On Computing the $\mathcal{H}_\infty$-norm of a Polynomial Matrix Fraction*

Didier Henrion[1,2]
Michael Šebek[3]
Martin Hromčík[3]

Abstract

We propose new algorithms to compute the $\mathcal{H}_\infty$-norm of a transfer function expressed as a fraction of polynomial matrices. The algorithms are based on optimization over linear matrix inequalities and do not require intensive gridding or conversion to state-space form.

Keywords

Linear Systems, $\mathcal{H}_\infty$-norm, Polynomial Matrices, LMI.

1 Introduction

This paper is an attempt to develop new algorithms to compute the $\mathcal{H}_\infty$-norm of a transfer function matrix represented as a fraction of polynomial matrices, or polynomial matrix fraction description (MFD). The motivations for doing so are numerous:

- The $\mathcal{H}_\infty$-norm of a stable transfer function, or maximum singular value over the stability boundary, arises often in control theory, especially in robust control theory, see [22] for a recent survey.

[1]Corresponding author. FAX: +33 5 61 33 69 69. E-mail: henrion@laas.fr

[2]Laboratoire d'Analyse et d'Architecture des Systèmes, Centre National de la Recherche Scientifique, 7 Avenue du Colonel Roche, 31 077 Toulouse, cedex 4, France.

[3]Center for Applied Cybernetics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic.

*This work was supported by the Ministry of Education of the Czech Republic under grant number LN003096.

- Fractions of polynomial matrices – or equivalently systems described by higher-order differential equations – arise naturally in several problems in control [14] and signal processing [1].
- Besides straightforward computationally intensive gridding, the only $\mathcal{H}_\infty$-norm computation algorithms we are aware of are based on state-space methods, i.e. they take state-space matrices as input data. These algorithms are shortly reviewed below. However, when a linear system is directly given as a polynomial MFD and one wants to compute its $\mathcal{H}_\infty$-norm, there is no alternative to converting first the MFD to state-space matrices, and then computing the $\mathcal{H}_\infty$-norm using state-space methods. This is clearly an additional burden.
- As a continuation of our previous research efforts, we are trying to develop a series of numerical algorithms to deal directly with polynomial matrices and solve various control problems via an algebraic, or polynomial approach [14]. The algorithms will be part of the new release 3.0 of a comprehensive package called the Polynomial Toolbox for MATLAB[1][20, 17].

Prior to describing the new algorithms, we now shortly review available state-space methods for $\mathcal{H}_\infty$-norm computation. Before 1989, a popular way to "compute" the $\mathcal{H}_\infty$-norm was to search for the maximum singular value over frequency either graphically (using the so-called singular value plot, or Bode magnitude plot for scalar transfer functions) or via a suitable frequency gridding. This approach still remains quite popular for its simplicity. Moreover, in the discrete-time case the relationship between the discrete Fourier transform theory and the evaluation of a polynomial at a point lying on the complex unit circle can be employed [2, 13]. As a result the efficient fast Fourier transform algorithm [2] can be utilized for computing the values of the involved transfer function over selected complex frequencies [20, 17] which makes the whole procedure much more effective. As pointed out in [3], obvious shortcomings are (a) determining the range and spacing of the frequencies to be checked, and (b) the large number of computations involved. Motivated by these issues, Boyd et al. [3] and Robel [19] independently proposed a systematic bisection method, using the relation between singular values of a transfer function and eigenvalues of a related Hamiltonian matrix. Bruinsma and Steinbuch [6] then presented an improved two-step algorithm based on lower bounds on the $\mathcal{H}_\infty$-norm. Quadratic convergence of the algorithm was proved in [4]. This algorithm is the core of all the MATLAB macros available for computing the $\mathcal{H}_\infty$-norm of a linear system:

- macros `hinfnorm` and `dhfnorm` of the μ-Analysis and Synthesis Toolbox 3.0.4
- macros `norminf` and `dnorminf` of the LMI Control Toolbox 1.0.5
- macro `normhinf` of the Robust Control Toolbox 2.0.6.

The algorithm was then improved using derivatives of the maximum singular value with respect to the frequency [21]. Latest refinements of this method, with extensions to the

[1] MATLAB is a trademark of The MathWorks, Inc. See `www.mathworks.com`

computation of complex stability radii of generalized eigenvalue problems, can be found in [9].

