Flow Control with (Min,+) Algebra

ISOLA'10

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Motivations

Flow control

- Arrival curve computation
  $\longrightarrow$ delay and backlog constraints
- Window flow control
  $\longrightarrow$ difference between data stream and acknowledgments stream

Outlines

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1. (Min,+) algebra and Network Calculus
   - (Min,+) algebra
   - Network Calculus modelling
   - Performance characteristics
2. Flow control
   - Arrival curve computation
   - Window flow control
3. Conclusions

Theory of (min,+) linear systems [1]

- Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena

[1] F. Baccelli *et al.*: Synchronisation and Linearity. Wiley and sons, 1992.

Theory of (min,+) linear systems [1]

- Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena
- Application areas: communication networks [2] [3], manufacturing systems, transportation systems [4]

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[1] F. Baccelli *et al.*: Synchronisation and Linearity. Wiley and sons, 1992.

[2] CS Chang: Performance guarantees. Springer, 2000.

[3] JY Le Boudec and P. Thiran: Network Calculus. Springer, 2001.

[4] B. Heidergott *et al.*: Max plus at work. Princeton University Press, 2006.

Idempotent semiring

Set $\mathcal{D}$ endowed with two inner operations [a]

- $\oplus \longrightarrow$ associative, commutative, idempotent ($a \oplus a = a$) neutral element ε
- $\otimes \longrightarrow$ associative, distributes over the sum neutral element e

[a] when $\otimes$ is commutative, $\mathcal{D}$ is said commutative

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Order relation

$$a = a \oplus b \quad \Leftrightarrow \quad a \succcurlyeq b$$

Example: idempotent semiring $\overline{\mathbb{R}}_{min}$

$$\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{-\infty, +\infty\}, min, +)$$

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$$\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{-\infty, +\infty\}, min, +)$$

Remark: order relation in $\overline{\mathbb{R}}_{min}$

$$5 \oplus 3 = 3 \quad \Leftrightarrow \quad 3 \succeq 5 \quad \Leftrightarrow \quad 3 \leq 5$$

Commutative idempotent semiring: $\{\mathcal{F}_0, \oplus, *\}$

Set [a] $\mathcal{F}_0$ endowed with

[a] non-decreasing functions $f : \mathbb{R} \mapsto \overline{\mathbb{R}}_{min}$ where $f(t) = 0$ for $t \leq 0$

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Set [a] $\mathcal{F}_0$ endowed with

- $\oplus \longrightarrow$ pointwise minimum
- $* \longrightarrow$ inf-convolution

$$f, g \in \mathcal{F}_0 \quad (f*g)(t) \triangleq \bigoplus_{\tau \geq 0} \{f(\tau) \otimes g(t-\tau)\} = \min_{\tau \geq 0} \{f(\tau) + g(t-\tau)\}$$

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Other operations in $\{\mathcal{F}_0, \oplus, *\}$

- deconvolution

$$(f \not{\circ} g)(t) \triangleq \bigwedge_{\tau \geq 0} \{f(\tau) - g(t - \tau)\} = \max_{\tau \geq 0} \{f(\tau) - g(\tau - t)\}$$

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- subadditive closure

$$f^{\star}(t) \triangleq \bigoplus_{\tau \geq 0} f^{\tau}(t) = \min_{\tau \geq 0} f^{\tau}(t) \quad \text{with} \quad f^{0}(t) = e$$

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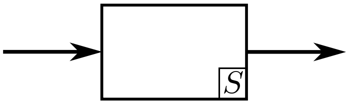
Fixed point theory: $f^\star$ is the optimal solution to $x = f * x \oplus e$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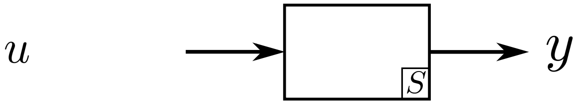
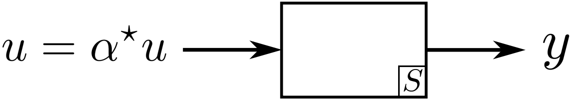


