Rank-one LMI Approach to Simultaneous Stabilization of Linear Systems*

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Abstract

Following a polynomial approach to control design, the co-stabilization by a fixed controller of a family of SISO linear systems is interpreted as an LMI feasibility problem with a rank-one constraint. An LMI relaxation algorithm and a potential reduction heuristic are then proposed for addressing this key optimization problem.

*This work was supported by the Barrande Project No. 97/005-97/026, by the Grant Agency of the Czech Republic under contract No. 102/97/0861, by the Ministry of Education of the Czech Republic under contract No. VS97/034 and by the French Ministry of Education and Research under contract No. 10-INSA-96.

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1 Introduction

We consider the problem of simultaneously stabilizing, or co-stabilizing, a family of single-input single-output (SISO) linear systems by one fixed controller of given order. This fundamental problem, recognized as one of the difficult open issues in linear system theory, arises for instance when trying to preserve stability of a control system under the failure of sensors, actuators or processors. Simultaneous stabilization of three or more systems was extensively studied in [3]. In [2], the question of knowing whether three linear systems are simultaneously stabilizable was shown to be rationally undecidable. Later on, co-stabilization was shown to belong to the wide range of robust control problems that are NP-hard [27, 4], i.e. that are very unlikely to be solved in polynomial time.

Since the co-stabilization problem does not seem to be tractable in its most general from, several researchers focused on devising simpler but approximate design methods. Obviously, this was done at the expense of conservatism. For instance, the simple linear programming (LP) approach proposed in [11] is based on stability conditions that are only necessary. In other words, the controller obtained through this method may actually not be co-stabilizing. Another technique was recently described in [1] that relies upon nonlinear programming (NLP). The co-stabilization problem is framed as a mathematical programming problem constrained by nonlinear necessary and sufficient stability conditions straightforwardly given by the Routh-Hurwitz criterion. Since the resulting problem is highly non-convex, convergence of the NLP scheme is never guaranteed. On the other hand, in contrast to the LP approach, the NLP algorithm – when converging – always yields a controller which is co-stabilizing. Finally, we point out another approach to simultaneous stabilization, recently published in [7]. Using state-space arguments, it is proved that the existence of a co-stabilizing feedback gain is equivalent to that of the solution of a set of coupled quadratic matrix inequalities. A heuristic iterative design algorithm based on linear matrix inequalities (LMIs) is then proposed.

Motivated by the results described in [11, 1], we propose an entirely different approach to simultaneous stabilization. Our technique, obtained in the framework of the polynomial approach to systems control [21], is believed to lend a new insight into this key control problem. We show that co-stabilization of a family of SISO linear systems actually amounts to solving a non-convex feasibility problem where all the non-convexity is concentrated into the constraint that a matrix subject to an LMI has rank one.

By formulating the co-stabilization problem as a rank constrained LMI problem (Section 2), we obtain tractable necessary or sufficient conditions for co-stabilization with a measure of the amount of conservatism. This leads us to an LMI relaxation procedure (Section 3). Then, we propose a heuristic design algorithm based on a potential reduction method (Section 4). Throughout the paper, applicability of these techniques are illustrated on simple examples.

The notations in this paper are fairly standard. $\mathrm{Tr}X$ and $\mathrm{rank}X$ respectively stand for the trace and the rank of matrix X, $\mathbf{I}_n$ stands for the identity matrix of size n, the notation $X > 0$ means that matrix X is positive definite and $x > 0$ means that all the components in vector x are positive. $p(\mathtt{s})$ represents a polynomial p in the indeterminate $\mathtt{s}$ but we shall drop this dependence on $\mathtt{s}$ whenever convenient.

2 Rank-One LMI Formulation

Let the real rational functions

$$P_i(\mathtt{s}) = \frac{n_i(\mathtt{s})}{d_i(\mathtt{s})}, \quad i = 1, \ldots, N, \quad \text{and} \quad C(\mathtt{s}) = \frac{n_c(\mathtt{s})}{d_c(\mathtt{s})}$$

be polynomial fraction descriptions for N proper plants and a proper controller. When controller C is placed in series with the plants and closed under unity feedback, C simultaneously internally stabilizes plants P_i if and only if characteristic polynomials

$$\delta_i(\mathtt{s}) = n_c(\mathtt{s})n_i(\mathtt{s}) + d_c(\mathtt{s})d_i(\mathtt{s}), \quad i = 1, \ldots, N$$

are Hurwitz stable. Given polynomials n_i and d_i, the problem of finding polynomials n_c and d_c such that polynomials $\delta_1, \ldots, \delta_N$ are stable is referred to as the *co-stabilization problem.* In this section, we use quadratic stability criteria in order to derive a rank-one LMI formulation of this problem.