Besides brute-force gridding, we are not aware of any $\mathcal{H}_\infty$-norm computation method that works directly on polynomial matrix coefficients and avoids conversion to state-space matrices as a preliminary pre-processing step. This paper is precisely an attempt to fill up this gap. We aim at providing a systematic and direct method to compute the $\mathcal{H}_\infty$-norm of a polynomial MFD without gridding or conversion to state-space model.

The basic tool we rely upon are linear matrix inequalities (LMIs, see [5] for an introduction and [7] for a recent overview). Relationships between $\mathcal{H}_\infty$-norm and LMIs are well-known in a state-space setting, especially in the scope of robust control: if (A, B, C, D) denotes the state-space description of a stable continuous-time system and γ is a given positive scalar, then the bounded-real lemma [5, Chapter 2] states that

$$\|D + C(sI - A)^{-1}B\|_\infty \leq \gamma$$

if and only if there exists a solution P to the LMI

$$\begin{bmatrix} A^\star P + PA + C^\star C & PB + C^\star D \\ B^\star P + D^\star C & D^\star D - \gamma^2 I \end{bmatrix} \preceq 0, \quad P = P^\star \succ 0,$$

where the star denotes conjugate transpose and $\succ$ (resp. $\succeq$) means positive definite (resp. semidefinite). One contribution of our paper is in showing that the same kind of results can be derived in a polynomial setting, when transfer function $H(s)$ is represented as a fraction[2] of two given polynomial matrices $N(s)$ and $D(s)$, i.e.

$$H(s) = N(s)D(s)^{-1}.$$

Our result can thus also be considered as a new illustration of the simultaneous use of LMIs and polynomial methods in control, recently surveyed in [11].

Finally, we must point out that in most of the cases, the numerical computation of the $\mathcal{H}_\infty$-norm of a discrete-time system is performed on an associated continuous-time system obtained via a bilinear transformation. This approach is not always numerically sound [22]. In contrast, the algorithm proposed in this paper can indifferently be applied to continuous-time or discrete-time systems.

2 Lower Bound

In this section, we show that computing a lower bound on the $\mathcal{H}_\infty$-norm of a polynomial MFD amounts to solving a quasi-convex generalized eigenvalue LMI problem (GEVP). Coefficient matrices of the LMI problem are directly obtained from matrix coefficients of the polynomial MFD.

[2]In this paper, only right MFDs are considered, since the $\mathcal{H}_\infty$-norm of a left MFD $H(s) = D(s)^{-1}N(s)$ is equal to the $\mathcal{H}_\infty$-norm of the transposed right MFD $H^\star(s) = N^\star(s)D(s)^{-\star}$.

We will handle continuous-time and discrete-time systems in a unified way. For this, we define

$$\mathcal{S} = \{s \in \mathbb{C} : \left[\begin{array}{c} 1 \\ s \end{array}\right]^\star S \left[\begin{array}{c} 1 \\ s \end{array}\right] \geq 0\}$$

as a closed stability region in the complex plane, where 2×2 Hermitian matrix S has one negative eigenvalue and one positive eigenvalue. We also define

$$\partial\mathcal{S} = \{s \in \mathbb{C} : \left[\begin{array}{c} 1 \\ s \end{array}\right]^\star S \left[\begin{array}{c} 1 \\ s \end{array}\right] = 0\}$$

as the one-dimensional boundary of stability region $\mathcal{S}$. Standard choices for $\partial\mathcal{S}$ are

