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Container of (min,+)-linear systems

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Abstract Based on the (min,+)-linear system theory, the work developed here takes the set membership approach as a starting point in order to obtain a container for ultimately pseudo-periodic functions representative of Discrete Event Dynamic Systems. Such a container, by approximating the exact system, ensures to entirely include it in a guaranteed way. To reach that point, the container introduced in this paper is given as an interval, the bounds of which are a convex function for the upper approximation and a concave function for the lower approximation. Thanks to the characteristics of the bounds, the aim is both to reduce data storage (that can be very high when exact functions are handled) and to reduce the algorithm complexity of the operations of sum, inf-convolution and subadditive closure. These operations are integrated into inclusion functions, the algorithms of which are of linear or quasi-linear complexity.

Keywords (Max,+) algebra · Discrete Event Dynamic Systems · Set membership approach · Algorithms · Computational complexity

1 Introduction

The theory of (max,+) algebra deals with the study of Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena, through the particular algebraic structure called *idempotent semiring* or *dioid* (Baccelli et al.

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1992). The areas of application of this theory are various. We can cite the production systems (Cottenceau et al. 2001), communication networks (Le Boudec and Thiran 2001; Chang 2000) and the transportation systems (Heidergott et al. 2006). More precisely, some control problems have already been solved in the context of production systems (Maia et al. 2003) and in the context of communication networks (Le Corronc et al. 2010). We can also recall that the theory of Network Calculus aims at analyzing and measuring the worst-case performance of a network.

These works on control and performance analysis share the feature that the underlying model relies on ultimately pseudo-periodic functions (denoted $\mathscr{F}_{cp}$ in this paper). Nowadays, some tools enable these kinds of functions to be handled (an overview for the Network Calculus is given in Boyer (2010)). A non-exhaustive list includes: the MinMaxGD toolbox (created by the LISA laboratory, see Cottenceau et al. (2000)), COINC software[1] and the DISCO toolbox for Network Calculus (see respectively Bouillard et al. (2009) and Schmitt and Zdarsky (2006)), and the RTC toolbox for an extension of the Network Calculus called Real-Time Calculus (see Wandeler and Thiele (2006)). The main operations of the (min,+) algebra such as the sum and the inf-convolution[2] are available in MinMaxGD, COINC, DISCO and RTC, whereas the operation of subadditive closure[3] is only available with MinMaxGD and COINC. Moreover, for MinMaxGD and COINC, the algorithms of these operations are described in Gaubert (1992) and Cottenceau (1999) for the former, and in Bouillard and Thierry (2008) for the latter. In these toolboxes, the complexity of sum and inf-convolution operations is linear or quasi-linear, whereas the one for subadditive closure tends to be polynomial. However, because of the characteristics of ultimately pseudo-periodic functions, the transient phenomena of handled functions can be significantly long. In this case, the amount of storage data overloads and the exact computations are not always possible within a reasonable time.

It can therefore be helpful to use alternative models with less complex algorithms and reduced data size. This paper follows this point of view by defining an original container for ultimately pseudo-periodic functions (see also the thesis of Le Corronc (2011)). The idea proposed here considers:

- a particular algebraic structure denoted $\mathscr{F}_{cp_{/\mathscr{L}}}$, built from the Legendre–Fenchel transform[4] $\mathscr{L}$ (specially well-suited for convex functions, see Rockafellar 1997; Baccelli et al. 1992; Fidler and Recker 2006),
- associated to the set membership approach (see Jaulin et al. (2001) and Moore (1979) for a general introduction, Litvinov and Sobolevskiī 2001; Lhommeau et al. 2005; Hardouin et al. 2009 in the semiring context).

More precisely, the upper bound of the container is the greatest element of the equivalence class modulo $\mathscr{L}$ of the approximated function: it is a convex function. Similarly, the lower bound of the container is a concave function that will contract this equivalence class since the approximated function necessarily belongs to the

[1] Name of a research project dealing with COmputational Issues in Network Calculus.

[2] Operation used for the concatenation of systems.

[3] Also called Kleene star operation and used for systems with closed-loop architecture.

[4] Also called the convex conjugate function.

container. In other words, the container is treated as the intersection between an interval of functions that contains the exact system, and the equivalence class of the approximated system modulo the Legendre–Fenchel transform. Thanks to the convex characteristics of the bounds of the container, their data representations only need small storage capacity.

Obviously, the downside is that such approximations provide results that are not exact. But the computations made for the proposed container guarantee to include the exact result as it is proposed in the set membership approach. Indeed, the operations of sum, inf-convolution and subadditive closure are integrated into inclusion functions that can be obtained by efficient algorithms thanks again to their convex characteristics. For instance, some existing results given in Le Boudec and Thiran (2001) and Schmitt and Zdarsky (2006), and leading to algorithms of linear complexity will be applied to the inclusion function of the inf-convolution. For the sum, since this operation is a minimum in (min,+) algebra, we will see that the complexity of its inclusion function is linear too. Finally, this paper develops new results by choosing a specific shape for the container, allowing us to deal with subadditive closure in an efficient way. Indeed, by applying factorization and simplification, the complexity of the inclusion function of the subadditive closure becomes quasi-linear.

The presentation of our approach is organized as follows: Firstly, Section 2 reminds us of the useful basis of (min,+)-linear systems. More precisely, some elements required for the study such as the idempotent semiring theory and the problem of transfer matrix computation will be introduced. Then, in Section 3, all the elements used to build the containers for these systems are presented with a canonical representation. Section 4 provides inclusion functions for sum, inf-convolution and subadditive closure operations, and outlines the underlying algorithms that handle them. Finally, in Section 5, some tests are provided in order to evaluate the toolbox called *ContainerMinMaxGD*[5], and to compare approximated computations to exact ones.

2 (Min,+)-linear systems

2.1 Reminder

We denote by $\mathbb{Z}_{\min}$ the set of integers with a min as $\oplus$ operator and the classical sum as $\otimes$ operator. On $\mathbb{Z}_{\min}$, the linear modeling of (min,+)-linear systems can be done through *counter functions*. More precisely, for an event labeled x, the function $x(t)$ defined from $\mathbb{Z}$ to $\mathbb{Z}_{\min}$ gives the cumulative number of events x that have occurred until time t. Therefore, this work considers systems described by the following state representation where $u(t)$, $x(t)$ and $y(t)$ are vectors of counter functions that respectively represent input events (events for which it is possible to control the occurrence), internal and output events:

$$\begin{cases} x(t) = A \otimes x(t-1) \oplus B \otimes u(t), \\ y(t) = C \otimes x(t). \end{cases} \tag{1}$$

[5] Created with the container and the algorithms described in this paper.

Moreover, as in the classical linear system theory, an input-output description of a (min,+)-linear system also exists. Indeed, by considering counter functions as event trajectories[6], the output y of a Single-Input Single-Output (SISO) system can be expressed as a convolution of the input u by a particular trajectory h called a *transfer function*. As in the classical theory, the transfer function of a system corresponds to the output due to a specific input that plays the role of "impulse". The transfer function can therefore be seen as the impulse response of a (min,+) system.

This kind of input-output behavior can also be handled through formal series where two operators of time-shift, denoted δ, and event-shift, denoted γ, are involved, and so the convolution is transformed into a formal series product. An example of this structure is idempotent semiring called $\mathscr{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ (see Cohen et al. (1989b) and Baccelli et al. (1992, Section 5.4.2)). In this framework, one of the most important features is that the behavior of (min,+)-linear systems can be handled thanks to finite and canonical representations that have periodic properties. Hence, the transfer series of such a system is an ultimately pseudo-periodic series that has a canonical representation.

In the literature, the state representation as well as the input-output model are well suited to describe the behavior of Timed Event Graphs[7] (TEG) with the "as soon as possible" firing rule as can be seen in Cottenceau et al. (2001). These models are also interesting to describe the behavior of some datagrams through a network as shown in Chang (2000) and Le Boudec and Thiran (2001).

Some useful software tools are available to handle such representations. The MinMaxGD toolbox computes the classical operations on periodic series of $\mathscr{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$. Also, the COINC software handles some piecewise affine pseudo-periodic functions with operations of min, max, (min,+)-convolution and subadditive closure.

2.2 Idempotent semiring theory

All the models introduced previously share some common algebraic features. In each case, the underlying algebraic structure is an idempotent semiring for which some reminders are given here (for more details, see Baccelli et al. 1992, Chapter 4; Gaubert 1992; Heidergott et al. 2006).

Definition 1 (Idempotent semiring) An idempotent semiring $\mathscr{D}$, also called dioid, is a set endowed with two inner operations denoted $\oplus$ and $\otimes$. The sum $\oplus$ is associative, commutative, idempotent (i.e. $\forall a \in \mathscr{D}, a \oplus a = a$) and admits a neutral element denoted ε. The product[8] $\otimes$ is associative, distributes over the sum and allows e to be a neutral element.

When $\otimes$ is commutative (i.e. $\forall a, b \in \mathscr{D}, a \otimes b = b \otimes a$), the idempotent semiring $\mathscr{D}$ is said to be commutative. In this case, an idempotent semiring is said to be

[6] Which can be seen as "signals" for DEDS.

[7] Subclass of Timed Petri Nets in which each place has exactly one upstream and one downstream transition.

[8] As in the usual algebra, operator $\otimes$ can be omitted: $ab = a \otimes b$.

complete if it is closed for infinite sums and if the product distributes over infinite sums too. In this case, the greatest element of $\mathscr{D}$ is denoted $\top$ (for *Top*) and represents the sum of all its elements ($\top = \bigoplus_{x \in \mathscr{D}} x$).

Furthermore, due to the idempotency of addition, a canonical order relation can be associated with $\mathscr{D}$ by the following equivalences: $\forall a, b \in \mathscr{D}$, $a \succcurlyeq b \Leftrightarrow a = a \oplus b$ and $b = a \wedge b$. Because of the lattice properties of a complete idempotent semiring, $a \oplus b$ is the least upper bound of $\mathscr{D}$ whereas $a \wedge b$ is its greatest lower bound.

Example 1 (Idempotent semirings $\overline{\mathbb{Z}}_{\max}$ and $\overline{\mathbb{R}}_{\max}$) The set $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\})$ endowed with the max operator as sum $\oplus$ and the addition as product $\otimes$ is a complete idempotent semiring where $\varepsilon = -\infty$, $e = 0$ and $\top = +\infty$. On $\overline{\mathbb{Z}}_{\max}$, the greatest lower bound $\wedge$ becomes the min operator. By similarity, the set $\overline{\mathbb{R}}_{\max} = (\mathbb{R} \cup \{-\infty, +\infty\})$ is a complete idempotent semiring with the same characteristics.

Example 2 (Idempotent semirings $\overline{\mathbb{Z}}_{\min}$ and $\overline{\mathbb{R}}_{\min}$) The set $\overline{\mathbb{Z}}_{\min} = (\mathbb{Z} \cup \{-\infty, +\infty\})$ endowed with the min operator as sum $\oplus$ and the addition as product $\otimes$ is a complete idempotent semiring where $\varepsilon = +\infty$, $e = 0$ and $\top = -\infty$. On $\overline{\mathbb{Z}}_{\min}$, the greatest lower bound $\wedge$ becomes the max operator. By similarity, the set $\overline{\mathbb{R}}_{\min} = (\mathbb{R} \cup \{-\infty, +\infty\})$ is a complete idempotent semiring with the same characteristics.

Remark 1 It is important to note that because of operator $\oplus$, the canonical order relation $\succcurlyeq$ on $\overline{\mathbb{Z}}_{\min}$ and $\overline{\mathbb{R}}_{\min}$ corresponds to the reverse of the natural order $\leq$:

$$3 \succcurlyeq 5 \quad \Leftrightarrow \quad 3 = 3 \oplus 5 = \min(3,5) \quad \Leftrightarrow \quad 3 \leq 5.$$

Theorem 1 (Baccelli et al. 1992, Theorem 4.75) *The implicit equation $x = ax \oplus b$ defined on a complete idempotent semiring $\mathscr{D}$ admits $x = a^\star b$ as the lowest solution:*

$$\forall a \in \mathscr{D}, \quad a^\star = \bigoplus_{i \geq 0} a^i \quad \textit{where} \quad a^{i+1} = a^i a \quad \textit{and} \quad a^0 = e.$$

This operator is called subadditive closure or Kleene star operator[9].

Numerous properties are associated with operator *. For instance, they are proposed in Gaubert (1992) and Cottenceau (1999) for the (max,+) algebra and more generally in Conway (1971) and Krob (1990) for the theory of rational identities. Those used in this paper are given below.

Property 1 If $\mathscr{D}$ is a commutative, complete idempotent semiring, then $\forall a, b \in \mathscr{D}$:

$$(ab^\star)^\star = e \oplus a(a \oplus b)^\star, \tag{2}$$

$$(a \oplus b)^\star = a^\star b^\star. \tag{3}$$

[9]This designation is different according to the context of use: Network Calculus or TEG.

Definition 2 (Homomorphism) A mapping Π from an idempotent semiring $\mathscr{D}$ into another one $\mathscr{C}$ is a homomorphism if $\forall a, b \in \mathscr{D}$:

$$\Pi(a \oplus b) = \Pi(a) \oplus \Pi(b) \text{ and } \Pi(\varepsilon) = \varepsilon,$$
$$\Pi(a \otimes b) = \Pi(a) \otimes \Pi(b) \text{ and } \Pi(e) = e.$$

Definition 3 (Congruence) In an idempotent semiring $\mathscr{D}$, a congruence is an equivalence relation denoted $\equiv$ that is compatible with operations $\oplus$ and $\otimes$, that is $\forall a, b, c \in \mathscr{D}$:

$$a \equiv b \quad \Rightarrow \quad \begin{cases} (a \oplus c) \equiv (b \oplus c), \\ (a \otimes c) \equiv (b \otimes c). \end{cases}$$

Definition 4 (Equivalence class) Let us consider an idempotent semiring $\mathscr{D}$ endowed with a congruence $\equiv$. The equivalence class of an element $a \in \mathscr{D}$ is denoted $[a]_\equiv$ and defined by:

$$[a]_\equiv \triangleq \{x \in \mathscr{D} \mid x \equiv a\}.$$

Lemma 1 (Baccelli et al. 1992, Lemma 4.24) *The quotient of an idempotent semiring $\mathscr{D}$ by a congruence $\equiv$ is an idempotent semiring denoted $\mathscr{D}_{/\equiv}$ and endowed with operations $\oplus$ and $\otimes$ defined as follows:*

$$[a]_\equiv \oplus [b]_\equiv \triangleq [a \oplus b]_\equiv,$$
$$[a]_\equiv \otimes [b]_\equiv \triangleq [a \otimes b]_\equiv.$$

Lemma 2 (Baccelli et al. 1992, Corollary 4.26) *If a mapping $\Pi : \mathscr{D} \mapsto \mathscr{C}$ is a homomorphism, then the relation $\overset{\Pi}{\equiv}$ defined below $\forall a, b \in \mathscr{D}$ is a congruence:*

$$\Pi(a) = \Pi(b) \quad \Leftrightarrow \quad a \overset{\Pi}{\equiv} b,$$

and the quotient of $\mathscr{D}$ by $\overset{\Pi}{\equiv}$ is simply denoted $\mathscr{D}_{/\Pi}$.

Definition 5 (Projector) A projector p is defined as a mapping from $\mathscr{D}$ to $\mathscr{D}$ such that:

$$p = p \circ p.$$

2.3 Modeling of (min,+)-linear systems

In the rest of this paper, the (min,+) modeling over the set of real numbers is chosen. Therefore, some counter functions from $\mathbb{R}$ to $\overline{\mathbb{R}}_{\min}$ (see Example 2) must be classified for the need of our study. Since these functions describe the cumulative number of events, they are without exception nondecreasing (*i.e.*, $\forall t_1 > t_2,\ f(t_1) \geq f(t_2)$).