2.1 Hermite-Fujiwara Matrix

Definition 1 *Given a pair of polynomials $p(\mathtt{s}) = p_0 + p_1\mathtt{s} + \cdots + p_m\mathtt{s}^m$ and $q(\mathtt{s}) = q_0 + q_1\mathtt{s} + \cdots + q_m\mathtt{s}^m$, the Bézoutian matrix of p and q is the m-by-m symmetric matrix $B(p,q)$ with entries*

$$b_{jk} = \sum_{l=j+k-m-1}^{j-1} (p_{j+k-l-1}q_l - q_{j+k-l-1}p_l)$$

where it is assumed that $p_l = q_l = 0$ if $l < 0$ or $l > m$.

Definition 2 *Given a polynomial $p(\mathtt{s}) = p_{\text{even}}(\mathtt{s}^2) + \mathtt{s}p_{\text{odd}}(\mathtt{s}^2)$ of degree $\leq m$ (some leading coefficients of p may be zero), define $p^\star(\mathtt{s}) = p_{\text{even}}(\mathtt{s}^2) - \mathtt{s}p_{\text{odd}}(\mathtt{s}^2)$. Denote by b_{jk} the entries of $B(p,p^\star)$, the Bézoutian matrix of p and $p^\star$. The Hermite-Fujiwara matrix of p of order m is the m-by-m matrix $H_m(p)$ with entries*

$$h_{jk} = (-1)^j b_{jk}, \quad j,k = 1, \ldots, m.$$

A fundamental property of the above matrix is captured by the following theorem, proved in [20, §3.1], [24, §1.3.3] or [12].

Theorem 1 *A polynomial p of degree m is Hurwitz stable if and only if the Hermite-Fujiwara matrix $H_m(p)$ is positive definite.*

Definition 3 *Given a pair of polynomials* $p(\mathtt{s}) = p_{\text{even}}(\mathtt{s}^2) + \mathtt{s}p_{\text{odd}}(\mathtt{s}^2)$ *and* $q(\mathtt{s}) = q_{\text{even}}(\mathtt{s}^2) + \mathtt{s}q_{\text{odd}}(\mathtt{s}^2)$ *of degree* $\leq m$ *(some leading coefficients of* p *or* q *may be zero), the Hermite-Fujiwara of* p *and* q *of order* m *is the* m*-by-*m *matrix*

$$H_m(p,q) = H_m(p_{\text{even}}(\mathtt{s}^2) + \mathtt{s}q_{\text{odd}}(\mathtt{s}^2)).$$

With this notation, the following property, easily proved by inspection [16], characterizes the Hermite-Fujiwara matrix of a linear combination of polynomials.

Lemma 1 *Given a real vector* $x = [x_0 \quad x_1 \ \cdots \ x_n]'$ *and* $n+1$ *polynomials* $p_0,\ p_1, \ldots, p_n$ *of degrees* $\leq m$*, we have*

$$H_m(\sum_{i=0}^{n} x_i p_i) = \sum_{j=0}^{n}\sum_{k=0}^{n} x_j x_k H_m(p_j, p_k).$$

2.2 BMI Formulation

In the sequel, it is assumed that the order of the desired co-stabilizing controller is fixed to $d = \deg d_c$, so

$$\begin{array}{rcl} n_c(\mathtt{s}) & = & x_0 + x_1\mathtt{s} + \cdots + x_d\mathtt{s}^d \\ d_c(\mathtt{s}) & = & x_{d+1} + x_{d+2}\mathtt{s} + \cdots + x_{2d+1}\mathtt{s}^d. \end{array}$$

