Rank-one LMI Approach to Stability of 2-D Polynomial Matrices[1]

Didier Henrion[2,3]
Michael Šebek[4,5]
Olivier Bachelier[6]

Abstract

New necessary and sufficient conditions are formulated for checking stability of a 2-D polynomial matrix. The conditions are expressed as a non-convex rank-one LMI problem derived from a formulation of the stability problem as a quadratic optimization problem. Sufficient stability conditions expressed as a convex LMI problem are readily derived from this formulation.

Keywords

2-D Systems, Polynomial Matrix, Stability, LMI.

1 Introduction

Since the early seventies, multi-dimensional (n-D), and in particular two-dimensional (2-D) systems have been attracting the attention of the scientific community working on the area of systems theory and signal processing. Several motivations underlie this research field. Complex industrial and physical processes are frequently of multi-dimensional nature (such as multipass processes in manufacturing, traffic control problems, image processing and information processes) [25]. In the framework of n-D data processing using

[1]This work was supported by the Barrande Project No. 97/005-97/026, by the Grant Agency of the Czech Republic under contract No. 102/99/1368 and by the Ministry of Education of the Czech Republic under contract No. VS97/034.

[2]Corresponding author. E-mail: `henrion@laas.fr`. FAX: +33 5 61 33 69 69.

[3]Laboratoire d'Architecture et d'Analyse des Systèmes - Centre National de la Recherche Scientifique, 7 avenue du Colonel Roche, 31 077 Toulouse, cedex 4, France.

[4]Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic.

[5]Trnka Laboratory of Automatic Control, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic.

[6]Laboratoire Automatique et Informatique Industrielle, Ecole Supérieure d'Ingénieurs de Poitiers, 40 avenue du Recteur Pineau, 86 022 Poitiers cedex, France.

parallel computing structures, the inspiring idea originated from the necessity of modeling dynamical phenomena parameterized by many independent variables (e.g. time and space coordinates). On the other hand, the class of multipass processes or repetitive dynamical systems has been identified as an important case of 2-D systems, with industrial application to the mining and steel processing industries and of central relevance to areas of control where repetition [32] and/or scanning is undertaken (as in image processing) [3, 4, 9]. More recently, n-D state space models have been applied in representing the processes of pollution and self-purification of a river stretch [15]. When investigating convolutional codes, n-D systems are viewed as "realizations" of encoders, decoders and syndrome generators [16]. Recent studies in iterative (trajectory) learning indicate the central role that multi-dimensional systems theory may have in the development of recursive/repetitive algorithms for "intelligent control" in the next decade.

Both basic approaches characteristic for linear time-invariant systems — geometric or state-space methods and algebraic, fractional or polynomial techniques — have been partly generalized to cope with 2-D or n-D systems and processes. The generalized polynomial approach naturally leads to 2-D or n-D polynomials and polynomial matrices, see [33, 34, 35, 36] and references therein. Although the stability of 2-D systems may sometimes depend also on the transfer function numerator, its investigation is often reduced to testing stability of 2-D polynomials and 2-D polynomial matrices. Even this appears to be a difficult task. First of all, different applications call for various different stability regions to be considered. Second, the problem nature is two-dimensional and hence even simple scalar 2-D polynomial stability tests either directly include a sort of a 2-D time consuming gridding or replace it by another involved technique [9, 23]. The matrix case is even worse as there are no direct tests available to check the stability of a 2-D polynomial matrix. The problem is usually transformed into the scalar case by evaluating the 2-D polynomial matrix determinant which by itself is a numerically difficult task [24, 26]. Alternatively, one can build a frequency dependent Lyapunov equation [37] or a constant 2-D Lyapunov equation [1, 20, 27]. However, in contrast with one-dimensional linear systems, there is always a gap between necessity and sufficiency of the 2-D Lyapunov stability conditions. Generally speaking, we can state that there is still a need for unified methods for checking stability of 2-D systems.

This paper is precisely aimed at overcoming the lack of sufficiently general methods for studying stability of 2-D polynomial matrices. The approach we pursue is based on an idea that originally appeared in [2]. The stability problem for 2-D polynomial matrices is first expressed as a quadratic optimization problem. Then, standard techniques are used to come up with a rank-one Linear Matrix Inequality (LMI, see [5]) formulation of this problem. The strengths of our method are enumerated below.

- The stability conditions are necessary and sufficient.
- 2-D polynomial matrices with real or complex coefficients can be handled.
- Covered stability regions include half-planes or disks but can be extended to more general regions [18].
- Several algorithms have been devised recently for solving approximately or exactly

rank-one LMI problems [8, 12, 38]. Moreover, the relaxation of the rank constraint directly yields convex LMI sufficient stability conditions.

- The stability conditions can be combined with other conditions of similar type arising in robust stability analysis problems, when some elements in the 2-D polynomial matrix are uncertain.

Note however that the design of algorithms for solving the rank-one LMI problem or the extension of our approach to robust stability analysis remain out of the scope of the present paper. These extensions will be covered elsewhere.