Definition 6 (Elementary function Δ_T^K) The elementary function denoted Δ_T^K and illustrated in Fig. 1, is the counter function defined by:

$$\Delta_T^K(t) = \begin{cases} K & \text{if } t \leq T, \\ +\infty & \text{otherwise.} \end{cases}$$

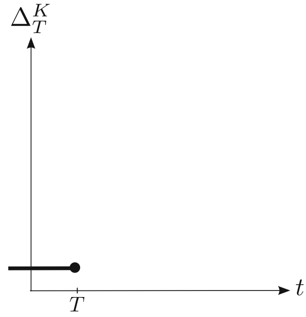


Fig. 1 An elementary function Δ_T^K

Definition 7 (Set $\mathscr{F}_c$) A function $f \in \mathscr{F}_c$ (see Fig. 2a) is a function that can be defined by an infinite sum (min) of elementary functions Δ_T^K, i.e.:

$$f = \bigoplus_{i=0}^{+\infty} \Delta_{t_i}^{k_i}.$$

Therefore, f is a piecewise constant function.

Definition 8 (Causality) Let f be a function of $\mathscr{F}_c$. Function f is said to be causal if:

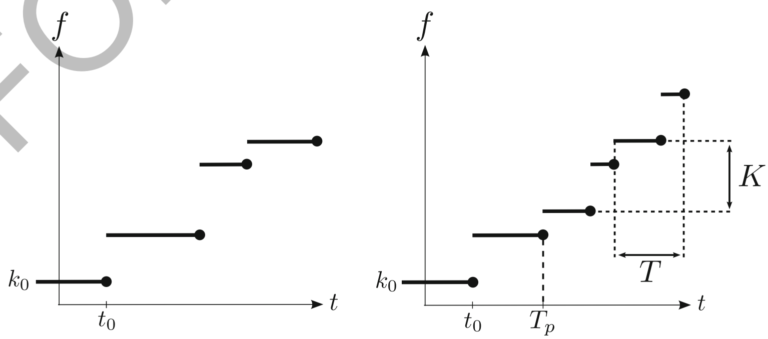
$$\begin{cases} f(t) = f(0) & \text{for } t < 0, \\ f(t) \geq 0 & \text{for } t \geq 0. \end{cases}$$

Remark 2 An elementary function Δ_T^K is causal if $K, T \geq 0$.

Definition 9 (Set $\mathscr{F}_{cp}$) A function $f \in \mathscr{F}_{cp}$ (see Fig. 2b) is a function of $\mathscr{F}_c$ that is in addition ultimately pseudo-periodic, i.e.:

$$\exists T_p \geq t_0, \exists K \in \mathbb{R}^+_{\min}, \exists T \in \mathbb{R}^+ \text{ such that } \forall t \geq T_p,\ \ f(t+T) = K \otimes f(t) = K + f(t),$$

where t_0 is the time of the first elementary function $\Delta_{t_0}^{k_0}$ of f. Hence $\mathscr{F}_{cp} \subset \mathscr{F}_c$.



(a) A function $f \in \mathscr{F}_c$

(b) An ultimately pseudo-periodic function $f \in \mathscr{F}_{cp} \subset \mathscr{F}_c$

Fig. 2 Examples of piecewise constant functions

Property 2 (Canonical representation) Each function of $\mathscr{F}_{cp}$ has a canonical form where T_p and T are minimums.

Property 3 (Asymptotic slope σ) Let f be a function of $\mathscr{F}_{cp}$, its asymptotic slope is defined by the ratio $\sigma(f) = K/T$.

According to this classification, trajectories of a (min,+)-linear system are naturally described by counter functions of $\mathscr{F}_c$ (i.e. nondecreasing piecewise constant functions). In this set, considered systems are described by the state Eq. 1 and recalled here by adding that $A \in \overline{\mathbb{Z}}_{\min}^{n\times n}$, $B \in \overline{\mathbb{Z}}_{\min}^{n\times p}$ and $C \in \overline{\mathbb{Z}}_{\min}^{q\times n}$ where n, p and q refer respectively to the state vector size, the input vector size and the output vector size:

$$\begin{cases} x(t) = A \otimes x(t-1) \oplus B \otimes u(t), \\ y(t) = C \otimes x(t). \end{cases}$$

For SISO systems (i.e. where $p = 1$ and $q = 1$), the development of the recurrent equations given by Eq. 1 leads to express output y as follows:

$$\begin{aligned} y(t) &= CBu(t) \oplus CABu(t-1) \oplus CA^2Bu(t-2) \oplus \ldots, \\ &= \bigoplus_{\tau \geq 0} CA^{\tau}Bu(t-\tau). \end{aligned} \tag{4}$$

In other words, the output is linked to the input by a convolution as defined below.

Definition 10 (Inf-convolution) Let $f(t)$ and $g(t)$ be two counter functions from $\mathbb{R}$ to $\overline{\mathbb{R}}_{\min}$. The (min,+)-convolution also called *inf-convolution* of f by g is the counter function defined below:

$$(f * g)(t) \triangleq \bigoplus_{\tau \geq 0}\{f(\tau) \otimes g(t-\tau)\} = \min_{\tau \geq 0}\{f(\tau) + g(t-\tau)\}.$$

Remark 3 The inf-convolution is a commutative operation that distributes over the sum $\oplus$. Its neutral element is denoted e and is defined by $e = \Delta_0^0$.

Thanks to this convolution product and according to Eq. 4, the input-output behavior of a (min,+)-linear system can be expressed as follows, where function h is called the transfer function:

$$y(t) = (h * u)(t) \quad \text{and} \quad h(\tau) = CA^{\tau}B.$$

By extension to the Multiple-Input Multiple-Output (MIMO) case, this input-output relation is described by:

$$Y(t) = (H * U)(t),$$

where matrix H is called the transfer matrix. The inf-convolution of two matrices $D^{d\times l}$ and $F^{l\times f}$ is defined as follows:

$$(D * F)_{ij} = \bigoplus_{k=1}^{l}\{D_{ik} * F_{kj}\}.$$

2.4 Transfer matrix computation

The transfer matrix of a MIMO system can be obtained from the state representation given by Eq. 1 as explained now. The time shifting between $x(t)$ and $x(t-1)$ and the event shifting contained in matrices A, B and C can also be expressed by inf-convolutions with elementary functions:

$$x(t-1) = \left(\Delta_1^0 * x\right)(t),$$
$$1 \otimes x(t) = \left(\Delta_0^1 * x\right)(t).$$

In other words, if $x \in \mathscr{F}_c$ describes a trajectory, then:

$$\Delta_1^0 * x = \text{trajectory } x \text{ shifted by 1 time unit},$$
$$\Delta_0^1 * x = \text{trajectory } x \text{ shifted by 1 event unit}.$$

So, by considering the idempotent semiring denoted $(\mathscr{F}_c, \oplus, *)$ of nondecreasing functions endowed with the min as sum and the inf-convolution as product, a different expression of Eq. 1 is obtained:

$$\begin{cases} x = A' * \Delta_1^0 * x \oplus B' * u, \\ y = C' * x, \end{cases} \tag{5}$$

where $A'_{ij} = \Delta_0^{A_{ij}}$, $B'_{ij} = \Delta_0^{B_{ij}}$, $C'_{ij} = \Delta_0^{C_{ij}}$, and x, u and y are vectors of functions in $\mathscr{F}_c$. Thanks to Theorem 1, on the semiring $(\mathscr{F}_c, \oplus, *)$, these equations are solved in order to lead to:

$$y = C' * \left(A' * \Delta_1^0\right)^\star * B' * u,$$

and so $y = H * u$ with:

$$H = C' * \left(A' * \Delta_1^0\right)^\star * B'. \tag{6}$$

Remark 4 (Subadditive closure $\Delta_T^{K\star}$) Let Δ_T^K, with $K, T > 0$, be an elementary function of the semiring $(\mathscr{F}_c, \oplus, *)$. Its subadditive closure $\Delta_T^{K\star}$ illustrated Fig. 3, is a function of $\mathscr{F}_{cp}$ such that:

$$\Delta_T^{K\star} = \bigoplus_{i \geq 0} \left(\Delta_T^K\right)^i \quad \text{where} \quad \left(\Delta_T^K\right)^{i+1} = \left(\Delta_T^K\right)^i * \left(\Delta_T^K\right) \quad \text{and} \quad \left(\Delta_T^K\right)^0 = e.$$

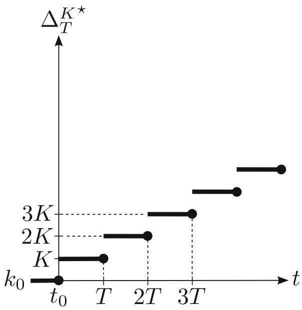


Fig. 3 Subadditive closure of an elementary function: $\Delta_T^{K\star} \in \mathscr{F}_{cp}$

Moreover $(\Delta_T^K)^i = \Delta_{iT}^{iK}$ so $\Delta_T^{K\star} = e \oplus \Delta_T^K \oplus \Delta_{2T}^{2K} \oplus \ldots$. The asymptotic slope of this subadditive closure is defined by $\sigma(\Delta_T^{K\star}) = K/T$. If $K, T \in \mathbb{N}$, then $\sigma(\Delta_T^{K\star}) \in \mathbb{Q}^+$.

The next result showing that a (min,+)-linear system has a transfer matrix that belongs to the sub-semiring $(\mathscr{F}_{cp}, \oplus, *)^{q\times p}$, is a key result for (min,+)-linear systems.

Theorem 2 (Baccelli et al. 1992, Theorem 5.39) *If matrices $A \in \overline{\mathbb{Z}}_{\min}^{n\times n}$, $B \in \overline{\mathbb{Z}}_{\min}^{n\times p}$ and $C \in \overline{\mathbb{Z}}_{\min}^{q\times n}$ of the state representation (Eq. 1) are positive, then $H = C' * (A' * \Delta_1^0)^\star * B'$ given by Eq. 6 is such that $\forall i, j$, H_{ij} is a causal ultimately pseudo-periodic function of $\mathscr{F}_{cp}$.*

Sketch of proof Firstly, according to the definition of system given in Eq. 5, transfer matrix H is obtained by doing a finite number of operations $\{\oplus, *, {}^\star\}$ on elementary functions Δ_T^K. Moreover, since elementary functions A'_{ij}, B'_{ij}, C'_{ij} and Δ_1^0 can be seen as functions of $\mathscr{F}_{cp}$, technical proof consists in verifying that the set $\mathscr{F}_{cp}$ is rationally closed (see for instance Gaubert 1992; Baccelli et al. 1992; Bouillard and Thierry 2008, Propositions 4 and 5). □

Remark 5 It is important to note that the elementary function denoted Δ_0^1 in the semiring $(\mathscr{F}_{cp}, \oplus, *)$ is nothing else but the γ shift operator of idempotent semiring $\mathscr{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$, and Δ_1^0 corresponds to the δ shift operator. In the context of $\mathscr{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$ modeling, the result of Theorem 2 is expressed by a more detailed result: the transfer matrix of a (min,+)-linear system that is rational (i.e. it can be expressed with a finite combination of $\{\gamma, \delta\}$ and $\{\oplus, \otimes, {}^\star\}$) is necessarily periodic and causal.

In conclusion of this section, due to Eq. 6, the computation of the behavior of a (min,+)-linear system relies on an efficient computation of operations such as sum $\oplus$, inf-convolution $*$ and subadditive-closure * of ultimately pseudo-periodic and causal functions.

3 Container of (min,+)-linear systems

3.1 Objectives

The exact computation of sum, inf-convolution and subadditive closure for ultimately pseudo-periodic and nondecreasing functions of $\mathscr{F}_{cp}$ can be really time and memory consuming (see for instance Cottenceau et al. 1998–2006; Gaubert 1992; Bouillard and Thierry 2008). The main objective of this work is to get some efficient algorithms to handle these functions. To achieve this objective, function $f \in \mathscr{F}_{cp}$ is not represented in an exact way[10], but is approximated by a set, more precisely by an interval of functions: $\mathbf{f} = [\,\underline{f}\,,\ \overline{f}\,] = \{f \in \mathscr{F}_{cp} \mid \underline{f} \preccurlyeq f \preccurlyeq \overline{f}\}$.

The operations between these sets have to be defined in order to contain the result in a guaranteed way. These operations are inspired from the set membership approach (Jaulin et al. 2001; Moore 1979) that proposes, for $\mathbf{f}$ and $\mathbf{g}$ two intervals of

[10]Which is made in the MinMaxGD toolbox.

functions, to carry out the computation on all the f and g that respectively belong to $[\,\underline{f}\,,\,\overline{f}\,]$ and $[\,\underline{g}\,,\,\overline{g}\,]$. Formally, the interval operations denoted $\diamond \in \{\oplus, *, ^{\star}\}$ are defined by:

$$\mathbf{f} \diamond \mathbf{g} = \{f \diamond g \mid f \in \mathbf{f} = [\,\underline{f}\,,\,\overline{f}\,] \text{ and } g \in \mathbf{g} = [\,\underline{g}\,,\,\overline{g}\,]\}.$$

In order to obtain efficient algorithms, a simple idea consists in doing the computation to get $\mathbf{f} \diamond \mathbf{g}$ by handling the bounds of the intervals. But this introduces pessimism concerning the computations and does not improve the complexity of the algorithms (we still handle ultimately pseudo-periodic functions). This is why we headed towards inclusion functions denoted $[\diamond] \in \{[\oplus], [*], ^{[\star]}\}$ that are such that:

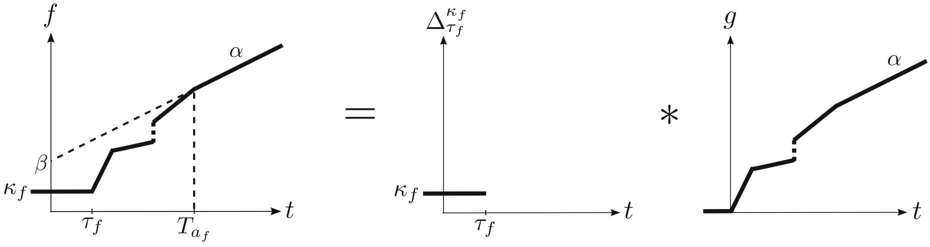
$$\mathbf{f}[\diamond]\mathbf{g} \supset \mathbf{f} \diamond \mathbf{g}.$$

In other words, the inclusion function $[\diamond]$ contains in a guaranteed way the result of $f \diamond g$, by intrinsically adding pessimism. Hence, the aim is to find inclusion functions with interesting algorithm complexity and allowing us to obtain intervals that are as small as possible.

The idea is to introduce containers denoted $[\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}}$ as intervals, the bounds of which are less memory consuming than functions of $\mathscr{F}_{cp}$ and that lead to algorithms with lower complexity than the ones used in MinMaxGD or in COINC. In this proposed container, the bound $\overline{f}$ corresponds to the greatest element of the equivalence class of f modulo the Legendre–Fenchel transform $\mathscr{L}$. Similarly, the bound $\underline{f}$ plays the role of the lower bound of this equivalence class. Finally, the functions $\underline{f}$ and $\overline{f}$ are piecewise affine, ultimately affine, and respectively concave and convex.

