Quadratic Matrix Inequalities and Stability of Polynomials

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Abstract

New relationships are enlightened between various quadratic matrix inequality conditions for stability of a scalar polynomial. It is namely shown how a recently published linear matrix inequality condition can be derived from Hermite and Lyapunov stability criteria.

Keywords

Stability, Polynomial, LMI.

1 Introduction

The problem of checking whether the roots of a polynomial belong to some region of the complex plane is of fundamental importance in control theory, because it is closely related to stability of dynamical systems. Standard methods for analyzing stability of polynomials are comprehensively described in [Gantmacher, 1959] or more recently in [Bhattacharyya et al., 1995]. The most popular stability analysis techniques are probably the determinantal Routh-Hurwitz or Schur-Cohn criteria, and Lyapunov's second approach.

Lyapunov-like techniques gradually gained a lot of interest among control people because of the development of efficient numerical procedures to solve either linear (Lyapunov) or quadratic (Riccati) matrix equalities or convex linear matrix inequalities (LMIs, see [Boyd et al., 1994]). This approach is well suited to solve stability analysis problems formulated in the state-space domain, and several extensions to address design problems are available, see e.g. [Skelton et al., 1998] for a good overview.

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Non-symmetric determinantal techniques for stability analysis, such as the Routh-Hurwitz or Schur-Cohn criteria, are generally more suited for educational purposes. They are limited to scalar systems and not really suited for design because they involve non-convex multivariable polynomial inequalities. The use of decision methods to handle these polynomial inequalities [Anderson et al., 1975] is possible only for problems of very small dimensions. However, recent results illustrate that their symmetric formulation, surveyed with admirable detail and historical perspective in [Lev-Ari et al., 1991], can shed new insights into control problems formulated in the algebra of polynomials [Kučera, 1979]. For instance, the well-known Kharitonov's theorem on stability of interval polynomials can be derived from properties of the symmetric Hermite criterion [Mansour and Anderson, 1993]. Links between determinantal criteria, Lyapunov's second approach and the Hermite criterion were surveyed in [Parks, 1977]. Similarly, an alternative proof of the Hermite criterion based on Lyapunov-like arguments was recently described in [Willems and Tempo, 1999]. Application of the Hermite criterion to the polynomial approach to the simultaneous stabilization of linear systems, a notorious robust design problem, was recently illustrated in [Henrion et al., 1999]. Finally, LMI optimization techniques were used in [Henrion et al., 2000] to derive a new condition of stability of a polynomial matrix. In the scalar case, this condition reduces to a matrix inequality that is linear in a Lyapunov-like matrix unknown, and quadratic in the polynomial coefficients.

The objective of this note is to further bridge the gap between state-space (linear matrix inequality or Lyapunov-like) techniques and polynomial (quadratic matrix inequality or Hermite-like) techniques in the case of stability analysis of a scalar polynomial. In the spirit of [Parks, 1977], we unveil several relationships existing between the Hermite criterion, the new LMI criterion proposed in [Henrion et al., 2000] and the standard Lyapunov inequality recently revisited in reference [Henrion and Meinsma, 2000].

2 Preliminaries

Given a polynomial

$$p(s) = p_0 + p_1 s + \cdots + p_k s^k$$

of degree k with complex coefficients, we define its coefficient vector

$$p = [\, p_0 \quad p_1 \quad \cdots \quad p_k \,]$$

as an element of $\mathbb{C}^{k+1}$, and the polynomial vector

$$\pi_k(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^k \end{bmatrix}$$

such that $p(s) = p\pi_k(s)$.