The outline of the paper is as follows. In Section 2, we state the 2-D stability problem to be solved. A non-convex rank-one LMI formulation of necessary and sufficient stability conditions is derived in Section 3. We briefly mention heuristic algorithms or global optimization algorithms that can be devised to solve this problem. We also show how convex LMI sufficient stability conditions can be derived. Numerical examples are provided in Section 4 to illustrate the method.

Notations: $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively. The star $\star$ denotes transpose conjugate. I_n denotes the identity matrix of dimension n. Trace$A$ is the trace of matrix A. Rank$A$ is the rank of matrix A. The matrix inequalities $A \succ B$ and $A \succeq B$ mean that matrix $A-B$ is positive definite and positive semidefinite, respectively.

2 Problem Statement

Suppose we are given a non-singular $n \times n$ complex 2-D polynomial matrix

$$A(s_1, s_2) = \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{d_2} A_{i_1 i_2} s_1^{i_1} s_2^{i_2}$$

with indeterminates s_1, s_2. We are also given two regions $\mathcal{D}_1$, $\mathcal{D}_2$ of the complex plane such that

$$\mathcal{D}_i = \{s \in \mathbb{C} : \alpha_i + \beta_i s + \beta_i^\star s^\star + \gamma_i s s^\star \geq 0\}$$

for $i = 1, 2$. In the above expression, complex scalars $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ are such that Hermitian matrix

$$B_i = B_i^\star = \begin{bmatrix} \alpha_i & \beta_i^\star \\ \beta_i & \gamma_i \end{bmatrix}$$

has one strictly positive eigenvalue and one strictly negative eigenvalue, for $i = 1, 2$. The interested reader is referred to [18] for a discussion about this particular choice of stability regions and possible extensions. In this paper, we restrict our attention to the following regions.

- Half-plane $\mathcal{D}_i = \{x + jy \in \mathbb{C} : ax + by + c \geq 0\}$ with $a, b, c \in \mathbb{R}$ and

$$B_i = \begin{bmatrix} 2c & a + jb \\ a - jb & 0 \end{bmatrix}.$$

- Disk $\mathcal{D}_i = \{s \in \mathbb{C} : |s - s_0| \leq r\}$ with $s_0 \in \mathbb{C}$, $r \in \mathbb{R}$, $r > 0$ and
$$B_i = \begin{bmatrix} r^2 - s_0 s_0^\star & s_0 \\ s_0^\star & -1 \end{bmatrix}.$$
- Disk complement $\mathcal{D}_i = \{s \in \mathbb{C} : |s - s_0| \geq r\}$ with $s_0 \in \mathbb{C}$, $r \in \mathbb{R}$, $r > 0$ and
$$B_i = \begin{bmatrix} -r^2 + s_0 s_0^\star & -s_0 \\ -s_0^\star & 1 \end{bmatrix}.$$

One can check that the matrices defined above satisfy our eigenvalue assumption. Note that some care must be taken with zeros at infinity when dealing with open stability regions such as the disk complement or the left half-plane, see [18, §3] for details.

A zero of a non-singular $n \times n$ 2-D polynomial matrix $A(s_1, s_2)$ is usually defined [36] as a complex couple $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ for which the rank of $A(s_1, s_2)$ drops below its normal value, i.e.
$$\mathrm{Rank} A(z_1, z_2) < n.$$
Matrix $A(s_1, s_2)$ is stable in region $\mathcal{D}_1 \times \mathcal{D}_2$ if and only if there is no zero of $A(s_1, s_2)$ in this region.

Without computing the determinant or the zeros of $A(s_1, s_2)$, we aim at finding necessary and sufficient conditions for checking stability of $A(s_1, s_2)$ in $\mathcal{D}_1 \times \mathcal{D}_2$.

3 Rank-one LMI Formulation

In this section, we show that stability of a 2-D polynomial matrix can be inferred from the optimal value of an LMI optimization problem where all the non-convexity is concentrated into the constraint that a matrix has rank-one. We briefly describe existing algorithms that can be used to solve this problem. We also point out that convex LMI sufficient stability conditions can easily be derived from this rank-one formulation.

3.1 Quadratic Optimization Problem

Couple $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ is a zero of non-singular 2-D polynomial matrix $A(s_1, s_2)$ if and only if there exists a non-zero vector $x \in \mathbb{C}^n$ such that
$$A(z_1, z_2)x = 0.$$
Consequently, matrix $A(s_1, s_2)$ is stable in region $\mathcal{D}_1 \times \mathcal{D}_2$ if and only if the optimal value μ of the quadratic optimization problem
$$\begin{array}{rcll} \mu & = & \min & x^\star A^\star(s_1, s_2) A(s_1, s_2) x \\ & & \text{s.t.} & s_1 \in \mathcal{D}_1 \\ & & & s_2 \in \mathcal{D}_2 \\ & & & x^\star x = 1 \end{array} \tag{1}$$
is strictly positive.

3.2 Rank-one LMI Problem

Now we show that solving quadratic optimization problem (1) amounts to solving a rank-one LMI optimization problem. We shall use the following important result.