It is important to note that in Network Calculus literature, convex and concave functions are often used in order to efficiently compute performance bounds during the analysis of a data network. For instance in Fidler and Recker (2006), the authors use the fact that in the convex analysis (Rockafellar 1997) the inf-convolution of convex functions corresponds to addition in the Legendre domain (a useful result for the concatenation of systems). However, they only propose to deal with an upper bound of the input/output behavior whereas here we offer to deal with a container enclosing the transfer relation in a guaranteed way. Another difference is that they make their computations in the Legendre domain while we always stay in the (min,+) domain and only use the properties of convex functions. Finally, one can find in Schmitt and Zdarsky (2006) the use of a lower bound for the system in addition to the classical upper bound. Indeed, this reference considers almost concave functions that allow them to introduce concave lower bounds of transfer relations and to propose an efficient computation for the inf-convolution. The definition of the lower bound for the container proposed here is inspired from this result. Nevertheless, they do not propose the computation of the subadditive closure as we offer to do here. Moreover, in the definition of our container, a canonical representation is provided in order to minimize the pessimism of its lower bound.

This section will be organized as follows. In Section 3.2, the classification of functions is completed by affine functions, since they will be used to frame functions of $\mathscr{F}_{cp}$. The Legendre–Fenchel transform $\mathscr{L}$ is defined in Section 3.3 and in Section 3.4, two operators of approximation are defined to approximate an exact function from above and from below. Finally, Section 3.4 will introduce the container we developed as an intersection between the interval of functions $[\,\underline{f}\,,\,\overline{f}\,]$ and the equivalence class of $\overline{f}$ modulo the Legendre–Fenchel transform $\mathscr{L}$.



(a) A function $f \in \mathscr{F}_{aa}$ (b) Which can be factorized by the inf-convolution $\Delta_{\tau_f}^{\kappa_f} * g$

Fig. 4 An ultimately affine function $f \in \mathscr{F}_{aa}$ and its factorization: $f = \Delta_{\tau_f}^{\kappa_f} * g$

3.2 Affine and ultimately affine functions

First of all, in order to build the container of functions of $\mathscr{F}_{cp}$, the classification of functions needs to be completed. Until now, only piecewise constant functions have been considered. Piecewise affine functions are now necessary. They are still defined from $\mathbb{R}$ to $\overline{\mathbb{R}}_{\min}$.

Definition 11 (Set $\mathscr{F}_{aa}$) A function $f \in \mathscr{F}_{aa}$ (see Fig. 4a) is a function that is constant on the interval $]-\infty, \tau_f]$, and:

- piecewise affine, i.e. composed of a finite number of intervals on which the function is affine[11],
- nondecreasing,
- ultimately affine from a time denoted T_{a_f}: $\exists T_{a_f} \geq \tau_f$ and $\exists \alpha, \beta \in \mathbb{R}$ such that $\forall t \geq T_{a_f},\ f(t) = \alpha t + \beta$,

on $]\tau_f, +\infty[$.

Property 4 (Asymptotic slope σ) Let f be a function of $\mathscr{F}_{aa}$, its asymptotic slope is defined by the one of its ultimately affine parts: $\sigma(f) = \alpha$.

Property 5 (Factorization of a function $f \in \mathscr{F}_{aa}$) A function $f \in \mathscr{F}_{aa}$ can be seen as the following inf-convolution illustrated by Fig. 4b:

$$f = \Delta_{\tau_f}^{\kappa_f} * g,$$

where κ_f, τ_f and $g \in \mathscr{F}_{aa}$ are given such that:

$$\begin{cases} \kappa_f = \lim\limits_{t \to -\infty} f(t), \\ \tau_f = \max\{\, t \mid f(t) = \kappa_f \}, \end{cases} \quad \text{and} \quad g(t) = f(t - \tau_f) - \kappa_f,$$

so $g(0) = 0$ and $\sigma(g) = \sigma(f)$. It means that a function $f \in \mathscr{F}_{aa}$ can always be seen as a function g (with $g(0) = 0$) shifted by an elementary function $\Delta_{\tau_f}^{\kappa_f}$: κ_f is the event shift and τ_f is the time shift.

[11]The affine parts are linked by *non-differentiable points*.

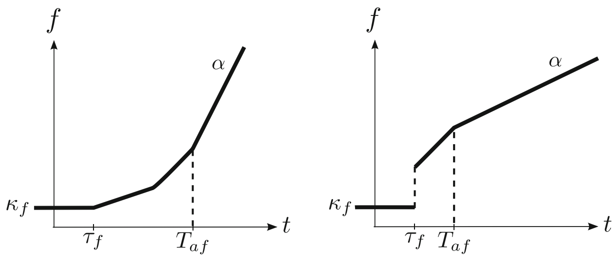


Fig. 5 Examples of ultimately affine functions

(a) A convex function $f \in \mathscr{F}_{\text{acx}} \subset \mathscr{F}_{\text{aa}}$

(b) A function $f \in \mathscr{F}_{\text{acv}} \subset \mathscr{F}_{\text{aa}}$ that is concave on $]\tau_f, +\infty[$

Definition 12 (Set $\mathscr{F}_{\text{acx}}$) A function $f \in \mathscr{F}_{\text{acx}}$ (see Fig. 5a) is a function of $\mathscr{F}_{\text{aa}}$ that is in addition convex[12].

Definition 13 (Set $\mathscr{F}_{\text{acv}}$) A function $f \in \mathscr{F}_{\text{acv}}$ (see Fig. 5b) is a function of $\mathscr{F}_{\text{aa}}$ that is in addition concave[13] on $]\tau_f, +\infty[$.

Remark 6 These kinds of concave functions can also be found in Schmitt and Zdarsky (2006) under the name *almost concave functions* or in Lenzini et al. (2006) where they are called *pseudoaffine* curves.

Definition 14 (Extremal point) In convex and concave functions, a non-differentiable point is called an *extremal point*.

Proposition 1 (Factorization of $f \in \mathscr{F}_{\text{acv}}$) *A function $f \in \mathscr{F}_{\text{acv}}$ can be factorized as follows (see Fig. 6):*

$$f = \Delta_{\tau_f}^{\kappa_f} * \Gamma_f, \tag{7}$$

where $\Gamma_f \in \mathscr{F}_{\text{acv}}$, $\Gamma_f(0) = 0$ and $\sigma(\Gamma_f) = \sigma(f)$.

Proof According to Property 5, $f = \Delta_{\tau_f}^{\kappa_f} * g$ with $g \in \mathscr{F}_{\text{acv}}$ and $g(0) = 0$. So $g = \Gamma_f$. □

The function Γ_f is the concave part of f shifted in the plane by the elementary function $\Delta_{\tau_f}^{\kappa_f}$. The following theorem about functions denoted Γ provides some useful equalities to deal with operations $*$ and $\star$ in the next section.

Theorem 3 (Le Boudec and Thiran 2001, Theorems 3.1.3, 3.1.6 and 3.1.9) *Let Γ_1 and Γ_2 be two functions of $\mathscr{F}_{\text{acv}}$ for which $\Gamma_1(0) = \Gamma_2(0) = 0$ (see Proposition 1), then:*

$$\Gamma_1 * \Gamma_2 = \Gamma_1 \oplus \Gamma_2, \tag{8}$$

[12] The epigraph of a convex function is a convex set.

[13] The hypograph of a concave function is a convex set.

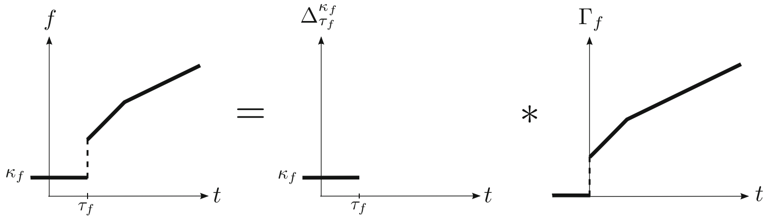


Fig. 6 Factorization of a concave function $f \in \mathscr{F}_{\text{acv}}$: $f = \Delta_{\tau_f}^{\kappa_f} * \Gamma_f$

with $\Gamma_1 \oplus \Gamma_2 \in \mathscr{F}_{\text{acv}}$ *and* $(\Gamma_1 \oplus \Gamma_2)(0) = 0$*. Moreover:*

$$\Gamma_1 = \Gamma_1^\star, \tag{9}$$

that is Γ_1 *is closed for the subadditive closure operation.*

3.3 Legendre–Fenchel transform

The construction of the container principally relies on the Legendre–Fenchel transform. This transform is well known in convex analysis (Rockafellar 1997), and already used in Network Calculus literature to efficiently compute performance bounds (Fidler and Recker 2006) or in the context of formal series of the idempotent semiring $\mathscr{M}_{\text{in}}^{\text{ax}}[\![\gamma, \delta]\!]$ (Cohen et al. 1989a; Burkard and Butkovič 2003). In this paper, this transform is applied to the set $\mathscr{F}_{cp}$ in order to reduce the computation complexity of operations involving these functions.

Definition 15 (Legendre–Fenchel transform $\mathscr{L}$) The Legendre–Fenchel transform applied to $f \in \mathscr{F}_{cp}$ is the mapping $\mathscr{L}$ defined from $(\mathscr{F}_{cp}, \oplus, *)$ to the idempotent semiring of convex functions[14] denoted $(\mathscr{D}_{convex}, \max, +)$ by:

$$\mathscr{L}(f)(s) \triangleq \sup_t \{s.t - f(t)\}.$$

Mapping $\mathscr{L}$ is a non injective homomorphism from $(\mathscr{F}_{cp}, \oplus, *)$ to $(\mathscr{D}_{convex}, \max, +)$, that is $\forall f, g \in (\mathscr{F}_{cp}, \oplus, *)$:

$$\mathscr{L}(f \oplus g) = \max(\mathscr{L}(f), \mathscr{L}(g)),$$
$$\mathscr{L}(f * g) = \mathscr{L}(f) + \mathscr{L}(g).$$

Definition 16 (Idempotent semiring $\mathscr{F}_{\text{cp}/\mathscr{L}}$) Let us consider the following equivalence relation $\forall f, g \in \mathscr{F}_{cp}$:

$$\mathscr{L}(f) = \mathscr{L}(g) \quad \Leftrightarrow \quad f \overset{\mathscr{L}}{\equiv} g,$$

where $\overset{\mathscr{L}}{\equiv}$ is a congruence (see Lemma 2). The quotient of $\mathscr{F}_{cp}$ by $\overset{\mathscr{L}}{\equiv}$ provides an idempotent semiring denoted $\mathscr{F}_{\text{cp}/\mathscr{L}}$ (see Lemma 1). An element of $\mathscr{F}_{\text{cp}/\mathscr{L}}$ is an

[14] $\mathscr{D}_{convex}$ is the set of convex functions endowed with the pointwise maximum as sum and the pointwise addition as product.

equivalence class modulo $\mathscr{L}$ denoted $[f]_{\mathscr{L}}$ containing all the functions of $\mathscr{F}_{cp}$ that have the same Legendre–Fenchel transform. Operations $\oplus$ and $\otimes$ of $\mathscr{F}_{\mathrm{cp}/\mathscr{L}}$ are defined below:

$$[f]_{\mathscr{L}} \oplus [g]_{\mathscr{L}} \triangleq [f \oplus g]_{\mathscr{L}},$$
$$[f]_{\mathscr{L}} \otimes [g]_{\mathscr{L}} \triangleq [f * g]_{\mathscr{L}}.$$

This idempotent semiring $\mathscr{F}_{\mathrm{cp}/\mathscr{L}}$ will play a central role for the definition of the convex upper bound of our container. Indeed, the following subsection goes back to the link between the Legendre–Fenchel transform of a function and its convex hull.

3.4 Operators of approximation

In order to build the container of a function $f \in \mathscr{F}_{cp}$, two operators of approximation are defined and will be used during the computation of the inclusion functions. The former is convex and approximates f from above (according to the $\preccurlyeq$ order), and the latter is concave and approximates f from below. We recall that the order $\preccurlyeq$ is not the natural order of functions but the canonical order on $(\mathscr{F}_{cp}, \oplus, *)$ (see Section 2.2 and Remark 1), that is in accordance with the literature.

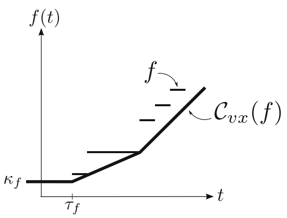
Remark 7 In regard with the notation of Property 5 and in order to help facilitate the understanding of this and the next section, the first elementary function $\Delta_{t_0}^{k_0}$ of a function $f = \bigoplus_{i=0}^{+\infty} \Delta_{t_i}^{k_i} \in \mathcal{F}_{cp}$ (see Definition 9), will be denoted $\Delta_{\tau_f}^{\kappa_f}$ in the sequel.

3.4.1 Convex approximation

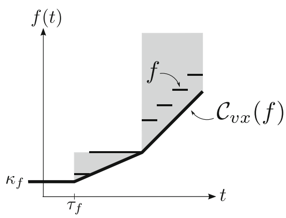
Definition 17 (Convex hull $\mathscr{C}_{vx}$) Let f be a function of $\mathscr{F}_{cp}$. The convex hull of f is denoted $\mathscr{C}_{vx}(f)$ (also called the convex approximation in this paper) and is the smallest convex function greater than f so $\mathscr{C}_{vx}(f) \succcurlyeq f$ and $\mathscr{C}_{vx}(f) \in \mathscr{F}_{\mathrm{acx}}$ (see Fig. 7a).

Property 6 Mapping $\mathscr{C}_{vx}$ is a projector, i.e. $\mathscr{C}_{vx}(f) = \mathscr{C}_{vx}(\mathscr{C}_{vx}(f))$ (see Definition 5). Moreover, the asymptotic slopes of $\mathscr{C}_{vx}(f)$ and f are equal: $\sigma(\mathscr{C}_{vx}(f)) = \sigma(f)$, and the extremal points of $\mathscr{C}_{vx}(f)$ belong to the function f. In particular $\Delta_{\tau_{\mathscr{C}_{vx}(f)}}^{\kappa_{\mathscr{C}_{vx}(f)}} = \Delta_{\tau_f}^{\kappa_f}$.

Fig. 7 Convex approximation of $f \in \mathscr{F}_{cp}$ and its equivalence class modulo the Legendre–Fenchel transform $\mathscr{L}$



(a) Convex approximation $\mathscr{C}_{vx}(f) \in \mathscr{F}_{\mathrm{acx}}$



(b) Equivalence class $[f]_{\mathscr{L}}$

Lemma 3 (Baccelli et al. 1992, Theorem 3.38) *Let $\mathscr{C}_{vx}(f)$ and $\mathscr{C}_{vx}(g)$ be the convex hulls of f and $g \in \mathscr{F}_{cp}$. Functions f and g have the same Legendre–Fenchel transform if, and only if, they have the same convex hull:*

$$\mathscr{L}(f) = \mathscr{L}(g) \quad \Leftrightarrow \quad \mathscr{C}_{vx}(f) = \mathscr{C}_{vx}(g) \quad \Leftrightarrow \quad [f]_{\mathscr{L}} = [g]_{\mathscr{L}}.$$

Property 7 Functions that have the same extremal points and the same asymptotic slope as f belong to the equivalent class $[\ f\ ]_{\mathscr{L}}$ as illustrated in Fig. 7b by the gray zone.