In this note we say that polynomial $p(s)$ is **stable** when all its roots are located in a given region

$$\{s \in \mathbb{C} \; : \; \sigma(s) = \pi_1^\star(s) S \pi_1(s) < 0\}$$

of the complex plane, where 2×2 Hermitian matrix

$$S = \begin{bmatrix} a & b \\ b^\star & c \end{bmatrix}$$

has one strictly positive eigenvalue and one strictly negative eigenvalue. Throughout the manuscript, the star denotes transpose conjugate. Standard choices for matrix S are

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for the left half-plane and continuous-time stability, and

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the unit disk and discrete-time stability. Other stability regions can be arbitrary half-planes, disk interiors or disk exteriors, see [Lev-Ari et al., 1991] or [Henrion et al., 2000].

Following [Lev-Ari et al., 1991], we define the **polynomial reflection** of $p(s)$ as the polynomial

$$\tilde{p}(s) = \left(\frac{b^\star + cs}{\sqrt{b^\star b - ac}}\right)^k p^\star\left(-\left(\frac{a+bs}{b^\star + cs}\right)^\star\right)$$

of degree k. Standard choices for $\tilde{p}(s)$ are

$$\tilde{p}(s) = p^\star(-s^\star) = p_0^\star - p_1^\star s + \cdots + (-1)^k p_k^\star s^k$$

for continuous-time stability ($a = 0, b = 1, c = 0$) and

$$\tilde{p}(s) = s^k p^\star(1/s^\star) = p_k^\star + p_{k-1}^\star s + \cdots + p_0^\star s^k$$

for discrete-time stability ($a = -1, b = 0, c = 1$). An important property of the polynomial reflection is that $p(s)$ and $\tilde{p}(s)$ are coprime if and only if $p(s)$ has no roots along the curve $\sigma(s) = 0$. When $p(s)$ is stable, then its reflection polynomial $\tilde{p}(s)$ is anti-stable, i.e. all its roots satisfy $\sigma(s) > 0$.

Associated with $p(s)$ we define the polynomials

$$\begin{aligned} n(s) &= \tfrac{\sqrt{2}}{2}(p(s) + \tilde{p}(s)) \\ d(s) &= \tfrac{\sqrt{2}}{2}(p(s) - \tilde{p}(s)) \end{aligned}$$

and the companion matrix

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -p_0/p_k & -p_1/p_k & \cdots & -p_{k-1}/p_k \end{bmatrix}$$

such that the determinant of $sI_k - A$ is equal to $p(s)$.

We define the Bezoutian of $p(s)$ as the quadratic form

$$B(s) = \frac{p^\star(s)p(s) - \tilde{p}^\star(s)\tilde{p}(s)}{\sigma(s)}.$$

In [Lev-Ari et al., 1991] it is proved that $B(s)$ is a polynomial in s and $s^\star$, i.e. $\sigma(s)$ divides $p^\star(s)p(s) - \tilde{p}^\star(s)\tilde{p}(s)$ and we can write

$$B(s) = \pi^\star_{k-1}(s) H \pi_{k-1}(s)$$

where the square matrix H of size k is referred to as the **Hermite matrix** of $p(s)$.

Let

$$R = \left[\begin{array}{ccccc} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{array}\right]$$

be a projection matrix of size $2k \times (k+1)$. Finally, the notation $P \succ 0$ (resp. $P \succeq 0$) means that matrix P is positive definite (resp. semidefinite) and $\otimes$ denotes the Kronecker product.

3 Main Result

Theorem 1 *The following statements are equivalent:*

(a) *$p(s)$ is stable*

(b) $H \succ 0$

(c) *there exists a matrix $P = P^\star \succ 0$ solving the LMI*

$$p^\star p \succ R^\star (S \otimes P) R \tag{1}$$

(d) *there exists a matrix $P = P^\star \succ 0$ solving the LMI*

$$n^\star d + d^\star n \succeq R^\star (S \otimes P) R \tag{2}$$

(e) *there exists a matrix $P = P^\star \succ 0$ solving the LMI*

$$\left[\begin{array}{c} I_n \\ A \end{array}\right]^\star (S \otimes P) \left[\begin{array}{c} I_n \\ A \end{array}\right] \prec 0 \tag{3}$$

(f) *the rational function*

$$g(s) = \frac{n(s)}{d(s)}$$

is strictly positive real, i.e. $n(s)$ and $d(s)$ are coprime polynomials and

$$g(s) + g^\star(s) > 0$$

when $\sigma(s) > 0$.