Let $n = 2d+1$ and collect all the design parameters into a column vector

$$x = [x_0 \quad x_1 \ \cdots \ x_n]'.$$

Finally, for $i = 1, \ldots, N$ and $j = 0, 1, \ldots, d$, we define the polynomials

$$\begin{array}{rcl} p_{ij}(\mathtt{s}) & = & \mathtt{s}^j n_i(\mathtt{s}) \\ p_{i(j+d+1)}(\mathtt{s}) & = & \mathtt{s}^j d_i(\mathtt{s}) \end{array}$$

so that closed-loop characteristic polynomial reads

$$\delta_i(\mathtt{s}) = \sum_{j=0}^{n} x_j p_{ij}(\mathtt{s}).$$

Let $m = \max_{i=1,\ldots,N} \deg \delta_i$. From Lemma 1, the Hermite-Fujiwara matrix of δ_i reads

$$H_m(\delta_i) = \sum_{j=0}^{n}\sum_{k=0}^{n} x_j x_k H_m(p_{ij}, p_{ik}).$$

For notational ease, let $H_{ijk} = H_m(p_{ij}, p_{ik})$. Since matrices H_{ijk} are known, the above matrix inequalities are bilinear in unknown design parameters x_j. In the control literature, such inequalities are known as bilinear matrix inequalities (BMIs). Recalling Theorem 1, the following necessary and sufficient conditions for co-stabilization have been shown.

Theorem 2 *Any vector x solution to the BMIs*

$$\sum_{j=0}^{n}\sum_{k=0}^{n} x_j x_k H_{ijk} > 0, \quad i = 1, \ldots, N \tag{1}$$

parametrizes a pair of polynomials n_c and d_c solving the co-stabilization problem.

In [15], the authors showed that many of the problems considered in the robust control literature can be formulated as BMIs. In view of Theorem 2, this is also the case for the co-stabilization problem. Unfortunately, BMIs are highly non-convex optimization problems and solving a general BMI was shown to be NP-hard [27]. It is therefore particularly relevant to characterize tractable approximation methods that may provide a solution or state that no solution exists.

2.3 LMI with Rank Constraint

A first step in this direction is formulating BMIs (1) as a special LMI problem where all the non-convexity is concentrated into a rank constraint. The motivation for doing so is that LMI problems are well-behaved convex problems for which many numerical methods have recently been developed, see [5] for a good overview.

Following [6], bilinear terms in (1) are collected into the entries

$$x_{jk} = x_j x_k, \quad j, k = 0, 1, \ldots, n$$

of a rank-one positive semidefinite matrix

$$X = xx' \geq 0 \tag{2}$$

so that BMIs (1) become a rank-one constrained LMI

$$\begin{array}{l} \sum_{j=0}^{n}\sum_{k=0}^{n} x_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ X \geq 0, \quad \operatorname{rank} X = 1. \end{array} \tag{3}$$

It must be underlined that the idea of formulating non-convex control problems as an LMI problem with a rank constraint is not new, see for instance [9]. In [26], the relationship between BMIs and rank constrained LMI problems was pointed out. Moreover, it is now recognized that BMIs are at the core of a very broad range of robust control problems [15]. Beyond the scope of systems control, rank constrained LMI problems also occur in mathematical programming and combinatorial optimization. In [25, 23, 6] it is shown that all of the problems with polynomial objective and polynomial constraints (including quadratic programming, binary programming, integer programming and job-shop scheduling, to mention just a few) may actually be formulated as rank-one LMI optimization problems.

3 LMI Relaxation

Now we show that the rank-one LMI formulation introduced in the previous section proves useful for deriving tractable necessary conditions and sufficient conditions for co-stabilization. Such conditions can dramatically simplify the design process. On the one hand, sufficient conditions may efficiently provide a co-stabilizing controller. On the other hand, necessary conditions may prevent using time-consuming heuristic or global optimization algorithms by readily stating that the plants are not co-stabilizable [11, 1].

As show below, upon interpreting (3) as the problem of measuring the radius of the intersection of given ellipsoids [26, 17], co-stabilization conditions are obtained through relaxations of the non-convex rank constraint. Most importantly, we are able to measure the amount of conservatism implied by these conditions.