Lemma 1 *Matrix $A(s_1, s_2)$ has the following Linear Fractional Representation (LFR)*

$$A(s_1, s_2) = A_{00} + L\Delta(I_{n(d_1+d_2)} - D\Delta)^{-1}R.$$

In the above equation

$$\Delta = \left[\begin{array}{cc} s_1 I_{nd_1} & 0 \\ 0 & s_2 I_{nd_2} \end{array}\right]$$

and

$$\begin{array}{rcl}
\left[\begin{array}{c||c} L & A_{00} \\ \hline\hline D & R \end{array}\right] & = & \left[\begin{array}{c} \mathcal{A} \\ \hline\hline \mathcal{Q} \end{array}\right] = \left[\begin{array}{c} \mathcal{A} \\ \hline\hline \mathcal{Q}_1 \\ \hline \mathcal{Q}_2 \end{array}\right] = \left[\begin{array}{c||c} \mathcal{R} & \mathcal{B} \end{array}\right] = \left[\begin{array}{c|c||c} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{B} \end{array}\right] \\
 & = & \left[\begin{array}{cccc|cccc||c}
I_n & 0 & \cdots & 0 & A_{0d_2} & \cdots & & A_{01} & A_{00} \\ \hline\hline
0 & I_n & & & A_{1d_2} & \cdots & & A_{11} & A_{10} \\
 & & \ddots & & & & & & \\
 & & & I_n & \vdots & & & \vdots & \vdots \\
 & & & 0 & A_{d_1d_2} & \cdots & & A_{d_11} & A_{d_10} \\ \hline
 & & & & 0 & I_n & & & 0 \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & I_n & 0 \\
 & & & & & & & 0 & I_n
\end{array}\right].
\end{array} \qquad (2)$$

Proof of Lemma 1 The LFR is obtained using the construction principle outlined in [40, §10.3]. Note that matrix D is strictly upper triangular, hence the above LFR is always well-posed. □

Note that the LFR of a 2-D polynomial matrix is neither unique nor necessarily minimal, i.e. of smallest possible dimension in Δ. As pointed out in [36, §10.4] and [40, §10.2], it is usually very hard to find a minimal realization for even a 2-D scalar polynomial. In order to avoid redundancy in the above representation, we may require that matrices $\mathcal{Q}_1$ and $\mathcal{R}_2$ have full rank. If this is not the case, we can perform a series of reductions as described in the Appendix. Performing this reduction is not necessary, but usually significantly reduces problem dimensions.

Let $r_1 = nd_1$ and $r_2 = nd_2$ denote the row dimensions of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively. In relation to the above LFR, we define vectors

$$p = \left[\begin{array}{c} p_1 \\ p_2 \end{array}\right] \in \mathbb{C}^{r_1+r_2}, \quad q = \left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] \in \mathbb{C}^{r_1+r_2}$$

such that

$$\begin{array}{rcl} A(s_1, s_2)x & = & A_{00}x + Lp \\ q & = & Rx + Dp \\ p & = & \Delta q. \end{array} \qquad (3)$$

Lemma 2 *Let B_i a Hermitian matrix and $\mathcal{D}_i$ the corresponding region as defined in Section 2. There exists vectors $p_i, q_i \in \mathbb{C}^{r_i}$ not both zero such that*

$$[q_i \quad p_i]B_i[q_i \quad p_i]^\star = \alpha_i q_i q_i^\star + \beta_i p_i q_i^\star + \beta_i^\star q_i p_i^\star + \gamma_i p_i p_i^\star \succeq 0 \tag{4}$$

if and only if $p_i = s_i q_i$ and $s_i \in \mathcal{D}_i$.

Proof The proof is adapted from Lemma C.1 in [10]. If $r_i = 1$, vectors p_i and q_i are scalars and the result readily follows. Now suppose that $r_i > 1$. Recall that matrix B_i has one positive eigenvalue and one negative eigenvalue. Therefore, for arbitrary vectors p_i and q_i, matrix (4) has generically one positive eigenvalue and one negative eigenvalue, the remaining $r_i - 2$ eigenvalues being zero. If we enforce matrix (4) to be positive semidefinite, then its negative eigenvalue vanishes, its rank drops down to one and vectors p_i, q_i become linearly dependent, i.e. there exists a scalar s_i such that $p_i = s_i q_i$. In this case, matrix

$$(\alpha_i + \beta_i s + \beta_i^\star s^\star + \gamma_i s s^\star) q_i q_i^\star$$

is positive semidefinite for any vector q_i. Therefore, scalar s_i belongs to region $\mathcal{D}_i$. Note that the case $p_i = 0$ and $q_i \neq 0$ corresponds to a scalar s_i at the origin. In a similar way, the case $p_i \neq 0$ and $q_i = 0$ corresponds to a scalar s_i at infinity. □

Define the rank-one positive semidefinite matrix

$$X = zz^\star = \left[\begin{array}{c} p \\ x \end{array}\right]\left[\begin{array}{c} p \\ x \end{array}\right]^\star \succeq 0 \tag{5}$$

and the projection matrices

$$\mathcal{P}_1 = \left[\begin{array}{cc|c} I_{r_1} & 0 & 0 \end{array}\right] \quad \mathcal{P}_2 = \left[\begin{array}{cc|c} 0 & I_{r_2} & 0 \end{array}\right] \quad \mathcal{X} = \left[\begin{array}{cc|c} 0 & 0 & I_n \end{array}\right].$$