Network elements

- Input and output flows u and y

$$\forall t, u(t) \geq y(t) \quad \Leftrightarrow \quad u \preccurlyeq y$$



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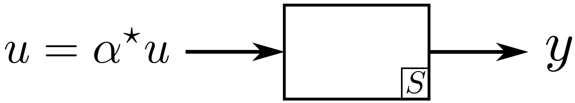
$$\forall t, u(t) \geq y(t) \quad \Leftrightarrow \quad u \preccurlyeq y$$

- Arrival curve $\alpha^\star$

$$u \leq \alpha u \quad \Leftrightarrow \quad u \succcurlyeq \alpha u \quad \Leftrightarrow \quad u = \alpha^\star u$$



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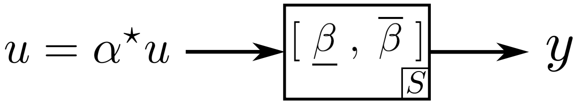
Sketch of proof



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Sketch of proof

- Service curve $[\,\underline{\beta}\,,\,\overline{\beta}\,]$ ([maximum service, minimum service])

$$\underline{\beta} u \;\preccurlyeq\; y \;\preccurlyeq\; \overline{\beta} u \quad \Leftrightarrow \quad y \in [\,\underline{\beta} u\,,\,\overline{\beta} u\,]$$

Delay $d(k)$ (waiting time of the k^{th} paquet) [5] [6]

$$\forall k, d(k)$$

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[6] Max Plus: Second order theory... CDC'91.

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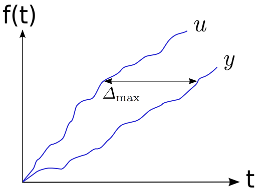

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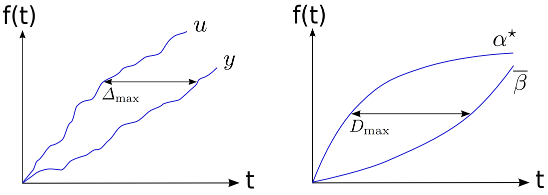

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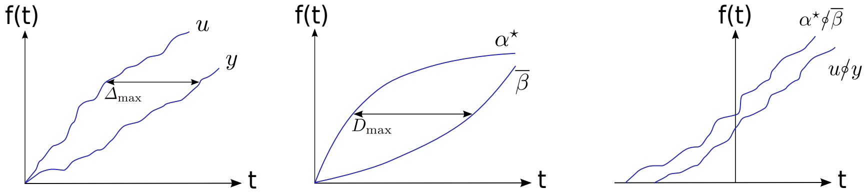

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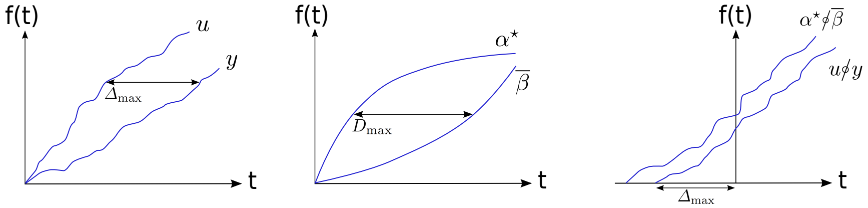

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$$\Delta_{\max} = \inf_{\Delta \geq 0} \{(u \oslash y)(-\Delta) \leq 0\}$$

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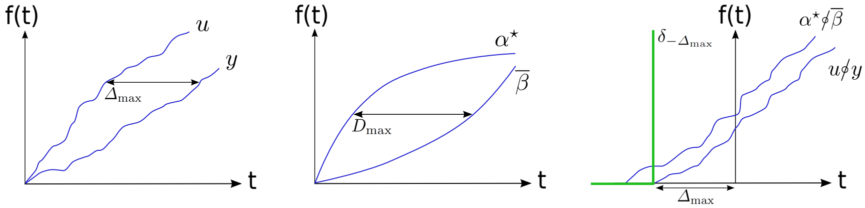

