

Flow Control with (Min,+) Algebra

Journées Nationales GdR GPL - Session AFSEC

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Motivations

Flow control

- Arrival curve computation
 - delay and backlog constraints
- Window flow control
 - difference between data stream and acknowledgments stream

Outlines

1 (Min,+) algebra and Network Calculus

- (Min,+) algebra
- Network Calculus modelling
- Performance characteristics

2 Flow control

- Arrival curve computation
- Window flow control

3 Conclusions

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3 Conclusions

Theory of $(\min, +)$ linear systems¹

- Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena

¹F. Baccelli *et al.*: Synchronization and Linearity. Wiley and sons, 1992.

Theory of $(\min, +)$ linear systems¹

- Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena
- Application areas: communication networks^{2 3}, manufacturing systems, transportation systems⁴

¹F. Baccelli *et al.*: Synchronization and Linearity. Wiley and sons, 1992.

²CS Chang: Performance guarantees. Springer, 2000.

³JY Le Boudec and P. Thiran: Network Calculus. Springer, 2001.

⁴B. Heidergott *et al.*: Max plus at work. Princeton University Press, 2006.

Idempotent semiring

Set \mathcal{D} endowed with two inner operations ^a

- $\oplus \rightarrow$ associative, commutative, idempotent ($a \oplus a = a$)
neutral element ε
- $\otimes \rightarrow$ associative, distributes over the sum
neutral element e

^a when \otimes is commutative, \mathcal{D} is said commutative

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Order relation

$$a = a \oplus b \Leftrightarrow a \succcurlyeq b$$

Example: idempotent semiring $\overline{\mathbb{R}}_{min}$

$$\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{-\infty, +\infty\}, \min, +)$$

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$$\overline{\mathbb{R}}_{min} = (\mathbb{R} \cup \{-\infty, +\infty\}, \min, +)$$

Remark: order relation in $\overline{\mathbb{R}}_{min}$

$$\min(5, 3) = 3 \Leftrightarrow 5 \oplus 3 = 3 \Leftrightarrow 3 \succcurlyeq 5 \Leftrightarrow 3 \leq 5$$

Commutative idempotent semiring: $\{\mathcal{F}_0, \oplus, *\}$

Set ^a \mathcal{F}_0 endowed with

^a non-decreasing functions $f : \mathbb{R} \mapsto \overline{\mathbb{R}}_{min}$ where $f(t) = 0$ for $t \leq 0$

Commutative idempotent semiring: $\{\mathcal{F}_0, \oplus, *\}$ Set ^a \mathcal{F}_0 endowed with

- $\oplus \rightarrow$ pointwise minimum
- $*$ \rightarrow inf-convolution

$$f, g \in \mathcal{F}_0 \quad (f * g)(t) \triangleq \bigoplus_{\tau \geq 0} \{f(\tau) \otimes g(\tau - t)\} = \min_{\tau \geq 0} \{f(\tau) + g(\tau - t)\}$$

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is a commutative idempotent semiring

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Other operations in $\{\mathcal{F}_0, \oplus, *\}$

- deconvolution

$$(f \oslash g)(t) \triangleq \bigwedge_{\tau \geq 0} \{f(\tau) - g(\tau - t)\} = \max_{\tau \geq 0} \{f(\tau) - g(\tau - t)\}$$

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Residuation theory: $x = f \oslash g$ is the greatest solution to $x * g \preccurlyeq f$

- subadditive closure

$$f^*(t) \triangleq \bigoplus_{\tau \geq 0} f^\tau(t) = \min_{\tau \geq 0} f^\tau(t) \quad \text{with} \quad f^0(t) = e$$

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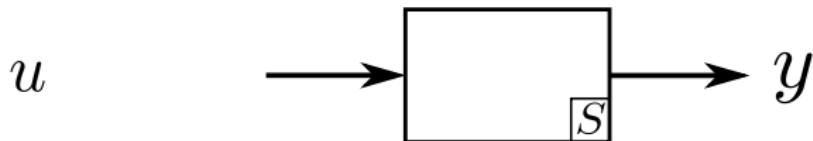
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Fixed point theory: f^* is the lowest solution to $x = f * x \oplus e$

