Euclidean division property, where q is a quotient, r is a remainder and $\gamma(\cdot)$ is a gauge.

We will define a gauge in the ring $\mathbb{R}_\Lambda(s)$ in the following way.

Definition 2.8 Let $f(s)$ be a nonzero element of $\mathbb{R}_\Lambda(s)$. We define the function $\gamma(f(s))$, called the degree of $f(s)$ and denoted $\deg_\Lambda f(s)$, as the number of zeros of $f(s)$ lying outside of Λ.

Remark 2.1 Observe that this definition agrees completely with the definition of the degree of an element in the rings $\mathbb{R}[s]$, $\mathbb{R}_p(s)$ and $\mathbb{R}_{ps}(s)$, where:

- If $a(s) \in \mathbb{R}[s]$ is a polynomial ($\Lambda = \infty$), then $\gamma(a(s)) = \deg\ a(s)$, which corresponds to the number of finite zeros of $f(s)$.
- If $f(s) = a(s)/b(s) \in \mathbb{R}_p(s)$ is a proper rational function ($\Lambda = \mathcal{C}$), then $\gamma(f(s)) = \deg\ b(s) - \deg\ a(s)$, which corresponds to the number of infinite zeros of $a(s)$.
- If $g(s) \in \mathbb{R}_{ps}(s)$ is a proper and stable rational function ($\Lambda = \mathcal{C}_-$), then $\gamma(g(s)) =$ number of unstable + infinite zeros of $g(s)$, i.e., number of zeros of $g(s)$ outside of $\mathcal{C}_-$.

Through this work, the degree of the polynomial $a(s) \in \mathbb{R}[s]$ will be denoted by $\deg\ a(s)$, the degree of the proper rational function $f(s) \in \mathbb{R}_p(s)$ will be denoted by $\deg_p\ f(s)$, and the degree of the proper and stable rational function $g(s) \in \mathbb{R}_{ps}(s)$ will be denoted by $\deg_{ps}\ g(s)$.

Example 2.2 Let $\mathbb{R}_p(s)$ be the ring of proper rational functions ($\Lambda = \mathcal{C}$), and consider the functions

$$f_1(s) = \frac{(s-1)(s+1)}{(s+2)}, \quad f_2(s) = \frac{(s-1)^2}{(s+2)(s+3)(s+4)}$$

It can be seen that $f_1(s) \notin \mathbb{R}_p(s)$ because it has a pole outside of Λ, namely the infinite pole, while $f_2(s) \in \mathbb{R}_p(s)$ and $\deg_p f_2(s) = 1$.

With this definition of $\deg_\Lambda\ f(s)$, the following is immediate.

Lemma 2.1 The units of the ring $\mathbb{R}_\Lambda(s)$ are the nonzero elements $f(s) \in \mathbb{R}_\Lambda(s)$ such that $\deg_\Lambda f(s) = 0$.

Proof. The fact that $\deg_\Lambda\ f(s) = 0$ implies that $f(s)$ has no zeros outside of Λ. Then there exists an element $g(s) := 1/f(s)$ which belongs to $\mathbb{R}_\Lambda(s)$, since the poles of $g(s)$ are the zeros of $f(s)$, and such that $f(s)g(s) = 1$. ■

Notice that the set of units of $\mathbb{R}_\Lambda(s)$ forms a field. In particular, for $\Lambda = \mathcal{C}^*$, any $f(s)$ is a unit of $\mathbb{R}_\Lambda(s) = \mathbb{R}(s)$, emphasizing the fact that the set of rational functions $\mathbb{R}(s)$ is also a field.

¿From the algebraic point of view, the following is a fundamental property of $\mathbb{R}_\Lambda(s)$.

Theorem 2.1 Given a symmetric set Λ, then $\mathbb{R}_\Lambda(s)$ is a Euclidean ring, where the gauge $\gamma(\cdot)$ is taken as in Definition 2.8. ■

Let $f(s),\ g(s) \in \mathbb{R}_\Lambda(s)$ for a given Λ, with $f(s) \neq 0$, $g(s) \neq 0$. From Definition 2.8, it can be seen that

$$\deg_\Lambda[f(s)g(s)] = \deg_\Lambda\ f(s) + \deg_\Lambda\ g(s),$$

then,

$$\deg_\Lambda\ f(s) \leq \deg_\Lambda\ [f(s)g(s)]. \tag{2.2}$$

¿From this, and following Definition 2.7, $\mathbb{R}_\Lambda(s)$ will be proved to be a Euclidean ring if Euclidean division applies, i.e., if there exist $q(s)$ and $r(s) \in \mathbb{R}_\Lambda(s)$ such that

$$f(s) = g(s)q(s) + r(s) \tag{2.3}$$

where

$$r(s) = 0, \quad \text{or} \quad \deg_\Lambda r(s) < \deg_\Lambda g(s). \tag{2.4}$$

In order to prove this, we need the following preliminary result (notice that in the following Lemma 2.2 we are considering elements in the polynomial ring $\mathbb{R}[s]$.)

Lemma 2.2 [14] Let $a(s)$, $b(s)$ and $w(s) \in \mathbb{R}[s]$ be given polynomials, where $a(s)$ and $b(s)$ are coprime. Then, there exist polynomials $\phi(s)$ and $\psi(s) \in \mathbb{R}[s]$ such that

$$a(s)\phi(s) + b(s)\psi(s) = w(s) \tag{2.5}$$

and, either,

$$\deg \phi(s) < \deg b(s), \quad \text{or} \quad \deg \psi(s) < \deg a(s).$$

■

Proof of Theorem 2.1

Let $f(s)$, $g(s) \in \mathbb{R}_\Lambda(s)$, with $g(s) \neq 0$. We will prove that there exist $q(s)$ and $r(s) \in \mathbb{R}_\Lambda(s)$ satisfying (2.3) and (2.4).

First, if $\deg_\Lambda g(s) = 0$, then $g(s)$ is a unit of $\mathbb{R}_\Lambda(s)$ and the required $q(s)$, $r(s) \in \mathbb{R}_\Lambda(s)$ are given by

$$q(s) = \frac{f(s)}{g(s)}, \qquad r(s) = 0.$$

Let us suppose that $\deg_\Lambda g(s) > 0$. We will differentiate between two cases depending whether or not Λ contains the point at infinity.

a) Suppose that Λ does not include the point at infinity, i.e., all the poles of $f(s)$ and $g(s)$ are finite (which is equivalent to say that $f(s)$ and $g(s)$ are proper rational functions.)

Let us write

$$f(s) = \frac{a_f(s)}{b_f(s)}, \qquad g(s) = \frac{a_g(s)}{b_g(s)},$$

for coprime polynomials $a_f(s)$, $b_f(s)$ and $a_g(s)$, $b_g(s)$.

Denote $n := \deg_\Lambda g(s)$ and factorize $a_g(s)$ as

$$a_g(s) = a_g^-(s) a_g^+(s)$$

where $a_g^-(s)$ contains the roots of $a_g(s)$ lying outside of Λ, and $a_g^+(s)$ contains the roots of $a_g(s)$ in Λ.

Since $b_f(s)$ and $a_g^-(s)$ are coprime, by Lemma 2.2 there exist $\phi(s)$ and $\psi(s) \in \mathbb{R}[s]$ such that

$$a_g^-(s)\phi(s) + b_f(s)\psi(s) = a_f(s)\pi^{n-1} \tag{2.6}$$

where $\deg \phi(s) < \deg b_f(s)$ and $\pi := s + \beta$ with $\beta \in \mathbb{R}$, and $-\beta \in \Lambda$.
Dividing both sides of (2.6) by $b_f(s)\pi^{n-1}$ yields

$$\frac{a_g^-(s)\phi(s)}{b_f(s)\pi^{n-1}} + \frac{\psi(s)}{\pi^{n-1}} = \frac{a_f(s)}{b_f(s)} \tag{2.7}$$

Let us define

$$q(s) = \frac{\phi(s)\pi}{b_f(s)}\, u(s), \qquad r(s) = \frac{\psi(s)}{\pi^{n-1}} \tag{2.8}$$

where

$$u(s) = \frac{b_g(s)}{a_g^+(s)\pi^n} \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$f(s) = g(s)q(s) + r(s).$$

With $q(s)$ defined as above, it is clear that it belongs to $\mathbb{R}_\Lambda(s)$ since $u(s)$ is a unit of $\mathbb{R}_\Lambda(s)$, $q(s)$ has no infinite poles (due to the fact that $\deg \phi(s) < \deg b_f(s)$), and the finite poles of $q(s)$ are in Λ.