Proof of the equivalence of (a) and (b):
It follows from Hermite's stability criterion, see Theorem 1 in [Lev-Ari et al., 1991]. It is not the purpose of this note to provide an alternative proof of this result, but Hermite's criterion and in particular the Hermite matrix H will prove instrumental to the derivation of the other proofs.

Proof of the equivalence of (b) and (d):
From the definition of the Bezoutian, it holds

$$\begin{array}{rcl} \sigma(s)B(s) & = & \sigma(s)\pi^\star_{k-1}(s)H\pi_{k-1}(s) \\ & = & p^\star(s)p(s) - \tilde{p}^\star(s)\tilde{p}(s) \\ & = & \frac{1}{2}(p^\star(s)+\tilde{p}^\star(s))(p(s)-\tilde{p}(s)) + \frac{1}{2}(p^\star(s)-\tilde{p}^\star(s))(p(s)+\tilde{p}(s)) \\ & = & n^\star(s)d(s) + d^\star(s)n(s) \\ & = & \pi^\star_k(s)(n^\star d + d^\star n)\pi_k(s). \end{array}$$

From the definition of $\sigma(s)$, we also have

$$\begin{array}{rcl} \sigma(s)B(s) & = & \sigma(s)\pi^\star_{k-1}(s)H\pi_{k-1}(s) \\ & = & \pi^\star_{k-1}(s)(aH + bsH + b^\star s^\star H + cs^\star sH)\pi_{k-1}(s) \\ & = & \left[\begin{array}{c} \pi_{k-1}(s) \\ s\pi_{k-1}(s) \end{array}\right]^\star (S\otimes H) \left[\begin{array}{c} \pi_{k-1}(s) \\ s\pi_{k-1}(s) \end{array}\right] \\ & = & \pi^\star_k(s)R^\star(S\otimes H)R\pi_k(s). \end{array}$$

Identification of the powers of the indeterminates s and $s^\star$ in these two expressions of the same two-variable polynomial leads to the equality

$$n^\star d + d^\star n = R^\star(S\otimes H)R \tag{4}$$

that is satisfied by the Hermite matrix H. Thus LMI (2) holds for the particular choice $P = H$. □

Proof of the equivalence of (a) and (c):
LMI (1) implies that

$$\pi^\star_k(s)p^\star p\pi_k(s) = p^\star(s)p(s) > \pi^\star_k(s)R^\star(S\otimes P)R\pi_k(s) = \sigma(s)\pi^\star_{k-1}(s)P\pi_{k-1}(s).$$

Since $P \succ 0$, it follows that $p(s)$ never vanishes when $\sigma(s) \geq 0$, i.e. that $p(s)$ is stable. Conversely, assume that $p(s)$ is stable. In view of the proof of the equivalence of (b) and (d), the choice $P = H \succ 0$ is feasible for the non-strict version of LMI (1), i.e.

$$p^\star p \succeq p^\star p - \tilde{p}^\star\tilde{p} = R^\star(S\otimes H)R.$$