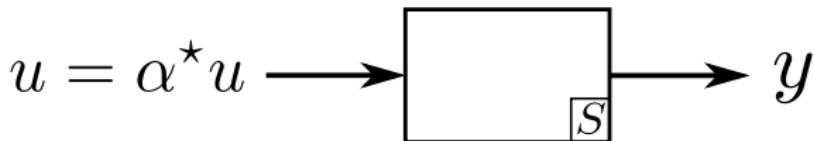




Network elements

- Input and output flows u and y

$$\forall t, u(t) \geq y(t) \Leftrightarrow u \preccurlyeq y$$



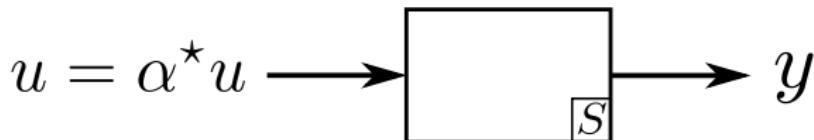
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$$u \leq \alpha u \Leftrightarrow u \succcurlyeq \alpha u \Leftrightarrow u = \alpha^* u$$



Network elements

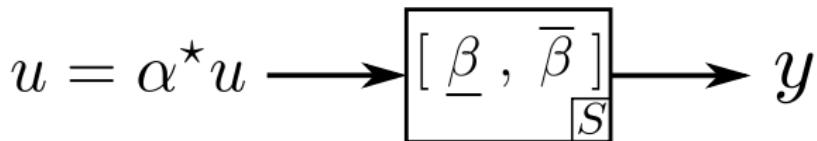
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Sketch of proof



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Sketch of proof

- Service curve $[\underline{\beta}, \bar{\beta}]$ ([maximum service, minimum service])

$$\underline{\beta}u \preccurlyeq y \preccurlyeq \bar{\beta}u \Leftrightarrow y \in [\underline{\beta}u, \bar{\beta}u]$$

Delay $d(k)$ (waiting time of the k^{th} paquet)^{5 6}

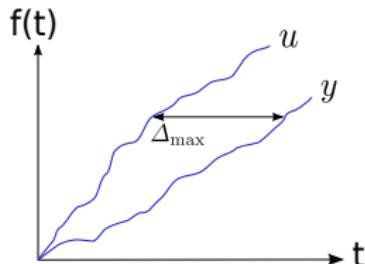
$$\forall k, d(k)$$

⁵A Bouillard et al.: Computation of a (min,+)... ValueTools'08.

⁶Max Plus: Second order theory... CDC'91.

Delay $d(k)$ (waiting time of the k^{th} paquet)^{5 6}

$$\forall k, d(k) \leq \Delta_{\max}$$

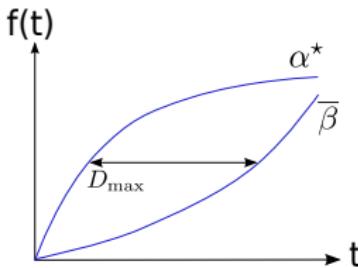
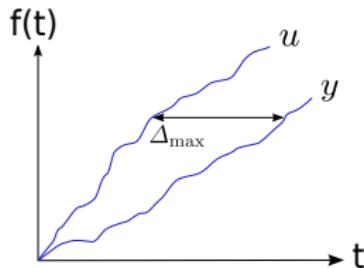


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$$\forall k, d(k) \leq \Delta_{\max} \leq D_{\max}$$

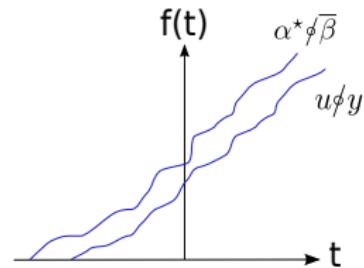
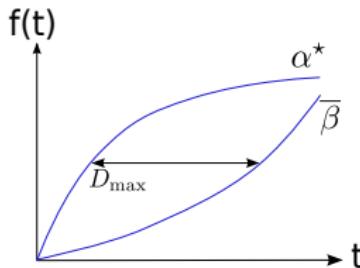
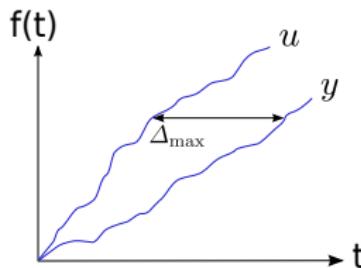