Recalling equations (2) and (3), it holds

$$\begin{array}{rclrcl} A(s_1, s_2)x & = & \mathcal{A}z & & & \\ q_1 & = & \mathcal{Q}_1 z & q_2 & = & \mathcal{Q}_2 z \\ p_1 & = & \mathcal{P}_1 z & p_2 & = & \mathcal{P}_2 z. \end{array} \tag{6}$$

Inequality (4) is thus an LMI in matrix X that we denote by a linear map

$$F_{\mathcal{D}_i}(X) = \alpha_i \mathcal{Q}_i X \mathcal{Q}_i^\star + \beta_i \mathcal{P}_i X \mathcal{Q}_i^\star + \beta_i^\star \mathcal{Q}_i X \mathcal{P}_i^\star + \gamma_i \mathcal{P}_i X \mathcal{P}_i^\star \tag{7}$$

associated with region $\mathcal{D}_i$.

Using equations (5), (6) and (7), an alternative formulation of problem (1) can now be derived. It reads

$$\begin{array}{rcll} \mu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \text{s.t.} & F_{\mathcal{D}_1}(X) \succeq 0 \\ & & & F_{\mathcal{D}_2}(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \mathrm{Trace}\mathcal{X}^\star\mathcal{X}X = 1 \\ & & & \mathrm{Rank}X = 1. \end{array} \tag{8}$$

We have shown the main result of this paper.

Theorem 1 *2-D polynomial matrix $A(s_1, s_2)$ is stable in region $\mathcal{D}_1 \times \mathcal{D}_2$ if and only if the optimal value μ of rank-one LMI problem (8) is strictly positive.*

3.3 Algorithms

Problem (8) is an LMI optimization problem with the additional constraint that a matrix has rank one. This rank constraint is unfortunately non-convex. It must be pointed out that rank-constrained LMIs frequently arise in control problems [11, 19] but also in mathematical programming and combinatorial optimization. In [30], it is shown that all of the problems with polynomial objective and polynomial constraints (including quadratic programming, binary programming, integer programming, to mention just a few) may actually be formulated as rank-one LMI optimization problems. More recently, it was shown that the problem of simultaneously stabilizing a set of SISO linear systems by one controller of given order can also be cast as a rank-one LMI problem [19]. In the same vein, the seemingly simple problem of stabilizing a given plant with a fixed-order compensator whose coefficients are linearly constrained can also be shown to be equivalent to solving a rank-one LMI problem, see [14, §1.8.2].

The motivation behind formulating the 2-D polynomial matrix stability problem as a rank-one LMI problem is in that several algorithms have been devised recently for solving rank-constrained LMIs. These algorithms are either heuristics or global optimization techniques.

A first heuristic was proposed in [8]. It is an extension of a potential reduction method for solving convex optimization problems. The approach is similar to the Newton algorithm proposed by Nesterov and Nemirovski [30], with the notable exception that the logarithmic barrier function is not convex and must be linearized in order to be minimized. Because of the non-convexity of the objective function, the algorithm may unfortunately get stuck in a local minimum. Successful applications of the potential reduction method were recently reported for LFT model reduction [22] and IQC synthesis [17].

A second heuristic was proposed in [12]. It is an extension of a linearization method to cone complementarity problems over LMIs. It is based on the Frank and Wolfe feasible direction method. The algorithm consists in a series of LMI optimization problems aimed at decreasing a non-linear objective function. Once the objective function is zero, a rank-one solution to the LMI is found. Despite the lack of convergence guarantee, this method has proved to been one of the best alternative to solve rank-constrained LMI problems, see the successful applications of this method to the problem of static output-feedback design [31].

Finally, global optimization techniques can also be used for solving rank-one LMI problems. Several algorithms were recently developed that take advantage of the special structure of the non-convexity, see [38] and references therein. Global convergence to a rank-one solution of the LMI is guaranteed, but this may be done at the expense of a considerable amount of computation.

3.4 Sufficient Stability Conditions

Since the condition $\mu > 0$ in rank-one LMI problem (8) is a necessary and sufficient condition for stability of 2-D polynomial matrix $A(s_1, s_2)$, it may be useful to consider the following convex LMI relaxation

$$\begin{array}{rcll} \nu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \mathrm{s.t.} & F_{\mathcal{D}_1}(X) \succeq 0 \\ & & & F_{\mathcal{D}_2}(X) \succeq 0 \\ & & & \mathrm{Trace}\mathcal{X}^\star\mathcal{X}X = 1 \\ & & & X = X^\star \succeq 0 \end{array} \tag{9}$$

where the non-convex rank-constraint has been dropped. Since the linear objective function is minimized in problem (9) over a feasible set that is larger than in problem (8), we directly come up with a sufficient stability condition.