- either the imaginary axis (continuous-time systems) for

$$S = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

- or the unit circle (discrete-time systems) for

$$S = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right].$$

By definition, a given scalar γ_{low} is a lower bound on $\|H(s)\|_\infty$ if and only if there exists a unitary vector v and a complex scalar $s \in \partial\mathcal{S}$ (or equivalently $s \in \mathcal{S}$ in virtue of the Maximum Modulus Theorem) such that

$$v^\star \left[\begin{array}{c} H(s) \\ I \end{array}\right]^\star \left[\begin{array}{cc} I & 0 \\ 0 & -\gamma_{\mathrm{low}}^2 I \end{array}\right] \left[\begin{array}{c} H(s) \\ I \end{array}\right] v \geq 0$$

or equivalently, if and only if

$$w^\star \left[\begin{array}{c} N(s) \\ D(s) \end{array}\right]^\star \left[\begin{array}{cc} I & 0 \\ 0 & -\gamma_{\mathrm{low}}^2 I \end{array}\right] \left[\begin{array}{c} N(s) \\ D(s) \end{array}\right] w \geq 0 \qquad (1)$$

for some non-zero vector w and some complex scalar $s \in \partial\mathcal{S}$. Note that the transformation $w = D(s)^{-1}v$ is well-defined since by assumption polynomial matrix $D(s)$ has no zero along $\partial\mathcal{S}$. Polynomial matrix

$$M(s) = \left[\begin{array}{c} N(s) \\ D(s) \end{array}\right] = \left[\begin{array}{c} N_0 \\ D_0 \end{array}\right] + \left[\begin{array}{c} N_1 \\ D_1 \end{array}\right] s + \cdots + \left[\begin{array}{c} N_d \\ D_d \end{array}\right] s^d$$

can be expressed via a linear fraction representation (LFR) as follows:

$$M(s) = M_0 + M_L\Delta(I - M_D\Delta)^{-1}M_R$$

where $\Delta = sI_{nd}$ and

$$\left[\begin{array}{c|c} M_0 & M_L \\ \hline M_R & M_D \end{array}\right] = \left[\begin{array}{c|cccccc} N_0 & N_1 & N_2 & \cdots & N_{d-1} & N_d \\ D_0 & D_1 & D_2 & \cdots & D_{d-1} & D_d \\ \hline I_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_n & 0 & & 0 & 0 \\ 0 & 0 & I_n & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n & 0 \end{array}\right].$$

In relation with the above LFR we can write

$$\begin{array}{rcl} M(s)w & = & M_0 w + M_L p \\ q & = & M_R w + M_D p \\ p & = & sq \end{array}$$

where the last equality holds for some non-zero vector q and some complex scalar $s \in \partial\mathcal{S}$ if and only if

$$\left[\begin{array}{cc} q & p \end{array}\right] S \left[\begin{array}{cc} q & p \end{array}\right]^\star = 0$$

see [18] or [16]. Denoting

$$\mathcal{N} = \left[\begin{array}{cccc} N_0 & N_1 & \cdots & N_d \end{array}\right], \quad \mathcal{D} = \left[\begin{array}{cccc} D_0 & D_1 & \cdots & D_d \end{array}\right]$$

and

$$x = \left[\begin{array}{c} w \\ p \end{array}\right]$$

inequality (1) can be written as

$$\left[\begin{array}{c} w \\ p \end{array}\right]^\star \left[\begin{array}{c} \mathcal{N} \\ \mathcal{D} \end{array}\right]^\star \left[\begin{array}{cc} I & 0 \\ 0 & -\gamma_{\mathrm{low}}^2 I \end{array}\right] \left[\begin{array}{c} \mathcal{N} \\ \mathcal{D} \end{array}\right] \left[\begin{array}{c} w \\ p \end{array}\right] = x^\star(\mathcal{N}^\star\mathcal{N} - \gamma_{\mathrm{low}}^2 \mathcal{D}^\star\mathcal{D})x \geq 0. \qquad (2)$$

Now define projection matrices

$$\mathcal{Q} = \left[\begin{array}{cc} I_{nd} & 0_{nd\times n} \end{array}\right], \quad \mathcal{P} = \left[\begin{array}{cc} 0_{nd\times n} & I_{nd} \end{array}\right]$$

and non-zero rank-one positive semi-definite matrix

$$X = xx^\star.$$

Existence of a non-zero vector x and a complex scalar $s \in \partial\mathcal{S}$ satisfying $p = sq$ and inequality (2) is then equivalent to the existence of a matrix X such that

$$\begin{array}{l} \mathrm{Trace}\, \mathcal{N}^\star\mathcal{N}X \geq \gamma_{\mathrm{low}}^2 \mathrm{Trace}\, \mathcal{D}^\star\mathcal{D}X \\ \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right]^\star = 0 \\ X = X^\star \succeq 0, \quad X \neq 0, \quad \mathrm{rank}\, X = 1 \end{array} \qquad (3)$$