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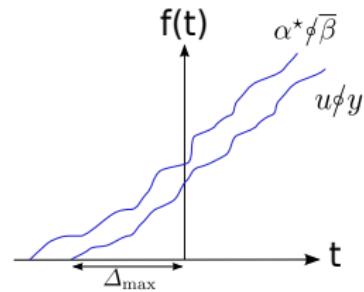
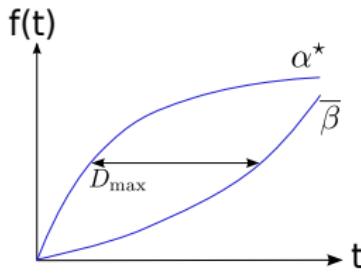
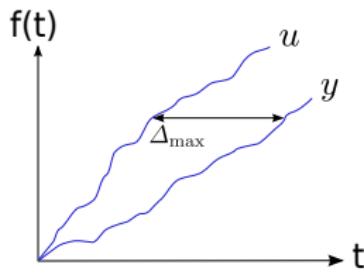


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where

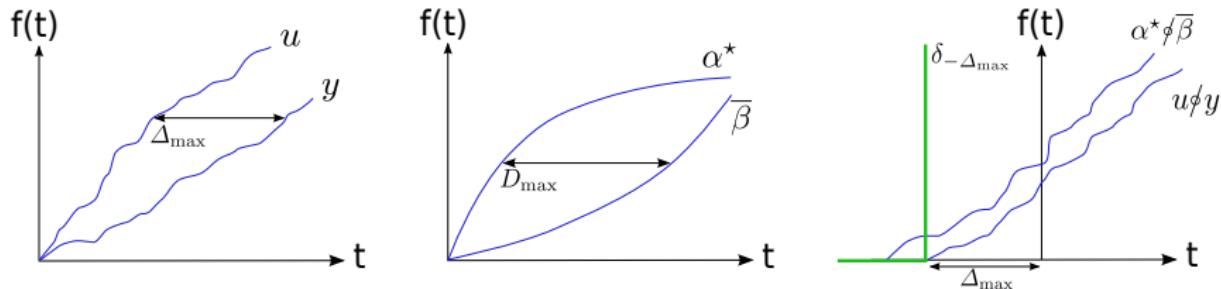
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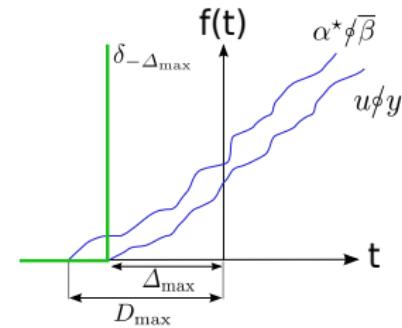
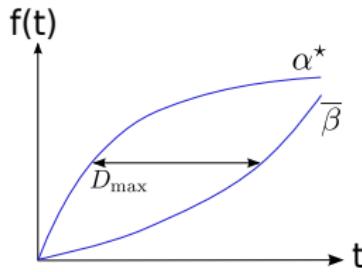
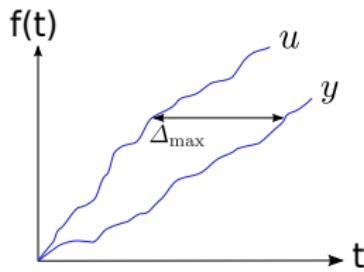
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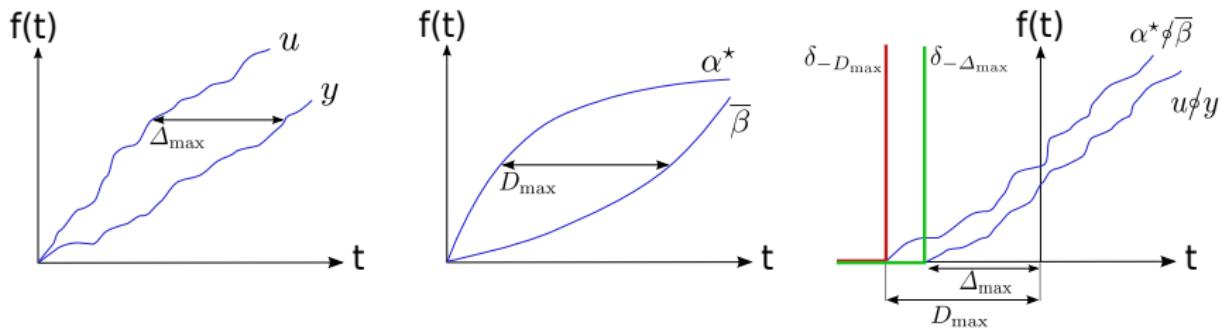
$$\Delta_{\max} = \inf_{\Delta \geq 0} \{(u \phi y)(-\Delta) \leq 0\} \quad \text{and} \quad D_{\max} = \inf_{D \geq 0} \{(\alpha^* \phi \bar{\beta})(-D) \leq 0\}$$

