Mode Sequences as Symbolic States in Abstractions of Incrementally Stable Switched Systems

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Abstract— **We present a novel approach to the computation of symbolic abstractions of incrementally stable switched systems. The main novelty consists in using mode sequences of given length as symbolic states for our abstractions. We show that the resulting symbolic models are approximately bisimilar to the original switched system and that an arbitrary precision can be achieved by considering sufficiently long mode sequences. The advantage of this approach over existing ones is double: firstly, the transition relation of the symbolic model admits a very compact representation under the form of a shift operator; secondly, our approach does not use lattices over the state-space and can potentially be used for higher dimensional systems. We provide a theoretical comparison with the lattice-based approach and present a simple criterion enabling to choose the most appropriate approach for a given switched system. Finally, we show an application to a model of road traffic for which we synthesize a schedule for the coordination of traffic lights under constraints of safety and fairness.**

I. Introduction

Over the recent years, there have been numerous studies about using discrete or symbolic abstractions for the control of hybrid systems (see e.g. [8] and the references therein). In particular, for switched systems, a specific class of hybrid systems well-suited to represent physical processes with various operation modes, the use of symbolic models related to the original system by an approximate equivalence relationship, namely approximate bisimulation, was proposed in [5] and [1], under an assumption of incremental stability. In these works, the computation of the symbolic models is based on the use of discrete (uniform or multi-scale) lattices approximating the state-space, and on the quantization of the dynamics of the switched system over these lattices.

By leaning on this notion of approximate bisimulation, this paper proposes an alternative approach to the computation of symbolic models for incrementally stable switched systems. The main novelty consists in using mode sequences of given length as symbolic states for our abstractions. Intuitively, a symbolic state represents the states that are reached by the switched system by applying the associated mode sequence. Then, the transition relation of the symbolic model can be naturally and elegantly described using a symbolic shift operator. We show that by considering sufficiently long mode sequences, these symbolic models can approximate arbitrarily accurately the original incrementally stable switched system. In addition, the fact that we do not explicitly discretize the state-space makes our approach potentially more suitable than [5] and [1] for higher dimensional systems. We provide a theoretical comparison with these approaches and present a simple criterion to choose the most appropriate approach for a given switched system.

Our work can be related to [7] and [4] where similar ideas of using input sequences as symbolic states in discrete abstractions can be found. In [7], for the class of controllable linear systems, it is shown that a symbolic model can be computed by identifying symbolic states with bounded input sequences. Contrarily to the present work, the resulting symbolic models are related to the original system by an exact bisimulation relation. In [4], input sequences are used as symbolic states resulting in abstractions related to stable linear systems by approximate bisimulation relations. The present work extends this approach by applying it to a broader class of systems.

The paper is organized as follows. Section II gives the mathematical background of transition systems (common framework used to model both switched and symbolic systems) and approximate bisimulation, whereas in Section III, the class of discrete-time switched systems and the notion of incremental stability are introduced. Then, in Section IV, we present our novel approach to the computation of symbolic models that are approximately bisimilar to an incrementally stable switched system and we compare this method with the lattice-based approach. Finally, Section V presents an example of application to a model of road traffic for which we synthesize a schedule for the coordination of traffic lights under constraints of safety and fairness.

II. Preliminaries

A. Transition systems

We use transition systems to describe switched systems as well as the symbolic models we build. This common framework allows us to compare their behaviors and evaluate the precision of approximation of the switched system by the symbolic model.

Definition 1: A *transition system* T is defined by the tuple $T = (X, U, \Delta, O, H)$ where:

- X is the set of states;
- U is the set of inputs (an input may also be called a label);
- $\Delta : X \times U \to 2^X$ is the transition relation that provides the successor(s) of a state for a given input;

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- Y is the set of ouputs (an output may also be called an observation);
- $H : X \to Y$ is the observation map.

A transition of T can be denoted by $x' \in \Delta(x,u)$ or $x \xrightarrow{u} x'$, and means that the system can evolve from state $x \in X$ to state $x' \in X$ under the action of input $u \in U$. Then, a *state trajectory* is a sequence of transitions $x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \dots$ and the associated *output trajectory* is the sequence of outputs $y_0 y_1 y_2 \dots$ where $y_k = H(x_k)$.