where $\otimes$ denotes the standard Kronecker product. Problem (3) is a rank-one LMI feasibility problem, a non-convex problem frequently arising in control, see [18, 16] or [10]. Now define the following convex relaxation

$$\begin{array}{l} \mathrm{Trace}\, \mathcal{N}^\star\mathcal{N}X \geq \gamma_{\mathrm{low}}^2 \mathrm{Trace}\, \mathcal{D}^\star\mathcal{D}X \\ \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array}\right]^\star = 0 \\ X = X^\star \succeq 0, \quad X \neq 0 \end{array} \qquad (4)$$

of problem (3) where the non-convex rank constraint has been dropped.

Lemma 1 *Rank-one LMI problem (3) is feasible if and only if LMI problem (4) is feasible.*

Proof Feasibility of problem (3) obviously implies feasibility of problem (4), so let us prove that the converse is also true. Let $X = \sum_{i=1}^{k} x_i x_i^\star$ be a positive semi-definite Hermitian matrix of rank k solving problem (4). In [18] or [16] it is proved that $\left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes x_i x_i^\star) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star = 0$ for all $i = 1, \ldots, k$. Since $\gamma_{\mathrm{low}}^2 \mathrm{Trace}\, \mathcal{D}^\star \mathcal{D} X \geq \mathrm{Trace}\, \mathcal{N}^\star \mathcal{N} X$ implies that there exists at least one index i such that $\gamma_{\mathrm{low}}^2 x_i^\star \mathcal{D}^\star \mathcal{D} x_i \geq x_i^\star \mathcal{N}^\star \mathcal{N} x_i$, problem (3) is also feasible and the lemma is proved. □

Lemma 1 states that γ_{low} is a given lower bound for $\|H(s)\|_\infty$ if and only if convex LMI problem (4) is feasible. Now if γ_{low} is allowed to vary, problem (4) becomes a quasi-convex generalized eigenvalue LMI problem (GEVP), see [5]. The $\mathcal{H}_\infty$-norm of $H(s)$ can thus be approached from below by maximizing γ_{low}.

Theorem 1 *Let $\hat{\mu}$ denote the optimum of the GEVP*

$$\begin{array}{rl} \min & \mu \\ \mathrm{s.t.} & \mu \mathrm{Trace}\, \mathcal{D}^\star \mathcal{D} X \geq -\mathrm{Trace}\, \mathcal{N}^\star \mathcal{N} X \\ & \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star = 0 \\ & X = X^\star \succeq 0, \quad X \neq 0. \end{array} \tag{5}$$

Then $\|H(s)\|_\infty = \sqrt{-\hat{\mu}}$.

Following arguments similar to that used in [12], it can be shown that $\mathrm{Trace}\, \mathcal{D}^\star \mathcal{D} X > 0$ for any feasible X, hence GEVP (5) is well-posed.

3 Upper Bound

In this section, we show that computing an upper bound on the $\mathcal{H}_\infty$-norm of a polynomial MFD amounts to solving a convex LMI optimization problem. As in Section 2, coefficient matrices of the LMI problem are directly obtained from matrix coefficients of the polynomial MFD.

Using the same notations as in Section 2, a given scalar γ_{up} is an upper bound for $\|H(s)\|_\infty$ if and only if the inequality

$$w^\star \left[\begin{array}{c} N(s) \\ D(s) \end{array} \right]^\star \left[\begin{array}{cc} -I & 0 \\ 0 & \gamma_{\mathrm{up}}^2 I \end{array} \right] \left[\begin{array}{c} N(s) \\ D(s) \end{array} \right] w \geq 0$$

holds for all non-zero vectors w and for all scalars $s \in \partial \mathcal{S}$ (or $s \in \mathcal{S}$ in virtue of the Maximum Modulus Theorem), or equivalently if and only if the optimum of the non-convex rank-constrained LMI problem