3.1 Ellipsoid Intersection Problem

Matrix inequalities (1) can be written as scalar inequalities

$$\sum_{j=0}^{n}\sum_{k=0}^{n} x_j x_k v' H_{ijk} v > 0, \quad i = 1, \ldots, N$$

where vector v describes the unit sphere $\mathbb{S}^m$. Equivalently, the above equation becomes

$$x' M_i(v) x > 0, \quad i = 1, \ldots, N, \quad \forall v \in \mathbb{S}^m$$

where

$$M_i(v) = \begin{bmatrix} v' H_{i00} v & \cdots & v' H_{i0n} v \\ \vdots & & \vdots \\ v' H_{in0} v & \cdots & v' H_{inn} v \end{bmatrix}.$$

Defining ρ as a positive real number such that ρ^{-1} is greater than all the eigenvalues of symmetric matrices $M_i(v) + M_i'(v)$ for all $i = 1, \ldots, N$ and $v \in \mathbb{S}^m$, we guarantee that symmetric matrix

$$Q_i(v) = \mathbf{I}_{n+1} - \rho[M_i(v) + M_i'(v)] = \begin{bmatrix} v' Q_{i00} v & \cdots & v' Q_{in0} v \\ \vdots & & \vdots \\ v' Q_{in0} v & \cdots & v' Q_{inn} v \end{bmatrix}$$

is always positive definite. In the above relation, we have $Q_{ijj} = \mathbf{I}_m - 2\rho H_{ijj}$ and $Q_{ijk} = -\rho(H_{ijk} + H_{ikj})$ for $j \neq k$. Scalar ρ can be found for instance upon overbounding matrices $M_i(v) + M_i'(v)$ by a symmetric interval matrix, and using the extremal eigenvalues results of [18].

In relation to this matrix, we define the set

$$\mathcal{E} = \{x : x' Q_i(v) x \leq 1, \ i = 1, \ldots, N, \ \forall v \in \mathbb{S}^m\}.$$

Since $Q_i(v)$ is always positive definite, $\mathcal{E}$ is the intersection of an infinite number of centered ellipsoids. Hence $\mathcal{E}$ is convex, centered about the origin and its radius is the maximal norm of any element in $\mathcal{E}$. The following theorem, proved in [16], provides an interpretation of co-stabilization as the problem of checking whether the radius of the intersection of ellipsoids is greater than one.

Theorem 3 *Any vector $x \in \mathcal{E}$ such that $x'x > 1$ parametrizes a pair of polynomials n_c and d_c solving the co-stabilization problem.*

The square of the radius of $\mathcal{E}$, denoted $p_{\rm opt}$, is solution to the optimization problem

$$\begin{array}{rcll} p_{\rm opt} & = & \max & x'x \\ & & \text{s.t.} & x \in \mathcal{E} \end{array} \tag{4}$$

where a convex function must be maximized over a convex set. Let $x_{\rm opt}$ denote the global optimum to problem (4). The following corollary is a straightforward byproduct of Theorem 3.

Corollary 1 *Vector $x_{\rm opt}$ parametrizes a pair of polynomials n_c and d_c solving the co-stabilization problem if and only if $p_{\rm opt} > 1$.*

In the literature, problem (4) is referred to as a concave minimization problem. Unsurprisingly, concave minimization is known to be a difficult global optimization problem [19, Chapter I, §2].

In this section, we are however not interested in solving (4) for its global optimum but rather seek tractable numerical methods for approximating $p_{\rm opt}$. For this purpose, use equation (2) to reformulate problem (4) as

$$\begin{array}{ll} \max & \mathrm{Tr}X \\ \text{s.t.} & \mathrm{Tr}XQ_i(v) \leq 1, \quad i = 1, \ldots, N, \quad \forall v \in \mathbb{S}^m \\ & X \geq 0, \quad \mathrm{rank}X = 1, \end{array}$$

or similarly, as the rank-one LMI problem

$$\begin{array}{ll} \max & \mathrm{Tr}X \\ \text{s.t.} & \sum_{j=0}^{n} \sum_{k=0}^{n} x_{jk} Q_{ijk} \leq \mathbf{I}_m, \quad i = 1, \ldots, N \\ & X \geq 0, \quad \mathrm{rank}X = 1. \end{array} \tag{5}$$