T is said to be *metric* if its output set Y is equipped with a metric d. It is said to be *deterministic* if for a given state and a given input, there is at most one successor, and it is said to be *finite* or *symbolic* if X and U are finite sets.

B. Approximate bisimulation relation

In the framework of metric transition systems, the notion of approximate bisimulation [3] allows us to verify that the distance between output trajectories of two transition systems T_1 and T_2 obtained for identical input sequences, is bounded by a given precision denoted ε.

Definition 2: Let $T_1 = (X_1, U, \Delta_1, Y, H_1)$ and $T_2 = (X_2, U, \Delta_2, Y, H_2)$ be two metric transition systems with the same input set U and the same output set Y equipped with a metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $\mathscr{R}_\varepsilon \subseteq X_1 \times X_2$ is said to be an *ε-approximate bisimulation relation* between T_1 and T_2 if for all $(x,q) \in \mathscr{R}_\varepsilon$:

(i) $d(H_1(x), H_2(q)) \leq \varepsilon$,
(ii) for all $u \in U$ and for all $x' \in \Delta_1(x,u)$, there exists $q' \in \Delta_2(q,u)$ such that $(x',q') \in \mathscr{R}_\varepsilon$,
(iii) for all $u \in U$ and for all $q' \in \Delta_2(q,u)$, there exists $x' \in \Delta_1(x,u)$ such that $(x',q') \in \mathscr{R}_\varepsilon$.

While an *exact bisimulation* requires perfect equality between the output trajectories, the objective here is to maintain the distance between them lower than a precision ε.

Remark 3: If transition relations Δ_1 and Δ_2 are such that for any given state and input, there is exactly one successor then conditions (ii) and (iii) are equivalent.

III. Switched systems

In this paper, we consider switched systems, a particular class of hybrid systems for which the state is continuous and the input is discrete. The set of inputs represents the different operation modes of the system (a continuous dynamics is associated with each mode); the input thus allows us to switch between several continuous dynamics. Approaches for computing approximately bisimilar symbolic models of incrementally stable switched systems have been proposed in [5] and [1], based on the use of uniform or multiscale lattices to approximate the state space. In this paper, we propose a novel approach which is not based on an explicit discretization of the state-space.

A. Discrete-time switched systems

For simplicity of the exposition, we consider in this paper discrete-time switched systems. However, similar to the work in [5], one can easily extend the approach to continuous-time switched systems and compute symbolic models for sampled versions of the system where the time-sampling parameter is a design parameter.

Definition 4: A *discrete-time switched system* Σ is defined by the triple $\Sigma = (\mathbb{R}^n, P, F)$ where:

- $\mathbb{R}^n$ is the state space;
- $P = \{0, \dots, m\}$ is the finite set of modes;
- $F = \{\Phi_0, \dots, \Phi_m\}$ is the collection of vector fields describing the continuous dynamics in each mode.

In Σ, a *switching signal* is a function $\mathbf{p} : \mathbb{N} \to P$. Given a switching signal $\mathbf{p}$, a *trajectory* of Σ is a function $\mathbf{x} : \mathbb{N} \to \mathbb{R}^n$ such that:

$$\forall t \in \mathbb{N},\ \mathbf{x}(t+1) = \Phi_{\mathbf{p}(t)}(\mathbf{x}(t)).$$

We denote by $(\mathbf{x}(t, x_0, \mathbf{p}))_{t\in\mathbb{N}}$ the trajectory of Σ associated to initial state x_0 and switching signal $\mathbf{p}$.

The dynamics of Σ can be embedded in a transition system $T(\Sigma) = (X, U, \Delta, Y, H)$ where the set of states is $X = \mathbb{R}^n$; the set of inputs is the set of modes $U = P$; the transition relation is given by:

$$x' \in \Delta(x,p) \iff x' = \Phi_p(x); \tag{1}$$

the set of outputs is $Y = \mathbb{R}^n$ and the observation map H is the identity map. $T(\Sigma)$ is deterministic and metric when $\mathbb{R}^n$ is equipped with some metric (e.g. given by some norm $\|.\|$). Moreover, Remark 3 applies to the transition system $T(\Sigma)$.

B. Incremental stability of switched systems

Incremental stability was shown to be a crucial property for the existence of approximately bisimilar symbolic models for switched systems [5].