Theorem 2 *2-D polynomial matrix $A(s_1, s_2)$ is stable in region $\mathcal{D}_1 \times \mathcal{D}_2$ if the optimal value ν of LMI problem (9) is strictly positive.*

Using standard semidefinite programming duality arguments [39], one can show that LMI optimization problem (9) is dual to LMI feasibility problem

$$\begin{array}{l} \mathcal{A}^\star\mathcal{A} \succ F^D_{\mathcal{D}_1}(P_1) + F^D_{\mathcal{D}_2}(P_2) \\ P_1 = P_1^\star \succ 0, \quad P_2 = P_2^\star \succ 0 \end{array} \tag{10}$$

where

$$F^D_{\mathcal{D}_i}(P_i) = \alpha_i \mathcal{Q}_i^\star P_i \mathcal{Q}_i + \beta_i \mathcal{Q}_i^\star P_i \mathcal{P}_i + \beta_i^\star \mathcal{P}_i^\star P_i \mathcal{Q}_i + \gamma_i \mathcal{P}_i^\star P_i \mathcal{P}_i$$

is a linear map dual to the map introduced in (7).

In contrast to rank-one LMI problem (8), LMI problems (9) and (10) are convex [5]. They can be solved very efficiently with the interior-point methods described in [30]. When these LMI problems feature complex coefficients, they can be transformed into LMI problems over reals with the help of using the standard manipulations described in [7].

4 Examples

In this section we provide two simple numerical examples illustrating the LMI techniques for checking stability of 2-D polynomial matrices. We only rely on convex LMI problem (9). For the examples at hand, we will show that the sufficient stability conditions of Theorem 2 prove to be necessary as well. However, this is not necessarily always the case.

Note that we do not illustrate the heuristics or global optimization techniques described in Paragraph 3.3 to solve non-convex LMI problem (8). This remains out of the scope of the paper and will be published elsewhere.

In the following examples, we consider stability in the unit disk, i.e.

$$\mathcal{D}_1 = \mathcal{D}_2 = \{s \in \mathbb{C} : ss^\star \leq 1\}$$

and

$$B_1 = B_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right].$$

In order to verify our results, we will use the following standard fact [36, §4.6].

Lemma 3 *Let $a, b, c \in \mathbb{R}$. 2-D polynomial $1 + as_1 + bs_2 + cs_1s_2$ is stable if and only if $|a + b| < 1 + c$ and $|a - b| < 1 - c$.*

In the sequel, the semidefinite programming problems are implemented with the user-friendly interface LMITOOL 2.0 for MATLAB [13] and solved with the interior-point methods of the SDPHA package 3.0 [6].

4.1 First Example

Consider the 2-D polynomial matrix

$$A(s_1, s_2) = \left[\begin{array}{cc} 4 + s_1 + s_2 + s_1^2 & s_1 + 4s_2 + s_1s_2 + s_2^2 + s_1^2s_2 \\ s_1 - 2s_2 & 1 + s_1s_2 - 2s_2^2 \end{array}\right].$$

LFR matrices (2) associated with $A(s_1, s_2)$ are given by

$$\left[\begin{array}{cccc|cccc||cc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 4 & 0 \\
0 & 1 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 1 \\
\hline\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

Following the reduction procedure described in the Appendix, we come up with the reduced LFR

$$\left[\begin{array}{cc|cc||cc}
1 & 0 & 1 & 4 & 4 & 0 \\
0 & 1 & -2 & 0 & 0 & 1 \\
\hline\hline
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

LMI optimization problem (9) reads

$$
\begin{array}{rcl}
\mu & = & \min \quad \mathrm{Trace} \begin{bmatrix} 1 & 0 & 1 & 4 & 4 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 1 & -2 & 5 & 4 & 4 & -2 \\ 4 & 0 & 4 & 16 & 16 & 0 \\ 4 & 0 & 4 & 16 & 16 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{bmatrix} X \\
& & \text{s.t.} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}^{\star} X \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\star} X \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0 \\
& & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\star} X \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\star} X \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0 \\
& & \mathrm{Trace} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} X = 1 \\
& & X = X^{\star} \succeq 0.
\end{array}
$$

The optimal solution to the above problem is $\mu = 0.0435 > 0$ for

$$
X = \begin{bmatrix} 0.2495 & 0.0241 & -0.0109 & 0.3087 & -0.3827 & -0.0852 \\ 0.0241 & 0.0234 & -0.0230 & 0.0146 & -0.0162 & -0.0957 \\ -0.0109 & -0.0230 & 0.0234 & 0.0023 & -0.0048 & 0.0950 \\ 0.3087 & 0.0146 & 0.0023 & 0.3928 & -0.4883 & -0.0423 \\ -0.3827 & -0.0162 & -0.0048 & -0.4883 & 0.6072 & 0.0446 \\ -0.0852 & -0.0957 & 0.0950 & -0.0423 & 0.0446 & 0.3928 \end{bmatrix} \succeq 0.
$$

Therefore, 2-D polynomial matrix $A(s_1, s_2)$ is stable.

Another possibility for checking stability of $A(s_1, s_2)$ could be to compute the determinant of $A(s_1, s_2)$. We obtain

$$\det A(s_1, s_2) = 4 + s_1 + s_2 + 2s_1 s_2.$$

In virtue of Lemma 3, one can check that this 2-D polynomial is stable indeed.