3.2 LMI Relaxation

Problem (5) is an LMI optimization problem with an additional non-convex constraint. As pointed out in [17], upper and lower bounds on the global optimum $p_{\rm opt}$ can be derived from the following convex LMI relaxation to (5), simply obtained upon dropping the rank constraint:

$$\begin{array}{ll} \max & \mathrm{Tr} X \\ \text{s.t.} & \sum_{j=0}^{n} \sum_{k=0}^{n} x_{jk} Q_{ijk} \leq \mathbf{I}_m, \quad i = 1, \ldots, N \\ & X \geq 0. \end{array} \tag{6}$$

In [17] we showed that the above relaxation is dual to the semidefinite Lagrangian relaxation, also known as the $\mathcal{S}$-procedure in the Russian control theory. We studied problem (4) where $\mathcal{E}$ was the intersection of a finite number of ellipsoids, but some of our results can be generalized without difficulty to the intersection of an infinite number of ellipsoids. This is captured by the following lemma, based on [17, Theorem 3].

Lemma 2 *Solve LMI relaxation (6) for an optimal solution X_{rel}. Compute w, the normalized eigenvector corresponding to λ, the maximal eigenvalue of X_{rel}, and let $x_{\mathrm{rel}} = \sqrt{\lambda} w$, $p_{\mathrm{rel}} = x'_{\mathrm{rel}} x_{\mathrm{rel}} = \lambda$. Then*

$$p_{\mathrm{rel}} \leq p_{\mathrm{opt}} \leq (n+1) p_{\mathrm{rel}}.$$

The lemma states that we can easily approximate the global optimum to non-convex problem (5), but only up to a factor $n+1$. In other words, we are able to measure the amount of conservatism inherent to the approximation method. For low values of n, this may result in useful necessary conditions and sufficient conditions for co-stabilization, as captured by the next corollaries.

Corollary 2 *(Sufficient condition for co-stabilization) – Let x_{rel} and p_{rel} be as in Lemma 2. If $p_{\mathrm{rel}} > 1$ then x_{rel} parametrizes a pair of polynomials n_c and d_c solving the co-stabilization problem.*

Corollary 3 *(Necessary condition for co-stabilization) – Let p_{rel} be as in Lemma 2. If $p_{\mathrm{rel}} < (n+1)^{-1}$ then the co-stabilization problem has no solution.*

Note that the necessary condition proposed in Corollary 3 still differs from the conditions given in [1, Theorem 1] or [11, §3]. Depending on the problem being treated, one condition may prove more useful than the others. One may also combine these conditions.

Now suppose that LMI relaxation (6) yields $(n+1)^{-1} \leq p_{\mathrm{rel}} \leq 1$ in Lemma 2, so that neither Corollary 2 nor Corollary 3 can be applied for finding a co-stabilizing controller or stating that no such a controller exists. In this case, one really requires addressing the rank constrained LMI problem (3), not an approximation thereof, in order to solve the co-stabilization problem. As underlined in Section 2, solving (3) amounts to solving BMIs (1). Several heuristic and global optimization algorithms were devised in [14] for finding solutions to BMIs. However, these methods are usually far too general. We will show in the next sections that the rank-one structure of problem (3) can further be exploited so as to derive more specific resolution methods.

3.3 Illustration

Let us illustrate the applicability of LMI relaxation (6) on a simple example. Suppose that we want to co-stabilize $N = 3$ first order plants

$$P_1(\mathrm{s}) = \frac{-1+\mathrm{s}}{2+3\mathrm{s}}, \quad P_2(\mathrm{s}) = \frac{4+2\mathrm{s}}{5-\mathrm{s}}, \quad P_3(\mathrm{s}) = \frac{3+\mathrm{s}}{1+\mathrm{s}}$$

with a controller of order $d = 1$. Note that plant P_2 is unstable. Following the notations defined above, we have $n = 3$ and $m = 2$. Hermite-Fujiwara matrices H_{ijk} are diagonal hence the first matrix inequalities in (3) yield scalar linear inequalities. They give rise to the following rank-one LMI feasibility problem (3)

$$\begin{bmatrix} -2 & 2 & -2 & -4 & 0 & -4 & 0 & 12 & 8 & 0 \\ 0 & -2 & 0 & 6 & -2 & 6 & -2 & 0 & 18 & 12 \\ 16 & 32 & 12 & 40 & 0 & 40 & 0 & -10 & 0 & 50 \\ 0 & 8 & 0 & -4 & 16 & -4 & 12 & 0 & 2 & -10 \\ 6 & 18 & 8 & 6 & 0 & 6 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 & 6 & 2 & 8 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_{00} \\ x_{01} \\ x_{02} \\ x_{03} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{22} \\ x_{23} \\ x_{33} \end{bmatrix} > 0,$$

$$X = X' = \begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{01} & x_{11} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix} \geq 0, \quad \mathrm{rank} X = 1.$$