Definition 5: A switched system Σ is said to be incrementally globally uniformly asymptotically stable (δ-GUAS) if there exists a $\mathscr{KL}$ function[1] β such that for all x_1, $x_2 \in \mathbb{R}^n$ and for all switching signals $\mathbf{p}$:

$$\forall t \in \mathbb{N},\ \|\mathbf{x}(t, x_1, \mathbf{p}) - \mathbf{x}(t, x_2, \mathbf{p})\| \leq \beta(\|x_1 - x_2\|, t).$$

In other words, Σ is incrementally stable if all the trajectories associated to the same switching signal $\mathbf{p}$ converge to the same trajectory, independently of their initial conditions. Hence, it follows that in an incrementally stable system, the past of its behavior is progressively forgotten.

Incremental stability can be proved with the help of Lyapunov functions.

Definition 6: A function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a δ-GUAS Lyapunov function for Σ if there exists two $\mathscr{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and a constant $0 < \lambda < 1$ such that for all $x_1, x_2 \in \mathbb{R}^n$:

$$\underline{\alpha}(\|(x_1 - x_2)\|) \leq V(x_1, x_2) \leq \overline{\alpha}(\|x_1 - x_2\|), \tag{2}$$

$$\forall p \in P, \qquad V(\Phi_p(x_1), \Phi_p(x_2)) \leq \lambda V(x_1, x_2). \tag{3}$$

[1] A continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathscr{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \to +\infty$ when $r \to +\infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathscr{KL}$ if for all $s \in \mathbb{R}_0^+$, $\beta(., s)$ is a $\mathscr{K}_\infty$ function and for all $r > 0$, $\beta(r, .)$ is strictly decreasing and $\beta(r, s) \to 0$ when $s \to +\infty$.

It is fairly obvious that the existence of a δ-GUAS Lyapunov function guarantees that the switched system Σ is δ-GUAS. In the following, we shall assume the existence of such a function. Similar to [5], we shall also make the supplementary assumption that there exists a $\mathscr{K}_\infty$ function γ such that for all $x_1, x_2, x_3 \in \mathbb{R}^n$:

$$|V(x_1,x_2) - V(x_1,x_3)| \leq \gamma(V(x_2,x_3)). \tag{4}$$

C. Switched affine systems

In the particular case where the dynamics in each mode is affine, that is:

$$\forall p \in P,\ \Phi_p(x) = A_p x + b_p, \tag{5}$$

with A_p a $n \times n$ matrix and $b_p \in \mathbb{R}^n$, one can search for a δ-GUAS Lyapunov function of the form:

$$V(x_1,x_2) = \sqrt{(x_1-x_2)^T M (x_1-x_2)} = ||x_1 - x_2||_M, \tag{6}$$

where M is a $n \times n$ positive definite symmetric matrix. Then, (3) holds if and only if:

$$\forall p \in P,\ A_p^T M A_p \leq \lambda^2 M. \tag{7}$$

Equations (2) and (4) clearly hold for any positive definite symmetric matrix M. Moreover, if $\mathbb{R}^n$ is equipped with the norm $\|.\|_M$ associated to the matrix M, then (2) and (4) hold with $\underline{\alpha}$, $\overline{\alpha}$, γ all equal to the identity map.

IV. Symbolic models of switched systems

In this section, we propose a new method for computing symbolic models for an incrementally stable switched system Σ. Contrarily to [5] or [1], we do not use lattices to approximate the state-space of the switched system.

In the proposed approach, the states of the symbolic model are identified with mode sequences of a given length. Indeed, because of the incremental stability property, for a given switching signal, all the trajectories of Σ converge to the same trajectory. Consequently, the latest applied modes are more important than earlier modes that lose importance over time. Then, the principle of approximation consists in fixing a *temporal horizon* $N \in \mathbb{N}$, and to consider that the current state of the symbolic model only depends on the last N applied modes. Therefore, we establish all the possible mode sequences of length N as the symbolic states of our symbolic model, a state being reached by the associated mode sequence. In the following, we propose a construction of a symbolic model $T_N(\Sigma)$ based on this idea and show that there exists an ε-approximate bisimulation relation between $T(\Sigma)$ and $T_N(\Sigma)$. Moreover, we show that any abitrary precision ε can be achieved by choosing N sufficiently large.

A. Mode sequences as symbolic states

Let $N \in \mathbb{N}$ be a temporal horizon, the detailed construction of the symbolic model $T_N(\Sigma) = (X_N, U, \Delta_N, Y, H_N)$ is as follows.