In the sequel we will show that we can always find a matrix $\tilde{P}$ such that

$$P = H + \tilde{P} \succ 0$$

solves LMI (1) strictly, i.e. such that the LMI

$$\tilde{p}^\star \tilde{p} \succ R^\star (S \otimes \tilde{P}) R$$

is satisfied. Invoking Theorem 2.3.12 in [Skelton et al., 1998], this is equivalent to the projected LMI

$$\tilde{N}^\star R^\star (S \otimes \tilde{P}) R \tilde{N} \prec 0$$

where

$$\tilde{N} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ -\tilde{p}_0/\tilde{p}_k & -\tilde{p}_1/\tilde{p}_k & \cdots & -\tilde{p}_{k-1}/\tilde{p}_k \end{bmatrix}$$

is a basis for the null-space of vector $\tilde{p}$. Defining

$$\tilde{A} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\tilde{p}_0/\tilde{p}_k & -\tilde{p}_1/\tilde{p}_k & \cdots & -\tilde{p}_{k-1}/\tilde{p}_k \end{bmatrix}$$

as the companion matrix associated with reflection polynomial $\tilde{p}(s)$, the above LMI can be written as a linear system of equations

$$\tilde{N}^\star R^\star (S \otimes \tilde{P}) R \tilde{N} = \begin{bmatrix} I_n \\ \tilde{A} \end{bmatrix}^\star (S \otimes \tilde{P}) \begin{bmatrix} I_n \\ \tilde{A} \end{bmatrix} = F(\tilde{P}) = Q \tag{5}$$

for some matrix $Q \prec 0$. Since $p(s)$ is stable, there is no eigenvalue of matrix $\tilde{A}$ on the stability boundary and the linear map F is one-to-one [Gantmacher, 1959]. Consequently, given any matrix $Q \prec 0$, there is a solution $\tilde{P}$ to linear system (5). Moreover, $\tilde{P} \prec 0$ because $\tilde{p}(s)$ is anti-stable. In particular, one can choose a solution $\tilde{P}$ arbitrarily close to zero such that $P = H + \tilde{P} \succ 0$ and LMI (1) is satisfied. □

Proof of the equivalence of (c) and (e):
A basis for the null-space of vector p is the matrix

$$N = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ -p_0/p_k & -p_1/p_k & \cdots & -p_{k-1}/p_k \end{bmatrix}.$$

As in the proof of the equivalence of (a) and (c), there exists a solution $P = P^\star \succ 0$ to LMI (1) if and only if there exists a solution $P = P^\star \succ 0$ to the projected LMI

$$N^\star R^\star (S \otimes P) R N = \begin{bmatrix} I_n \\ A \end{bmatrix}^\star (S \otimes P) \begin{bmatrix} I_n \\ A \end{bmatrix} \prec 0$$

which is precisely Lyapunov inequality (3). $\square$

Proof of the equivalence of (a) and (f):
If $p(s)$ is stable then

$$n^\star(s)d(s) + d^\star(s)n(s) = \sigma(s)\pi^\star_{k-1}(s)H\pi_{k-1}(s) > 0$$

when $\sigma(s) > 0$. Since $p(s)$ and $\tilde{p}(s)$ are coprime, $n(s)$ and $d(s)$ are also coprime and the real number

$$g(s) + g^\star(s) = \frac{n(s)}{d(s)} + \frac{n^\star(s)}{d^\star(s)} = \frac{n^\star(s)d(s) + d^\star(s)n(s)}{d^\star(s)d(s)}$$

is positive when $\sigma(s) > 0$. Conversely, if $g(s)$ is strictly positive real, then

$$n^\star(s)d(s) + d^\star(s)n(s) = p^\star(s)p(s) - \tilde{p}^\star(s)\tilde{p}(s) > 0$$

hence $p^\star(s)p(s) > 0$ when $\sigma(s) > 0$. Coprimeness of $n(s)$ and $d(s)$ implies coprimeness of $p(s)$ and $\tilde{p}(s)$ and therefore $p(s)$ has no roots along the curve $\sigma(s) = 0$. Consequently, $p(s)$ is stable. $\square$