According to Lemma 3, it is then possible to determine the equivalence modulo the Legendre–Fenchel transform $\mathscr{L}$ by using the convex hull. As a consequence, the computations modulo the transform $\mathscr{L}$ are equivalent to the computations modulo the convex hull. Formally, we have $\forall f, g \in \mathscr{F}_{cp}$:

$$[f]_{\mathscr{L}} \oplus [g]_{\mathscr{L}} = [f \oplus g]_{\mathscr{L}} \quad \Leftrightarrow \quad \mathscr{C}_{vx}(\mathscr{C}_{vx}(f) \oplus \mathscr{C}_{vx}(g)) = \mathscr{C}_{vx}(f \oplus g), \tag{10}$$

$$[f]_{\mathscr{L}} \otimes [g]_{\mathscr{L}} = [f * g]_{\mathscr{L}} \quad \Leftrightarrow \quad \mathscr{C}_{vx}(\mathscr{C}_{vx}(f) * \mathscr{C}_{vx}(g)) = \mathscr{C}_{vx}(f * g), \tag{11}$$

$$[f]^{\star}_{\mathscr{L}} = [f^{\star}]_{\mathscr{L}} \quad \Leftrightarrow \quad \mathscr{C}_{vx}(\mathscr{C}_{vx}(f)^{\star}) = \mathscr{C}_{vx}(f^{\star}). \tag{12}$$

Theorem 4 (Baccelli et al. 1992, Theorem 6.19) *Let f be a function of $\mathscr{F}_{cp}$ and $[f]_{\mathscr{L}}$ be its equivalence class in the idempotent semiring $\mathscr{F}_{cp_{/\mathscr{L}}}$ (see Definition 16). Function $\mathscr{C}_{vx}(f) \in \mathscr{F}_{acx}$ is the greatest representative of $[f]_{\mathscr{L}}$, i.e.:*

$$[\,\mathscr{C}_{vx}(f)\,]_{\mathscr{L}} = [f]_{\mathscr{L}} \quad \textit{and} \quad \forall g \in [f]_{\mathscr{L}}, \quad g \preccurlyeq \mathscr{C}_{vx}(f).$$

Hence, thanks to Theorem 4, we obtain a method to perform computations on the idempotent semiring $\mathscr{F}_{cp_{/\mathscr{L}}}$, even if in practice the Legendre–Fenchel transform $\mathscr{L}$ of a function $f \in \mathscr{F}_{cp}$ will never be explicitly computed. Indeed, each equivalence class of $\mathscr{F}_{cp_{/\mathscr{L}}}$ has a canonical representative that is the convex hull of the functions of the class. Therefore, if we make the computations modulo the convex hull, we carry out the computations in the quotient dioid $\mathscr{F}_{cp_{/\mathscr{L}}}$, we simplify the results and we still conserve their equivalence class modulo $\mathscr{L}$.

3.4.2 Concave approximation

First of all, let us recall that a function $f \in \mathscr{F}_{cp}$ is constant on $]-\infty, \tau_f]$ (see Remark 7) and then nondecreasing piecewise constant on $]\tau_f, +\infty[$. Its concave hull denoted $conc(f)$, i.e. the greatest concave function lower than f (according to the order $\preccurlyeq$), is necessarily the function $\varepsilon : t \mapsto +\infty$. Hence, this concave hull is not useful to provide a lower approximation of f.

However, a lower bound of f can be defined as a function of $\mathscr{F}_{acv}$ that is constant on $]-\infty, \tau_f]$ and concave on $]\tau_f, +\infty[$. This projection in $\mathscr{F}_{acv}$ is called a concave approximation and corresponds to that defined as an almost concave function in Schmitt and Zdarsky (2006).

Definition 18 (Concave approximation $\mathscr{C}_{cv}$) Let f be a function of $\mathscr{F}_{\text{cp}}$. The concave approximation of f illustrated Fig. 8 is denoted $\mathscr{C}_{cv}(f)$ and defined by:

$$\mathscr{C}_{cv}(f)(t) \triangleq \begin{cases} f(t) & \text{for } t \le \tau_f, \\ conc(f)(t) & \text{for } t > \tau_f, \end{cases}$$

where $\tau_f = \max\{t \mid f(t) = f(-\infty)\}$. So $\mathscr{C}_{cv}(f) \preccurlyeq f$ and $\mathscr{C}_{cv}(f) \in \mathscr{F}_{\text{acv}}$.

Property 8 Mapping $\mathscr{C}_{cv}$ is a projector, *i.e.* $\mathscr{C}_{cv}(f) = \mathscr{C}_{cv}(\mathscr{C}_{cv}(f))$. Moreover, $\sigma(\mathscr{C}_{cv}(f)) = \sigma(f)$ and $\Delta^{\kappa_{\mathscr{C}_{cv}(f)}}_{\tau_{\mathscr{C}_{cv}(f)}} = \Delta^{\kappa_f}_{\tau_f}$.

Remark 8 It must be noted that this concave approximation $\mathscr{C}_{cv}$ is not symmetrical with the convex one $\mathscr{C}_{vx}$. More precisely, the equivalences of Eqs. 10–12 are not verified. However, $\mathscr{C}_{cv}$ is an isotone mapping so the following properties are satisfied. Let f and g be two functions of $\mathscr{F}_{\text{cp}}$:

$$\begin{cases} \mathscr{C}_{cv}(f) \preccurlyeq f \\ \mathscr{C}_{cv}(g) \preccurlyeq g \end{cases} \quad \Rightarrow \quad \begin{cases} \mathscr{C}_{cv}(f) \oplus \mathscr{C}_{cv}(g) \preccurlyeq f \oplus g, \\ \mathscr{C}_{cv}(f) * \mathscr{C}_{cv}(g) \preccurlyeq f * g, \\ \mathscr{C}_{cv}(f)^\star \preccurlyeq f^\star, \end{cases}$$

and

$$\mathscr{C}_{cv}(\mathscr{C}_{cv}(f) \oplus \mathscr{C}_{cv}(g)) \preccurlyeq \mathscr{C}_{cv}(f \oplus g),$$
$$\mathscr{C}_{cv}(\mathscr{C}_{cv}(f) * \mathscr{C}_{cv}(g)) \preccurlyeq \mathscr{C}_{cv}(f * g),$$
$$\mathscr{C}_{cv}(\mathscr{C}_{cv}(f)^\star) \preccurlyeq \mathscr{C}_{cv}(f^\star).$$

3.5 Definition of the container

The objective of this section is to build an approximation of ultimately pseudo-periodic functions of $\mathscr{F}_{\text{cp}}$, such that the computations on the approximated function are more efficient than on the original ones. In order to do this, the container defined below is an interval of functions associated with an equivalence class modulo $\mathscr{L}$.

Definition 19 (Set $\mathbf{F}$ of containers) The set of containers considered in the sequel is the set denoted $\mathbf{F}$ and defined by:

$$\mathbf{F} \triangleq \{\, [\, \underline{f}\,,\ \overline{f}\,]_{\mathscr{L}} \mid \underline{f} \in \mathscr{F}_{\text{acv}}\,,\ \overline{f} \in \mathscr{F}_{\text{acx}}\,,\ \sigma(\underline{f}) = \sigma(\overline{f})\,\},$$

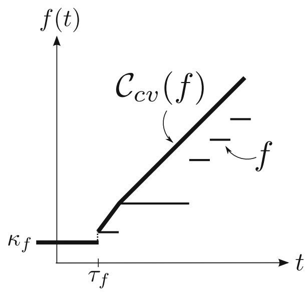
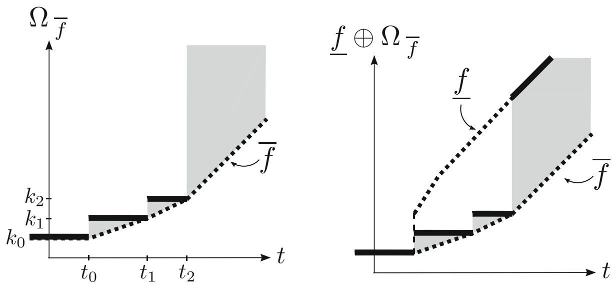


Fig. 8 Concave approximation of $f \in \mathscr{F}_{\text{cp}}$: $\mathscr{C}_{cv}(f) \in \mathscr{F}_{\text{acv}}$



(a) Lower bound $\Omega_{\overline{f}}$ of the equivalent class $[\,\overline{f}\,]_{\mathscr{L}}$

(b) Smallest element $\underline{f} \oplus \Omega_{\overline{f}}$ of the container

Fig. 9 Role of the lower bound $\Omega_{\overline{f}}$ during the construction of a container $\mathbf{f} = [\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}} \in \mathbf{F}$.

with $[\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}}$ the subset defined as follows:

$$\begin{aligned}[\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}} &\triangleq [\,\underline{f}\,,\,\overline{f}\,] \cap [\,\overline{f}\,]_{\mathscr{L}},\\ &= \{\, f \mid \underline{f} \preccurlyeq f \preccurlyeq \overline{f}\,,\ [f]_{\mathscr{L}} = [\,\overline{f}\,]_{\mathscr{L}}\}.\end{aligned}$$

So, a container of $\mathbf{F}$ is a subset of an interval $[\,\underline{f}\,,\,\overline{f}\,]$, the bounds $\underline{f}$ and $\overline{f}$ of which are respectively concave[15] and convex. The elements of $[\,\underline{f}, \overline{f}\,]_{\mathscr{L}}$ are equivalent to $\overline{f}$ modulo the Legendre–Fenchel transform $\mathscr{L}$. This means that $\forall f \in [\,\underline{f}, \overline{f}\,]_{\mathscr{L}}$, $\overline{f} = \mathscr{C}_{vx}(f)$ (with $\mathscr{C}_{vx}$ the convex approximation given in Definition 17).

3.5.1 Canonical representation of a container of F

Among the containers of set $\mathbf{F}$, we propose to define a canonical one. Its definition is based on a lower bound of the equivalence class $[\,\overline{f}\,]_{\mathscr{L}}$. Hence we first introduce this lower bound.

Definition 20 (Lower bound $\Omega_{\overline{f}}$) Let $[\,\overline{f}\,]_{\mathscr{L}}$ be an equivalence class of the semiring $\mathscr{F}_{\mathrm{cp}/\mathscr{L}}$ and $\overline{f} \in \mathscr{F}_{\mathrm{acx}}$ be its greatest element, that is $\overline{f} = \mathscr{C}_{vx}(\overline{f})$ (see Theorem 4). Function $\Omega_{\overline{f}} \in \mathscr{F}_c$ defined below is a lower bound of $[\,\overline{f}\,]_{\mathscr{L}}$:

$$\Omega_{\overline{f}} \triangleq \bigoplus_{i=0}^{n} \Delta_{t_i}^{k_i} \quad \text{and} \quad \forall t > t_n,\ \Omega_{\overline{f}}(t) = +\infty, \tag{13}$$

where pairs (t_i, k_i) are the coordinates of the n extremal points of $\overline{f}$. Therefore:

$$\forall f \in [\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}},\ f \succcurlyeq \Omega_{\overline{f}} \quad \text{and} \quad \Delta_{t_0}^{k_0} = \Delta_{\tau_{\overline{f}}}^{\kappa_{\overline{f}}}.$$

This lower bound is illustrated in Fig. 9a in which the gray zone represents the equivalent class $[\,\overline{f}\,]_{\mathscr{L}}$.

[15] On $]\tau_f, +\infty[$.

Remark 9 Even if function $\Omega_{\overline{f}}$ is a lower bound of $[\,\overline{f}\,]_{\mathscr{L}}$, it must be noted that it does not have the same asymptotic slope than $\overline{f}$, indeed according to Eq. 13 the asymptotic slope of $\Omega_{\overline{f}}$ is infinite. Therefore $\Omega_{\overline{f}}$ does not belong to equivalence class $[\,\overline{f}\,]_{\mathscr{L}}$ (see Property 7 and Theorem 4).

This lower bound $\Omega_{\overline{f}}$ leads to the following implication: if $f \in [\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}}$, then $f \succcurlyeq \underline{f} \oplus \Omega_{\overline{f}}$. Therefore, as it is illustrated in Fig. 9b, $\underline{f} \oplus \Omega_{\overline{f}}$ is the smallest element of the container $[\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}}$, and interval $[\,\underline{f} \oplus \Omega_{\overline{f}}\,,\,\overline{f}\,]$ (the gray zone in Fig. 9b) contains the function f. Consequently, according to Definition 19, a same container of $\mathbf{F}$ can be represented by different intervals, as long as $\underline{f}_1 \oplus \Omega_{\overline{f}} = \underline{f}_2 \oplus \Omega_{\overline{f}}$. In other words, one can have (see Fig. 10):

$$[\,\underline{f}_1\,,\,\overline{f}\,] \neq [\,\underline{f}_2\,,\,\overline{f}\,] \quad \text{while} \quad [\,\underline{f}_1\,,\,\overline{f}\,]_{\mathscr{L}} = [\,\underline{f}_2\,,\,\overline{f}\,]_{\mathscr{L}}.$$

In order to avoid such ambiguities, a canonical representation for these containers is defined by doing the concave approximation of $\underline{f} \oplus \Omega_{\overline{f}}$.

Definition 21 (Canonical representation of a container) The canonical representation of a container $\mathbf{f} = [\,\underline{f}\,,\,\overline{f}\,]_{\mathscr{L}} \in \mathbf{F}$ is written as follows:

$$[\,\mathscr{C}_{cv}\big(\underline{f} \oplus \Omega_{\overline{f}}\big)\,,\,\overline{f}\,]_{\mathscr{L}} = [\,\underline{f}'\,,\,\overline{f}\,]_{\mathscr{L}},$$

with $\Delta_{\tau_{\underline{f}'}}^{\kappa_{\underline{f}'}} = \Delta_{\tau_{\overline{f}}}^{\kappa_{\overline{f}}}$, and $\mathscr{C}_{cv}$ the concave approximation given in Definition 18. In Fig. 10, this canonical representation is given by the container $[\,\underline{f}_3\,,\,\overline{f}\,]_{\mathscr{L}}$ since $\underline{f}_3 = \mathscr{C}_{cv}\big(\underline{f}_1 \oplus \Omega_{\overline{f}}\big) = \mathscr{C}_{cv}\big(\underline{f}_2 \oplus \Omega_{\overline{f}}\big)$.

Remark 10 This canonical representation will also be the one chosen for the software representation. Indeed, in addition to offering the advantage of representing unambiguously a container of $\mathbf{F}$, it also allows useless points of $\underline{f}$ to be removed. In the example of Fig. 10, the points of $\underline{f}_1$ and $\underline{f}_2$ located below $\underline{f}_3$ can so be removed in order to keep only the canonical representation $[\,\underline{f}_3\,,\,\overline{f}\,]_{\mathscr{L}}$.