The set of states is $X_N = P^N$, that is the states are all the possible sequences of modes of length N. The set of inputs is the set of modes $U = P$ and the set of outputs is $Y = \mathbb{R}^n$; these are the same as in $T(\Sigma)$. This will allow us to compare the transition systems $T(\Sigma)$ and $T_N(\Sigma)$ in the approximate bisimulation framework (see Definition 2).

As for the transition relation, let $w \in X_N$ be a state of $T_N(\Sigma)$ such that $w = p_1 p_2 \dots p_{N-1} p_N$. This state is reached in the symbolic model, from any state, by applying the finite sequence of inputs $p_1 p_2 \dots p_{N-1} p_N$. Then, the transition relation takes the form of a shift operator, that is:

$$w' \in \Delta_N(p,w) \iff w' = p_2 \dots p_{N-1} p_N p.$$

Hence, it appears that the state of the symbolic model is uniquely determined by the last N applied modes. Also, it appears that similar to the transition systems $T(\Sigma)$, Remark 3 applies to the transition system $T_N(\Sigma)$.

It remains to define the observation map H_N. For that purpose, we use a *source state* $x_s \in \mathbb{R}^n$ for our symbolic model. For any $w \in X_N$, $w = p_1 p_2 \dots p_{N-1} p_N$, the associated output $H_N(w)$ is denoted y_w and is defined by:

$$y_w = \Phi_w(x_s) = \Phi_{p_N} \circ \Phi_{p_{N-1}} \circ \dots \circ \Phi_{p_2} \circ \Phi_{p_1}(x_s). \tag{8}$$

An example of symbolic model $T_N(\Sigma)$ with $N = 3$ and $P = \{0,1\}$, encompassing 8 states, is illustrated Fig. 1.

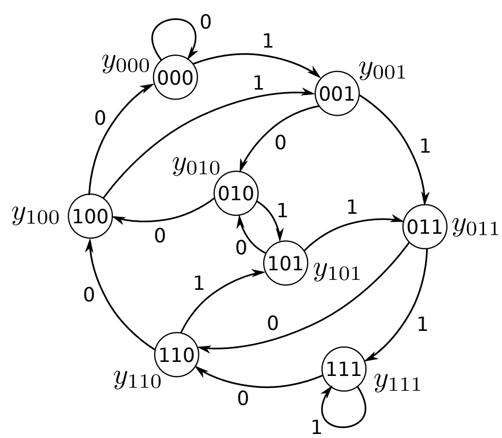

Fig. 1: Example of symbolic model $T_N(\Sigma)$ with $N = 3$ and $P = \{0,1\}$.

B. Rounding error

Since the transition relation in $T(\Sigma)$ is given in (1) by $x' = \Phi_p(x)$, $T_N(\Sigma)$ would be an exact representation of $T(\Sigma)$ if we had $y_{w'} = \Phi_p(y_w)$ for all $w \in X_N$, $p \in P$ and $w' \in \Delta_N(p,w)$. However, with a finite number of states w, the equality can generally not be reached and a rounding error ζ for each transition is unavoidable (see Fig. 2). So, we establish an upper bound on these rounding errors.

In the following, let us assume that there exists a δ-GUAS Lyapunov function V for Σ. Let η be the maximal rounding error measured by the δ-GUAS Lyapunov function V for all possible sequences $w \in X_N$ and all modes $p \in P$:

$$\eta = \max_{\substack{p,w \\ w' \in \Delta_N(p,w)}} V(\Phi_p(y_w), y_{w'}). \tag{9}$$

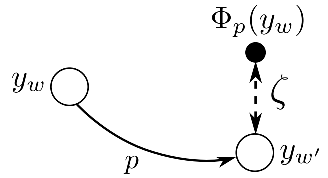

Fig. 2: Rounding error ζ between $y_{w'}$ and $\Phi_p(y_w)$.