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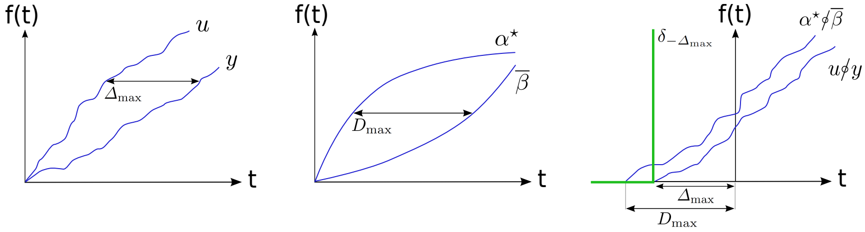

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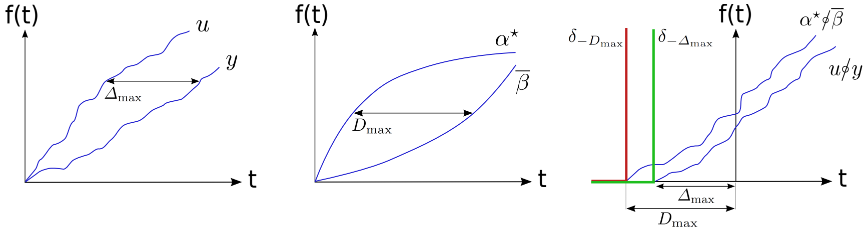

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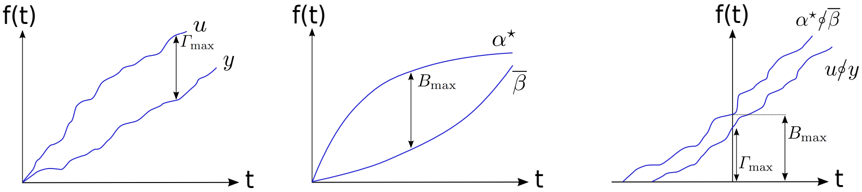
Backlog $b(t)$ (amount of paquets at time t) [7] [8]

$$\forall t, b(t) \quad \leq \quad \Gamma_{\max} \quad \leq \quad B_{\max}$$



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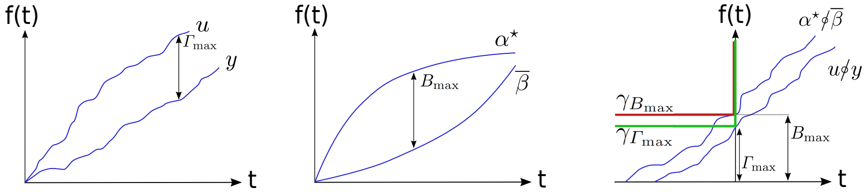

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   - (Min,+) algebra
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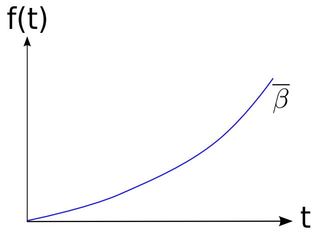
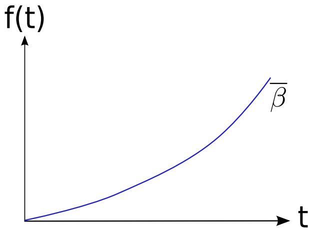
Goal

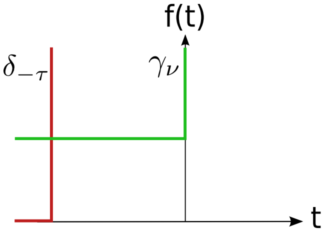
- Minimum service $\overline{\beta}$ known



Goal

- Minimum service $\overline{\beta}$ known
- Fixed worst end-to-end delay τ or backlog ν





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- Minimum service $\overline{\beta}$ known
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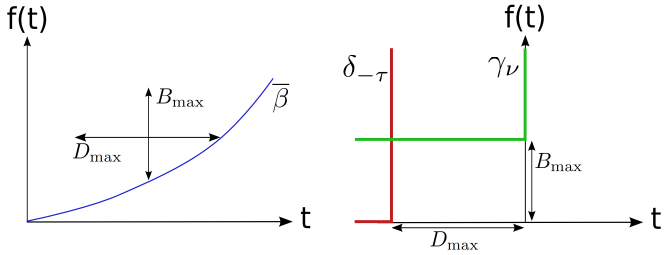
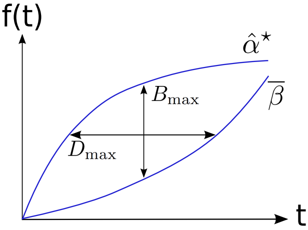
- Upper bounds $D_{\mathsf{max}} = \tau$ and $B_{\mathsf{max}} = \nu$