Next, as $\mathbb{R}_\Lambda(s)$ is a ring, it follows that

$$r(s) = f(s) - g(s)q(s) \in \mathbb{R}_\Lambda(s).$$

Furthermore, $r(s)$ is a proper rational function with $n-1$ zeros, including finite and infinite ones, and consequently

$$\deg_\Lambda\ r(s) \leq n - 1 < \deg_\Lambda\ g(s).$$

Notice that in this case the remainder $r(s)$ given by (2.8) is not unique, but depends on the term π.

b) Suppose that Λ includes the point at infinity. Write again

$$g(s) = \frac{a_g(s)}{b_g(s)} = \frac{a_g^+(s)a_g^-(s)}{b_g(s)}$$

where $a_g^-(s)$ contains the finite zeros of $g(s)$ outside of Λ. Thus,

$$n := \deg_\Lambda\ g(s) = deg\ a_g^-(s).$$

Now, since $a_g^-(s)$ and $b_f(s)$ are coprime, by Lemma 2.2 there exist $\phi(s), \psi(s) \in \mathbb{R}[s]$ such that

$$a_g^-(s)\phi(s) + b_f(s)\psi(s) = a_f(s) \tag{2.10}$$

where $\deg \psi(s) < \deg a_g^-(s)$.

Dividing (2.10) by $b_f(s)$ we get

$$\frac{a_g^-(s)\phi(s)}{b_f(s)} + \psi(s) = \frac{a_f(s)}{b_f(s)}$$

So, if we define

$$q(s) = \frac{\phi(s)b_g(s)}{b_f(s)a_g^+(s)}, \qquad r(s) = \psi(s) \tag{2.11}$$

it follows that $f(s) = g(s)q(s) + r(s)$.

¿From (2.11), it can be seen that $q(s) \in \mathbb{R}_\Lambda(s)$. Furthermore, $r(s) \in \mathbb{R}_\Lambda(s)$ and

$$\deg_\Lambda\ r(s) = \deg \psi^-(s) \leq \deg \psi(s) < \deg a_g^-(s) = \deg_\Lambda\ g(s).$$

Then

$$\deg_\Lambda\ r(s) < \deg_\Lambda\ g(s)$$

and this completes the proof. ■

As far as the sources is concerned, Lemma 2.2 is taken from [14], and the first part of the proof of Theorem 2.1 is also inspired on [14] where it is proved that the set of proper and stable rational functions is a Euclidean ring.

In order to illustrate the division algorithm for Λ-generalized polynomials presented in the proof of Theorem 2.1, consider the following example.

Example 2.3 Let $\Lambda = \mathcal{C}_-$, then $\mathbb{R}_\Lambda(s)$ is the ring of proper and stable rational functions $\mathbb{R}_{ps}(s)$. Let $f(s),\ g(s) \in \mathbb{R}_{ps}(s)$ be given by

$$f(s) = \frac{s-4}{(s+1)^3}, \quad g(s) = \frac{(s+3)(s-3)}{(s+2)^2},$$

and suppose that we want to divide $f(s)$ by $g(s)$. Then, there must exist $q(s), r(s) \in \mathbb{R}_{ps}(s)$ such that

$$f(s) = g(s)q(s) + r(s)$$

where $r(s) = 0$, or $\deg_{ps}\ r(s) < \deg_{ps}\ g(s)$.

Following the proof of Theorem 2.1 part a), we have

$$n = \deg_{ps}\ g(s) = 1$$

$$a_g(s) = \underbrace{(s+3)}_{a_g^+(s)}\underbrace{(s-3)}_{a_g^-(s)}$$

Then, the polynomials $\phi(s), \psi(s)$ such that

$$a_g^-(s)\phi(s) + b_f(s)\psi(s) = a_f(s)\pi^{n-1}$$

with $\deg \phi(s) < \deg b_f(s)$, are given by

$$\phi(s) = \frac{1}{64}(s^2+6s+85), \quad \psi(s) = -\frac{1}{64},$$

which implies that

$$q(s) = \frac{1}{64}\frac{(s+2)^2(s^2+6s+85)}{(s+3)(s+1)^3}, \quad r(s) = -\frac{1}{64}$$

are the Λ-generalized polynomials we are looking for.

Before ending this section, some more concepts are further introduced.

Definition 2.9 Let $\mathbb{R}_\Lambda(s)$ be a ring of Λ-generalized polynomials for a given Λ, and let $f(s)$ and $h(s) \in \mathbb{R}_\Lambda(s)$. Then $h(s)$ will be said to divide $f(s)$, denoted $h(s) \mid f(s)$, (or that $f(s)$ is a multiple of $h(s)$) if there exists $x(s) \in \mathbb{R}_\Lambda(s)$ such that $f(s) = h(s)\,x(s)$.

Lemma 2.3 Let $h(s),\ f(s) \in \mathbb{R}_\Lambda(s)$. Then $h(s)$ divides $f(s)$ if and only if each zero of $h(s)$ outside of Λ is also a zero of $f(s)$, at least with the same multiplicity.

Proof. Necessity. The claim is trivial from the fact that in

$$f(s) = h(s)x(s)$$

no cancellation of zeros outside of Λ of $h(s)$ are produced in the product $h(s)x(s)$.

Sufficiency. Factorize $f(s)$ as

$$f(s) = h(s)\bar{x}(s)$$

for some rational function $\bar{x}(s)$. We can see that $\bar{x}(s)$ has no poles outside of Λ because $f(s),\ h(s) \in \mathbb{R}_\Lambda(s)$, and the zeros of $h(s)$ outside of Λ are contained in $f(s)$. Thus, necessarily $\bar{x}(s) \in \mathbb{R}_\Lambda(s)$.

Example 2.4 Let $\mathbb{R}_\Lambda(\mathrm{s})$ be the ring of proper and stable rational functions, and consider the following rational functions in $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$:

$$f_1(s) = \frac{(s+2)(s-1)}{(s+1)^2}, \quad f_2(s) = \frac{(s+1)(s-2)}{(s+4)^3}, \quad f_3(s) = \frac{(s-1)(s-2)}{(s+1)^4}$$

According to Lemma 2.3, the only divisibility relations between these functions are $f_1(s) \mid f_3(s)$ and $f_2(s) \mid f_3(s)$, which can be easily verified.

Remark 2.2 From Lemma 2.3, it can be seen that if $h(s) \mid f(s)$ then

$$\deg_\Lambda\, h(s) \leq \deg_\Lambda\, f(s). \tag{2.12}$$

A special case occurs in the ring of proper rational functions $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$, where condition (2.12) is not only necessary but also sufficient for $h(s)$ to divide $f(s)$. For this kind of Λ-generalized polynomials, if $\deg_\Lambda\, h(s) \leq \deg_\Lambda\, f(s)$, then the zeros outside of $\Lambda(:= \mathcal{C})$ of $h(s)$ (zeros at infinity) are also zeros of $f(s)$. Then, we have the following result.

Corollary 2.1 Let $h(s)$, $f(s) \in \mathbb{R}_{\mathrm{p}}(\mathrm{s})$ be proper rational functions. Then $h(s)$ divides $f(s)$ if and only if $\deg_{\mathrm{p}}\, h(s) \leq \deg_{\mathrm{p}}\, f(s)$. ■

Definition 2.10 The function $h(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ is said to be a greatest common divisor (gcd) of $f(s)$ and $g(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ if

- $h(s)$ is a common divisor of $f(s)$ and $g(s)$, i.e., $h(s) \mid f(s)$ and $h(s) \mid g(s)$, and
- any other common divisor of $f(s)$ and $g(s)$ is a divisor of $h(s)$.

A gcd of a set of more than 2 functions can be defined in an analogous manner.

Observe that if $h(s)$ is a gcd of $f(s)$ and $g(s)$, then $h_1(s) := h(s)u(s)$ is also a gcd of $f(s)$ and $g(s)$, where $u(s)$ is a unit of $\mathbb{R}_\Lambda(\mathrm{s})$. Thus, a greatest common divisor of two generalized polynomials is unique up to multiplication by units of the ring.

Remark 2.3 The relevant information in determining the gcd of the functions $f_i(s)$ is the set of zeros outside of Λ which are common to all these elements. Indeed, let $h(s)$ be a gcd of a finite set of Λ-generalized polynomials $f_1(s)$, $f_2(s)$, …, $f_k(s)$. From Lemma 2.3 and Definition 2.10 it can be seen that

$$\deg_\Lambda\, h(s) = n,$$

where n is the number of zeros outside of Λ which are common to these functions $f_1(s)$, $f_2(s)$, …, $f_k(s)$.

Example 2.5 Let $\mathbb{R}_\Lambda(\mathrm{s})$ be the ring of proper rational functions $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$, and let $f_1(s)$, $f_2(s)$ and $f_3(s) \in \mathbb{R}_{\mathrm{p}}(\mathrm{s})$ be given by

$$f_1(s) = \frac{s+2}{(s+1)^3}, \quad f_2(s) = \frac{s-8}{s^2(s-2)^3}, \quad f_3(s) = \frac{1}{s^3}$$

The zeros outside of Λ common to these functions are two zeros at infinity (or one zero at infinity of order 2.) Then, the set of gcd's of these functions is characterized by

$$h(s) = \frac{1}{s^2}\, u(s)$$

where $u(s)$ is a unit of $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$.