Lemma 7: Let us assume that there exists a δ-GUAS Lyapunov function V for Σ and let $N \in \mathbb{N}$ be a temporal horizon. Then, the following inequality holds:

$$\eta \leq \lambda^N \max_p V(\Phi_p(x_s), x_s). \tag{10}$$

Proof. Let $w \in X_N$, $w = p_1 p_2 \dots p_{N-1} p_N$ with $p \in P$, and let $w' \in \Delta_N(p, w)$. According to (8), outputs y_w and $y_{w'}$ can be written as a composition of Φ_p from source state x_s, that is $y_w = \Phi_{p_N} \circ \dots \circ \Phi_{p_2} \circ \Phi_{p_1}(x_s)$ and $y_{w'} = \Phi_p \circ \Phi_{p_N} \circ \dots \Phi_{p_2}(x_s)$. Then, starting from:

$$V(\Phi_p(y_w), y_{w'}) =$$
$$V(\Phi_p \circ \Phi_{p_N} \circ \dots \circ \Phi_{p_2} \circ \Phi_{p_1}(x_s), \Phi_p \circ \Phi_{p_N} \circ \dots \Phi_{p_2}(x_s)),$$

and thanks to (3) satisfied by the δ-GUAS Lyapunov function, the following inequality holds:

$$V(\Phi_p \circ \Phi_{p_N} \circ \dots \circ \Phi_{p_2} \circ \Phi_{p_1}(x_s), \Phi_p \circ \Phi_{p_N} \circ \dots \Phi_{p_2}(x_s))$$
$$\leq \lambda V(\Phi_{p_N} \circ \dots \circ \Phi_{p_2} \circ \Phi_{p_1}(x_s), \Phi_{p_N} \circ \dots \circ \Phi_{p_2}(x_s)).$$

Then, by iteration:

$$V(\Phi_p(y_w), y_{w'}) \leq \lambda^N V(\Phi_{p_1}(x_s), x_s).$$

Since this holds for all $w \in X_N$ and all $p \in P$, it follows from (9) that (10) holds.

□

It should be noted that since $\lambda \in (0,1)$, η can be made arbitrarily small by choosing N sufficiently large. Finally, we can observe that the choice of source point x_s has also an influence on the value of η. Then, in order to obtain η as small as possible, the objective is to minimize the largest rounding error between x_s and $\Phi_p(x_s)$ for all points in $\mathbb{R}^n$ and for all modes p:

$$x_s = \min_{x \in \mathbb{R}^n} \max_p V(\Phi_p(x), x). \tag{11}$$

C. Approximate bisimulation relation

In this section, we establish our main approximation result which shows that the symbolic model $T_N(\Sigma)$ is approximately bisimilar to the transition system $T(\Sigma)$ describing the dynamics of the switched system.

Theorem 8: Let us assume that there exists a δ-GUAS Lyapunov function V for Σ and that (4) holds. Let $N \in \mathbb{N}$ be a temporal horizon and η be given by (9), let ε satisfy the following inequality:

$$\underline{\alpha}(\varepsilon) \geq \frac{\gamma(\eta)}{1-\lambda}. \tag{12}$$

Then, the relation:

$$\mathscr{R}_\varepsilon = \{(x, w) \in X \times X_N |\, V(x, y_w) \leq \underline{\alpha}(\varepsilon)\}$$

is an ε-approximate bisimulation between $T(\Sigma)$ and $T_N(\Sigma)$.

Proof Let $(x, w) \in \mathscr{R}_\varepsilon$, let us remark that we have:

$$\begin{aligned} \|H(x) - H_N(w)\| &= \|x - y_w\| \\ &\leq \underline{\alpha}^{-1}(V(x, y_w)) \;\leq\; \varepsilon. \end{aligned}$$

Hence, the first condition of Definition 2 holds.

Remark 3 applies to $T(\Sigma)$ and $T_N(\Sigma)$, therefore the second and third condition of Definition 2 are equivalent. Let $p \in P$ and $x' \in \Delta(p, x)$, $w' \in \Delta_N(p, w)$. According to (4) and (9), we have:

$$\begin{aligned} V(x', y_{w'}) &\leq V(x', \Phi_p(y_w)) + \gamma(V(\Phi_p(y_w), y_{w'})) \\ &\leq V(x', \Phi_p(y_w)) + \gamma(\eta). \end{aligned}$$

Then, since $x' = \Phi_p(x)$ and by (3), we obtain:

$$\begin{aligned} V(x', y_{w'}) &\leq V(\Phi_p(x), \Phi_p(y_w)) + \gamma(\eta) \\ &\leq \lambda V(x, y_w) + \gamma(\eta), \end{aligned}$$

and so:

$$V(x', y_{w'}) \leq \lambda \underline{\alpha}(\varepsilon) + \gamma(\eta) \leq \underline{\alpha}(\varepsilon),$$

since $V(x, y_w) \leq \underline{\alpha}(\varepsilon)$ and by (12). Therefore, it follows that $(x', w') \in \mathscr{R}_\varepsilon$ and $\mathscr{R}_\varepsilon$ is an ε-approximate bisimulation relation between $T(\Sigma)$ and $T_N(\Sigma)$.