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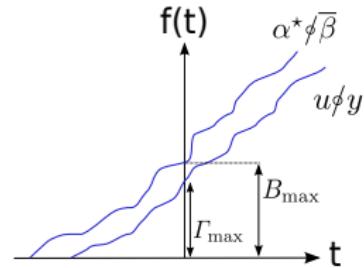
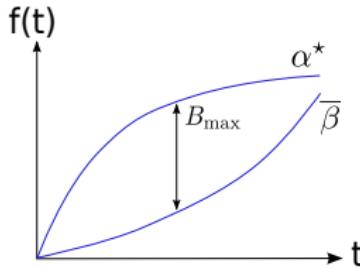
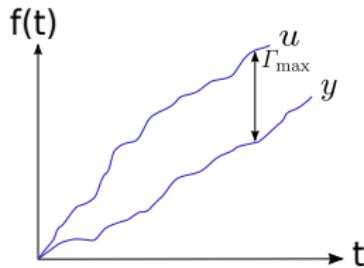
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Backlog $b(t)$ (amount of paquets at time t) ^{7 8}

$$\forall t, b(t) \leq \Gamma_{\max} \leq B_{\max}$$



where

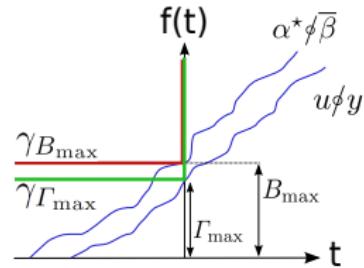
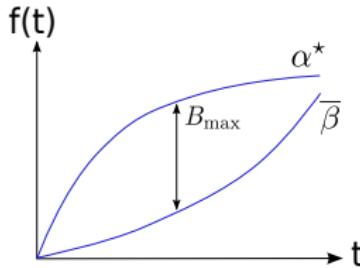
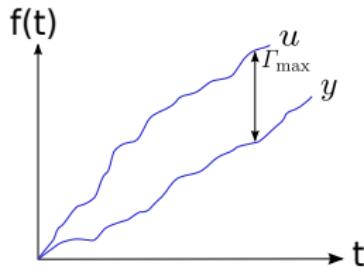
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- (Min,+) algebra
- Network Calculus modelling
- Performance characteristics