$$\begin{array}{rl} \min & \mathrm{Trace}\, (\gamma_{\mathrm{up}}^2 \mathcal{D}^\star \mathcal{D} - \mathcal{N}^\star \mathcal{N}) X \\ \mathrm{s.t.} & \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star = 0 \\ & X = X^\star \succeq 0, \quad X \neq 0, \quad \mathrm{rank}\, X = 1 \end{array}$$

is non-negative. Using the same arguments as in the proof of Lemma 1, this is equivalent to the non-negativeness of the optimum of the convex LMI problem

$$\begin{array}{ll} \min & \mathrm{Trace}\,(\gamma_{\mathrm{up}}^2 \mathcal{D}^\star \mathcal{D} - \mathcal{N}^\star \mathcal{N}) X \\ \mathrm{s.t.} & \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star = 0 \\ & X = X^\star \succeq 0, \quad X \neq 0 \end{array} \tag{6}$$

where the rank constraint has been dropped.

Lemma 2 *The optimum in problem (6) is non-negative if and only if there exists a solution P to the LMI feasibility problem*

$$\begin{array}{l} \gamma_{\mathrm{up}}^2 \mathcal{D}^\star \mathcal{D} - \mathcal{N}^\star \mathcal{N} \succeq \left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right] (S \otimes P) \left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right]^\star \\ P = P^\star. \end{array} \tag{7}$$

Proof Suppose that problem (6) has a non-negative optimum. Therefore, for all $P = P^\star$ and all non-zero $X = X^\star \succeq 0$ it holds $\mathrm{Trace}\,(\gamma_{\mathrm{up}}^2 \mathcal{D}^\star \mathcal{D} X - \mathcal{N}^\star \mathcal{N} X) \geq \mathrm{Trace}\,(\left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star) P = \mathrm{Trace}\,(\left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right] (S \otimes P) \left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right]^\star) X$. Existence of P satisfying matrix inequality (7) readily follows. Conversely, if there exists $P = P^\star$ such that (7) holds then the previous inequality is satisfied for all non-zero matrices $X = X^\star \succeq 0$. Consequently, for all non-zero $X = X^\star \succeq 0$ such that $\left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right] (S \otimes X) \left[\begin{array}{cc} \mathcal{Q} & \mathcal{P} \end{array} \right]^\star = 0$, the optimum value in problem (7) is non-negative. $\square$

The $\mathcal{H}_\infty$-norm of $H(s)$ can thus be approached from above by minimizing γ_{up}.

Theorem 2 *Let $\hat{\mu}$ denote the optimum of LMI optimization problem*

$$\begin{array}{ll} \min & \mu \\ \mathrm{s.t.} & \mu \mathcal{D}^\star \mathcal{D} - \mathcal{N}^\star \mathcal{N} \succeq \left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right] (S \otimes P) \left[\begin{array}{cc} \mathcal{Q}^\star & \mathcal{P}^\star \end{array} \right]^\star \\ & P = P^\star. \end{array} \tag{8}$$

Then $\|H(s)\|_\infty = \sqrt{\hat{\mu}}$.

4 Numerical Examples

Both LMI problems (5) and (8) can be solved efficiently with the LMI Control Toolbox 1.0.5 for MATLAB. Polynomial matrices are handled with the Polynomial Toolbox 2.0 for MATLAB [20, 17]. The relative accuracy of the computations is set to 10^{-8}.