One important idea in Theorem 1 can be summarized as follows. The polynomial $p(s)$ is stable if and only if both LMIs

$$\begin{array}{rcl} p^\star p & \succ & R^\star(S \otimes P)R, \quad H \succ P \succ 0 \\ \tilde{p}^\star \tilde{p} & \succ & R^\star(S \otimes \tilde{P})R, \quad H \succ -\tilde{P} \succ 0 \end{array}$$

are feasible. Limit choices for P and $\tilde{P}$ correspond to non-strict versions of the LMIs, namely

$$\begin{array}{rcl} p^\star p & \succeq & R^\star(S \otimes P)R, \quad\ \ P = H \ \text{ or } \ P = 0 \\ \tilde{p}^\star \tilde{p} & \succeq & R^\star(S \otimes \tilde{P})R, \quad -\tilde{P} = H \ \text{ or } \ \tilde{P} = 0. \end{array}$$

Moreover, the central choice $P - \tilde{P} = H \succ 0$ corresponds to the equality

$$p^\star p - \tilde{p}^\star \tilde{p} = n^\star d + d^\star n = R^\star(S \otimes H)R.$$

Note that the equivalence of (a) and (c) was proved in Theorem 1 in [Henrion et al., 2000] as follows. First, the problem of checking stability of a polynomial is formulated as a quadratic optimization problem. Second, an equivalent LMI optimization problem with a non-convex rank constraint is derived using linear fractional representations. Third, it is shown that the rank constraint is irrelevant. Fourth and last, the LMI formulation (1) is obtained via standard semi-definite programming duality arguments. Above we provide an alternative, constructive, Hermite-based proof of this result.

The same kind of ideas based on rank-constrained LMIs were used in the reference [Henrion and Meinsma, 2000] to prove the equivalence of (a) and (e). Here too, we give another proof of this result in the case that matrix A is the companion matrix associated with polynomial $p(s)$. When A is another matrix with characteristic polynomial $p(s)$, we can use a similarity transformation to prove the same result.

In [Parks, 1977] it is recalled how the choice $P = H \succ 0$ satisfies a non-strict version of Lyapunov inequality, leading to a Lyapunov proof of Hermite's stability criterion. Here we go further and show how matrix $\tilde{P} \prec 0$ can be found such that $P = H + \tilde{P} \succ 0$ satisfies the strict Lyapunov inequality.

Note also that the equivalence of (a) and (f) is well-known [Anderson and Bitmead, 1977], especially in the context of signal processing, where positive realness is often invoked when dealing with stability of filters.

In view of the above proof of equivalence of (b) and (d), LMI (2) is actually equivalent to the matrix equation (4) where a matrix $P = P^\star \succ 0$ is sought. It is a linear system of $k(k+1)/2$ equations with $k(k+1)/2$ unknowns that admits a unique solution $P = H$ which is positive definite if and only if polynomial $p(s)$ is stable.

Defining the projection matrix

$$\tilde{R} = \left[\begin{array}{ccccccc} 1 & & & 0 & 0 & 1 & & \\ & \ddots & & 0 & 0 & & \ddots & \\ & & 1 & 0 & 0 & & & 1 \end{array}\right]$$

of size $k \times 2(k+1)$, it can be shown using standard duality arguments [Henrion et al., 2000] that LMI (1) is feasible if and only if the optimum value μ of the LMI optimization problem

$$\begin{array}{rcll} \mu & = & \min & pXp^\star \\ & & \text{s.t.} & \tilde{R}(S \otimes X)\tilde{R}^\star \succeq 0 \\ & & & X \succeq 0, \quad X \neq 0 \end{array} \tag{6}$$

is strictly positive. Since the criterion cannot be negative, LMI (1) is infeasible if and only if the LMI problem

$$\begin{array}{l} pXp^\star = 0 \\ \tilde{R}(S \otimes X)\tilde{R}^\star \succeq 0 \\ X \succeq 0, \quad X \neq 0 \end{array}$$

is feasible.