It is suggested in Remark 2.3 that a gcd of a set of functions can be obtained analytically by inspection of the zeros of these functions. We will show that this can be also done in an algorithmic way.

The algorithm to obtain a gcd for polynomials is well known in the literature (see, for instance [7]) and it is based on the principle of polynomial division. Since we have already shown that Euclidean division also applies for Λ-generalized polynomials, this procedure can be readily extended to cope with this general case.

Let $f(s)$ and $g(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ be two nonzero Λ-generalized polynomials. Since $\mathbb{R}_\Lambda(\mathrm{s})$ is a Euclidean ring, there exist $q_1(s)$ and $r_1(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ such that

$$f(s) = g(s)q_1(s) + r_1(s)$$

where $r_1(s) = 0$, or $\deg_\Lambda\, r_1(s) < \deg_\Lambda\, g(s)$.

Suppose that $r_1(s) \neq 0$ and consider now the pair $g(s)$, $r_1(s)$. Again, there exist $q_2(s)$ and $r_2(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ such that

$$g(s) = r_1(s)q_2(s) + r_2(s)$$

where $r_2(s) = 0$, or $\deg_\Lambda\, r_2(s) < \deg_\Lambda r_1(s)$.

If we take now the pair $r_1(s)$, $r_2(s)$ and continue applying this procedure we will arrive at some step i where $r_i(s) = 0$ and $r_{i-1}(s) \neq 0$ since $f(s)$ and $g(s)$ are of finite degree.

Basically, what we are doing with this algorithm on the pair of Λ-generalized polynomials involved at any step is to subtract multiples of the lower degree Λ-generalized polynomial from the other until one of them is reduced to zero. Then, not surprisingly, $r_{i-1}(s) \neq 0$ will be the gcd of $f(s)$ and $g(s)$.

If we arrange $f(s)$ and $g(s)$ as a row matrix

$$\left[\begin{array}{cc} f(s) & g(s) \end{array}\right]$$

the operations described above amount to find a 2×2 Λ-generalized matrix

$$V(s) = \left[\begin{array}{cc} p(s) & r(s) \\ q(s) & s(s) \end{array}\right]$$

such that

$$\left[\begin{array}{cc} f(s) & g(s) \end{array}\right] \left[\begin{array}{cc} p(s) & r(s) \\ q(s) & s(s) \end{array}\right] = \left[\begin{array}{cc} h(s) & 0 \end{array}\right]$$

where $h(s)$ is a gcd of $f(s)$ and $g(s)$.

It is also clear from this algorithm that if $h(s)$ is a gcd of $f(s)$ and $g(s)$ we can always write

$$f(s)p(s) + g(s)q(s) = h(s) \tag{2.13}$$

for some $p(s)$ and $q(s) \in \mathbb{R}_\Lambda(\mathrm{s})$.

Definition 2.11 Let $h(s)$ be a gcd of $f(s)$ and $g(s) \in \mathbb{R}_\Lambda(\mathrm{s})$. Then $f(s)$ and $g(s)$ are called coprime if $h(s)$ is a unit of $\mathbb{R}_\Lambda(\mathrm{s})$.

The following result is immediate from Remark 2.3.

Lemma 2.4 The Λ-generalized polynomials $f(s)$ and $g(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ are coprime if they do not have any common zeros outside of Λ. ■

Another characterization of coprimeness for Λ-generalized polynomials is the following Bézout identity, which is the natural extension of the well known Bézout identity for polynomials (see for instance [7]).

Lemma 2.5 The Λ-generalized polynomials $f(s)$ and $g(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$ are coprime if and only if there exist $x(s)$ and $y(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$ such that

$$f(s)x(s) + g(s)y(s) = 1 \tag{2.14}$$

Proof. Necessity. From (2.13) there exist $p(s)$ and $q(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$ such that

$$f(s)p(s) + g(s)q(s) = h(s),$$

where $h(s)$ is a gcd of $f(s)$ and $g(s)$. Since $f(s)$ and $g(s)$ are coprime, then $h(s)$ is a unit of $\mathrm{IR}_\Lambda(\mathrm{s})$. Set $x(s) = p(s)h^{-1}(s)$ and $y(s) = q(s)h^{-1}(s)$. Thus

$$f(s)x(s) + g(s)y(s) = 1$$

where $x(s),\ y(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$.

Sufficiency. Let $h(s)$ be any gcd of $f(s)$ and $g(s)$, i.e.,

$$f(s) = h(s)\bar{f}(s), \qquad g(s) = h(s)\bar{g}(s)$$

for some $\bar{f}(s),\ \bar{g}(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$. Under the assumption that (2.14) holds, we can write

$$h(s)\{\bar{f}(s)x(s) + \bar{g}(s)y(s)\} = 1$$

so that $h^{-1}(s) = \bar{f}(s)x(s) + \bar{g}(s)y(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$.

Thus, $f(s)$ and $g(s)$ are coprime since $h(s)$ is a unit of $\mathrm{IR}_\Lambda(\mathrm{s})$. ■

3 Λ-Generalized polynomial matrices

Matrices whose entries are Λ-generalized polynomials will be studied in this section. As in the case of Λ-generalized polynomials, we will see that these matrices have analogous properties to those of polynomial matrices. Definitions and results about elementary operations over the ring of Λ-generalized polynomials and matrix equivalence are presented, as well as canonical forms (Hermite form and Smith form) related to matrix equivalence and elementary operations. These canonical forms can be considered as the natural extensions of similar concepts for polynomial matrices, and they will be used in Section 4 to study the problem of decoupling with Λ-stability of linear multivariable systems. Actually, many classical results related to polynomial matrices, like coprimeness, polynomial matrix equations, etc., which are not considered here, can be obtained as straightforward extensions for Λ-generalized polynomial matrices.

The set of $m \times n$ matrices whose entries belong to $\mathrm{IR}_\Lambda(\mathrm{s})$ will be denoted by $\mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$, and elements of $\mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$ will be written in upper case letters.

Addition and multiplication of Λ-generalized matrices is defined in the usual way, i.e., let $A(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$, $B(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$, and $C(s) \in \mathrm{IR}_\Lambda^{\mathrm{n}\times\mathrm{l}}(\mathrm{s})$, then:

The sum of $A(s)$ and $B(s)$ is the $m \times n$ matrix

$$X(s) = A(s) + B(s)$$

where $x_{ij}(s) = a_{ij}(s) + b_{ij}(s)$.

The product of $A(s)$ and $C(s)$ is the $m \times l$ matrix

$$Y(s) = A(s)C(s)$$

where

$$y_{ij}(s) = \sum_{k=1}^{n} a_{ik}(s) b_{kj}(s).$$

With addition and multiplication of Λ-generalized matrices as defined above, the set of square matrices which belong to $\mathbb{R}_\Lambda^{m\times m}(s)$ is readily seen to be a ring with respect to these operations, however not commutative because matrix multiplication is not commutative.

Definition 3.1 A matrix $U(s) \in \mathbb{R}_\Lambda^{m\times m}(s)$ is said to be a unit of the ring $\mathbb{R}_\Lambda^{m\times m}(s)$ if there exists $V(s) \in \mathbb{R}_\Lambda^{m\times m}(s)$ such that

$$U(s)V(s) = V(s)U(s) = I_m \tag{3.1}$$

Units of $\mathbb{R}_\Lambda^{m\times m}(s)$ will be called $\mathbb{R}_\Lambda(s)$-unimodular matrices. Then, the units of $\mathbb{R}_\Lambda^{m\times m}(s)$ for polynomial, proper, and proper and stable rational matrices will be called, respectively, $\mathbb{R}[s]$-unimodular, $\mathbb{R}_p(s)$-unimodular and $\mathbb{R}_{ps}(s)$-unimodular matrices.

Lemma 3.1 A nonsingular square matrix $U(s) \in \mathbb{R}_\Lambda^{m\times m}(s)$ is $\mathbb{R}_\Lambda(s)$-unimodular if and only if det $U(s)$ is a unit of $\mathbb{R}_\Lambda(s)$, i.e., a nonzero rational function whose degree is zero.

Proof. Necessity. From (3.1) we have that

$$U^{-1}(s) = \frac{\text{Adj } U(s)}{\det U(s)} \in \mathbb{R}_\Lambda^{m\times m}(s),$$

then $(\det U(s))^{-1} \in \mathbb{R}_\Lambda(s)$, i.e., det $U(s)$ is a unit of $\mathbb{R}_\Lambda(s)$.