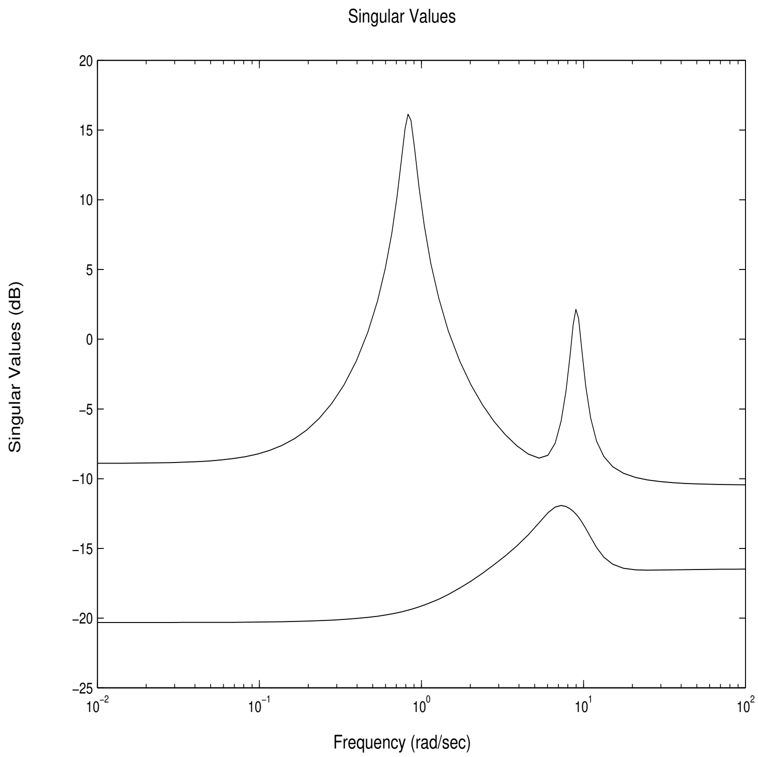

Figure 1: Singular value plot of the first example system.

4.1 First Example

Consider the continuous-time stable MIMO system $H(s) = N(s)D(s)^{-1}$ studied in [3] where

$$N(s) = \begin{bmatrix} -1.398487234 & 0.035936066 \\ -0.753204082 & -0.074480347 \end{bmatrix} + \begin{bmatrix} -0.114403422 & 0.502460582 \\ -0.187316241 & 0.694165229 \end{bmatrix} s$$
$$+ \begin{bmatrix} -0.033192482 & 0.178428698 \\ 0.000000000 & -0.090744891 \end{bmatrix} s^2$$

and

$$D(s) = \begin{bmatrix} -4.546556842 & 0.001339245 \\ 4.469627733 & 0.832831822 \end{bmatrix} + \begin{bmatrix} -0.086300454 & 0.088835731 \\ 0.068597796 & 0.103120792 \end{bmatrix} s$$
$$+ \begin{bmatrix} -0.110641607 & 0.594762328 \\ 0.000000000 & 0.604965943 \end{bmatrix} s^2.$$

The singular value plot of the system is represented in Figure 1.

We first apply Theorem 1 and solve GEVP (5) with macro `gevp` of the LMI Control Toolbox. In order to enhance convergence of the GEVP solver, we compute with macro `feasp` a warm start solution to LMI feasibility problem (3). The initial guess on lower bound γ_{low} is obtained as in [6, Section 4] as the maximum singular value of $H(s)$ evaluated at $s = j\omega$ where ω is the norm of the less damped zero of polynomial matrix $D(s)$, in

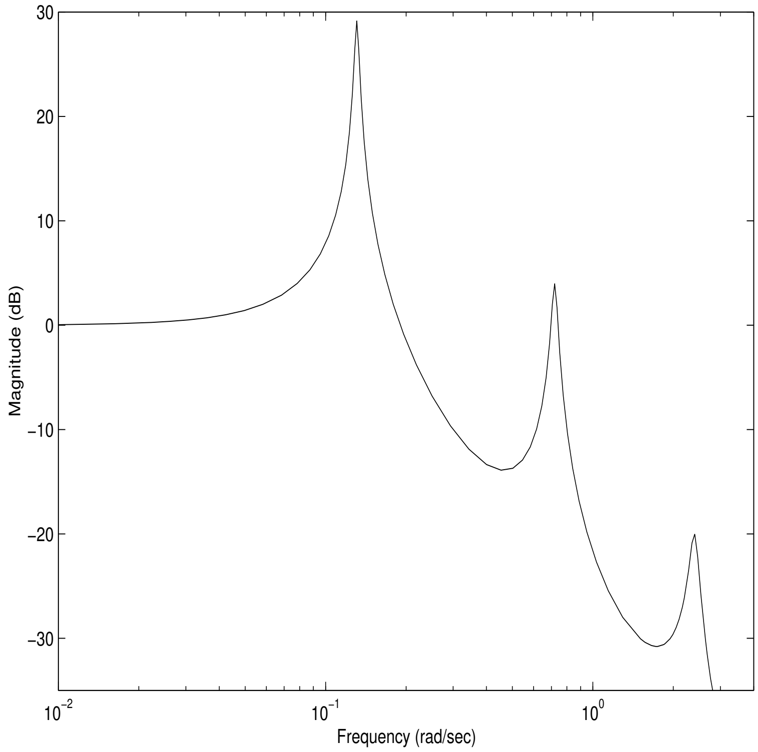

Figure 2: Bode magnitude plot of the second example system.