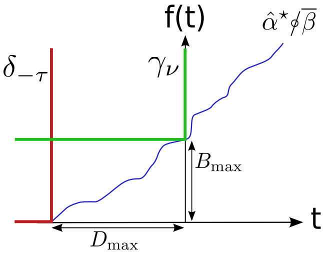


Goal

- Minimum service $\overline{\beta}$ known
- Fixed worst end-to-end delay τ or backlog ν
- Upper bounds $D_{\max} = \tau$ and $B_{\max} = \nu$
- Computation of the minimal constraint $\hat{\alpha}^{\star}$ s.t.

$$\hat{\alpha}^{\star} \oslash \overline{\beta} \succeq \delta_{-\tau} \quad \text{and} \quad \hat{\alpha}^{\star} \oslash \overline{\beta} \succeq \gamma_{\nu}$$





Proposition 1 (time performance)

$$\hat{\alpha}^{\star}$$

[9] CA. Maia *et al.*: Optimal closed-loop control... Automatic Control, 2003.

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Sketch of proof

- $\alpha^\star \not\phi \overline{\beta} \succcurlyeq \delta_{-\tau} \quad \Leftrightarrow \quad \alpha^\star \succcurlyeq \delta_{-\tau}\overline{\beta}$

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- $a^{\star} = \bigwedge\{x^{\star} \mid x = x^{\star}, x^{\star} \succcurlyeq a\}$ (see lemma 1 in [9])

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Proposition 2 (data performance)

$$\hat{\alpha}^\star = \bigwedge \{\alpha^\star \mid \alpha^\star \succcurlyeq \gamma_\nu\overline{\beta}\} = (\gamma_\nu\overline{\beta})^\star$$

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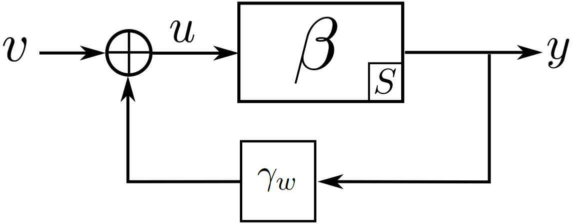
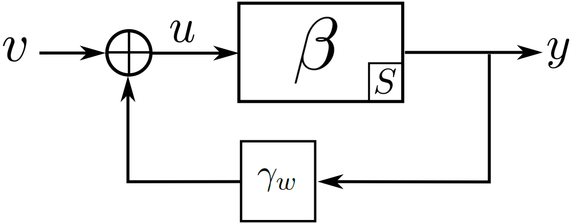
Window Flow Control

- Communication network S



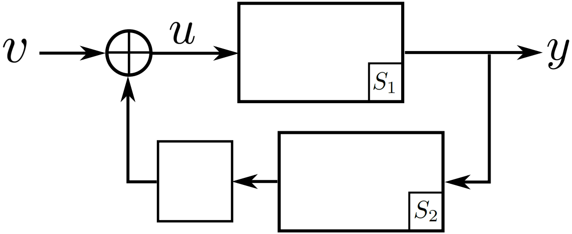
Window Flow Control

- Communication network S
- Window size $w \rightarrow \gamma_w$
- Limit the amount of data



Goal

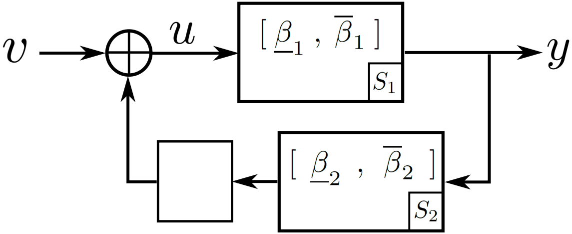
- Data stream S_1 and acknowledgments stream S_2 [10]



[10] R Agrawal *et al.*: Performance bounds... Transactions on Networking, 1999.