4 Illustration

4.1 Continuous-time third order polynomials

We will first illustrate our results in the simple case of a continuous-time polynomial of degree $k = 3$ with real coefficients. Associated with polynomial

$$p(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3$$

is the polynomial reflection

$$\tilde{p}(s) = p_0 - p_1 s + p_2 s^2 - p_3 s^3$$

and the polynomials

$$\begin{array}{rcl} n(s) & = & \sqrt{2}(p_0 + p_2 s^2) \\ d(s) & = & \sqrt{2}(p_1 s + p_3 s^3) \end{array}$$

which respectively correspond to the even and odd parts of $p(s)$. LMI (2) reads

$$\begin{bmatrix} 0 & 2p_0p_1 & 0 & 2p_0p_3 \\ 2p_0p_1 & 0 & 2p_1p_2 & 0 \\ 0 & 2p_1p_2 & 0 & 2p_2p_3 \\ 2p_0p_3 & 0 & 2p_2p_3 & 0 \end{bmatrix} \succeq \begin{bmatrix} 0 & p_{11} & p_{21} & p_{31} \\ p_{11} & 2p_{21} & p_{31}+p_{22} & p_{32} \\ p_{21} & p_{31}+p_{22} & 2p_{32} & p_{33} \\ p_{31} & p_{32} & p_{33} & 0 \end{bmatrix}$$

where p_{ij} denotes entry (i,j) in symmetric matrix P. Equivalently, P is solution to the linear system of equations (4) which is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{22} \\ p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = \begin{bmatrix} 2p_0p_1 \\ 0 \\ 2p_1p_2 \\ 2p_0p_3 \\ 0 \\ 2p_2p_3 \end{bmatrix}.$$

The Hermite matrix of $p(s)$ solves the above system, i.e.

$$P = H = \begin{bmatrix} 2p_0p_1 & 0 & 2p_0p_3 \\ 0 & 2p_1p_2 - 2p_0p_3 & 0 \\ 2p_0p_3 & 0 & 2p_2p_3 \end{bmatrix}$$

and matrix H is positive definite if and only if the roots of $p(s)$ are located in the open left half-plane.

4.2 Discrete-time third order polynomials

When $p(s)$ is a discrete-time polynomial of degree $k = 3$ with real coefficients, we define its polynomial reflection

$$\tilde{p}(s) = p_3 + p_2 s + p_1 s^2 + p_0 s^3$$

and the polynomials

$$\begin{array}{rcl} n(s) & = & \frac{\sqrt{2}}{2}((p_0+p_3)(1+s^3) + (p_1+p_2)(s+s^2)) \\ d(s) & = & \frac{\sqrt{2}}{2}((p_0-p_3)(1-s^3) + (p_1-p_2)(s-s^2)). \end{array}$$

LMI (2) reads

$$\begin{bmatrix} p_0^2 - p_3^2 & p_0p_1 - p_2p_3 & p_0p_2 - p_1p_3 & 0 \\ p_0p_1 - p_2p_3 & p_1^2 - p_2^2 & 0 & -p_0p_2 + p_1p_3 \\ p_0p_2 - p_1p_3 & 0 & -p_1^2 + p_2^2 & -p_0p_1 + p_2p_3 \\ 0 & -p_0p_2 + p_1p_3 & -p_0p_1 + p_2p_3 & -p_0^2 + p_3^2 \end{bmatrix} \succeq$$
$$\begin{bmatrix} -p_{11} & -p_{21} & -p_{31} & 0 \\ -p_{21} & -p_{22}+p_{11} & -p_{32}+p_{21} & p_{31} \\ -p_{31} & -p_{32}+p_{21} & -p_{33}+p_{22} & p_{32} \\ 0 & p_{31} & p_{32} & p_{33} \end{bmatrix}$$

where p_{ij} denotes entry (i,j) in symmetric matrix P. Equivalently, P is solution to the linear system of equations (4) which is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{22} \\ p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} = \begin{bmatrix} p_3^2 - p_0^2 \\ p_2 p_3 - p_0 p_1 \\ p_2^2 - p_1^2 \\ p_1 p_3 - p_0 p_2 \\ p_2 p_3 - p_0 p_1 \\ p_3^2 - p_0^2 \end{bmatrix}.$$