4.2 Second Example

Consider the 2-D polynomial matrix

$$A(s_1, s_2) = \begin{bmatrix} 1 + a^2 s_1 & (ab + c)s_1 \\ (ab - c)s_2 & 1 + b^2 s_2 \end{bmatrix}$$

parametrized in $a, b, c \in \mathbb{R}$.

Let $c = \sqrt{3}/2$. We build the associated LFR matrices and solve LMI problem (9) for different values of a and b. Values of criterion μ as a function of a and b are reported in Figure 2.

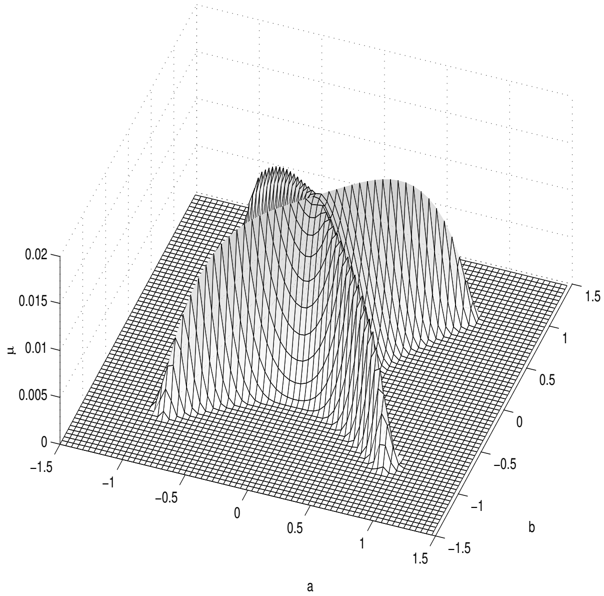

Figure 2. Optimal criterion μ as a function of parameters a and b.

In Figure 3, we represented the domain of parameters a and b for which $\mu > 0$. For these values, polynomial matrix $A(s_1, s_2)$ is stable.

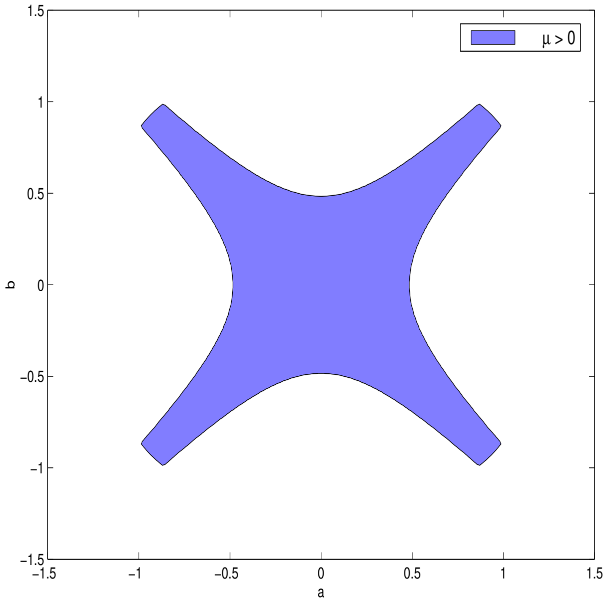

Figure 3. Values of a and b for which $\mu > 0$.

These results can be verified analytically. The determinant of $A(s_1, s_2)$ reads

$$\det A(s_1, s_2) = 1 + a^2 s_1 + b^2 s_2 + c^2 s_1 s_2.$$

Recalling Lemma 3, the stability region in Figure 3 corresponds to the intersection of disk

$$\{a, b \in \mathbb{R} : a^2 + b^2 < 7/4\}$$

with hyperbolic region

$$\{a, b \in \mathbb{R} : -1/4 < a^2 - b^2 < 1/4\}.$$

5 Conclusion

We have proposed new and general rank-one LMI conditions for stability of a given 2-D polynomial matrix. Contrary to most of the existing methods, the stability conditions are necessary and sufficient. No determinant or zero computation is required. As another advantage, we can easily handle matrices with complex coefficients. The main shortcoming of the method is that rank-one LMI problems are non-convex. No efficient algorithm is known for solving these problems, but several heuristics and global optimization algorithms are proposed for addressing rank-one LMI problems [8, 12] and this is an active and promising research area, see [38] and references therein. Morover, a tractable convex formulation of sufficient stability conditions is readily derived from the rank-one LMI formulation.

The most promising direction for further research is probably the extension of our approach to uncertain 2-D polynomial matrices and robust stability analysis. Several attempts were recently reported to address this problem [28, 21, 29]. In a forthcoming paper, we will show that our stability conditions can naturally be combined with other LMIs representing norm-bounded, positive-real or interval uncertainties.

Acknowledgments

We are grateful to Laurent El Ghaoui, University of California at Berkeley, for providing a preprint of the paper [2]. The first author would like to thank Denis Arzelier and Dimitry Peaucelle, LAAS-CNRS, Toulouse, and Jamal Daafouz, CRAN-ENSEM, Nancy, for many interesting discussions.

Appendix

Suppose we are given a 2-D polynomial matrix $A(s_1, s_2)$ and its LFR as in Lemma 1. The following procedure reduces the LFR such that matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{R}_1$, $\mathcal{R}_2$ have full rank.