Goal

- Data stream S_1 and acknowledgments stream S_2 [10]
- Interval of service $[\ \underline{\beta}\ ,\ \overline{\beta}\ ]$



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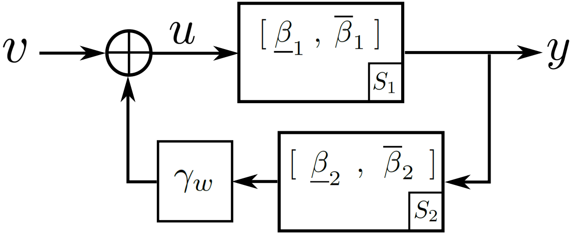
Goal

- Data stream S_1 and acknowledgments stream S_2 [10]
- Interval of service $[\ \underline{\beta}\ ,\ \overline{\beta}\ ]$
- Computation of the minimal window size $\hat{\gamma}_w$ s.t.

closed-loop behavior (S_1 and S_2) $=$ open-loop behavior (S_1)



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Interval of service [$\underline{\text{maximum service}}$, $\overline{\text{minimum service}}$]

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- Open-loop system (S_1)

$$[\ \underline{\beta}_1\ ,\ \overline{\beta}_1\ ]$$

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- Open-loop system (S_1)

$$[\ \underline{\beta}_1\ ,\ \overline{\beta}_1\ ]$$

- Closed-loop system (S_1 and S_2)

$$[\ \underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star\ ,\ \overline{\beta}_1(\gamma_w \overline{\beta}_2 \overline{\beta}_1)^\star\ ]$$

Proposition 3

$$\hat{\gamma_w} = \bigoplus \{\gamma_w \mid \underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \overline{\beta}_1(\gamma_w \overline{\beta}_2 \overline{\beta}_1)^\star \preccurlyeq \overline{\beta}_1\}$$

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$$\begin{aligned}\hat{\gamma_w} &= \bigoplus\{\gamma_w \mid \underline{\beta}_1(\gamma_w\underline{\beta}_2\underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \overline{\beta}_1(\gamma_w\overline{\beta}_2\overline{\beta}_1)^\star \preccurlyeq \overline{\beta}_1\} \\ &= (\underline{\beta}_1 \mathbin{\lozenge\!\!\backslash} \underline{\beta}_1 \mathbin{\phi\!\!/} (\underline{\beta}_2\underline{\beta}_1)) \ \wedge \ (\overline{\beta}_1 \mathbin{\lozenge\!\!\backslash} \overline{\beta}_1 \mathbin{\phi\!\!/} (\overline{\beta}_2\overline{\beta}_1))\end{aligned}$$

Proposition 3

$$\begin{aligned}\hat{\gamma_w} &= \bigoplus\{\gamma_w \mid \underline{\beta}_1(\gamma_w\underline{\beta}_2\underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \overline{\beta}_1(\gamma_w\overline{\beta}_2\overline{\beta}_1)^\star \preccurlyeq \overline{\beta}_1\} \\ &= (\underline{\beta}_1 \mathbin{\rlap{$\diamond$}\backslash} \underline{\beta}_1 \mathbin{\rlap{$\diamond$}/} (\underline{\beta}_2\underline{\beta}_1)) \wedge (\overline{\beta}_1 \mathbin{\rlap{$\diamond$}\backslash} \overline{\beta}_1 \mathbin{\rlap{$\diamond$}/} (\overline{\beta}_2\overline{\beta}_1))\end{aligned}$$

Sketch of proof (lower bound of the interval) [11]

Because of properties of deconvolution $\mathbin{\rlap{$\diamond$}\backslash}$ and subadditive closure $\star$

$$\underline{\beta}_1(\gamma_w\underline{\beta}_2\underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1$$

[11] B Cottenceau *et al.*: Model reference control... Automatica, 2001.