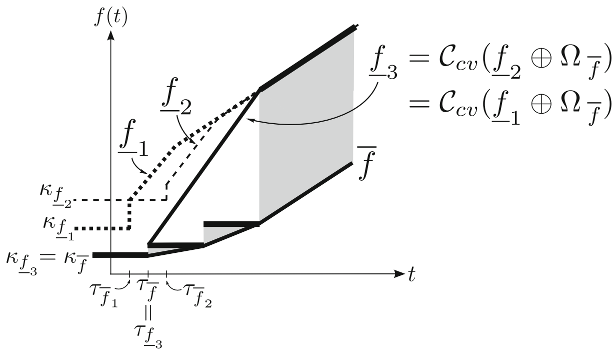
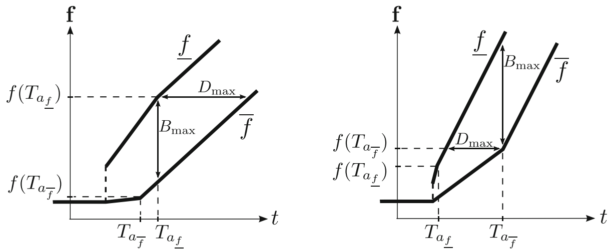


Fig. 10 Canonical representation $[\,\underline{f}_3\,,\,\overline{f}\,]_{\mathscr{L}}$ of the identical containers $[\,\underline{f}_1\,,\,\overline{f}\,]_{\mathscr{L}}$ and $[\,\underline{f}_2\,,\,\overline{f}\,]_{\mathscr{L}}$

Fig. 11 Maximal uncertainty $\Sigma_{\mathbf{f}} = \{ D_{\max} , B_{\max} \}$ of a container $\mathbf{f} \in \mathbf{F}$



(a) Case $\underline{f}(T_{a_{\underline{f}}}) > \overline{f}(T_{a_{\overline{f}}})$ and $\max\{T_{a_{\underline{f}}}, T_{a_{\overline{f}}}\} = T_{a_{\underline{f}}}$

(b) Case $\underline{f}(T_{a_{\underline{f}}}) \leq \overline{f}(T_{a_{\overline{f}}})$ and $\max\{T_{a_{\underline{f}}}, T_{a_{\overline{f}}}\} = T_{a_{\overline{f}}}$

3.5.2 Maximal uncertainty of a container of F

Thanks to the canonical form of a container, $\mathbf{f} = [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}} \in \mathbf{F}$ for which $\Delta^{\kappa_{\underline{f}}}_{\tau_{\underline{f}}} = \Delta^{\kappa_{\overline{f}}}_{\tau_{\overline{f}}}$ and $\sigma(\underline{f}) = \sigma(\overline{f})$ (see Definitions 21 and 19), the maximal distances in time and event domains between $\underline{f}$ and $\overline{f}$ are finite and the loss of precision due to approximations is bounded. This uncertainty is defined below.

Definition 22 (Maximal uncertainty $\Sigma_{\mathbf{f}}$ of a container **f**) Let $\mathbf{f} = [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ be a container of **F**. Its maximal uncertainty denoted $\Sigma_{\mathbf{f}}$ is defined as follows:

$$\Sigma_{\mathbf{f}} = \{ D_{\max}\ ,\ B_{\max} \},$$

where $D_{\max}$ and $B_{\max}$ (see Fig. 11) are respectively called the *delay* and the *backlog* of the container[16]. The former element of $\Sigma_{\mathbf{f}}$ corresponds to the maximal distance between $\underline{f}$ and $\overline{f}$ in the time domain:

$$D_{\max} \triangleq \inf_{\tau \geq 0}\{\tau \mid \underline{f}(t_0) \leq \overline{f}(t_0 + \tau)\} \quad \text{where} \quad t_0 = \begin{cases} T_{a_{\underline{f}}} & \text{if } \underline{f}(T_{a_{\underline{f}}}) > \overline{f}(T_{a_{\overline{f}}}), \\ T_{a_{\overline{f}}} & \text{otherwise.} \end{cases}$$

The latter element corresponds to the maximal distance between $\underline{f}$ and $\overline{f}$ in the event domain:

$$B_{\max} \triangleq \underline{f}(t_0) - \overline{f}(t_0) \quad \text{where} \quad t_0 = \max\{T_{a_{\underline{f}}}, T_{a_{\overline{f}}}\}.$$

These computations, according to Bouillard et al. (2007, Lemmas 3 et 4), are quite simple because of the convex characteristics of functions $\underline{f}$ and $\overline{f}$. Indeed, since the bounds $\underline{f}$ and $\overline{f}$ are ultimately affine from $T_{a_{\underline{f}}}$ and $T_{a_{\overline{f}}}$ (see Definition 11), and with identical asymptotic slope, it is enough to know where are their last points before their ultimately affine parts i.e. $(T_{a_{\underline{f}}}, \underline{f}(T_{a_{\underline{f}}}))$ and $(T_{a_{\overline{f}}}, \overline{f}(T_{a_{\overline{f}}}))$, and to compare the obtained coordinates. So, the maximal uncertainty $\Sigma_{\mathbf{f}}$ can be computed from these points as illustrated in Fig. 11.

[16]These designations come from the Network Calculus.

4 Operations between containers: inclusion functions

In this section, we consider operations between the containers of **F** (see Definition 19). These operations between two elements **f** and **g** $\in$ **F** are monotonic, nondecreasing operations $\diamond \in \{\oplus, *, ^\star\}$, and are defined in Section 3.1 by:

$$[\ \underline{f} \diamond \underline{g}\ ,\ \overline{f} \diamond \overline{g}\ ].$$

Unfortunately, **F** is not closed for these operations. Hence, we will use inclusion functions that are internals to **F**, as in the set membership approach (Moore 1979; Jaulin et al. 2001).

Definition 23 (Inclusion functions of operators $\{\oplus, *, ^\star\}$) Let $\mathbf{f} = [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ and $\mathbf{g} = [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}}$ be two containers belonging to **F** and $\diamond \in \{\oplus, *, ^\star\}$ be a set of operations. Inclusion functions of these operators for containers of **F** denoted:

$$[\diamond] \in \{[\oplus], [*], ^{[\star]}\} \quad \text{are such that} \quad \begin{cases} \mathbf{f}[\diamond]\mathbf{g} \supset \mathbf{f} \diamond \mathbf{g}, \\ \mathbf{f}[\diamond]\mathbf{g} \in \mathbf{F}. \end{cases} \tag{14}$$

In order to ensure condition $\mathbf{f}[\diamond]\mathbf{g} \in \mathbf{F}$ and in particular for functions $[\oplus]$ and $^{[\star]}$, the set $\mathbf{f} \diamond \mathbf{g}$ will be upper and lower rounded by the convex and concave approximations $\mathscr{C}_{vx}$ and $\mathscr{C}_{cv}$. These approximations are interesting since they lead to operations having a linear or quasi-linear complexity as it will be shown in the sequel.

Moreover, depending on the needs of operations, the canonical form of container $[\mathscr{C}_{cv}\big(\underline{f} \oplus \Omega_{\overline{f}}\big)\ ,\ \overline{f}]_{\mathscr{L}}$ (see Definition 21) will be used for functions $[\oplus]$ and $[*]$, whereas for function $^{[\star]}$, the interval without the concave approximation $[\underline{f} \oplus \Omega_{\overline{f}}\ ,\ \overline{f}]$ will be picked. This choice is relevant in order to obtain in all cases a weak level of complexity.

4.1 Convex and concave approximations $\mathscr{C}_{vx}$ and $\mathscr{C}_{cv}$

Firstly, the complexity of the algorithm giving the convex and concave approximations of a function $f \in \mathscr{F}_{\text{aa}}$ is proposed below.

Proposition 2 (Algorithmic complexity of $\mathscr{C}_{vx}$ and $\mathscr{C}_{cv}$) *Let N_f (respectively N_g) be the number of non-differentiable points of f (respectively g) in $\mathscr{F}_{\text{aa}}$. The computation of $\mathscr{C}_{vx}(f)$ (respectively $\mathscr{C}_{cv}(g)$) is of linear complexity, namely $\mathscr{O}(N_f)$ (respectively $\mathscr{O}(N_g)$).*

Proof Convex and concave approximations rely on known algorithms of convex hull computation in the computational geometry (Graham 1972). In the worst case, only two scans of the list of non-differentiable points are required. □

Remark 11 According to Definitions 18 and 17, let us recall that:

- $\mathscr{C}_{cv}(f) \in \mathscr{F}_{\text{acv}}, \mathscr{C}_{cv}(f) \preccurlyeq f, \sigma(\mathscr{C}_{cv}(f)) = \sigma(f)$ and $\Delta^{K_{\mathscr{C}_{cv}(f)}}_{\tau_{\mathscr{C}_{cv}(f)}} = \Delta^{K_f}_{\tau_f}$,
- $\mathscr{C}_{vx}(f) \in \mathscr{F}_{\text{acx}}, \mathscr{C}_{vx}(f) \succcurlyeq f, \sigma(\mathscr{C}_{vx}(f)) = \sigma(f)$ and $\Delta^{K_{\mathscr{C}_{vx}(f)}}_{\tau_{\mathscr{C}_{vx}(f)}} = \Delta^{K_f}_{\tau_f}$,
- $\mathscr{C}_{cv}$ and $\mathscr{C}_{vx}$ are projectors.

By the way, these projections can be applied to either functions of $\mathscr{F}_{cp}$ or functions of $\mathscr{F}_{aa}$.

4.2 Inclusion function of the sum: $[\oplus]$

Prior to studying inclusion function $[\oplus]$, it must be remembered that operator $\oplus$ is order-preserving, i.e. $f \preccurlyeq g \Rightarrow a \oplus f \preccurlyeq a \oplus g$ and as a consequence $\underline{f} \oplus \underline{g} \preccurlyeq \overline{f} \oplus \overline{g}$.

Proposition 3 (Inclusion function $[\oplus]$) *Let* $\mathbf{f} = [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ *and* $\mathbf{g} = [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}}$ *be two containers of* $\mathbf{F}$ *given in their canonical forms. The operation denoted* $[\oplus]$ *and defined by:*

$$\mathbf{f}[\oplus]\mathbf{g} = [\ \underline{\mathbf{f}[\oplus]\mathbf{g}}\ ,\ \overline{\mathbf{f}[\oplus]\mathbf{g}}\ ] \triangleq [\ \mathscr{C}_{cv}(\underline{f} \oplus \underline{g})\ ,\ \mathscr{C}_{vx}(\overline{f} \oplus \overline{g})\ ]_{\mathscr{L}},$$

is an inclusion function for the sum $\oplus$*, i.e.:*

$$\forall f \in [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}} \text{ and } \forall g \in [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}} \quad \Rightarrow \quad \begin{cases} f \oplus g \ \in \ \mathbf{f}[\oplus]\mathbf{g}, \\ \mathbf{f}[\oplus]\mathbf{g} \ \in \ \mathbf{F}. \end{cases}$$

Proof According to Remark 11 and since $\oplus$ is order-preserving, $\forall f \in [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ and $\forall g \in [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}}$, $\mathscr{C}_{cv}(\underline{f} \oplus \underline{g}) \preccurlyeq f \oplus g \preccurlyeq \mathscr{C}_{vx}(\overline{f} \oplus \overline{g})$, and projections $\mathscr{C}_{cv}(\underline{f} \oplus \underline{g})$ and $\mathscr{C}_{vx}(\overline{f} \oplus \overline{g})$ respectively belong to the sets $\mathscr{F}_{acv}$ and $\mathscr{F}_{acx}$. Then, since $\sigma(\underline{f}) = \sigma(\overline{f})$ and $\sigma(\underline{g}) = \sigma(\overline{g})$, that the sum $\oplus$ is the minimum operation, and that $\sigma(\mathscr{C}_{cv}(\underline{f} \oplus \underline{g})) = \sigma(\underline{f} \oplus \underline{g})$ (*ibid* for $\overline{f}$, $\overline{g}$ and $\mathscr{C}_{vx}$), then $\sigma(\mathscr{C}_{cv}(\underline{f} \oplus \underline{g})) = \min(\sigma(\underline{f}), \sigma(\underline{g})) = \min(\sigma(\overline{f}), \sigma(\overline{g})) = \sigma(\mathscr{C}_{vx}(\overline{f} \oplus \overline{g}))$. □

This inclusion function $[\oplus]$ does not necessarily provide a canonical result. This requires the concave approximation of $(\underline{\mathbf{f}[\oplus]\mathbf{g}} \oplus \Omega_{\overline{\mathbf{f}[\oplus]\mathbf{g}}})$ and we thus obtain the following container:

$$[\ \mathscr{C}_{cv}(\underline{\mathbf{f}[\oplus]\mathbf{g}} \oplus \Omega_{\overline{\mathbf{f}[\oplus]\mathbf{g}}})\ ,\ \overline{\mathbf{f}[\oplus]\mathbf{g}}\ ]_{\mathscr{L}} = [\ \mathscr{C}_{cv}(\mathscr{C}_{cv}(\underline{f} \oplus \underline{g}) \oplus \Omega_{\mathscr{C}_{vx}(\overline{f} \oplus \overline{g})})\ ,\ \mathscr{C}_{vx}(\overline{f} \oplus \overline{g})\ ]_{\mathscr{L}}.$$

Proposition 4 (Algorithmic complexity of $[\oplus]$) *Let* $N_{\underline{f}}$, $N_{\overline{f}}$, $N_{\underline{g}}$ *and* $N_{\overline{g}}$ *be the number of extremal points of respectively* $\underline{f}$, $\overline{f}$, $\underline{g}$ *and* $\overline{g}$. *The computation of* $\mathbf{f}[\oplus]\mathbf{g}$ *is of linear complexity, namely* $\mathscr{O}(N_{\underline{f}} + N_{\overline{f}} + N_{\underline{g}} + N_{\overline{g}})$.

Proof The minimum of two ultimately affine functions is of linear complexity since it requires in the worst case only one scan of extremal points for each function. Moreover, according to Proposition 2 the complexity of convex and concave approximations is linear. □

4.3 Inclusion function of the inf-convolution: $[*]$

The inf-convolution $*$ is order-preserving, hence $\underline{f} * \underline{g} \preccurlyeq \overline{f} * \overline{g}$.

Proposition 5 *If* $\underline{f}$ *and* $\underline{g} \in \mathscr{F}_{\mathrm{acv}}$, *then* $\underline{f} * \underline{g} \in \mathscr{F}_{\mathrm{acv}}$.

Proof Thanks to the factorization of functions of $\mathscr{F}_{acv}$:

$$\begin{aligned} \underline{f} * \underline{g} &= \left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Gamma_{\underline{f}}\right) * \left(\Delta_{\tau_{\underline{g}}}^{\kappa_{\underline{g}}} * \Gamma_{\underline{g}}\right) && \text{see Eq. 7,} \\ &= \left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Delta_{\tau_{\underline{g}}}^{\kappa_{\underline{g}}}\right) * \left(\Gamma_{\underline{f}} * \Gamma_{\underline{g}}\right) && \text{see Remark 3,} \\ &= \left(\Delta_{\tau_{\underline{f}}+\tau_{\underline{g}}}^{\kappa_{\underline{f}}+\kappa_{\underline{g}}}\right) * \left(\Gamma_{\underline{f}} \oplus \Gamma_{\underline{g}}\right) && \text{see Eq. 8,} \\ &= \Delta_{\tau_{\underline{f}*\underline{g}}}^{\kappa_{\underline{f}*\underline{g}}} * \Gamma_{\underline{f}*\underline{g}}. \end{aligned} \tag{15}$$

According to Theorem 3, the sum of concave functions $\Gamma_{\underline{f}}$ and $\Gamma_{\underline{g}}$ belongs to $\mathscr{F}_{acv}$. Hence, $\underline{f} * \underline{g}$ is also a function of $\mathscr{F}_{acv}$ with $\Gamma_{\underline{f}*\underline{g}} = \Gamma_{\underline{f}} \oplus \Gamma_{\underline{g}}$ and $\Delta_{\tau_{\underline{f}*\underline{g}}}^{\kappa_{\underline{f}*\underline{g}}} = \Delta_{\tau_{\underline{f}}+\tau_{\underline{g}}}^{\kappa_{\underline{f}}+\kappa_{\underline{g}}}$. Let us note that a similar result can be found in Schmitt and Zdarsky (2006, Lemma 2) and in Pandit et al. (2006, Theorem 3.2). □

Proposition 6 *If $\underline{f}$ and $\underline{g} \in \mathscr{F}_{acx}$, then $\underline{f} * \underline{g} \in \mathscr{F}_{acx}$.*

Proof According to Le Boudec and Thiran (2001, Theorem 3.1.6) and Bouillard et al. (2008, Theorem 5), if a and b are convex functions of $\mathscr{F}_{acx}$, so is $a * b$. Moreover, the computation of $a * b$ is obtained by putting end-to-end the different linear pieces of a and b, sorted by nondecreasing slopes. □