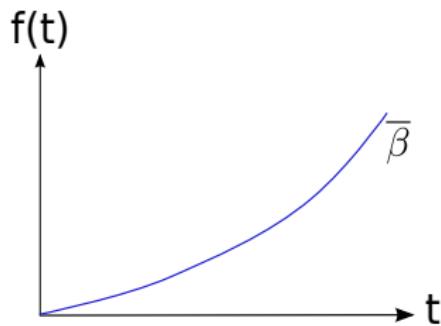
2 Flow control

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3 Conclusions

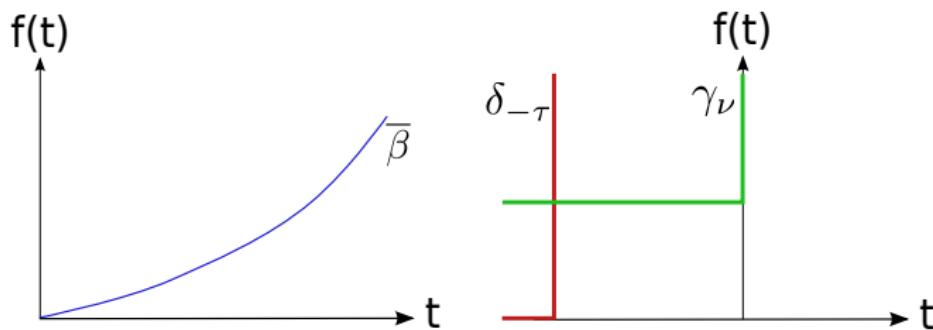
Goal

- Minimum service $\bar{\beta}$ known



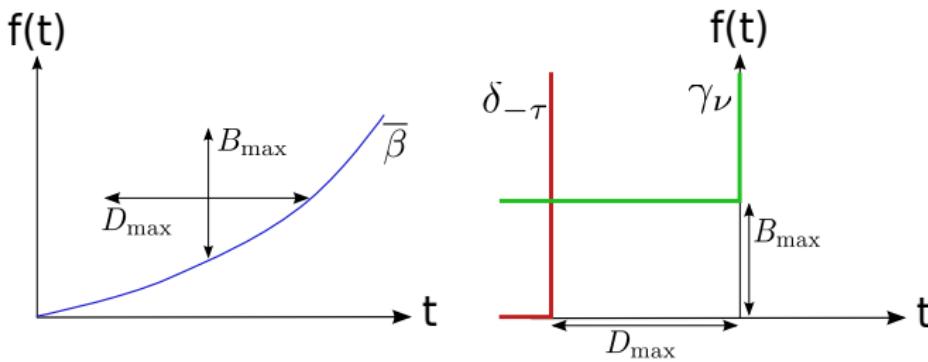
Goal

- Minimum service $\bar{\beta}$ known
- Fixed worst end-to-end delay τ or backlog ν



Goal

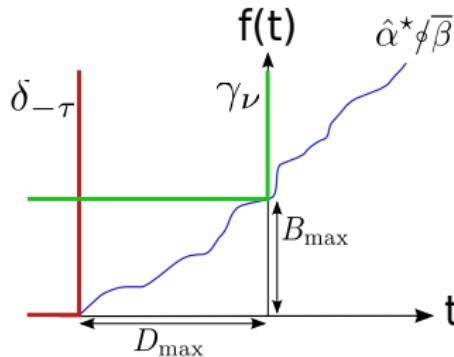
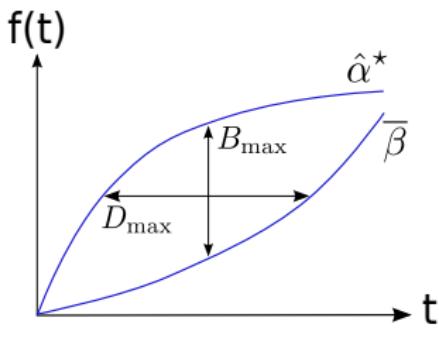
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- Upper bounds $D_{\max} = \tau$ and $B_{\max} = \nu$



Goal

- Minimum service $\bar{\beta}$ known
- Fixed worst end-to-end delay τ or backlog ν
- Upper bounds $D_{\max} = \tau$ and $B_{\max} = \nu$
- Computation of the minimal constraint $\hat{\alpha}^*$ s.t.

$$\hat{\alpha}^* \not\ll \bar{\beta} \succcurlyeq \delta_{-\tau} \quad \text{and} \quad \hat{\alpha}^* \not\ll \bar{\beta} \succcurlyeq \gamma_\nu$$



Proposition 1 (time performance)

$$\hat{\alpha}^*$$

⁹ CA. Maia et al.: Optimal closed-loop control... Automatic Control, 2003.