turn computed with macro `roots` of the Polynomial Toolbox. We get 6.440511058 as an initial lower bound. The lower bound obtained when solving GEVP (5) is

$$\gamma_{\rm low} = 6.440516511$$

which is approximately equal to 16 dB.

Then, we apply Theorem 2 and solve LMI optimization problem (6). Using LMI Control Toolbox macro `mincx` we obtain

$$\gamma_{\rm up} = 6.440516533.$$

4.2 Second Example

Consider the identified stable SISO model $H(s) = N(s)/D(s)$ of a flexible arm featuring very low damped oscillatory modes, given in [15, Table I] with no load:

$$\begin{array}{rcl} N(s) & = & 0.000000000 + 0.006401809s + 0.014593640s^2 - 0.069754510s^3 \\ & & +0.044055750s^4 + 0.038248090s^5 - 0.007055882s^6 \\ D(s) & = & 1.000000000 - 2.104942000s + 1.048514000s^2 + 0.338362200s^3 \\ & & +0.460007800s^4 - 1.514238000s^5 + 0.798784900s^6. \end{array}$$

The Bode magnitude plot of the system is represented in Figure 2.

The GEVP solver available in the LMI Control Toolbox (macro `gevp`) unexpectedly featured poor accuracy and slow convergence. We did not understand properly this phenomenon, so we implemented a bisection algorithm to solve feasibility problem (5) for successive values of μ via repeated calls of LMI Control Toolbox macro `feasp`.

We first solve LMI optimization problem (8) to obtain

$$\gamma_{\rm up} = 28.664397997 \, (\approx 29\,{\rm dB})$$

as an upper bound on $\|H(s)\|_\infty$. This bound is then used to start the bisection algorithm together with the initial lower bound 28.663636505 obtained via zeros of $D(s)$ as in the first example. After 17 iterations, the bisection algorithm applied to GEVP (5) returned

$$\gamma_{\rm low} = 28.664397991$$

as a lower bound on the $\mathcal{H}_\infty$-norm of $H(s)$.

5 Conclusion

We have shown that both a lower bound and an upper bound on the $\mathcal{H}_\infty$-norm of a transfer function can be computed via LMI optimization problems whose coefficients are those of the polynomial matrices of a MFD of the transfer function. As far as we know, the $\mathcal{H}_\infty$-norm computation methods proposed here are the first that avoid conversion from MFD to state-space model or brute-force frequency gridding. The algorithms will be implemented in the new release 3.0 of the Polynomial Toolbox for MATLAB [17].

Numerical properties (computational load, accuracy, convergence) of the algorithms proposed in this paper strongly depend on numerical properties of the software used to solve LMI problems, as illustrated by the poor accuracy and slow convergence of the GEVP solver in Example 4.2. This point deserves further investigation and intensive computational experiments must be carried out. An alternative algorithm for computing the $\mathcal{H}_\infty$-norm of polynomial matrix fractions following the Hamiltonian-based approach of [19, 3, 6] is also under study. Finally, it should be insightful to extend the ideas of this paper to the synthesis of (sub)optimal $\mathcal{H}_\infty$ controllers, in the spirit of [8].

References

[1] A. Ahlen and M. Sternad "Optimal Filtering Problems", Chapter 5 in K. Hunt (Editor) "Polynomial Methods in Optimal Control and Filtering", *Control Engineering Series, Peter Peregrinus*, London, 1993.

[2] D. Bini and V. Pan "Polynomial and Matrix Computations. Volume 1: Fundamental algorithms", *Birkhäuser*, Boston, Massachusetts, 1994.

[3] S. Boyd, V. Balakrishnan and P. Kabamba "A Bisection Method for Computing the $\mathcal{H}_\infty$-norm of a Transfer Matrix and Related Problems", *Mathematics of Control, Signals, and Systems*, Vol. 2, pp. 207–219, 1989.