Proposition 7 (Inclusion function $[*]$) *Let* $\mathbf{f} = [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ *and* $\mathbf{g} = [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}}$ *be two containers of* $\mathbf{F}$ *given in their canonical forms. The operation denoted* $[*]$ *and defined by:*

$$\mathbf{f}[*]\mathbf{g} = [\ \underline{\mathbf{f}[*]\mathbf{g}}\ ,\ \overline{\mathbf{f}[*]\mathbf{g}}\ ] \triangleq [\ \underline{f} * \underline{g}\ ,\ \overline{f} * \overline{g}\ ]_{\mathscr{L}},$$

is an inclusion function for the inf-convolution $*$*, i.e.:*

$$\forall f \in [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}} \text{ and } \forall g \in [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}} \quad \Rightarrow \quad \begin{cases} f * g \ \in \ \mathbf{f}[*]\mathbf{g}, \\ \mathbf{f}[*]\mathbf{g} \ \in \ \mathbf{F}. \end{cases}$$

Proof Proof is found firstly thanks to the order-preserving of operator $*$ where $\forall f \in [\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ and $\forall g \in [\ \underline{g}\ ,\ \overline{g}\ ]_{\mathscr{L}}$: $\underline{f} * \underline{g} \preccurlyeq f * g \preccurlyeq \overline{f} * \overline{g}$; secondly thanks to Propositions 5 and 6 where $\underline{f} * \underline{g} \in \mathscr{F}_{acv}$ and $\overline{f} * \overline{g} \in \mathscr{F}_{acx}$. Finally, for the asymptotic slopes $\sigma(\underline{f} * \underline{g})$ and $\sigma(\overline{f} * \overline{g})$:

- thanks to Proposition 1 and to Eq. 15, $\sigma(\underline{f} * \underline{g}) = \sigma(\Gamma_{\underline{f}} \oplus \Gamma_{\underline{g}})$. Since $\Gamma_{\underline{f}} \oplus \Gamma_{\underline{g}} = \min(\Gamma_{\underline{f}}, \Gamma_{\underline{g}})$, that $\sigma(\Gamma_{\underline{f}}) = \sigma(\underline{f})$ and that $\sigma(\Gamma_{\underline{g}}) = \sigma(\underline{g})$, then $\sigma(\underline{f} * \underline{g}) = \min(\sigma(\underline{f}), \sigma(\underline{g}))$,
- thanks to the proof of Proposition 6, the computation of $\overline{f} * \overline{g}$ is obtained by putting end-to-end the different linear pieces of $\overline{f}$ and $\overline{g}$. Since $\overline{f}$ and $\overline{g}$ are ultimately affine, this handling stops when the lowest asymptotic slope between $\overline{f}$ and $\overline{g}$ is reached, i.e. until $\min(\sigma(\overline{f}), \sigma(\overline{g}))$.

Therefore, since $\sigma(\underline{f}) = \sigma(\overline{f})$ and $\sigma(\underline{g}) = \sigma(\overline{g})$, then $\sigma(\underline{f} * \underline{g}) = \min(\sigma(\underline{f}), \sigma(\underline{g})) = \min(\sigma(\overline{f}), \sigma(\overline{g})) = \sigma(\overline{f} * \overline{g})$. □

Again, this inclusion function $[*]$ does not necessarily provide a canonical result. The canonical form of $\mathbf{f}[*]\mathbf{g}$ is given by:

$$[\, \mathscr{C}_{cv}(\underline{\mathbf{f}[*]\mathbf{g}} \oplus \Omega_{\overline{\mathbf{f}[*]\mathbf{g}}})\,,\ \overline{\mathbf{f}[*]\mathbf{g}}\,]_{\mathscr{L}} = [\, \mathscr{C}_{cv}((\underline{f} * \underline{g}) \oplus \Omega_{\overline{f}*\overline{g}})\,,\ \overline{f}*\overline{g}\,]_{\mathscr{L}}.$$

Proposition 8 (Algorithmic complexity of $[*]$) *Let $N_{\underline{f}}$, $N_{\overline{f}}$, $N_{\underline{g}}$ and $N_{\overline{g}}$ be the number of extremal points of respectively $\underline{f}$, $\overline{f}$, $\underline{g}$ and $\overline{g}$. The computation of $\mathbf{f}[*]\mathbf{g}$ is of linear complexity, namely $\mathscr{O}(N_{\underline{f}} + N_{\overline{f}} + N_{\underline{g}} + N_{\overline{g}})$.*

Proof According to the proof of Proposition 6, the computation of $\overline{f} * \overline{g}$ is obtained by putting end-to-end the different linear pieces of $\overline{f}$ and $\overline{g}$, sorted by nondecreasing slopes. Hence, the computation only requires a simple scan of functions[17]. Regarding the lower bound, the inf-convolution is defined as in Eq. 15 by:

$$\underline{f} * \underline{g} = \left(\Delta_{\tau_{\underline{f}}+\tau_{\underline{g}}}^{\kappa_{\underline{f}}+\kappa_{\underline{g}}}\right) * \left(\Gamma_{\underline{f}} \oplus \Gamma_{\underline{g}}\right),$$

it is enough to make the minimum of the concave parts with a linear complexity, and the shift in the plane by the elementary function $\Delta_{\tau_{\underline{f}}+\tau_{\underline{g}}}^{\kappa_{\underline{f}}+\kappa_{\underline{g}}}$ in a constant time, namely in $\mathscr{O}(1)$. □

4.4 Inclusion function of the subadditive closure: $^{[\star]}$

The subadditive closure * is order-preserving, hence $\underline{f}^{\star} \preccurlyeq \overline{f}^{\star}$.

Lemma 4 (Asymptotic slope of $\mathscr{C}_{vx}(\overline{f}^{\star})$) *The asymptotic slope of $\mathscr{C}_{vx}(\overline{f}^{\star})$ is given below:*

$$\sigma(\mathscr{C}_{vx}(\overline{f}^{\star})) = \min\left(\sigma(\overline{f}), \min_{i=0}^{n}\left(\frac{k_i}{t_i}\right)\right),$$

where pairs (t_i, k_i) are the n extremal points of $\overline{f}$.

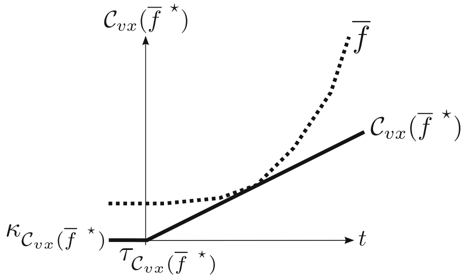
Proof Thanks to the characteristics of both the convex approximation and the subadditive closure, function $\mathscr{C}_{vx}(\overline{f}^{\star})$ belongs to the set $\mathscr{F}_{\text{acx}}$ with only one extremal point. Hence, according to Property 5, $\mathscr{C}_{vx}(\overline{f}^{\star}) = \left(\Delta_{\tau_{\mathscr{C}_{vx}(\overline{f}^{\star})}}^{\kappa_{\mathscr{C}_{vx}(\overline{f}^{\star})}}\right) * g$ where $\kappa_{\mathscr{C}_{vx}(\overline{f}^{\star})} = 0$, $\tau_{\mathscr{C}_{vx}(\overline{f}^{\star})} = 0$ and g is a half line with $\sigma(\mathscr{C}_{vx}(\overline{f}^{\star}))$ as its slope (see Fig. 12), then, the slope of g comes either from the computations k_i/t_i or from $\sigma(\overline{f})$. □

Proposition 9 (Inclusion function $^{[\star]}$) *Let $\mathbf{f} = [\, \underline{f}\,,\ \overline{f}\,]_{\mathscr{L}}$ be a container of* $\mathbf{F}$. *The operation denoted $^{[\star]}$ and defined by:*

$$\mathbf{f}^{[\star]} = [\, \underline{\mathbf{f}^{[\star]}}\,,\ \overline{\mathbf{f}^{[\star]}}\,],$$

[17] The sorting of the function slopes is assumed to be made by the data structure used for their representation.

Fig. 12 Upper bound $\overline{f}$ of a container $\mathbf{f} \in \mathbf{F}$ and the convex approximation of its subadditive closure $\mathscr{C}_{vx}(\overline{f}^{\,\star})$



with $\underline{\mathbf{f}^{[\star]}} \triangleq \bigoplus_{i=0}^{n} \mathscr{C}_{cv}(\Delta_{t_i}^{k_i\star}) \oplus \mathscr{C}_{cv}\left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \left(\mathscr{C}_{cv}(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star}) \oplus \Gamma_{\underline{f}}\right)\right)$ *and* $\overline{\mathbf{f}^{[\star]}} \triangleq \mathscr{C}_{vx}(\overline{f}^{\,\star})$, *is an inclusion function for the subadditive closure* $\star$, *i.e.:*

$$\forall f \in [\,\underline{f}\,,\ \overline{f}\,]_{\mathscr{L}} \quad \Rightarrow \quad \begin{cases} f^\star \in \mathbf{f}^{[\star]}, \\ \mathbf{f}^{[\star]} \in \mathbf{F}, \end{cases}$$

where pairs (t_i, k_i) *of* $\underline{\mathbf{f}^{[\star]}}$ *are the n extremal points of* $\overline{f}$, $\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}}$ *is the elementary function of* $\underline{f}$ *and* $\Gamma_{\underline{f}}$ *is the concave part of* $\underline{f}$ *(see Proposition 1).*

Proof Firstly, since $\overline{\mathbf{f}^{[\star]}} = \mathscr{C}_{vx}(\overline{f}^{\,\star})$, then $\overline{\mathbf{f}^{[\star]}} \in \mathscr{F}_{acx}$. Moreover, $\forall f \in [\,\underline{f}\,,\ \overline{f}\,]_{\mathscr{L}}$, $f \preccurlyeq \overline{f} \Rightarrow f^\star \preccurlyeq \overline{f}^{\,\star} \Rightarrow f^\star \preccurlyeq \mathscr{C}_{vx}(f^\star) \preccurlyeq \mathscr{C}_{vx}(\overline{f}^{\,\star})$ so $f^\star \preccurlyeq \overline{\mathbf{f}^{[\star]}}$.

Secondly, for the computation of $\underline{\mathbf{f}^{[\star]}}$, contrary to inclusion functions $[\oplus]$ and $[*]$, Proposition 9 does not perform the computation of $\mathbf{f}^{[\star]}$ with the canonical form of the container but with the following interval:

$$\mathbf{f} = [\,\underline{f} \oplus \Omega_{\overline{f}}\,,\ \overline{f}\,].$$

Let us recall that $\underline{f} \oplus \Omega_{\overline{f}}$ is the smallest element of the container. Hence, if $f \in [\,\underline{f}\,,\ \overline{f}\,]_{\mathscr{L}}$, then $f \in [\,\underline{f} \oplus \Omega_{\overline{f}}\,,\ \overline{f}\,]$ and $\underline{f} \oplus \Omega_{\overline{f}} \preccurlyeq f \Rightarrow (\underline{f} \oplus \Omega_{\overline{f}})^\star \preccurlyeq f^\star$. The development of $(\underline{f} \oplus \Omega_{\overline{f}})^\star$ is detailed below:

$$\begin{aligned}
(\underline{f} \oplus \Omega_{\overline{f}})^\star &= \left(\left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Gamma_{\underline{f}}\right) \oplus \left(\bigoplus_{i=0}^{n} \Delta_{t_i}^{k_i}\right)\right)^\star && \text{see Eqs. 7 and 13,} \\
&= \left(\Delta_{t_0}^{k_0} \oplus \Delta_{t_1}^{k_1} \oplus \ldots \oplus \Delta_{t_n}^{k_n} \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Gamma_{\underline{f}}\right)^\star, && \\
&= \left(\Delta_{t_0}^{k_0} \oplus \Delta_{t_1}^{k_1} \oplus \ldots \oplus \Delta_{t_n}^{k_n} \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Gamma_{\underline{f}}^{\star}\right)^\star && \text{see Eq. 9,} \\
&= \Delta_{t_0}^{k_0\star} * \Delta_{t_1}^{k_1\star} * \ldots * \Delta_{t_n}^{k_n\star} * \left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Gamma_{\underline{f}}^{\star}\right)^\star && \text{see Eq. 3,} \\
&= \Delta_{t_0}^{k_0\star} * \Delta_{t_1}^{k_1\star} * \ldots * \Delta_{t_n}^{k_n\star} * \left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star} * \Gamma_{\underline{f}}\right) && (16) \\
& && \text{see Eqs. 2, 3 and 9.}
\end{aligned}$$

Then, by introducing concave approximations $\mathscr{C}_{cv}$ in Eq. 16, we will approach this computation from below (indeed $\mathscr{C}_{cv}(f) \preccurlyeq f$) that so becomes:

$$\mathscr{C}_{cv}\left(\Delta_{t_0}^{k_0\star}\right) * \mathscr{C}_{cv}\left(\Delta_{t_1}^{k_1\star}\right) * \ldots * \mathscr{C}_{cv}\left(\Delta_{t_n}^{k_n\star}\right) * \mathscr{C}_{cv}\left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \mathscr{C}_{cv}\left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star}\right) * \Gamma_{\underline{f}}\right). \quad (17)$$

Furthermore, the following functions are closed for the subadditive closure operation:

– the concave approximation of the subadditive closure of elementary functions: $\mathscr{C}_{cv}(\Delta_{t_i}^{k_i\star})$ (see Fig. 13),
– the concave approximation of a function containing e: $\mathscr{C}_{cv}(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \mathscr{C}_{cv}(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star}) * \Gamma_{\underline{f}})$,
– the concave function: $\Gamma_{\underline{f}}$.

So, by applying Theorem 3, the definition of the lower bound $\underline{\mathbf{f}}^{[\star]}$ is given by:

$$\begin{aligned}\underline{\mathbf{f}}^{[\star]} &\triangleq \mathscr{C}_{cv}\left(\Delta_{t_0}^{k_0\star}\right) \oplus \mathscr{C}_{cv}\left(\Delta_{t_1}^{k_1\star}\right) \oplus \ldots \oplus \mathscr{C}_{cv}\left(\Delta_{t_n}^{k_n\star}\right) \oplus \mathscr{C}_{cv}\left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \left(\mathscr{C}_{cv}\left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star}\right) \oplus \Gamma_{\underline{f}}\right)\right), \\ &= \bigoplus_{i=0}^{n} \mathscr{C}_{cv}\left(\Delta_{t_i}^{k_i\star}\right) \oplus \mathscr{C}_{cv}\left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \left(\mathscr{C}_{cv}\left(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star}\right) \oplus \Gamma_{\underline{f}}\right)\right). \end{aligned} \tag{18}$$

Since $(\underline{f} \oplus \Omega_{\overline{f}})^\star \preccurlyeq f^\star$ and since the concave approximations applied to $(\underline{f} \oplus \Omega_{\overline{f}})^\star$ approximate the functions from below ($\mathscr{C}_{cv}(f) \preccurlyeq f$), then $\underline{\mathbf{f}}^{[\star]} \preccurlyeq (\underline{f} \oplus \Omega_{\overline{f}})^\star \preccurlyeq f^\star$. The function $\underline{\mathbf{f}}^{[\star]}$ is composed of sums of concave functions, therefore $\underline{\mathbf{f}}^{[\star]} \in \mathscr{F}_{acv}$.