Sufficiency. Conversely, if det $U(s)$ is a unit of $\mathbb{R}_\Lambda(s)$, then $U^{-1}(s) = \frac{\text{Adj } U(s)}{\det U(s)} \in \mathbb{R}_\Lambda^{m\times m}(s)$. ∎

3.1 Elementary operations

Definition 3.2 Given a matrix $A(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$, elementary column operations on $A(s)$ (over $\mathbb{R}_\Lambda(s)$) are defined as

i) changing the ith and jth columns,
ii) multiplying the ith column by a unit of $\mathbb{R}_\Lambda(s)$, and,
iii) adding to the ith column the jth column multiplied by any element of $\mathbb{R}_\Lambda(s)$.

Elementary row operations are defined with the obvious changes.

The relationship between elementary column operations and unimodular matrices is as follows: Given $A(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$, any set of elementary column operations on $A(s)$ is equivalent to postmultiplying $A(s)$ by a $\mathbb{R}_\Lambda(s)$-unimodular matrix $U(s) \in \mathbb{R}_\Lambda^{n\times n}(s)$, and vice versa.

Let us be more precise. Let $A(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$. We will use the notation

$$k: \quad c_i \longrightarrow c_i + \alpha(s)c_j \tag{3.2}$$

to indicate that in step k, a column operation is performed on $A_{k-1}(s)$ in such a way that column i will be replaced by the sum of column i plus column j multiplied by $\alpha(s)$, where $\alpha(s) \in \mathbb{R}_\Lambda(s)$, $k > 0$, and $A_0(s) := A(s)$. The resulting matrix will be called $A_k(s)$.

Following this notation the elementary column operations of Definition 3.2 can be represented, respectively, as

i) $c_i \longleftrightarrow c_j$,

ii) $c_i \longrightarrow u(s)\ c_i$, where $u^{-1}(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$, and
iii) $c_i \longrightarrow c_i + \alpha(s)\ c_j$, where $\alpha(s) \in \mathrm{IR}_\Lambda(\mathrm{s})$, $i \neq j$.

To keep track of these operations, let $U_k(s)$ denote the matrix obtained by performing on $U_{k-1}(s)$ the same elementary operation as on $A_{k-1}(s)$, where $U_0(s) := I_n$. Then, it results that

$$A(s)U_k(s) = A_k(s). \tag{3.3}$$

In other words, $U_k(s)$ is the unimodular matrix which is equivalent to the set of elementary column operations carried out on $A(s)$.

Analogously, any set of elementary row operations on $A(s)$ is equivalent to premultiplying $A(s)$ by a $\mathrm{IR}_\Lambda(\mathrm{s})$-unimodular matrix and vice versa.

3.2 Matrix equivalence

Definition 3.3 Let $A(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$ and $B(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$ be Λ-generalized polynomial matrices. Then, $A(s)$ and $B(s)$ are said to be, respectively, *a)* right equivalent, *b)* left equivalent, *c)* equivalent, if there exist $\mathrm{IR}_\Lambda(\mathrm{s})$-unimodular matrices $U(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{m}}(\mathrm{s})$ and $V(s) \in \mathrm{IR}_\Lambda^{\mathrm{n}\times\mathrm{n}}(\mathrm{s})$, such that

a) $A(s) = B(s)V(s)$
b) $A(s) = U(s)B(s)$
c) $A(s) = U(s)B(s)V(s)$.

Left and right equivalence will be also referred as one-sided equivalence, while c) will be called two-sided equivalence, or simple, equivalence.

The relations *a*), *b*) and *c*) are readily seen to be equivalence relations, since they satisfy the properties

- $A(s) \sim A(s)$
- if $A(s) \sim B(s)$, then $B(s) \sim A(s)$
- if $A(s) \sim B(s)$ and $B(s) \sim C(s)$, then $A(s) \sim C(s)$,

where $A(s) \sim B(s)$ means that $A(s)$ is equivalent to $B(s)$.

Definition 3.4 The set of matrices that are equivalent to $A(s)$ in the sense of Definition 3.3 will be called the orbit of $A(s)$, and denoted $\Omega(A)$.

One of the outstanding features of matrix equivalence is the existence of a particular matrix in the set $\Omega(A)$ which characterizes the properties common to all its elements. From the relationship between elementary operations and unimodular matrices, it turns out that this matrix can be computed from any element of $\Omega(A)$ by performing elementary operations on it, the type of which depends on the kind of equivalence between the elements of $\Omega(A)$, i.e., right equivalence amounts to elementary column operations, left equivalence to elementary row operations, and equivalence to elementary row and column operations.

Thus, any matrix $A(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$ can be "reduced" by elementary operations over $\mathrm{IR}_\Lambda(\mathrm{s})$ to a standard form, which will be proved to be a canonical form for A(s).

There exist two basic types of standard forms depending whether we use equivalence or one-sided matrix equivalence. The latter case is considered in the following result, where for simplicity we suppose that the matrix $A(s)$ has full row rank.

Theorem 3.1 Let $A(s) \in \mathrm{IR}_\Lambda^{\mathrm{m}\times\mathrm{n}}(\mathrm{s})$ with rank $A(s) = m$. Then there exists a $\mathrm{IR}_\Lambda(\mathrm{s})$-unimodular matrix

$U(s) \in \mathbb{R}_\Lambda^{n\times n}(s)$ and a matrix $H(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$, called the generalized (column) Hermite form of $A(s)$ and unique up to units of $\mathbb{R}_\Lambda(s)$, such that

$$A(s)U(s) = H(s) \tag{3.4}$$

where

$$H(s) = \begin{bmatrix} h_{11}(s) & & (0) & \vdots & \\ \vdots & \ddots & & \vdots & (0) \\ h_{1m}(s) & \dots & h_{mm}(s) & \vdots & \end{bmatrix} \tag{3.5}$$

and, for $i > j$,

$$h_{ij}(s) = 0, \quad \text{or} \quad \deg_\Lambda\ h_{ij}(s) < \deg_\Lambda\ h_{ii}(s). \tag{3.6}$$

Proof. The proof is constructive. The matrix $H(s)$ is obtained by elementary column operations on $A(s)$ and using the division algorithm for Λ-generalized polynomials. The procedure is essentially the same as in the case of the Hermite form of polynomial matrices (see [4].) ∎

It seems misleading to call $H(s)$ a canonical form for $A(s)$, when it happens that it is not completely unique but that it is unique up to units of $\mathbb{R}_\Lambda(s)$. We can overcome this difficulty by introducing the concept of a monic Λ-generalized polynomial. This notion, however, has to be established for every particular case of Λ-generalized polynomials. We will define it for the rings we are mainly interested in this work.

Definition 3.5 Consider the rings $\mathbb{R}[s]$, $\mathbb{R}_p(s)$, and $\mathbb{R}_{ps}(s)$, then:

- A polynomial will be taken to be monic as usually understood, i.e., $f(s) \in \mathbb{R}[s]$ is monic if the coefficient corresponding to its highest power is equal to 1.
- The proper rational function $f(s) \in \mathbb{R}_p(s)$ will be called monic if it is of the form $f(s) = \frac{1}{s^n}$, where $n = \deg_p\ f(s)$.
- The proper and stable rational function $f(s) \in \mathbb{R}_{ps}(s)$ will be called monic if it is of the form $f(s) = \frac{\epsilon(s)}{(s+\alpha)^n}$ where $\epsilon(s) \in \mathbb{R}[s]$ is a monic polynomial with only unstable roots (antistable polynomial), α is a positive real number, and $n = \deg_{ps}\ f(s)$.

The basic idea behind this definition is that from any $f(s) \in \mathbb{R}_\Lambda(s)$ we can obtain the simplest element (a monic one) by extracting units of $\mathbb{R}_\Lambda(s)$ from it.

Regarding Theorem 3.1, if we further specify that $h_{ii}(s)$ in (3.5) are monic Λ-generalized polynomials, then $H(s)$ is a unique matrix. We will consider that this is the case when speaking about the Hermite form of some matrix.

Considering left equivalence of matrices and making the necessary changes in Theorem 3.1 we can establish a generalized row Hermite form, which will not be discussed here. Two-sided equivalence, or simply equivalence of matrices, leads to the following result.

Theorem 3.2 Let $A(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$ with rank $A(s) = r \le \min(m, n)$. Then, there exist $\mathbb{R}_\Lambda(s)$-unimodular matrices $V_1(s) \in \mathbb{R}_\Lambda^{m\times m}(s)$ and $V_2(s) \in \mathbb{R}_\Lambda^{n\times n}(s)$ such that

$$V_1(s)A(s)V_2(s) = \Psi(s) \tag{3.7}$$

where

$$\Psi(s) = \begin{bmatrix} \operatorname{diag}\{\psi_i(s)\}_{i=1}^r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.8}$$

and $\psi_i(s)$ divides $\psi_{i+1}(s)$, i.e.,

$$\psi_i(s) \mid \psi_{i+1}(s), \qquad i = 1, \dots, r-1. \tag{3.9}$$

$\blacksquare$

The Λ-generalized polynomials $\psi_i(s)$ are unique up to multiplication by units of $\mathbb{R}_\Lambda(s)$. We will consider them to be monic, where as we stated previously, the definition of a monic Λ-generalized polynomial depends on the ring $\mathbb{R}_\Lambda(s)$, and they will be called the invariant factors of $A(s)$. Under this assumption the matrix $\Psi(s)$, hereafter called the Λ-generalized Smith form, or simply the Smith form of $A(s)$, is a canonical form for $A(s)$.