Proposition 3

$$
\begin{aligned}
\hat{\gamma_w} &= \bigoplus \{\gamma_w \mid \underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \overline{\beta}_1(\gamma_w \overline{\beta}_2 \overline{\beta}_1)^\star \preccurlyeq \overline{\beta}_1\} \\
&= (\underline{\beta}_1 \circ\!\!\!\!\backslash\, \underline{\beta}_1 \,/\!\!\!\!\circ\, (\underline{\beta}_2 \underline{\beta}_1)) \wedge (\overline{\beta}_1 \circ\!\!\!\!\backslash\, \overline{\beta}_1 \,/\!\!\!\!\circ\, (\overline{\beta}_2 \overline{\beta}_1))
\end{aligned}
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Sketch of proof (lower bound of the interval) [11]

Because of properties of deconvolution $\circ\!\!\!\!\backslash$ and subadditive closure $\star$

$$
\begin{aligned}
\underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \quad &\Leftrightarrow \quad (\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \circ\!\!\!\!\backslash\, \underline{\beta}_1 \\
&\Leftrightarrow \quad \gamma_w \underline{\beta}_2 \underline{\beta}_1 \preccurlyeq \underline{\beta}_1 \circ\!\!\!\!\backslash\, \underline{\beta}_1 \\
&\Leftrightarrow \quad \gamma_w \preccurlyeq \underline{\beta}_1 \circ\!\!\!\!\backslash\, \underline{\beta}_1 \,/\!\!\!\!\circ\, (\underline{\beta}_2 \underline{\beta}_1)
\end{aligned}
$$

[11] B Cottenceau *et al.*: Model reference control... Automatica, 2001.

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$$
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&= {\color{red}(\underline{\beta}_1 \backslash\!\!\!\circ\, \underline{\beta}_1 /\!\!\!\circ\, (\underline{\beta}_2 \underline{\beta}_1))} \wedge (\overline{\beta}_1 \backslash\!\!\!\circ\, \overline{\beta}_1 /\!\!\!\circ\, (\overline{\beta}_2 \overline{\beta}_1))
\end{aligned}
$$

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\underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 &\Leftrightarrow (\gamma_w \underline{\beta}_2 \underline{\beta}_1)^\star \preccurlyeq \underline{\beta}_1 \backslash\!\!\!\circ\, \underline{\beta}_1 \\
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[11] B Cottenceau *et al.*: Model reference control... Automatica, 2001.

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Outlines

1. (Min,+) algebra and Network Calculus
   - (Min,+) algebra
   - Network Calculus modelling
   - Performance characteristics
2. Flow control
   - Arrival curve computation
   - Window flow control
3. Conclusions

What have we done?

Traffic regulation and performance guarantee: flow control

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- Optimal arrival curve $\hat{\alpha}^{\star}$
  $\longrightarrow$ respect of delay or backlog constraints

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Traffic regulation and performance guarantee: flow control

- Optimal arrival curve $\hat{\alpha}^{\star}$
  $\longrightarrow$ respect of delay or backlog constraints
- Optimal window size $\hat{\gamma_w}$
  $\longrightarrow$ window flow control

Thank you for your attention ...

Questions ?

Sketch of proof

Why $u \succcurlyeq \alpha u \Leftrightarrow u = \alpha^\star u$?

Back

Sketch of proof

Why $u \succcurlyeq \alpha u \Leftrightarrow u = \alpha^\star u$?

- isotony of the inf-convolution $(a \succcurlyeq b \Leftrightarrow ac \succcurlyeq bc)$

$$u \succcurlyeq \alpha u \;\Rightarrow\; \alpha u \succcurlyeq (\alpha^2 u) \;\Rightarrow\; (\alpha^2 u) \succcurlyeq (\alpha^3 u) \;\Rightarrow\; \ldots$$

Sketch of proof

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- order relation of $\mathcal{F}_0$ $(a \succcurlyeq b \Leftrightarrow a = a \oplus b)$

$$u = u \oplus (\alpha u) \oplus (\alpha^2 u) \oplus \dots = \bigoplus_{n \geq 0} \alpha^n u$$

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So

$$u \succcurlyeq \alpha u \quad \Leftrightarrow \quad u = \alpha^\star u$$

and

$$\alpha \text{ (and } \alpha^\star\text{) is an arrival curve for } u \text{ iff } u = \alpha^\star u$$

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