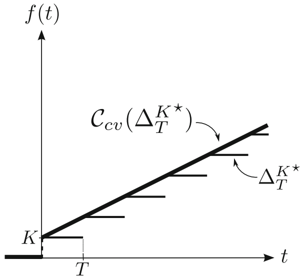
Lastly, regarding the asymptotic slopes, let us first deal with the one of $\underline{\mathbf{f}}^{[\star]}$. The subadditive closure of elementary functions $\Delta_{t_i}^{k_i\star}$ (from $\overline{f}$) provides an ultimately pseudo-periodic function of $\mathscr{F}_{cp}$ (see Remark 4) with $\sigma(\Delta_{t_i}^{k_i\star}) = k_i/t_i$ as asymptotic slope. Moreover, according to Cottenceau et al. (1998–2006), let f and g be two functions of $\mathscr{F}_{cp}$, then $\sigma(f * g) = \min(\sigma(f), \sigma(g))$. Finally, $\sigma(\Gamma_{\underline{f}}) = \sigma(\underline{f}) = \sigma(\overline{f})$ (see Proposition 1 and Definition 19). So, the asymptotic slope of $\underline{\mathbf{f}}^{[\star]}$ is given by:

$$\sigma(\underline{\mathbf{f}}^{[\star]}) = \sigma((\underline{f} \oplus \Omega_{\overline{f}})^\star) = \min\left(\sigma(\overline{f}), \min_{i=0}^{n}\left(\frac{k_i}{t_i}\right)\right),$$

where pairs (t_i, k_i) are the n extremal points of $\overline{f}$. According to Lemma 4, this asymptotic slope is the same as the one of $\overline{\mathbf{f}}^{[\star]}$, i.e. $\sigma(\underline{\mathbf{f}}^{[\star]}) = \sigma(\overline{\mathbf{f}}^{[\star]})$. □

To conclude, the algorithm complexity of the computation of these bounds are given below.

Fig. 13 Concave approximation of the subadditive closure of an elementary function: $\mathscr{C}_{cv}(\Delta_T^{K\star}) \in \mathscr{F}_{acv}$.



Proposition 10 (Algorithmic complexity of $\overline{\mathbf{f}^{[\star]}}$) *Let $N_{\overline{f}}$ be the extremal point number of $\overline{f}$. The computation of $\overline{\mathbf{f}^{[\star]}}$ is of linear complexity, namely $\mathscr{O}(N_{\overline{f}})$.*

Proof According to Lemma 4, the computation of $\overline{\mathbf{f}^{[\star]}}$ only requires some research of its asymptotic slope by checking the $N_{\overline{f}}$ extremal points of $\overline{f}$ and the asymptotic slope $\sigma(\overline{f})$. □

Remark 12 One can see that in the proof of Proposition 9, the computation of $\underline{\mathbf{f}^{[\star]}}$ does not come from Eq. 17 but from Eq. 18 with concave approximations. These approximations are necessary to obtain an efficient algorithm for this inclusion function of the subadditive closure. Indeed, the computation of $(\underline{f} \oplus \Omega_{\overline{f}})^\star$ involves inf-convolutions of ultimately pseudo-periodic functions of $\mathscr{F}_{cp}$ that are very memory- and time-consuming. Formally, let $f = \Delta_{t_0}^{k_0\,\star}$ and $f' = \Delta_{t_1}^{k_1\,\star}$ be two subadditive closures of elementary functions with $\sigma(f) < \sigma(f')$. These functions are stair case functions of the set $\mathscr{F}_{cp}$, and since they have different asymptotic slopes, the inf-convolution of f by f' is given by:

$$f * f' = \left(e \oplus \Delta_{t_1}^{k_1} \oplus \ldots \oplus \Delta_{(K-1)t_1}^{(K-1)k_1}\right) * \left(\Delta_{t_0}^{k_0}\right)^\star,$$

with K a positive integer conditioning from which moment function f is permanently above function f' (see Cottenceau et al. (1998–2006) and Le Corronc (2011, Theorem A.28)). The algorithm complexity of this computation is linear depending on the size of K, namely in $\mathscr{O}(K)$, but without conditions on how large the value of K is. The algorithm complexity of this computation can not be controlled and the tentatives undertaken in order to achieve simple and efficient computations become useless by using this method.

Proposition 11 (Algorithmic complexity of $\underline{\mathbf{f}^{[\star]}}$) *Let n be the number of elementary functions $\Delta_{t_i}^{k_i}$ of $(\underline{f} \oplus \Omega \overline{f})$ and $N_{\Gamma_{\underline{f}}}$ be the number of extremal points of $\Gamma_{\underline{f}}$. The computation of $\underline{\mathbf{f}^{[\star]}}$ is in $\mathscr{O}((n + N_{\Gamma_{\underline{f}}}) log(n + N_{\Gamma_{\underline{f}}}))$.*

Proof In order to prove this proposition, we will divide Eq. 18 into two parts.

– Firstly, let us consider $\bigoplus_{i=0}^{n} \mathscr{C}_{cv}(\Delta_{t_i}^{k_i\,\star})$.

According to Proposition 4, the sum of two functions of $\mathscr{F}_{acv}$ is of linear complexity. But, since the computation of $\underline{\mathbf{f}^{[\star]}}$ needs to make the sum of n functions, the complexity will be normally extended to n^2. Nevertheless, thanks to algorithms such as "divide and conquer" where recursion is used in order to break down a problem into sub-problems until these become simple enough to be solved directly, this complexity can be reduced to a quasi-linear problem. Then, if a two by two min of $\mathscr{C}_{cv}(\Delta_{t_i}^{k_i\,\star})$ is made, this part of the equation is solved in $\mathscr{O}(n \log(n))$.

– Secondly, for the part $\mathscr{C}_{cv}\left(e \oplus \Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}} * \left(\mathscr{C}_{cv}(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\,\star}) \oplus \Gamma_{\underline{f}}\right)\right)$.

The sum of $\mathscr{C}_{cv}(\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}\star})$ and $\Gamma_{\underline{f}}$ is made in linear time, namely $\mathscr{O}(N_{\Gamma_{\underline{f}}})$ where $N_{\Gamma_{\underline{f}}}$ is the number of extremal points of $\Gamma_{\underline{f}}$. Then, the shift of $\Delta_{\tau_{\underline{f}}}^{\kappa_{\underline{f}}}$ is in $\mathscr{O}(1)$ as well as the addition of e. The concave approximation does not modify this result since its computation is in linear time (see Proposition 2). So, the complexity of this part of equation is in $\mathscr{O}(N_{\Gamma_{\underline{f}}})$.

Therefore, the complexity of evaluating the expression in Eq. 18 is in $\mathscr{O}((n + N_{\Gamma_{\underline{f}}}) log(n + N_{\Gamma_{\underline{f}}}))$. $\square$

To summarize, this section has shown that all inclusion functions $[\diamond] \in \{[\oplus], [*], [^\star]\}$ applied to containers of **F** can be computed with a linear complexity for the sum and the inf-convolution, and a quasi-linear complexity for the subadditive closure. Of course, the results of computations also belong to the set **F**. These advantageous algorithmic complexities are possible only thanks to the convex characteristics of the bounds of the containers.

It is also important to keep in mind that in all these proposed inclusion functions, the upper bound $\overline{f}$ is the canonical representative of the equivalence class $[\ \overline{f}\ ]_{\mathscr{L}}$ of the elements of the container. Therefore, one can see these operations on the containers of **F** as operations on the semiring $\mathscr{F}_{cp}{}_{/\mathscr{L}}$, for which the handled equivalence classes are restricted: a container $[\ \underline{f}\ ,\ \overline{f}\ ]_{\mathscr{L}}$ only describes the elements of the equivalence class of $[\ \overline{f}\ ]_{\mathscr{L}}$ greater than $\underline{f}$. Finally, since in the container the equivalence class of the exact system is also preserved ($[\ \overline{f}\ ]_{\mathscr{L}} = [f]_{\mathscr{L}}$), some of its important characteristics are kept such as the asymptotic slope and the extremal points of the upper bound that really belong to the exact system.

5 Tests and applications

The container and the algorithms of inclusion functions introduced in this paper have been implemented in a toolbox called *ContainerMinMaxGD*. It is a set of C++ classes, available at the following address: http://www.istia.univ-angers.fr/~euriell.lecorronc/Recherche/softwares.php.

Below, several tests of this toolbox are proposed. They have been carried out by using a computer with the following configuration: 2.9 GB of memory and 4 processors (Intel(R) Core(TM) i7 CPU L640 @ 2.13 GHz).

5.1 Pessimism of computations and gain in memory consumption

In order to evaluate the pessimism introduced by the inclusion functions and the performance according to memory consumption, we consider two containers **S** and **R** of **F**. The former is the container built from the exact computation $S = f \diamond g$ (with $f, g \in \mathscr{F}_{cp}$ two exact systems), that is $\mathbf{S} = [\ \mathscr{C}_{cv}(S)\ ,\ \mathscr{C}_{vx}(S)\ ]_{\mathscr{L}}$. The latter contains the result of the computation $\mathbf{R} = \mathbf{f}[\diamond]\mathbf{g}$ (with $\mathbf{f}, \mathbf{g} \in \mathbf{F}$ two containers that respectively contain f and g). So $S \in \mathbf{S} \subset \mathbf{R}$.

A first criterion deals with pessimism due to the approximated computations, in other words, how close are the approximated computations to the real ones? The inclusion functions handle containers that are composed of convex and concave approximations hence, to evaluate the pessimism they introduced, we will compare $\mathbf{S}$, the container of S, and $\mathbf{R}$, the result of the inclusion functions.

Definition 24 (Pessimism of computations) The pessimism between $\mathbf{S}$ and $\mathbf{R}$ is defined by the formula:

$$\frac{\Sigma_{\mathbf{R},\mathbf{S}}}{\Sigma_{\mathbf{R}}},$$

where $\Sigma_{\mathbf{R},\mathbf{S}}$ is the maximal distance between $\mathbf{R}$ and $\mathbf{S}$, and $\Sigma_{\mathbf{R}}$ is the maximal uncertainty of $\mathbf{R}$. These quantities are illustrated in Fig. 14.

This indicator can be completed by another criterion about the gain in memory consumption we can do with these approximated computations. Indeed, as shown previously, the complexity of the computations depends of the number of points in the containers. We can thereby observe the difference of the number of points between the exact system S and the container $\mathbf{R}$ obtained with inclusion functions.

Definition 25 (Ratio of memory consumption saved) The memory consumption saved between S and $\mathbf{R}$ is evaluated by the following ratio:

$$1 - \frac{N_{\mathbf{R}}}{N_S},$$

where $N_{\mathbf{R}}$ is the number of points of the container $\mathbf{R}$, that is the number of extremal points of $\underline{R}$ and $\overline{R}$, and N_S is the number of points of $S = f \diamond g$, this being the number of its elementary functions until the periodic part plus the number of points necessary for one periodicity has been reached. These quantities are also illustrated in Fig. 14, with circles for the points of S (the last two circle points are for the periodicity in this example) and triangles for the points of $\mathbf{R}$. In this case, the ratio is equal to $1 - 4/8 = 50\ \%$.

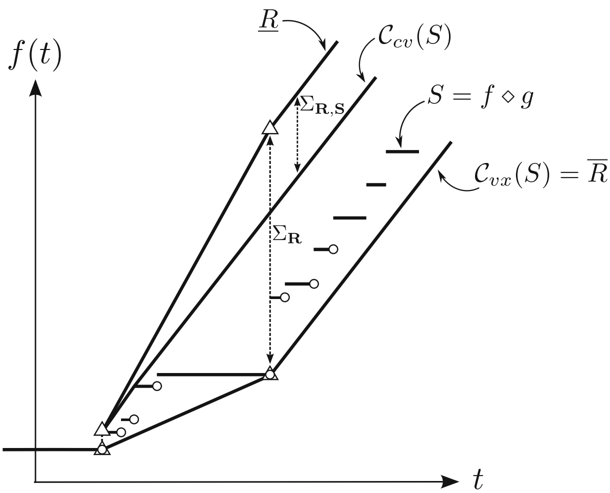


Fig. 14 Pessimism and memory consumption between the approximated computations $\mathbf{R}$ and the exact ones $S \in \mathbf{S}$

Remark 13 One can note that since the bound $\overline{f}$ of a container is the greatest element of $[f]_{\mathscr{L}}$ (the equivalent class of f modulo the Legendre–Fenchel transform), the pessimism between **R** and **S** does not come from this upper approximation: $\overline{R} = \mathscr{C}_{vx}(S)$ (see Fig. 14). However, the error between these two containers comes from the inequalities obtained with concave approximations: $\underline{R} \preccurlyeq \mathscr{C}_{cv}(S)$ (see Remark 8).

Below we propose three examples to analyze the evolution of these two criteria.

5.2 Examples of application

5.2.1 Computation of a transfer function

Firstly, we give first an example of the computation of a transfer function by comparing the toolbox ContainerMinMaxGD versus the toolbox MinMaxGD where functions of $\mathscr{F}_{cp}$ are used.

For this example, let us consider a SISO system ($p = q = 1$) described by the following state representation:

$$\begin{cases} x = A * x \oplus B * u, \\ y = C * x. \end{cases}$$

Matrices A, B and C are given such that:

$$A = \begin{pmatrix} \Delta^{39}_{19} & \Delta^{45}_{29} & \Delta^{22}_{14} & \Delta^{20}_{40} \\ \Delta^{1}_{23} & \Delta^{27}_{5} & \Delta^{36}_{20} & \Delta^{12}_{1} \\ \Delta^{9}_{48} & \Delta^{43}_{46} & \Delta^{22}_{35} & \Delta^{39}_{32} \\ \Delta^{10}_{6} & \Delta^{27}_{27} & \Delta^{32}_{30} & \Delta^{32}_{9} \end{pmatrix}, \quad B = \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} \quad \text{and} \quad C = (\varepsilon \;\; \varepsilon \;\; \varepsilon \;\; e),$$

where the entries of matrices belong to sub-semiring $(\mathscr{F}_{cp}, \oplus, *)$. Thanks to Theorem 1, transfer matrix $H^{1\times 1}$ of the system is obtained by:

$$H = C * A^{\star} * B.$$

On the one hand, with MinMaxGD, the computations are made from the exact values of A, B and C with classical operations of (min,+) algebra i.e. $\{\oplus, *, ^{\star}\}$. On the other hand, with ContainerMinMaxGD, each entry of matrices is lower and upper approximated by a container belonging to **F**. The handled matrices are $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{F}$ and the performed operations are inclusion functions $\{[\oplus], [*], ^{[\star]}\}$. We can note that entries of matrices are elementary functions, hence there is no uncertainty[18] in these containers: $\mathscr{C}_{cv}(\Delta^K_T) = \mathscr{C}_{vx}(\Delta^K_T) = \Delta^K_T$. Therefore, the computation of transfer matrix **H** is:

$$\mathbf{H} = \mathbf{C}[*]\mathbf{A}^{[\star]}[*]\mathbf{B} = [\; \underline{H} \;,\; \overline{H} \;].$$

[18]The construction of a container **f** from one elementary function Δ^K_T provides two identical bounds $\underline{f} = \overline{f}$ with only one extremal point of (T, K) coordinates, and an infinite asymptotic slope $\sigma(\underline{f}) = \sigma(\overline{f}) = +\infty$.