Definition 3.6 The determinantal divisors of $A(s) \in \mathbb{R}_\Lambda^{m\times n}(s)$ are defined as

$$\Delta_i(s) := \text{monic gcd of all } i \times i \text{ minors of } A(s), \quad i = 1, \ldots, r, \tag{3.10}$$

where $r = \text{rank } A(s)$.

Lemma 3.2 Let $\psi_i(s)$ and $\Delta_i(s)$ be, respectively, the invariant factors and determinantal divisors of $A(s)$. Then,

$$\psi_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)} \tag{3.11}$$

where $\Delta_0(s) := 1$.

Proof. Since $A(s)$ is equivalent to $\Psi(s)$ they have both the same greatest common divisors of minors of any order. Then

$$\begin{aligned}
\Delta_1(s) &= \psi_1(s)\\
\Delta_2(s) &= \psi_1(s)\psi_2(s)\\
&\vdots\\
\Delta_r(s) &= \psi_1(s)\psi_2(s)\cdots\psi_r(s)
\end{aligned}$$

i.e.,

$$\psi_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}$$

where $\Delta_0(s) := 1$. $\blacksquare$

Lemma 3.2 provides a second method for computing the Λ-generalized Smith form of a matrix $A(s)$, the first one being by elementary row and column operations on $A(s)$.

Example 3.1 Let $\mathbb{R}_\Lambda(s)$ be the ring of proper and stable rational functions $\mathbb{R}_{ps}(s)$, and let the matrix $A(s) \in \mathbb{R}_{ps}^{2\times 2}(s)$ be

$$A(s) = \begin{bmatrix} \frac{s-1}{(s+1)(s+2)^2} & 0 \\ \frac{1}{(s+1)(s+2)(s+3)} & \frac{s+1}{(s+4)^5} \end{bmatrix}.$$

The Hermite column form of this matrix, for $\pi = s+1$, is given by

$$H(s) = \begin{bmatrix} \frac{s-1}{(s+1)(s+2)^2} & 0 \\ \frac{1}{(s+1)(s+2)(s+3)} & \frac{s+1}{(s+4)^5} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{(s+2)^2}{(s+1)^2} & 0 \\ \frac{(s+4)^5}{(s+1)^4(s+3)} & \frac{(s+4)^5}{(s+1)^5} \end{bmatrix}}_{U(s)} = \begin{bmatrix} \frac{s-1}{(s+1)^3} & 0 \\ \frac{1}{(s+1)^3} & \frac{1}{(s+1)^4} \end{bmatrix}$$

where the $\mathbb{R}_{ps}(s)$-unimodular matrix $U(s)$ represents elementary column operations on $A(s)$.

The Smith form of $A(s)$ is given by

$$\Psi(s) = V_1(s)A(s)V_2(s) = \begin{bmatrix} \frac{1}{\pi^2} & 0 \\ 0 & \frac{s-1}{\pi^5} \end{bmatrix}$$

where the $\mathbb{R}_{ps}(s)$-unimodular matrices $V_1(s)$ and $V_2(s)$, given by

$$V_1(s) = \begin{bmatrix} \frac{s(s+2)}{(s+1)(s+3)} & 1 \\ -\frac{(s+2)(s+4)^5}{(s+3)(s+1)^6} & \frac{(s-1)(s+4)^5}{(s+1)^6} \end{bmatrix}, \quad V_2(s) = \begin{bmatrix} \frac{6(s+1)^2}{\pi^2} & -\frac{6(s+1)^8}{(s+4)^5\pi^5} \\ 0 & \frac{(s+1)^5}{\pi^5} \end{bmatrix},$$

represent, respectively, elementary row and column operations on $A(s)$.

The Smith form of $A(s)$ can be also computed using Lemma 3.2 as follows. The determinantal divisors of $A(s)$ are

$$\Delta_1(s) = \frac{1}{\pi^2}, \quad \Delta_2(s) = \frac{s-1}{\pi^7}$$

Thus,

$$\psi_1(s) = \frac{1}{\pi^2}, \quad \psi_2(s) = \frac{s-1}{\pi^5}$$

are the invariant factors of $A(s)$, as before.

As it can be seen from this example, elementary operations over a ring $\mathbb{R}_\Lambda(s)$ to obtain the Hermite and the Smith forms can be very involved. For the computation of the Smith form using Lemma 3.2, obtaining the determinantal divisors of the matrix amounts to obtain the greatest common divisors of minors of the matrix, which can be also very complicated.

4 Application to decoupling with Λ-stability

Roughly speaking, decoupling of dynamic systems implies independence between the inputs and the outputs of the system, and from the practical point of view it is of interest to achieve decoupling because it is often desirable to control the outputs of the system independently.

In this work, we are interested in the row-by-row decoupling of linear multivariable systems by static state feedback. This problem has been extensively studied since the 1960's, and it has been solved for the case of regular state feedback (see for instance [3, 1]), which is the kind of state feedback that has to be applied to decouple systems with the same number of inputs and outputs (square systems). The regular decoupling problem with stability by static state feedback has been solved in [9, 12].

We will tackle in this part of the paper the problem of decoupling with Λ-stability for linear multivariable square systems, which will be defined as the problem of decoupling plus the modes of the closed-loop system being located at a specific symmetric region Λ of the complex plane. The problem statement is presented in Section 4.1. In Section 4.2 we define the Λ-stable interactor via the Hermite column form of the system transfer function over the ring $\mathbb{R}_\Lambda(s)$, and show that this matrix is the part of the system that is invariant under the action of a state feedback which preserves Λ-stability. The main results and some illustrative examples are presented in Section 4.3. The structural necessary and sufficient conditions for this problem to have a solution will be stated in terms of the Λ-stable interactor of the system, and in terms of the row and global zero structure of the system out of the region Λ. A constructive procedure to obtain a state feedback which solves the problem is also provided.

4.1 Problem statement

We consider in this work square linear multivariable controllable systems described by

$$(A, B, C) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^p$ are, respectively, the state, input and output vectors of the system.

Definition 4.1 The system (A, B, C) is said to be row by row decouplable by static state feedback if there exists a state feedback

$$(F, G) : \quad u(t) = Fx(t) + Gv(t)$$

where $F \in \mathbb{R}^{p \times n}$ and $G \in \mathbb{R}^{p \times p}$ are constant matrices, with G nonsingular, and $v(t)$ is a new input vector, such that the input $v_i(t)$ controls the output $y_i(t)$, $i = 1, \ldots, p$, without affecting the other outputs.

Let Λ be a symmetric subset of the complex plane $\mathcal{C}$.

Definition 4.2 The system (A, B, C) is said to be row by row decouplable with Λ-stability by static state feedback, if it is decouplable and the eigenvalues of the matrix $(A + BF)$ are located in Λ.

Observe that Λ is taken here as a subset of the complex plane $\mathcal{C}$, and not as a subset of the extended complex plane $\mathcal{C}^*$ as in Section 2, since the eigenvalues of the matrix $(A + BF)$ are all in $\mathcal{C}$.

From the input-output point of view, the previous formulation is equivalent to the existence of a state feedback (F, G) such that the transfer function $T_{F,G}(s)$ of the closed loop system $(A + BF, BG, C)$ is of the form

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \text{ diag } \{w_1(s), \ldots, w_p(s)\}$$

and the eigenvalues of the matrix $(A+BF)$ are located in Λ, which implies also that $w_i(s) \in \mathbb{R}_\Lambda(s)$, $i = 1, \ldots, p$.

The eigenvalues of the matrix $(A + BF)$ are the modes of the closed-loop system $(A + BF, BG, C)$. The poles of the system are a subset of the system modes, and both sets coincide if the system is controllable and observable.

The dynamics of the closed loop system $(A + BF, BG, C)$ depends on the position of the eigenvalues of the matrix $(A + BF)$, thus the importance of locating these eigenvalues. Decoupling assures that every input is controlling every output independently; the modes location assures required system dynamics, for instance stability or faster response.

We can suppose without loss of generality that the modes of the system (the eigenvalues of matrix A), are in Λ, i.e., that (A, B, C) is Λ-stable; if not, there always exists a preliminary state feedback which places the modes in Λ, since we are considering the system (A, B, C) to be controllable. If the modes of the system are in Λ, then the entries of the system transfer function $T(s) = C(sI - A)^{-1}B$ are Λ-generalized polynomials, because they are rational functions whose poles are in Λ. Then, $T(s)$ can be considered as a Λ-generalized polynomial matrix. This consideration will be helpful when obtaining the column Hermite form and the Smith form of $T(s)$ over the ring $\mathbb{R}_\Lambda(s)$, and so that the conditions for decoupling with Λ-stability can be directly tested on the transfer function of the system.