Step 1. If $\mathcal{Q}_i$ is rank deficient for some $i = 1, 2$, let U be the lower-left triangular matrix with ones along the diagonal such that the linearly dependent rows in $\mathcal{Q}$ are forced to zero in matrix $U\mathcal{Q}$. Let J be the set of indices of these rows in $\mathcal{Q}$. Replace matrices L,R and D by LU^{-1},UR and UDU^{-1}, respectively. Suppress rows J in R,D,Δ and suppress columns J in L,D,Δ. Update matrices $\mathcal{R}$ and $\mathcal{Q}$ accordingly.

Step 2. If $\mathcal{R}_i$ is rank deficient for some $i = 1, 2$, let V be the upper-right triangular matrix with ones along the diagonal such that the linearly dependent columns in $\mathcal{R}$ are forced to zero in matrix $\mathcal{R}V$. Let J be the set of indices of these columns in $\mathcal{R}$. Replace matrices L,R and D by LV,$V^{-1}R$ and $V^{-1}DV$, respectively. Suppress columns J in L,D,Δ and suppress rows J in R,D,Δ. Update matrices $\mathcal{R}$ and $\mathcal{Q}$ accordingly.

Step 3. Go to Step 1 until no reduction can be achieved.

The above procedure results in a reduced LFR of $A(s_1, s_2)$ with full rank matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{R}_1$, $\mathcal{R}_2$. The proof follows by careful inspection. The key point is in noting that matrix Δ commutes with matrix $U = V^{-1}$, i.e. $\Delta U = U\Delta$. Therefore, Δ is not affected by the successive similarity transformations. Redundant components in p_i and q_i corresponding to zero rows in $\mathcal{Q}_i$ or zero columns in $\mathcal{R}_i$ can therefore be removed without altering the LFR of $A(s_1, s_2)$.

References

[1] B. D. O. Anderson, P. Agathoklis, E. I. Jury, M. Mansour, "Stability and the Matrix Lyapunov Equation for Discrete Two-dimensional Systems", *IEEE Transactions on Circuits and Systems*, Vol. 33, pp. 261–267, 1986.

[2] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, "Robust Semidefinite Programming", Chapter 6, pp. 139–162 in H. Wolkowicz, R. Saigal, L. Vandenberghe (Editors) "Handbook of Semidefinite Programming" *Kluwer Academic Publishers*, Boston, Massachusetts, 2000.

[3] N. K. Bose, "Applied Multidimensional System Theory", *Van Nostrand Reinhold Comp.*, New York, 1982.

[4] N. K. Bose, "Multidimensional System Theory", *D. Reidel Publishing Comp.*, Dordrecht, 1984.

[5] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, "Linear Matrix Inequalities in System and Control Theory", *SIAM*, Philadelphia, Pennsylvania, 1994.

[6] N. Brixius, R. Sheng, F. A. Potra "SDPHA: a MATLAB Implementation of Homogeneous Interior-Point Algorithms for Semidefinite Programming", *Optimization Methods and Software*, Vol. 11/12, pp. 583–596, 1999.

[7] M. Chilali, "Méthodes LMI pour l'Analyse et la Synthèse Multi-Critères", *Ph. D. Thesis*, UFR Mathématiques de la Décision, Paris IX Dauphine, France, 1996.

[8] J. David, B. De Moor, "Designing Reduced Order Output Feedback Controllers Using a Potential Reduction Method", *Proceedings of the American Control Conference*, AACC, Baltimore, Maryland, pp. 845–849, 1994.

[9] D. E. Dudgeon, R. M. Mersereau, "Multidimensional Digital Signal Processing", Prentice-Hall Signal Processing Series, *Prentice Hall*, Englewood Cliffs, New Jersey, 1984.

[10] L. El Ghaoui, "Robustness of Linear Systems to Parameter Variations", *Ph. D. Thesis*, Department of Aeronautics and Astronautics, Stanford University, California, 1990.

[11] L. El Ghaoui, P. Gahinet, "Rank Minimization under LMI Constraints: A Framework for Output Feedback Problems", *Proceedings of the European Control Conference*, Groningen, The Netherlands, 1993.

[12] L. El Ghaoui, F. Oustry, M. Ait Rami, "A Cone Complementarity Linearization Algorithm for Static Output Feedback and Related Problems", *IEEE Transactions on Automatic Control*, Vol. 42, No. 8, pp. 1171–1176, 1997.

[13] L. El Ghaoui, J. L. Commeau, "LMITOOL 2.0 Package: An Interface to Solve LMI Problems", *E-Letters on Systems, Control and Signal Processing*, Issue 125, 1999.

[14] L. El Ghaoui, S. I. Niculescu (Editors) "Advances in Linear Matrix Inequality Methods in Control", *SIAM*, Philadelphia, Pennsylvania, 1999.

[15] E. Fornasini, "A 2-D Systems Approach to River Pollution Modelling", *Multidimensional Systems and Signal Processing*, Vol. 2, pp. 233–265, 1991.