□

As a consequence of Lemma 7 and Theorem 8, it is clear that the precision ε of the symbolic model $T_N(\Sigma)$ can be made arbitrarily small by choosing N sufficiently large.

D. Comparison with the lattice-based approach

In [5], an approach for computing approximately bisimilar symbolic models for incrementally stable switched systems has been proposed. It is based on the approximation of the state space $\mathbb{R}^n$ by the following lattice:

$$[\mathbb{R}^n]_\nu = \left\{ z \in \mathbb{R}^n \mid z_i = k_i \frac{2\nu}{\sqrt{n}},\ k_i \in \mathbb{Z}, i = 1, \dots, n \right\},$$

where $\nu > 0$ is a state sampling parameter. Then, the symbolic model of the switched system Σ is given by the transition system $T_\nu(\Sigma) = (X_\nu, U, \Delta_\nu, Y, H_\nu)$ where the set of states is $X_\nu = [\mathbb{R}^n]_\nu$, the set of inputs is the set of modes $U = P$ and the set of outputs is $Y = \mathbb{R}^n$. The transition relation is given by a quantization of the dynamics of Σ over the lattice $[\mathbb{R}^n]_\nu$, that is for $z, z' \in X_\nu$, $p \in P$:

$$z' \in \Delta_\nu(p, z) \iff \|z' - \Phi_p(z)\| \leq \nu.$$

Finally, the observation map is the natural inclusion map: $H_\nu(z) = z$. The following result, presented here in its discrete-time version, is established in [5].

Theorem 9: Let us assume that there exists a δ-GUAS Lyapunov function V for Σ and that (4) holds. Let $\nu > 0$

be a state sampling parameter, let ε satisfy the following inequality:

$$\underline{\alpha}(\varepsilon) \geq \frac{\gamma(\nu)}{1-\lambda}. \tag{13}$$

Then, the relation:

$$\mathscr{R}_\varepsilon = \{(x,z) \in X \times X_\nu |\, V(x,z) \leq \underline{\alpha}(\varepsilon)\}$$

is an ε-approximate bisimulation between $T(\Sigma)$ and $T_\nu(\Sigma)$.

In the following, we provide a discussion and a simple criterion for helping to decide which is more appropriate for a given switched system.

One can see from (12) and (13) that the accuracy of the symbolic models $T_N(\Sigma)$ and $T_\nu(\Sigma)$ are the same if $\eta = \nu$. Using for η the estimate given by (10), one has that the accuracy are the same if $\nu = \lambda^N \eta_0$ where $\eta_0 = \max_p V(\Phi_p(x_s), x_s)$.

As for the number of symbolic states in the abstractions, there are $|P|^N$ states in $T_N(\Sigma)$ (where $|P|$ is the number of modes) while the number of states in $T_\nu(\Sigma)$, when restraining the dynamics on a compact subset of $\mathbb{R}^n$, is given by C/ν^d (where C is a positive number proportional to the volume of the compact subset and d is the dimension of the system). Then, for $N \in \mathbb{N}$, the symbolic model $T_\nu(\Sigma)$ achieving the same accuracy as $T_N(\Sigma)$ has $C/(\lambda^N \eta_0)^d$ states.

Therefore, this means that it is more convenient to use $T_N(\Sigma)$ rather than $T_\nu(\Sigma)$ as soon as:

$$|P|^N \leq \frac{C}{\eta_0^d} \left(\lambda^{-d}\right)^N.$$

This is asymptotically the case whenever:

$$|P|\lambda^d \leq 1. \tag{14}$$

The previous inequality provides a simple criterion helping to decide which approach is more appropriate for approximating a given switched system. This criterion gives a relation between the number of modes, the dimension of the switched system and the contraction rate of the δ-GUAS Lyapunov function. In particular, it appears that the approach based on using mode sequences has interest for systems in higher-dimension and with few modes.