Table 1 Script files for computations in toolboxes MinMaxGD and ContainerMinMaxGD

// Script for the example // with Scilab/MinMaxGD	// Script for the example // with ContainerMinMaxGD
A = smatrix(4,4);	MCserie A
A(1,1) = series([39,19]);	A(1,1) = (19 39);
A(1,2) = series([45,29]);	A(1,2) = (29 45);
A(1,3) = series([22,14]);	A(1,3) = (14 22);
A(1,4) = series([20,40]);	A(1,4) = (40 20);
A(2,1) = series([1,23]);	A(2,1) = (23 1);
A(2,2) = series([27,5]);	A(2,2) = (5 27);
A(2,3) = series([36,20]);	A(2,3) = (20 36);
A(2,4) = series([12,1]);	A(2,4) = (1 12);
A(3,1) = series([9,48]);	A(3,1) = (48 9);
A(3,2) = series([43,46]);	A(3,2) = (46 43);
A(3,3) = series([22,35]);	A(3,3) = (35 22);
A(3,4) = series([39,32]);	A(3,4) = (32 39);
A(4,1) = series([10,6]);	A(4,1) = (6 10);
A(4,2) = series([27,27]);	A(4,2) = (27 27);
A(4,3) = series([32,30]);	A(4,3) = (30 32);
A(4,4) = series([32,9])	A(4,4) = (9 32)
B = smatrix(4,1);	MCserie B
B(1,1) = series(e)	B(1,1) = (0 0);
	B(4,1) = (eps)
C = smatrix(1,4);	MCserie C
C(1,4) = series(e)	C(1,4) = (0 0)
H = C * stargd(A) * B	MCserie H
	H = C * Star(A) * B

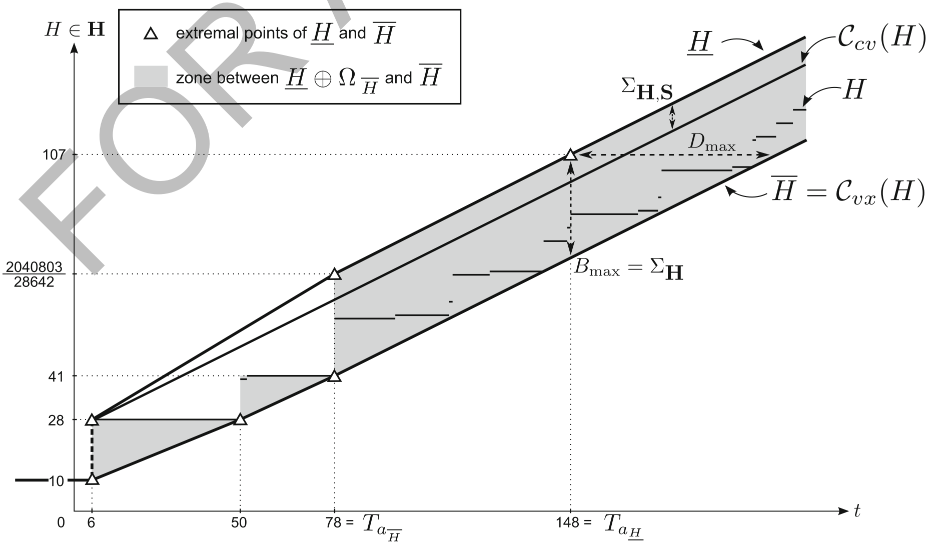


Fig. 15 Transfer matrix $H^{1\times 1} \in \mathscr{F}_{cp}$ and its container $\mathbf{H} = [\,\underline{H}\,,\ \overline{H}\,] \in \mathbf{F}$

Table 1 describes the lines written with toolboxes MinMaxGD and ContainerMinMaxGD, and the results obtained are illustrated in Fig. 15.

First of all, the pessimism of container $\mathbf{H}$ versus the container that approximates the exact system $\mathbf{S} = [\ \mathscr{C}_{cv}(H)\ ,\ \mathscr{C}_{vx}(H)\ ]$ (see Definition 24) is:

$$\frac{\Sigma_{\mathbf{H},\mathbf{S}}}{\Sigma_{\mathbf{H}}} = \frac{8}{31} = 25.8\ \%,$$

i.e. $\mathbf{H}$ is 25.8 % greater than the convex and concave approximations of the exact computation. We can therefore note that the computation with MinMaxGD produces a matrix H with a sum of 80 elementary functions Δ_T^K. Here are some of them (during the transient and the periodic parts) as well as its asymptotic slope:

$$H = \Delta_6^{10} \oplus \Delta_{50}^{28} \oplus \Delta_{52}^{40} \oplus \Delta_{78}^{41} \oplus \ldots \oplus \left(\Delta_{494}^{255} \oplus \ldots \oplus \Delta_{550}^{285}\right) * \Delta_{62}^{31^\star}, \quad \sigma(H) = 31/62.$$

Concerning the elements of the container $\mathbf{H}$, its bounds $\underline{H}$ and $\overline{H}$ have many fewer points than H. Indeed, they have respectively 4 and 3 extremal points given below by their pairs (time, event):

$$\begin{cases} \underline{H} = \{(6, 10)\ ;\ (6^+, 28)\ ;\ \left(78, \dfrac{2040803}{28642}\right)\ ;\ (148, 107)\}, \\ \overline{H} = \{(6, 10)\ ;\ (50, 28)\ ;\ (78, 41)\}. \end{cases}$$

So, according to the number of points, the memory consumption saved (see Definition 25) is:

$$1 - \frac{N_{\mathbf{H}}}{N_H} = 1 - \frac{7}{80} = 91.25\ \%.$$

Then, $\underline{H}$ and $\overline{H}$ and H have the same first elementary function:

$$\Delta_{\tau_{\underline{H}}}^{\kappa_{\underline{H}}} = \Delta_{\tau_{\overline{H}}}^{\kappa_{\overline{H}}} = \Delta_{\tau_H}^{\kappa_H} = \Delta_6^{10},$$

and the asymptotic slope of $\mathbf{H}$ corresponds to the reduced form to that of H:

$$\sigma(\underline{H}) = \sigma(\overline{H}) = 1/2.$$

Finally, the maximal uncertainty $\Sigma_{\mathbf{H}}$ of $\mathbf{H}$ is also provided by the toolbox. Here, we use Definition 22 which gives us:

$$\underline{H}(T_{a_{\underline{H}}}) = 107 > \overline{H}(T_{a_{\overline{H}}}) = 41 \quad \text{and} \quad \max\{T_{a_{\underline{H}}}, T_{a_{\overline{H}}}\} = T_{a_{\underline{H}}} = 148.$$

Hence:

$$\Sigma_{\mathbf{H}} = \{\ D_{\max} = 62\ ,\ B_{\max} = 31\ \}.$$

To conclude this example, let us recall that the exact system H truly belongs to the grey zone i.e. $\underline{H} \oplus \Omega_{\overline{H}} \preccurlyeq H \preccurlyeq \overline{H}$.

5.2.2 Subadditive closure of a matrix

The second example comes from Olsder et al. (1998) and proposes to compute the subadditive closure of a given matrix $A \in (\mathscr{F}_{cp}, \oplus, *)^{10\times 10}$. This matrix is a benchmark with numerous interconnections among its elements when its subadditive closure is computed. Indeed, even if matrix A does not have many elements different from ε, matrix $A^\star$ is full of functions of $\mathscr{F}_{cp}$, elements Δ_T^K of which are given with $T \in [\ 0\ ,\ 975\ ]$ and $K \in [\ 0\ ,\ 21\ ]$. The entries of matrix A that are not function ε are the following ones:

$$\begin{array}{llll} A(1,4) = \Delta_{58}^{1}, & A(2,1) = \Delta_{61}^{2}, & A(2,8) = e, & A(3,2) = \Delta_{81}^{1}, \\ A(3,9) = e, & A(4,3) = \Delta_{86}^{2}, & A(5,8) = \Delta_{58}^{1}, & A(6,4) = e, \\ A(6,5) = \Delta_{61}^{1}, & A(7,6) = \Delta_{35}^{1}, & A(7,10) = e, & A(8,7) = \Delta_{36}^{1}, \\ A(9,3) = e, & A(9,10) = \Delta_{69}^{1}, & A(10,7) = e, & A(10,9) = \Delta_{69}^{2}. \end{array}$$

Then, let $\mathbf{A} \in \mathbf{F}^{10\times 10}$ be the container of A. Again, since matrix A contains only elementary functions, there is no uncertainty between $\mathbf{A}$ and A, i.e. $\mathscr{C}_{cv}(A) = \mathscr{C}_{vx}(A) = \mathbf{A}$. The computation of $\mathbf{A}^{[\star]}$ is made with the ContainerMinMaxGD toolbox whereas the computation of $A^\star$ is made with MinMaxGD. Finally, let $\mathbf{S} \in \mathbf{F}$ be the container that approximates exact matrix $A^\star$, i.e. $\mathbf{S} = [\ \mathscr{C}_{cv}(A^\star)\ ,\ \mathscr{C}_{vx}(A^\star)\ ]_{\mathscr{L}}$.

For this example, we can observe that the average of pessimism of container $\mathbf{A}^{[\star]}$ versus the container approximating the exact system $\mathbf{S}$ is:

$$\frac{\Sigma_{\mathbf{A}^{[\star]},\mathbf{S}}}{\Sigma_{\mathbf{A}^{[\star]}}} = 55.07\ \%.$$

This pessimistic result must be carefully linked to the average of the uncertainty $\Sigma_{\mathbf{A}^{[\star]}}$ of $\mathbf{A}^{[\star]}$ which is:

$$\Sigma_{\mathbf{A}^{[\star]}} = \{\ D_{\max} = 286\ ,\ B_{\max} = 6\ \}.$$

Therefore, even if container $\mathbf{A}^{[\star]}$ is about 55 % larger than $A^\star$, pessimism is only of 3.3 in the event domain and 157.3 in the time domain. Finally, regarding the number of points of $\mathbf{A}^{[\star]}$ and $A^\star$, the average ratio of memory consumption saved is:

$$1 - \frac{N_{\mathbf{A}^{[\star]}}}{N_{A^\star}} = 98\ \%.$$

5.2.3 More generally

In this example, we propose to generalize the computation of pessimism and the gain in memory consumption of our containers. Indeed, here we carry out the subadditive closure of matrices with various sizes, and we observe the average of error as well as the gain in the number of points between the container obtained with inclusion functions and either the container of the result obtained with exact computations for the pessimism, or directly the exact system for the number of points.

Let $A \in (\mathscr{F}_{cp}, \oplus, *)^{n\times n}$ and $\mathbf{A} \in \mathbf{F}^{n\times n}$ be two square matrices. The entries of A are either elementary functions Δ_T^K, where T and K are randomly-chosen integers in interval $[1, 10]$, or function $\varepsilon : t \mapsto +\infty$. Matrix $\mathbf{A}$ is the container of A, that is each entry of $\mathbf{A}$ is the container obtained from the respective entry of A. Hence, there is still no uncertainty in $\mathbf{A}$. Then, let $\mathbf{R}, \mathbf{S} \in \mathbf{F}^{n\times n}$ and $S \in (\mathscr{F}_{cp}, \oplus, *)^{n\times n}$ be three square matrices. The first contains the result of computation $\mathbf{R} = \mathbf{A}^{[\star]}$, and the second is the matrix container built from computation $S = A^\star$, that is $\mathbf{S} = [\ \mathscr{C}_{cv}(S)\ , \mathscr{C}_{vx}(S)\ ]_{\mathscr{L}}$. So $S \in \mathbf{S} \subset \mathbf{R}$. The experiment is carried out 5 times with a matrix size ranging from 2×2 to 60×60.

For these tests, the average of pessimism observed between **R** and **S** is:

$$\frac{\Sigma_{\mathbf{R},\mathbf{S}}}{\Sigma_{\mathbf{R}}} = 27\ \%,$$

that is **R** is 27 % larger than **S**. According to the gain in the number of points between the exact system S and the container **R**, we can observe:

$$1 - \frac{N_{\mathbf{R}}}{N_S} = 71\ \%,$$

that is **R** saves 71 % of space memory in comparison with the exact representation.

5.3 Experimental complexity

A final test of the ContainerMinMaxGD toolbox is about the practical complexity of the computation of the subadditive closure of a square matrix depending on its size.

To this end, let **A** be a square matrix of $\mathbf{F}^{n\times n}$. The entries of **A** are either elementary functions Δ_T^K, where T and K are integers randomly chosen in the interval $[1, 5]$, or the function $\varepsilon : t \mapsto +\infty$. The computation of $\mathbf{A}^{[\star]}$ is made with a size of matrix ranging from 2×2 to 60×60, and the average of the CPU times is noted.

The CPU time for the computation of $\mathbf{A}^{[\star]}$, depending on the size of **A**, is illustrated in Fig. 16. It appears that the practical complexity is approximately in $\mathscr{O}(n^3 \log n)$, with n the size of the matrix. Moreover, we can see that for example, the average CPU time of the computation of a 50×50 matrix is about 200 s.

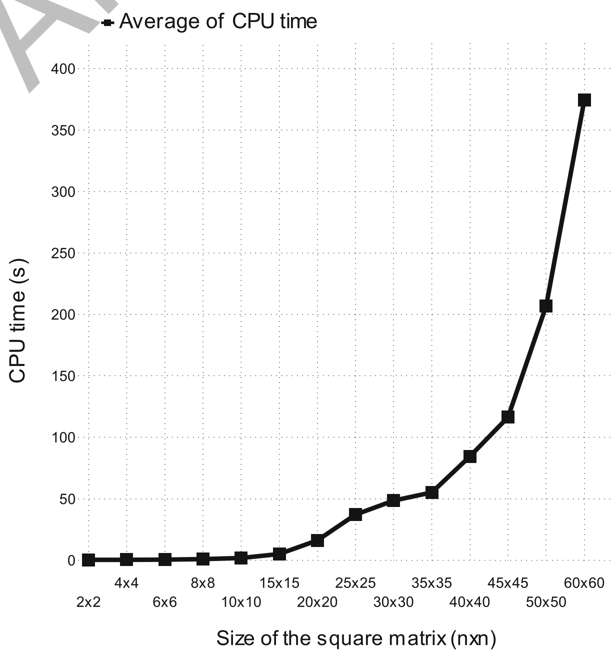


Fig. 16 CPU time of the computation of $\mathbf{A}^{[\star]}$, depending on the size of **A**

6 Conclusion

This paper has focused on the computation of the transfer function h for (min,+)-linear systems. More precisely, since the exact computations can be time and memory consuming, we introduced an approximated approach of the exact system h via a container $\mathbf{h} \in \mathbf{F}$ such that:

$$\mathbf{h} = [\,\underline{h}\,,\,\overline{h}\,]_{\mathscr{L}} = [\,\underline{h}\,,\,\overline{h}\,] \cap [\,\overline{h}\,]_{\mathscr{L}} \quad \text{and} \quad [\,\overline{h}\,]_{\mathscr{L}} = [h]_{\mathscr{L}},$$

where $[h]_{\mathscr{L}}$ is the equivalent class of h modulo the Legendre–Fenchel transform $\mathscr{L}$. The bounds $\underline{h}$ and $\overline{h}$ are two ultimately affine functions with convex characteristics. This work has also been inspired by the set membership approach since the main operations of (min,+) algebra, i.e. the sum, the inf-convolution and the subadditive closure, have been integrated into inclusion functions $[\diamond] \in \{[\oplus], [*], {}^{[\star]}\}$ in which only the bounds of the intervals are handled.

Despite the approximations, since the equivalence class modulo the transform $\mathscr{L}$ of the exact system is preserved, some of its important characteristics are kept, such as the asymptotic slope and the extremal points of the upper bound that really belong to the exact system. Furthermore, the convex characteristics of the bounds of the interval allow us to reduce both algorithm complexity of the computations made over these systems, and the amount of data storage. Indeed, the algorithmic complexity of inclusion functions is linear for the sum and the inf-convolution, and quasi-linear for the subadditive closure.

Finally, the container and its algorithms have been implemented in a toolbox developed in C++, called ContainerMinMaxGD. The proposed tests of this toolbox have demonstrated the performance and the computational advantage of these containers by comparison with exact solutions.

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