4.2 State feedback with Λ-stability

Let $u(t) = Fx(t) + Gv(t)$ be a static state feedback applied on the Λ-stable system (A, B, C), and such that the closed-loop system $(A + BF, BG, C)$ is also Λ-stable. The closed-loop transfer function is given by

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG.$$

After some manipulations we obtain

$$\begin{aligned} T_{F,G}(s) &= C(sI - A)^{-1}B[I - F(sI - A)^{-1}B]^{-1}G \\ &= T(s)R(s) \end{aligned}$$

where $T(s) = C(sI - A)^{-1}B$ is the transfer function of the system (A, B, C), and

$$R(s) := [I - F(sI - A)^{-1}B]^{-1}G.$$

Since the closed-loop system is supposed to be Λ-stable, then $R(s)$ must be clearly a Λ-generalized polynomial matrix. Further, from

$$\begin{aligned} R^{-1}(s) &= G^{-1}[I - F(sI - A)^{-1}B] \\ &= \tfrac{1}{\det(sI-A)}\, G^{-1}[I - F\ \mathrm{Adj}(sI - A)B] \end{aligned}$$

it can be seen that $R^{-1}(s)$ is also a Λ-generalized polynomial matrix, since the roots of $\det(sI - A)$ are in Λ. Then, we have the following result.

Lemma 4.1 Let $T(s)$ be the transfer function of a Λ-stable system (A, B, C). Then, the effect of a static state feedback $u(t) = Fx(t) + Gv(t)$, G nonsingular, which preserves Λ-stability can be represented in transfer function terms as a $\mathbb{R}_\Lambda(\mathrm{s})$-unimodular matrix postmultiplying $T(s)$. ■

This fact establishes a natural restriction on the type of feedback we can use while trying to achieve decoupling with Λ-stability: For our purposes, the state feedback $u(t) = Fx(t) + Gv(t)$ will be said to be an admissible state feedback if its effect on the system (A, B, C) can be represented as a $\mathbb{R}_\Lambda(\mathrm{s})$-unimodular precompensator $R(s)$ acting on the system transfer function $T(s)$.

This can be considered as the matrix interpretation of the fact that we are neither allowed to introduce Λ-unstable poles (poles out of Λ) nor to cancel out Λ-unstable zeros in order to keep the internal Λ-stability of the closed-loop system.

At this stage, it is important to consider the information of the system (A, B, C) which remains invariant under the action of $\mathbb{R}_\Lambda(\mathrm{s})$-unimodular compensation, and consequently, invariant under the action of an admissible state feedback.

4.2.1 The Λ-stable interactor

Since the action of an admissible state feedback on (A, B, C) can be represented as multiplication of $T(s)$ on the right by a $\mathbb{R}_\Lambda(\mathrm{s})$-unimodular matrix, the information of the system that is invariant under such a feedback is contained in the column Hermite form of $T(s)$ over the ring $\mathbb{R}_\Lambda(\mathrm{s})$ (see Theorem 3.1).

Lemma 4.2 Let $T(s)$ be the transfer function of (A, B, C). Then, there exist a $\mathbb{R}_\Lambda(\mathrm{s})$-unimodular matrix $U(s) \in \mathbb{R}_\Lambda^{\mathrm{p}\times\mathrm{p}}(\mathrm{s})$ and a nonsingular lower triangular matrix $\Phi_\Lambda^{-1}(s) \in \mathbb{R}_\Lambda^{\mathrm{p}\times\mathrm{p}}(\mathrm{s})$, unique up to units of the ring $\mathbb{R}_\Lambda(\mathrm{s})$, such that

$$T(s)U(s) = \Phi_\Lambda^{-1}(s) = \begin{bmatrix} \varphi_{11}(s) & & (0) \\ \vdots & \ddots & \\ \varphi_{p1}(s) & \dots & \varphi_{pp}(s) \end{bmatrix}, \tag{4.1}$$

where the rational functions $\varphi_{ij}(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ satisfy, for $i > j$,

$$\varphi_{ij}(s) = 0, \text{ or } \deg_\Lambda \varphi_{ij}(s) < \deg_\Lambda \varphi_{ii}(s), \tag{4.2}$$

and they are of the form

$$\varphi_{ii}(s) = \frac{\alpha_{ii}(s)}{\pi^{k_{ii}}} \tag{4.3}$$

and for $i > j$

$$\varphi_{ij}(s) = \frac{\alpha_{ij}(s)}{\pi^{k_{ij}}}, \tag{4.4}$$

where $\alpha_{ii}(s)$ is an antistable polynomial, or polynomial with only Λ-unstable roots (roots out of Λ), $\pi = s + \beta$ is a stable term ($-\beta \in \Lambda$), $\alpha_{ij}(s) \in \mathbb{R}[s]$ is a polynomial, and k_{ii}, k_{ij} are positive integers. The matrix $U(s)$ represents elementary column operations on $T(s)$ over $\mathbb{R}_\Lambda(s)$. ■

The rational matrix $\Phi_\Lambda(s)$, which is the inverse of $\Phi_\Lambda^{-1}(s)$, will be called the Λ-stable interactor of the system, since this matrix can be considered as the extension of the classic interactor [15] for the region Λ. Note, as stated in Lemma 4.2, that the matrix $\Phi_\Lambda(s)$ is not unique but it is unique up to units of the ring $\mathbb{R}_\Lambda(s)$. For a fixed $\pi = s + \beta$, $\Phi_\Lambda(s)$ is unique, and having this in mind is why we called it "the" stable interactor of the system. Actually, the algebraic properties of $\Phi_\Lambda(s)$ do not depend on the choice of π, nor the results stated in the next sections based on these properties.

The matrix $\Phi_\Lambda(s)$ is in general a rational matrix having only Λ-unstable poles. This can be seen from the fact that the numerator of the determinant of $\Phi_\Lambda^{-1}(s)$ is the product of the antistable polynomials $\alpha_{ii}(s)$, $i = 1, \ldots, p$. Observe that if (A, B, C) has no finite zeros out of Λ, then $\Phi_\Lambda(s)$ is a polynomial matrix.

The Λ-stable interactor is a state feedback invariant which contains the Λ-unstable structural information of the system, i.e., zeros out of Λ of the system, including infinite zeros.

4.2.2 Feedback realization of precompensators

It was shown before that the effect of an admissible state feedback is equivalent in transfer function terms as acting a $\mathbb{R}_\Lambda(s)$-unimodular precompensator on the system transfer function $T(s)$. The converse problem for a proper compensator is the following: A given proper compensator $P(s)$ is said to be feedback realizable on the system (A, B, C) if there exists a state feedback (F, G) such that $P(s) = [I - F(sI - A)^{-1}B]^{-1}G$.

The following result, used in the proof of Theorem 4.1, states the conditions for a proper compensator to be realizable.

Lemma 4.3 [5] Let the matrices $N_1(s)$ and $D(s)$ be a right coprime matrix fraction description (MFD) of the system (A, B, I_n) with $D(s)$ column reduced, and let $P(s)$ be a nonsingular compensator. Then $P(s)$ is state feedback realizable on (A, B, I_n) if and only if

- $P(s)$ is biproper, and
- $P^{-1}(s)D(s)$ is a polynomial matrix.

■

4.3 Main results

The following result presents the conditions for decoupling with Λ-stability.

Theorem 4.1 The controllable and Λ-stable square system (A, B, C) is decouplable with Λ-stability if and only if its associated Λ-stable interactor $\Phi_\Lambda(s)$ is a diagonal matrix.
Proof. Necessity. Suppose that (A, B, C) is decouplable with Λ-stability. Then there exists a state feedback (F, G) such that

$$\begin{aligned} T_{F,G}(s) &= C(sI - A - BF)^{-1}BG \\ &= T(s)[I - F(sI - A)^{-1}B]^{-1}G = W(s) \end{aligned}$$

Since $[I - F(sI - A)^{-1}B]^{-1}G$ is a $\mathbb{R}_\Lambda(s)$-unimodular matrix, and $W(s) = \mathrm{diag}\{w_1(s), \ldots, w_p(s)\}$, then it follows that $\Phi_\Lambda(s)$ is diagonal.

Sufficiency. Supposing that $\Phi_\Lambda(s)$ is diagonal, we will prove this part by showing that the $\mathbb{R}_\Lambda$(s)-unimodular matrix $U(s)$ appearing in (4.1) is feedback realizable. Let $N_1(s)$ and $D(s)$ be a right coprime MFD of the system (A, B, I_n) with $D(s)$ column reduced. According to Lemma 4.3, $U(s)$ will be proved to be feedback realizable if the product

$$U^{-1}(s)D(s)$$

is polynomial.