In order to build matrices Q_{ijk} as in Section 3.1, the interval matrix maximal eigenvalue theorem of [18] yields $\rho = 9.135 \cdot 10^{-3}$. Using a standard LMI solver [13], the optimal solution to LMI relaxation (6) reads

$$X_{\mathrm{rel}} = \begin{bmatrix} 0.0541 & 0.1545 & 0.1519 & 0.0957 \\ 0.1545 & 0.4414 & 0.4341 & 0.2735 \\ 0.1519 & 0.4341 & 0.4270 & 0.2690 \\ 0.0957 & 0.2735 & 0.2690 & 0.1695 \end{bmatrix} \geq 0.$$

The singular values of X_{rel} are approximately 1.09, $4.56 \cdot 10^{-5}$, $1.29 \cdot 10^{-5}$ and $7.62 \cdot 10^{-7}$ hence the above matrix has almost rank one. After a simple eigenvector computation, Lemma 2 yields

$$x_{\mathrm{rel}} = [0.2325 \quad 0.6644 \quad 0.6534 \quad 0.4117]'.$$

The sufficient condition of Corollary 2 holds since $p_{\mathrm{rel}} = x'_{\mathrm{rel}} x_{\mathrm{rel}} = 1.09 > 1$, hence

$$C(\mathrm{s}) = \frac{0.2325 + 0.6644\mathrm{s}}{0.6534 + 0.4117\mathrm{s}}$$

is guaranteed to be a co-stabilizing controller for plants P_1, P_2 and P_3.

4 Potential Reduction

In this section, we focus on a first heuristic for tackling problem (3). It hinges upon an extension of a potential reduction method for solving convex optimization problems, and was first proposed in [8]. Convergence to the global optimum is not guaranteed but the method may sometimes be useful. The principle lying behind the approach is as follows.

4.1 Rank minimization

Instead of solving (3) for a matrix X of rank one, we will try to solve the equivalent problem

$$\begin{array}{ll} \min & \mathrm{rank} X \\ \text{s.t.} & \sum_{j=0}^{n} \sum_{k=0}^{n} x_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ & X \geq 0. \end{array} \tag{7}$$

As soon as the above criterion equals one, a solution X to problem (3) is found. As stated in Section 2.3, minimization of the rank of a matrix subject to convex LMI constraint is a long-standing problem in control theory. Its most important application resides in the design of reduced order controllers and static output feedback stabilization [9, 10, 22].

In [8], the authors proposed an extension of a potential reduction method – an interior point method for convex optimization – for solving rank minimization problems. The approach for solving (7) is similar to the Newton algorithm proposed by Nesterov and Nemirovskii [23], with the notable exception that the objective function is not convex and must be linearized in order to be minimized. Because of the non-convexity of the objective function, the algorithm may unfortunately get stuck in a local minimum. Technical details are not reproduced here for conciseness. The interested reader may consult [8].

4.2 First illustration

We now illustrate the applicability of the potential reduction method on the basic problem of co-stabilizing $N = 3$ first order unstable plants

$$P_1(\mathrm{s}) = \frac{1-\mathrm{s}}{2-3\mathrm{s}}, \quad P_2(\mathrm{s}) = \frac{2+\mathrm{s}}{3-\mathrm{s}}, \quad P_3(\mathrm{s}) = \frac{1+3\mathrm{s}}{1-\mathrm{s}}$$

with a controller of order $d = 1$. Following the approach of Section 3, we selected $\rho = 0.03202$ and LMI relaxation (6) provided us with the value $p_{\mathrm{rel}} = 0.9690$. Unfortunately, neither Corollary 2 nor Corollary 3 can be used for finding a co-stabilizing controller or proving that the plants are not simultaneously stabilizable. An alternative technique is required.