The Hermite matrix of $p(s)$ solves the above system, i.e.

$$P = H = \begin{bmatrix} p_3^2 - p_0^2 & p_2 p_3 - p_0 p_1 & p_1 p_3 - p_0 p_2 \\ p_2 p_3 - p_0 p_1 & p_2^2 + p_3^2 - p_0^2 - p_1^2 & p_2 p_3 - p_0 p_1 \\ p_1 p_3 - p_0 p_2 & p_2 p_3 - p_0 p_1 & p_3^2 - p_0^2 \end{bmatrix}$$

and matrix H is positive definite if and only if the roots of $p(s)$ are located in the open unit disk.

4.3 Numerical Example

Let $p(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$ be a stable continuous-time polynomial. The Hermite matrix of $p(s)$ is given by

$$H = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 16 & 0 \\ 2 & 0 & 6 \end{bmatrix} \succ 0.$$

With the choice $P = H$ feasible for LMI (2), equality (4) reads

$$n^\star d + d^\star n = R^\star \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} R = \begin{bmatrix} 0 & 6 & 0 & 2 \\ 6 & 0 & 18 & 0 \\ 0 & 18 & 0 & 6 \\ 2 & 0 & 6 & 0 \end{bmatrix}.$$

With the help of the LMI Toolbox for MATLAB [The Mathworks, 1998], we obtain

$$P = \begin{bmatrix} 3.1858 & 2.7348 & 1.0603 \\ 2.7348 & 7.8099 & 2.7348 \\ 1.0603 & 2.7348 & 3.1858 \end{bmatrix} \succ 0$$

as a feasible solution of LMI (1) and LMI (3). Note that $P = H + \tilde{P}$ for the choice

$$Q = \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{21} & q_{22} & q_{32} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} -1.8793 & 2.7396 & -2.8984 \\ 2.7396 & -10.939 & 7.5173 \\ -2.8984 & 7.5173 & -11.416 \end{bmatrix} \prec 0$$

in the linear system (5) given by

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} \tilde{p}_{11} \\ \tilde{p}_{21} \\ \tilde{p}_{22} \\ \tilde{p}_{31} \\ \tilde{p}_{32} \\ \tilde{p}_{33} \end{bmatrix} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \\ q_{31} \\ q_{32} \\ q_{33} \end{bmatrix},$$

so that

$$\tilde{P} = \begin{bmatrix} -2.8142 & 2.7348 & -0.9397 \\ 2.7348 & -8.1901 & 2.7348 \\ -0.9397 & 2.7348 & -2.8142 \end{bmatrix} \prec 0.$$

Now let $p(s) = (s+1)^2(s-3) = -3 - 5s - s^2 + s^3$ be an unstable continuous-time polynomial. The Hermite matrix of $p(s)$ is given by

$$H = \begin{bmatrix} 30 & 0 & -6 \\ 0 & 16 & 0 \\ -6 & 0 & -2 \end{bmatrix}$$

and it is indefinite. We can check that LMI (1) is infeasible since the optimal value of the dual LMI optimization problem (6) is equal to $\mu = 0$ for the choice

$$X = xx^\star = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}^\star = \begin{bmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 81 \\ 9 & 27 & 81 & 243 \\ 27 & 81 & 243 & 729 \end{bmatrix} \succeq 0$$

corresponding to a vector $x = \pi_3(3)$ such that $px = 0$. Consequently, LMIs (2) and (3) are also infeasible.

5 Conclusion

We have clarified the connections existing between the scalar version of the new LMI formulation of the stability of a polynomial matrix recently published in [Henrion et al., 2000] and standard stability criteria based on Hermite quadratic forms, Lyapunov's inequality and strict positive realness.