From (4.1)

$$T(s)U(s) = CN_1(s)D^{-1}(s)U(s) = \Phi_\Lambda^{-1}(s),$$

and from this, we get

$$\Phi_\Lambda(s)CN_1(s) = U^{-1}(s)D(s). \tag{4.5}$$

Since the left hand side of the last equation has only poles out of Λ, and the right hand side has only poles in Λ, then both sides of (4.5) must be polynomial.

Given that $U^{-1}(s)D(s)$ is polynomial, and taking $D(s)$ column reduced, then there exists a constant solution X, Y with X no singular to the polynomial matrix equation [8]

$$XD(s) + YN_1(s) = U^{-1}(s)D(s) \tag{4.6}$$

Thus, a state feedback (F, G) which decouples the system (A, B, C) and preserves internal Λ-stability is obtained as $F = -X^{-1}Y$, $G = X^{-1}$. ■

Remark 4.1 From the fact that the $\mathbb{R}_\Lambda$(s)-unimodular matrix $U(s)$ in (4.1) is feedback realizable, then $\Phi_\Lambda^{-1}(s)$ (diagonal or not) can be considered as the transfer function of the closed-loop system $(A+BF, BG, C)$, whose Λ-stable interactor is $\Phi_\Lambda(s)$. Thus, the matrix $U(s)$ can be regarded as the input-output result of a static state feedback which assigns poles of the system at the positions of Λ-stable zeros producing cancellation, and the remaining poles being located at $s = -\beta \in \Lambda$. In this way, $\Phi_\Lambda^{-1}(s)$ is the resulting transfer function of $(A+BF, BG, C)$.

The necessary and sufficient conditions for decoupling with Λ-stability, stated in Theorem 4.1 in terms of the Λ-stable interactor of the system, can also be expressed in terms of the row and global zero structure of the system out of Λ, which will be defined as follows.

Definition 4.3 Let n_i be the number of zeros out of Λ (Λ-unstable zeros), finite or infinite, multiplicities included, of the ith row of $T(s)$. The integers $\{n_1, n_2, \ldots, n_p\}$ will be called the row zero structure of the system out of Λ.

Remark 4.2 Let $\Phi_\Lambda^{-1}(s)$ be the Hermite column form of $T(s)$ as given by (4.1). Because of property (4.2), and since $U(s)$ in (4.1) is $\mathbb{R}_\Lambda$(s)-unimodular, it can be seen that if $\varphi_{ii}(s)$ is the only entry different from zero of the ith row of $\Phi_\Lambda^{-1}(s)$, then

$$n_i = \deg_\Lambda \varphi_{ii}(s) = k_{ii},$$

otherwise

$$n_i < \deg_\Lambda \varphi_{ii}(s) = k_{ii}.$$

The integer n_i can actually be obtained directly from the elements of $T(s)$ as follows. If $g_i(s)$ is the greatest common divisor over $\mathbb{R}_\Lambda$(s) of all entries of the ith row of $T(s)$ (see Section 2.2), then

$$n_i = \deg_\Lambda g_i(s), \quad i = 1, \ldots, p.$$

The global zero structure of the system out of Λ will be defined via the Smith form of the system transfer function (see Theorem 3.2).

Let $\Psi(s)$ be the Smith form of the system transfer function $T(s)$ over the ring $\mathbb{R}_\Lambda(s)$ for a given Λ, i.e., there exist $\mathbb{R}_\Lambda(s)$-unimodular matrices $V_1(s)$ and $V_2(s)$ such that

$$V_1(s)T(s)V_2(s) = \Psi(s) = \text{diag}\{\psi_i(s)\}_{i=1}^p \tag{4.7}$$

where the rational functions $\psi_i(s) \in \mathbb{R}_\Lambda(s)$ satisfy the divisibility conditions over $\mathbb{R}_\Lambda(s)$,

$$\psi_i(s) \mid \psi_{i+1}(s), \quad i = 1, \ldots, p-1,$$

and they are of the form

$$\psi_i(s) = \frac{\epsilon_i(s)}{\pi^{n'_i}}$$

where $\epsilon_i(s)$ is a polynomial with only Λ-unstable roots, $\pi = s + \beta$ is a stable term $(-\beta \in \Lambda)$, and n'_i is a positive integer.

Definition 4.4 The global zero structure of the system out of Λ will be defined as the set of integers $\{n'_1, n'_2, \ldots, n'_p\}$, which correspond to the degree of the rational functions $\psi_i(s)$ in the Smith form of the system transfer function $T(s)$, i.e.,

$$n'_i = \deg_\Lambda \ \psi_i(s), \quad i = 1, \ldots, p.$$

Lemma 4.3 Let $\Phi_\Lambda^{-1}(s)$ and $\Psi(s)$ be, respectively, the column Hermite form and the Smith form of $T(s)$ over $\mathbb{R}_\Lambda(s)$ as given by (4.1) and (4.7). Then

$$\sum_{i=1}^p k_{ii} = \sum_{i=1}^p n'_i \tag{4.8}$$

Proof. From (4.1) and (4.7), and due to the fact that the degree of the determinant of a $\mathbb{R}_\Lambda(s)$-unimodular matrix is zero (see Lemma 3.1), it can be seen that

$$\deg_\Lambda \ \det \ \Phi_\Lambda^{-1}(s) = \deg_\Lambda \ \det \ \Psi(s).$$

Then (4.8) results from the previous equation. ■

Theorem 4.2 The Λ-stable interactor of the system is diagonal (equivalently: the system is decouplable with Λ-stability) if and only if the row and global zero structure of the system out of Λ is the same, i.e., if and only if

$$\sum_{i=1}^p n_i = \sum_{i=1}^p n'_i \tag{4.9}$$

Proof. Suppose that $\Phi_\Lambda(s)$ is diagonal. Then, from Remark 4.2

$$n_i = \deg_\Lambda \ \varphi_{ii}(s) = k_{ii}, \quad i = 1, \ldots, p.$$

Thus, from Lemma 4.3 we have (4.9).

Sufficiency. Suppose that (4.9) holds, but that $\Phi_\Lambda(s)$ is not diagonal. From Remark 4.2 we have that $n_i < k_{ii}$ for some $i = 1, \ldots, p$, implying also that

$$\sum_{i=1}^p n_i < \sum_{i=1}^p k_{ii}$$

Following Lemma 4.3, this means that

$$\sum_{i=1}^{p} n_i < \sum_{i=1}^{p} n'_i$$

contradicting our assumption. ∎

Next, a couple of examples will illustrate the main results.

Example 4.1 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -1 & 1 & 1 & 4 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

whose transfer function is

$$T(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ \frac{1}{(s+1)^4} & \frac{s-1}{(s+1)^3} \end{bmatrix}$$

Let us suppose that we want to decouple this system by static state feedback, such that the closed-loop system is internally stable (i.e., the eigenvalues of the matrix $(A + BF)$ are in the open left half complex plane $\mathcal{C}_-$). It can be observed that the Λ-stable interactor of this system for $\Lambda = \mathcal{C}_-$ ($\mathbb{R}_\Lambda(s)$=$\mathbb{R}_{ps}(s)$) is not a diagonal matrix, because the entry $(2, 1)$ of $T(s)$ can not be cancelled out by an elementary column operation over $\mathbb{R}_{ps}(s)$, thus the system is not decouplable with stability. In terms of the row and global zero structure of the system out of $\Lambda = \mathcal{C}_-$ (see Theorem 4.2), we have that

$$n_1 = 2, \quad n_2 = 2,$$
$$n'_1 = 2, \quad n'_2 = 3,$$

and (4.9) does not hold. The reason for this is that the row infinite zeros and the global infinite zeros coincide, but not the row and global unstable zeros, since $s = 1$ is an unstable global zero but it is not a row zero of $T(s)$.

Since the row infinite zeros $\{2, 2\}$ and the global infinite zeros $\{2, 2\}$ coincide, this system is decouplable, but not decouplable with stability. Indeed, the decoupling compensator

$$P(s) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{(s+1)(s-1)} & \frac{s+1}{s-1} \end{bmatrix}$$

is feedback realizable, and the corresponding state feedback realizing $P(s)$ can be obtained from a constant solution X, Y with X nonsingular to the polynomial matrix equation $XD(s) + YN_1(s) = P^{-1}(s)D(s)$, where $N_1(s)$, $D(s)$ is a right coprime MFD of (A, B, I_n) with $D(s)$ column reduced. Then the state feedback, with

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

produces the decoupled system $(A + BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix}$$

but the closed-loop system is not internally stable, since one of the eigenvalues of the matrix $(A+BF)$ is equal to 1.