Using a MATLAB[1] implementation of the potential reduction method described in [8], we obtained after 55 iterations (5.9 seconds of CPU time on a Sun SparcStation 4) a rank-one

[1] MATLAB is a trademark of the MathWorks, Inc.

solution

$$X = \begin{bmatrix} 11.5726 & 8.8626 & -4.9922 & -3.9054 \\ 8.8626 & 6.7872 & -3.8231 & -2.9909 \\ -4.9922 & -3.8231 & 2.1535 & 1.6847 \\ -3.9054 & -2.9909 & 1.6847 & 1.3180 \end{bmatrix} \approx xx' = \begin{bmatrix} 3.4019 \\ 2.6052 \\ -1.4675 \\ -1.1480 \end{bmatrix} \begin{bmatrix} 3.4019 \\ 2.6052 \\ -1.4675 \\ -1.1480 \end{bmatrix}'$$

to problem (7). The above vector x is computed by a simple singular value decomposition. It yields the co-stabilizing controller

$$C(\mathtt{s}) = \frac{3.4019 + 2.6052\mathtt{s}}{-1.4675 - 1.1480\mathtt{s}}.$$

4.3 Second Illustration

As another illustration of the potential reduction method, we consider the following parameter-varying system

$$P(\mathtt{s}) = \frac{1 + \sqrt{\theta} + \mathtt{s}}{3\theta + 11 + (\theta^2 - 10)\mathtt{s} + \mathtt{s}^2}, \quad 3 \leq \theta \leq 11$$

proposed in [7]. We aim at simultaneously stabilizing with a constant feedback ($d = 0$) three plants at operating points $\theta_0 = 7$, $\theta_1 = 3$ and $\theta_2 = 11$.

We obtained after 23 iterations (1.1 seconds of CPU time on a Sun SparcStation 4) a rank-one solution

$$X = \begin{bmatrix} 17.6360 & 0.7301 \\ 0.7301 & 0.0302 \end{bmatrix} \approx xx' = \begin{bmatrix} 4.1995 \\ 0.1739 \end{bmatrix} \begin{bmatrix} 4.1995 \\ 0.1739 \end{bmatrix}'$$

to problem (7). A co-stabilizing static feedback is therefore

$$C(\mathtt{s}) = \frac{4.1995}{0.1739} = 24.1489.$$

5 Conclusion

In this paper we presented a new approach to simultaneous stabilization of a family of linear systems. We established the equivalence between co-stabilization and a special instance of a global optimization problem: the rank-one LMI feasibility problem, whose geometry can be exploited for designing approximation algorithms or heuristics. Note that, as in our preliminary work [16], we obtained this rank constrained LMI feasibility problem while pursing a polynomial, or algebraic approach to control design [21]. This is in contrast with previously available results that were rather concentrating on state-space arguments [9, 15]. In addition to that, our paper features some numerical techniques for addressing the rank-one LMI feasibility problem.

Our work hints of several potential directions for future research. We strongly believe that further work must be devoted to the study of algorithms for solving the rank-one LMI problem. Indeed, this key problem appears to encompass in an elegant way a huge range of practical robust control design issues, but also difficult problems arising in combinatorial optimization. Investigations are underway for showing that, when pursuing a polynomial approach, the problems of static output feedback stabilization and robust pole placement into a disk are also amenable to this unifying treatment.

It should also be insightful to assess the applicability to our problem of the various rank minimization heuristics proposed in the literature. For instance, the cutting plane technique for unary programming proposed in [25] may prove useful. Recent theoretical results on the rank minimization problem [22] also deserve to be considered.

Finally, we are now studying deterministic global optimization techniques [19] taking advantage of the special structure of the rank-one LMI problem or, equivalently, of the ellipsoidal intersection problem: convexity of the criterion, convexity of the feasible set. Related results will be issued in forthcoming reports.

Acknowledgments

The authors would like to thank Laurent El Ghaoui (Ecole Nationale Supérieure des Techniques Avancées, Paris) and Denis Arzelier (LAAS-CNRS, Toulouse) for helpful suggestions. They are also grateful to Arnaud Hiret (Ecole Supérieure d'Electricité, Gif-sur-Yvette) for providing a MATLAB implementation of the potential reduction method.

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