V. Application to a model of road traffic

A. Model of road traffic

Now, let us see an example of a switched system and its symbolic model obtained by the method described in this paper.

The chosen switched system Σ illustrated Fig. 3 is the model of a road divided in 5 cells of 250 meters each with 2 entries and 2 ways out. The two entries are controlled by traffic lights, denoted f_1 and f_2, that enable (green light) or not (red light) the vehicles to pass.

In Σ, the dynamic we want to observe is the density of traffic φ_i, given in vehicles per cell, for each cell i of the road. The state of Σ is the 5-dimensions vector $x = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)$ and its set of modes is $P = \{0, 1, 2\}$ (so the number of modes is $|P| = 3$) where:

- mode 0 means f_1 green and f_2 green;
- mode 1 means f_1 green and f_2 red;
- mode 2 means f_1 red and f_2 green.

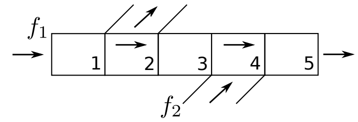

Fig. 3: Example of a switched system Σ that models a road divided in 5 cells with 2 entries and 2 ways out.

Then, let T be a discrete time interval in hours (h), l_i be the length of a cell in kilometers (km), and v_i be the flow speed of the vehicles in kilometers per hour (km/h). Inspired by the work of [2], the model of a simple cell (*eg.* cells 3 and 5) is described by:

$$\varphi_i(t+1) = \varphi_i(t) - \frac{Tv_i}{l_i}\varphi_i(t) + \frac{Tv_{i-1}}{l_{i-1}}\varphi_{i-1}(t).$$

For cells 1 and 4, add the number of vehicles that can go in, and for cell 2 substract the number of vehicles that can go out. Moreover, in order to respect the conditions of stability given in [6], inequality $v_i T \leq l_i$ has to be satisfied. Since the length of a cell is fixed at 0.25 km, this is for instance achieved with a flow speed of 70 km/h and the discrete time interval equal to 10 seconds. During this time interval, we define that 6 vehicles pass the entry controlled by green light f_1, 8 vehicles pass the entry controlled by green light f_2, and one quarter of vehicles that leave cell 1 goes out on the first exit (ratio denoted q_2).

Finally, in Σ, each mode can be described by affine functions as in (5). Matrices A_0, A_1, and A_2 are identical and given by:

$$A_0 = \begin{pmatrix} 1-\frac{Tv_1}{l_1} & 0 & 0 & 0 & 0 \\ \frac{Tv_1}{l_1} & 1-\frac{Tv_2}{l_2}-q_2 & 0 & 0 & 0 \\ 0 & \frac{Tv_2}{l_2} & 1-\frac{Tv_3}{l_3} & 0 & 0 \\ 0 & 0 & \frac{Tv_3}{l_3} & 1-\frac{Tv_4}{l_4} & 0 \\ 0 & 0 & 0 & \frac{Tv_4}{l_4} & 1-\frac{Tv_5}{l_5} \end{pmatrix},$$

and associated vectors b_p are the followings $b_0 = [6\ 0\ 8\ 0\ 0]'$, $b_1 = [6\ 0\ 0\ 0\ 0]'$, $b_2 = [0\ 0\ 8\ 0\ 0]'$. Also, one can find a δ-GUAS Lyapunov function V for Σ of the form given in (6) where the corresponding matrix M is:

$$M = \begin{pmatrix} 2.0789 & 0.6874 & 0.1471 & -0.0100 & 0.0111 \\ 0.6874 & 1.6367 & 0.4236 & 0.0240 & -0.0114 \\ 0.1471 & 0.4236 & 1.5768 & -0.0212 & 0.0215 \\ -0.0100 & 0.0240 & -0.0212 & 1.5376 & -0.3876 \\ 0.0111 & -0.0114 & 0.0215 & -0.3876 & 1.2796 \end{pmatrix},$$

and because of (7), $\lambda = 0.8$. As for the choice of the source point x_s given in (11), we use the *fminimax* Matlab function that provides $x_s = [3.8570\ 3.3750\ 3.3750\ 8.5177\ 8.5177]'$.

Remark 10: For that system, we have $|P|\lambda^d = 0.98$ so the criterion given by (14) indicates that the approach based on the mode sequences is more appropriate than the approach based on lattices.