[4] S. Boyd, V. Balakrishnan "A Regularity Result for the Singular Values of a Transfer Matrix and a Quadratically Convergent Algorithm for Computing its L_∞-norm", *Systems and Control Letters*, Vol. 15, pp. 1–17, 1990.

[5] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan "Linear Matrix Inequalities in System and Control Theory", *SIAM Studies in Applied Mathematics*, Philadelphia, Pennsylvania, 1994.

[6] N. A. Bruinsma and M. Steinbuch "A Fast Algorithm to Compute the $\mathcal{H}_\infty$-norm of a Transfer Function Matrix", *Systems and Control Letters*, Vol. 14, pp. 287–293, 1990.

[7] L. El Ghaoui and S. I. Niculescu (Editors) "Advances in Linear Matrix Inequality Methods in Control", *SIAM Advances in Control and Design*, Philadelphia, Pennsylvania, 1999.

[8] P. Gahinet and P. Apkarian "A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control", *International Journal of Robust and Nonlinear Control*, Vol. 4, pp. 421–448, 1994.

[9] Y. Genin, P. Van Dooren and V. Vermaut "Convergence of the Calculation of $\mathcal{H}_\infty$-norms and Related Questions", *Proceedings of the Mathematical Theory of Networks and Systems Symposium*, pp. 629–632, Padova, Italy, 1998.

[10] D. Henrion, S. Tarbouriech, M. Šebek "Rank-one LMI Approach to Simultaneous Stabilization of Linear Systems", *Systems and Control Letters*, Vol. 38, No. 2, pp. 79–89, 1999.

[11] D. Henrion "LMIs for Linear Systems Control by Polynomial Methods", *Proceedings of the IFAC Symposium on Robust Control Design*, Prague, Czech Republic, June 2000.

[12] D. Henrion and G. Meinsma "Rank-one LMIs and Lyapunov's Inequality", *Proceedings of the IEEE Conference on Decision and Control*, Sydney, Australia, December 2000.

[13] M. Hromčík and M. Šebek "New Algorithm for Polynomial Matrix Determinant Based on FFT", *Proceedings of the European Control Conference*, Karlsruhe, Germany, 1999.

[14] V. Kučera "Discrete Linear Control: The Polynomial Approach", *John Wiley and Sons*, Chichester, England, 1979.

[15] I. D. Landau, J. Langer, D. Rey and J. Barnier "Robust Control of a 360° Flexible Arm Using the Combined Pole Placement/Sensitivity Function Shaping Method", *IEEE Transactions on Control Systems Technology*, Vol. 4, No. 4, pp. 369–383, 1996.

[16] G. Meinsma, Y. Shrivastava and M. Fu "A Dual Formulation of Mixed μ and the Losslessness of (D, G)-scaling", *IEEE Transactions on Automatic Control*, Vol. 42, No. 7, pp. 1032–1036, 1997.

[17] Polyx, Ltd. "The Polynomial Toolbox for MATLAB", 2000. See the web page at `www.polyx.cz`.

[18] A. Rantzer "On the Kalman-Yakubovich-Popov Lemma", *Systems and Control Letters*, Vol. 28, pp. 7–10, 1996.

[19] G. Robel "On Computing the Infinity Norm", *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, pp. 882–884, 1989.

[20] M. Šebek, H. Kwakernaak, D. Henrion and S. Pejchová "New Algorithms for Polynomial Matrices used in the Polynomial Toolbox 2.0 for MATLAB", *Proceedings of the IEEE Conference on Decision and Control*, pp. 3661–3668, Tampa, Florida, 1998.

[21] J. Sreedhar, P. Van Dooren and A. L. Tits "A Fast Algorithm to Compute the Real Structured Stability Radius", in R. Jeltsch and M. Mansour (Editors) "Stability Theory: Proceedings of Hurwitz Centenary Conference", pp. 219–230, *Birkhauser*, 1996.

[22] A. A. Stoorvogel "Numerical Problems in Robust and $\mathcal{H}_\infty$ Optimal Control", *NICONET Report*, No. 1999-13, Katholieke Universiteit Leuven, Belgium, 1999.