This work can be considered as a tentative preliminary step toward the generalization of Hermite criterion to polynomial matrices. It seems that no nice multivariable extension of Hermite criterion is available in the control literature, but some attempts were reported in [Anderson and Bitmead, 1977] and [Kraus et al., 1996]. It is hoped that some progress may be made in this direction, using the ideas exposed in this note, since the LMI formulation proposed in [Henrion et al., 2000] is also valid for polynomial matrices.

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References

[Anderson et al., 1975] B. D. O. Anderson, N. K. Bose and E. I. Jury "Output Feedback Stabilization and Related Problems - Solution via Decision Methods", *IEEE Transactions on Automatic Control*, Vol. 20, No. 1, pp. 53–66, 1975.

[Anderson and Bitmead, 1977] B. D. O. Anderson and R. R. Bitmead "Stability of Matrix Polynomials", *International Journal of Control*, Vol. 26, No. 2, pp. 235–247, 1977.

[Bhattacharyya et al., 1995] S. P. Bhattacharyya, H. Chapellat and L. H. Keel "Robust Control: The Parametric Approach", *Prentice Hall*, Upper Saddle River, New Jersey, 1995.

[Boyd et al., 1994] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan "Linear Matrix Inequalities in System and Control Theory", *SIAM Studies in Applied Mathematics*, Philadelphia, Pennsylvania, 1994.

[Gantmacher, 1959] F. R. Gantmacher "The Theory of Matrices", *Chelsea*, New York, 1959.

[Henrion et al., 1999] D. Henrion, S. Tarbouriech and M. Šebek "Rank-one LMI Approach to Simultaneous Stabilization of Linear Systems", *Systems and Control Letters*, Vol. 38, No. 2, pp. 79-89, 1999.

[Henrion and Meinsma, 2000] D. Henrion and G. Meinsma "Rank-one LMIs and Lyapunov's Inequality", *IEEE Transactions on Automatic Control*, Vol. 46, No. 8, pp. 1285–1288, 2001.

[Henrion et al., 2000] D. Henrion, O. Bachelier and M. Šebek "D-stability of Polynomial Matrices", *International Journal of Control*, Vol. 74, No. 8, pp. 845–856, 2001.

[Kraus et al., 1996] F. J. Kraus, M. Mansour and M. Šebek "Hurwitz Matrix for Polynomial Matrices", International Series of Numerical Mathematics, Vol. 121, pp. 67–74, *Birkhäuser Verlag*, Basel, Switzerland, 1996.

[Kučera, 1979] V. Kučera "Discrete Linear Control: The Polynomial Approach", *John Wiley and Sons*, Chichester, 1979.

[Lev-Ari et al., 1991] H. Lev-Ari, Y. Bistritz and T. Kailath "Generalized Bezoutians and Families of Efficient Zero-Location Procedures", *IEEE Transactions on Circuits and Systems*, Vol. 38, No. 2, pp. 170–186, 1991.

[The Mathworks, 1998] The MathWorks, Inc. "LMI Control Toolbox for Matlab", Release 1.0.5, 1998.

[Mansour and Anderson, 1993] M. Mansour and B. D. O. Anderson "Kharitonov's Theorem and the Second Method of Lyapunov", *Systems and Control Letters*, Vol. 20, pp. 39–47, 1993.

[Parks, 1977] P. C. Parks "A New Proof of Hermite's Stability Criterion and a Generalization of Orlando's Formula", *International Journal of Control*, Vol. 26, No. 2, pp. 197–206, 1977.

[Skelton et al., 1998] R. E. Skelton, T. Iwasaki and K. Grigoriadis "A Unified Algebraic Approach to Linear Control Design", *Taylor and Francis*, London, 1998.

[Willems and Tempo, 1999] J. C. Willems and R. Tempo "The Kharitonov Theorem with Degree Drop", *IEEE Transactions on Automatic Control*, Vol. 44, No. 11, pp. 2218–2220, 1999.