[16] E. Fornasini, M. E. Valcher, "Algebraic Aspects of 2-D Convolutional Codes", *IEEE Transactions on Information Theory*, Vol. 40, No. 4, pp. 1068–1082, 1994.

[17] A. Helmerssonn, "IQC Synthesis Based on Inertia Constraints", *Technical Report*, No. 1994, Department of Electrical Engineering, Linköping University, Sweden, 1998.

[18] D. Henrion, O. Bachelier, M. Šebek, "$\mathcal{D}$-Stability of Polynomial Matrices", *Research Report*, No. 99180, LAAS-CNRS, Toulouse, France, 1999. Submitted for publication.

[19] D. Henrion, S. Tarbouriech, M. Šebek, "Rank-one LMI Approach to Simultaneous Stabilization of Linear Systems", *Systems and Control Letters*, Vol. 38, No. 2, pp. 79–89, 1999.

[20] T. Hinamoto, "2-D Lyapunov Equation and Filter Design Based on the Fornasini-Marchesini Second Model", *IEEE Transactions on Circuits and Systems*, Part I, Vol. 40, No. 2, pp. 102–110, 1993.

[21] T. Hinamoto, "Stability of 2-D Discrete Systems Described by the Fornasini-Marchesini Second Model", *IEEE Transactions on Circuits and Systems*, Part I, Vol. 44, No. 3, pp. 254–257, 1997.

[22] A. Hiret, C. Valentin-Charonnel, G. Duc, J. P. Bonnet, "Different Multidimensional Reduction Algorithms for the LFT Model of a Missile", *Proceedings of the IMACS International Conference CESA*, Hammamet, Tunisia, 1998.

[23] E. I. Jury, "Stability of Multidimensional Systems and Related Problems", Chapter 3 in S. Tzafestas, "Multidimensional Systems, Techniques and Applications", *Marcel Dekker*, New York, 1985.

[24] E. I. Jury, "Modified Stability Table for 2-D Digital Filters", *IEEE Transactions on Circuits and Systems*, Vol. 35, pp. 116–119, 1988.

[25] T. Kaczorek, "2-D Linear Systems", Lecture Notes in Control and Information Sciences, *Springer Verlag*, Englewood Cliffs, New Jersey, 1985.

[26] A. Kanellakis, S. Tzafestas, N. Theodorou, "Stability Tests for 2-D Systems Using the Schwarz Form and the Inners Determinants", *IEEE Transactions on Circuits and Systems*, Vol. 38, No. 9, pp. 1071–1077, 1991.

[27] W. S. Lu, "On a Lyapunov Approach to Stability Analysis of 2-D Digital Filters", *IEEE Transactions on Circuits and Systems*, Part I, Vol. 41, No. 10, pp. 665–669, 1994.

[28] W. S. Lu, "On Robust Stability of 2-D Discrete Systems", *IEEE Transactions on Automatic Control*, Vol. 40, No. 3, pp. 502–506, 1995.

[29] N. E. Mastorakis, "A Method for Computing the 2-D Stability Margin", *IEEE Transactions on Circuits and Systems*, Part II, Vol. 45, No. 3, pp. 376–379, 1998.

[30] Y. Nesterov, A. Nemirovski, "Interior-Point Polynomial Methods in Convex Programming", *SIAM*, Philadelphia, Pennsylvania, 1994.

[31] M. C. de Oliveira and J. C. Geromel, "Numerical Comparison of Output Feedback Methods", *Proceedings of the American Control Conference*, AACC, pp. 72–76, Albuquerque, New Mexico, 1997.

[32] E. Rogers, D. Owens, "2-D Transfer Function Based Controller Design for Differential Linear Repetitive Processes", *Proceedings of the 2nd IFAC Workshop on System Structure and Control*, Prague, Czech Republic, pp. 420–423, 1992.

[33] M. Šebek, "On 2-D Pole Placement", *IEEE Transactions on Automatic Control*, Vol. 30, No. 8, pp. 819–822, 1985.

[34] M. Šebek, "n-D Matrix Polynomial Equations", *IEEE Transactions on Automatic Control*, Vol. 33, No. 5, pp. 499–502, 1988.

[35] M. Šebek, "2-D Exact Model Matching", *IEEE Transactions on Automatic Control*, Vol. 37, No. 2, pp. 215–217, 1992.

[36] M. Šebek, "Multi-Dimensional Systems: Control Via Polynomial Techniques", *DrSc. Dissertation*, Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague, 1994.

[37] D. Šiljak, "Stability Criteria for Two Variable Polynomials", *IEEE Transactions on Circuits and Systems*, Vol. 22, pp. 185–189, 1975.

[38] H. D. Tuan, P. Apkarian, Y. Nakashima "A New Lagragian Dual Global Optimization Algorithm for Solving Bilinear Matrix Inequalities", *Proceedings of the American Control Conference*, AACC, San Diego, California, 1999.

[39] L. Vandenberghe, S. Boyd, "Semidefinite Programming", *SIAM Review*, Vol. 38, pp. 49–95, 1996.

[40] K. Zhou, J. C. Doyle, K. Glover, "Robust and Optimal Control", *Prentice Hall*, Upper Saddle River, New Jersey, 1996.