B. Numerical tests

According to these caracteristics, symbolic model $T_N(\Sigma)$ that approximates switched system Σ is computed as described in Subsection IV-A.

The first test we made deals with precision ε of the approximate bisimulation between $T_N(\Sigma)$ and $T(\Sigma)$. Different values of ε are so obtained by taking an increasing size N of mode sequences from 10 to 14. The results are illustrated in Table I where $\Pi = |P|^N$ is the state number of $T_N(\Sigma)$, η is the rounding error defined in (9), and ε is the precision of the abstraction given in Theorem 8. We can note that we get a high precision quickly ($\varepsilon = 0.0939$ for $N = 10$) in comparison to the values of densities, despite a number of states not very large ($\Pi = 59\ 049$).

TABLE I: Computation of precision ε between $T_N(\Sigma)$ and $T(\Sigma)$, for different sizes of mode sequences N and under safety and fairness constraints.

N	10	12	14
Π	59 049	531 441	4 782 969
Π_s	35 189	305 765	2 655 400
η	0.0187	0.0019	0.00017
ε	0.0939	0.0095	0.00085

In addition to these results, we also deal with the control of $T_N(\Sigma)$ thanks to constraints of *safety* and *fairness*. The former constraint is established in order to regulate the number of vehicles on the road and to keep the density of traffic lower than a dangerous value. For a flow speed of 70 km/h, this maximal density is fixed at 15 vehicles per cell. The latter constraint is a way to alternate the accesses between the 2 entries and to allow only 3 identical consecutive modes of red light. This ensures fairness between modes 1 and 2. Under these two constaints, safe and controllable states Π_s of $T_N(\Sigma)$, given in Table I, are computed by a fixed-point algorithm and we guarantee that all the transitions of the controlled system keep the states in Π_s.

Finally, we propose a schedule for the coordination of traffic lights for the controlled system. The idea is to favor mode 0 (two green lights) in order to give access on the road to the maximum number of vehicles. For instance, when $N = 14$, we obtain the following sequence of modes:

$$0-0-0-2-1-0-0-2-1-0-0-2-1-2,$$

that is applied from state $w_0 = 00021002100212$. Then, after state w_0, the schedule passes by state $w_1 = 00210021002120$ to state $w_{13} = 20002100210021$ and is cyclically repeated. The corresponding traffic densities of the cells are illustrated in Fig. 4.

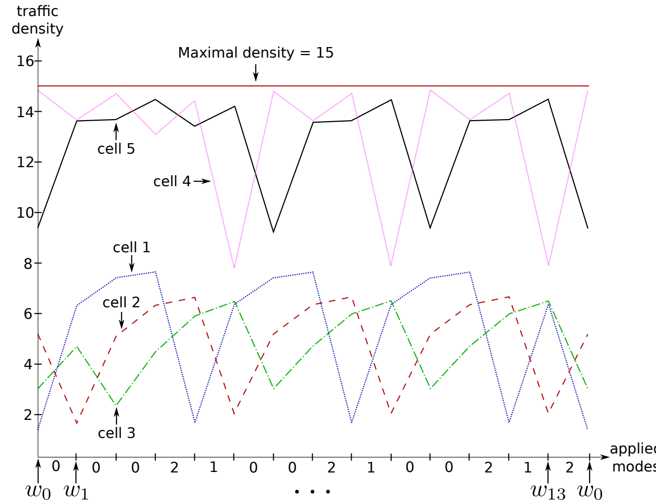

Fig. 4: Evolution of traffic densities of the cells under the schedule $0-0-0-2-1-0-0-2-1-0-0-2-1-2$.

VI. Conclusion

In this paper, a method for the computation of symbolic abstractions of incrementally stable switched systems has been studied.

Contrarily to methods of state-space discretization by lattices, the symbolic model we build takes mode sequences of the switched system as symbolic states. The abstraction obtained is shown to be approximately bisimilar to the original system with an arbitrary precision ε by considering sufficiently long mode sequences.

We experiment this method on a model of a road where the traffic densities of each cell (the states) are regulated by the traffic lights (the inputs). The system is controlled under constraints of safety, the density must be lower than a dangerous value in each cell, and fairness, the green lights have to be equally distributed between the two entries.

In future work, we should address the problem of developing efficient controller synthesis techniques exploiting the specific structure of the transition relation of this kind of symbolic models.

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