Example 4.2 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -5 & -6 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & -5 & -9 & -6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

whose transfer function is

$$T(s) = C(sI-A)^{-1}B = \begin{bmatrix} \frac{1}{(s+2)^2} & 0 \\ \frac{s-1}{(s+2)^4} & \frac{(s-1)(s+1)}{(s+2)^3} \end{bmatrix}$$

Again, let us suppose that we want to decouple this system by static state feedback, such that the closed-loop system is internally stable. The Λ-stable interactor of this system for $\Lambda = \mathcal{C}_-$ ($\mathbb{R}_\Lambda(s)=\mathbb{R}_{ps}(s)$) is a diagonal matrix

$$\Phi_\Lambda(s) = \begin{bmatrix} \frac{1}{\pi^2} & 0 \\ 0 & \frac{s-1}{\pi^2} \end{bmatrix}^{-1} = \begin{bmatrix} \pi^2 & 0 \\ 0 & \frac{\pi^2}{s-1} \end{bmatrix}$$

or equivalently, the row and global infinite and unstable zeros of the system coincide, meaning that the system is decouplable with stability.

From Lemma 4.2, and with $\pi = s+1$, we have that

$$U(s) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{(s+1)(s+2)} & \frac{s+2}{s+1} \end{bmatrix}$$

Then, from a constant solution X, Y, with X nonsingular, to the polynomial matrix equation $XD(s)+YN_1(s) = U^{-1}(s)D(s)$, where $N_1(s)$, $D(s)$ is a right coprime MFD of (A, B, I_n) with $D(s)$ column reduced (see the sufficiency part of Theorem 4.1), we obtain the state feedback

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which produces the decoupled and stable system $(A+BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = C(sI-A-BF)^{-1}BG = \begin{bmatrix} \frac{1}{(s+2)^2} & 0 \\ 0 & \frac{s-1}{(s+2)^2} \end{bmatrix}$$

Actually, this system can be decoupled with stability with closed-loop transfer function

$$T_{F,G}(s) = \begin{bmatrix} \frac{1}{(s+\alpha_1)(s+\alpha_2)} & 0 \\ 0 & \frac{s-1}{(s+\alpha_3)(s+\alpha_4)} \end{bmatrix}$$

where $\alpha_{1,2,3,4}$ are arbitrary positive real numbers.

Let us suppose now that we want to decouple this system with Λ-stability, where Λ is the region shown in Fig. 4.1, starting at $s = -3$, and forming an angle of $\pm 45°$ with the real negative axis.

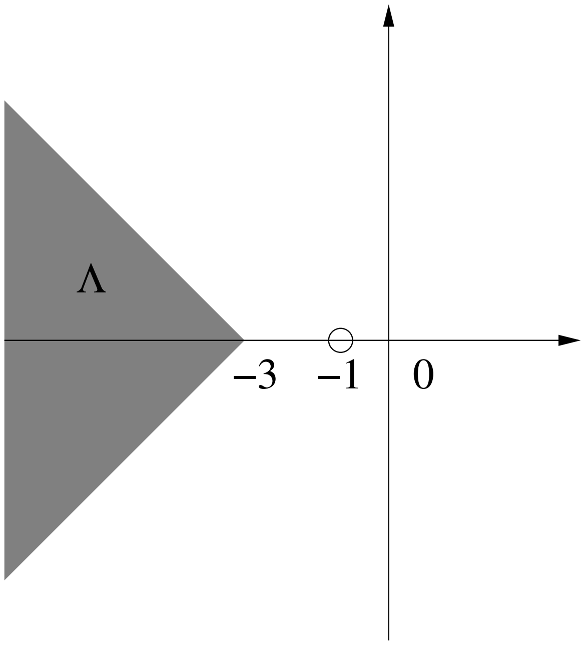

Fig. 4.1. Region Λ

The row zero structure of the system out of Λ is

$$n_1 = 2, \quad n_2 = 2,$$

and the global zero structure out of Λ is

$$n_1' = 2, \quad n_2' = 3.$$

Since these structures do not coincide due to the fact that the global zero at $s = -1$ is not a row zero of the system, then decoupling with Λ-stability by static state feedback can not be achieved. This can also be seen from the fact that the Λ-stable interactor of this system is not a diagonal matrix. Notice that the angle of the region Λ with respect to the real axis plays a role only if the system has complex finite zeros with imaginary part different from zero, which is not the case in this example.

Summarizing: decoupling as such, i.e., each system output being controlled independently by one and only one input regardless of the position of the closed-loop modes, can be achieved if and only if the row infinite zero structure of the system is the same as the global infinite zero structure. If the system is decouplable, decoupling with closed-loop modes in a particular region Λ of the complex plane can be achieved if and only if the global finite zeros out of Λ of the system, multiplicities included, are also row finite zeros of the system.

Observe that if a system (A, B, C) with no finite zeros is decouplable, then it can be decoupled with closed-loop modes located at any finite position of the complex plane. If a decouplable system has finite zeros, then for the system to be decouplable with Λ-stability, the region Λ must contain every global zero that it is not a row zero of the system. For instance, in Example 4.2, the only global finite zero that it is not a row finite zero is $s = -1$; then the system can be decoupled with Λ-stability if and only if the region Λ contains this point. The angle of the region Λ with respect to the real axis is important only in the case that the system has complex zeros with imaginary part different from zero.

Theorems 4.1 and 4.2 state the necessary and sufficient conditions for decoupling with Λ-stability, Theorem

4.1 in terms of the Λ-stable interactor of the system, and Theorem 4.2 in terms of the global and row zero structure of the system out of Λ. From the practical point of view, since elementary operations in a particular ring Λ could be very involved, in general is simpler to test the conditions of Theorem 4.2 than to obtain the Λ-stable interactor of the system. The row zero structure of the system out of Λ can be obtained for instance from the gcd $g_i(s)$ of the i-th row of $T(s)$ as explained in Remark 4.2, and the global zero structure of the system out of Λ can be obtained from the Smith form of $T(s)$ using the determinantal divisors of $T(s)$, avoiding thus the use of elementary operations in $\mathbb{R}_\Lambda(s)$. This procedure, however, can also be very complicated. If the system (A, B, C) is decouplable with Λ-stability, a decoupling state feedback can be computed as indicated in the proof of Theorem 4.1.

Since the symmetric region Λ of the complex plane can be arbitrarily chosen, it is to be expected that previous results on decoupling can be deduced from our conditions. Indeed, the conditions for decoupling of square linear multivariable systems, and decoupling with stability can be easily deduced from our solution. For these two cases, the conditions of Theorem 4.2 can be interpreted as follows:

- Consider the decoupling problem with closed-loop modes in any finite position. In this case $\Lambda = \mathcal{C}$ and the region out of Λ is $\Lambda' = \infty$. Then the system (A, B, C) is decouplable if and only if the row and global infinite zero structure of the system is the same [3, 1].
- In the decoupling problem with stability, it is required that all modes of the closed-loop system are in the left hand complex plane, i.e., $\Lambda = \mathcal{C}_-$ and $\Lambda' = \mathcal{C}_+ \cup \infty$. Then the system is decouplable in this case if and only if the row and global zero structure of the system out of Λ is the same, i.e., if and only if the row and global infinite zeros, plus the row and global zeros in $\mathcal{C}_+$ are the same [9, 12].

5 Conclusions

In this paper we have examined algebraic properties of Λ-generalized polynomials. It has been shown that widely used sets of rational functions, like polynomials, proper, and proper and stable rational functions can be regarded as Λ-generalized polynomials for a suitable region Λ. It has been also proved that the set of Λ-generalized polynomials is a Euclidean ring, and some other results concerning division and coprimeness of Λ-generalized matrices, as well as canonical forms for Λ-generalized polynomial matrices have been presented.

We believe our approach is more general and comprehensive in the sense that comprises generalized polynomials as defined in previous works [10, 11, 14, 13]. In these references, generalized polynomials are limited to proper rational functions with poles in a region of the complex plane. With the inclusion of the point at infinity, and considering a region of the extended complex plane, a wider set of rational functions, like polynomials and nonproper rational functions, can also be considered as particular cases of Λ-generalized polynomials.

An application of the concepts to the problem of decoupling with Λ-stability of linear square multivariable systems has been also presented. The structural necessary and sufficient conditions for this problem to have a solution are stated in terms of the Λ-stable interactor of the system or in terms of the row and global zero structure of the system out of Λ.

It is hoped that these concepts will be applied in a near future to solve in a unified way other multivariable control problems besides decoupling.

Throughout the paper we have focused mainly on algebraic properties of Λ-generalized polynomials, using elementary operations (see Section 3.1) as the key tool to derive canonical forms such as the Hermite form (Theorem 3.1) or Smith form (Theorem 3.2). As shown in Theorems 4.1 and 4.2, obtention of these canonical forms is a cornerstone of the design algorithm to achieve multivariable decoupling. For the same reason as Gaussian elimination without pivoting can be numerically unstable in the field of real or complex-valued matrices, it is now recognized that elementary operations on polynomial or rational objects are not numerically sound. An important future research topic is therefore the development of numerical stable, efficient algorithms to deal with Λ-generalized polynomials. Preliminary work in this direction was described in [6] in the special case of polynomial matrices.

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