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$$\hat{\alpha}^* = \bigwedge \{\alpha^* \mid \alpha^* \succcurlyeq \delta_{-\tau} \bar{\beta}\}$$

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Proposition 1 (time performance)

$$\hat{\alpha}^* = \bigwedge \{\alpha^* \mid \alpha^* \succcurlyeq \delta_{-\tau}\bar{\beta}\} = (\delta_{-\tau}\bar{\beta})^*$$

9 CA. Maia et al.: Optimal closed-loop control... Automatic Control, 2003.

Proposition 1 (time performance)

$$\hat{\alpha}^\star = \bigwedge \{ \alpha^\star \mid \color{red}{\alpha^\star \succcurlyeq \delta_{-\tau} \bar{\beta}} \} = (\delta_{-\tau} \bar{\beta})^\star$$

Sketch of proof

- $$\bullet \quad \alpha^* \not\sim \overline{\beta} \succcurlyeq \delta_{-\tau} \quad \Leftrightarrow \quad \alpha^* \succcurlyeq \delta_{-\tau} \overline{\beta}$$

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Sketch of proof

- $\alpha^* \not\succcurlyeq \bar{\beta} \Leftrightarrow \alpha^* \succcurlyeq \delta_{-\tau} \bar{\beta}$
- $a^* = \bigwedge \{x^* \mid x = x^*, x^* \succcurlyeq a\}$ (see lemma 1 in ⁹)

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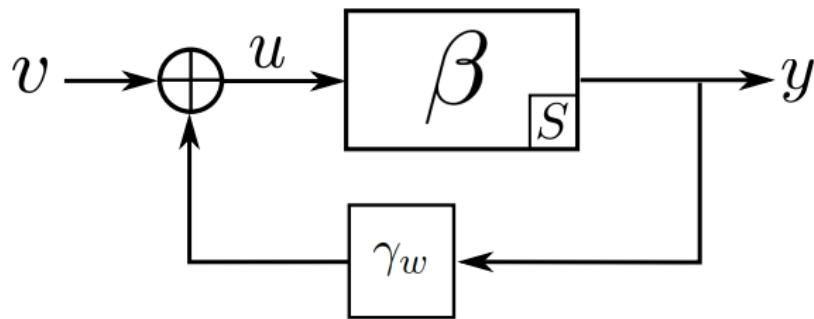
Proposition 2 (data performance)

$$\hat{\alpha}^* = \bigwedge \{\alpha^* \mid \alpha^* \succcurlyeq \gamma_\nu \bar{\beta}\} = (\gamma_\nu \bar{\beta})^*$$

⁹ CA. Maia *et al.*: Optimal closed-loop control... Automatic Control, 2003.

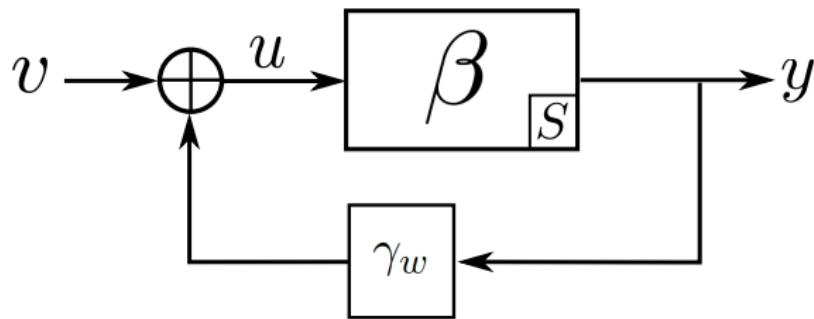
Window Flow Control

- Communication network S



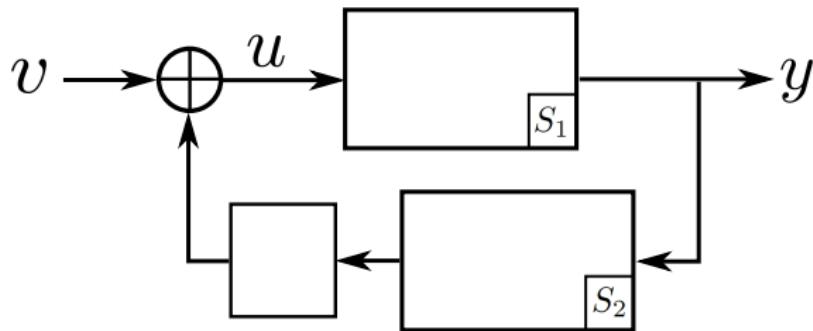
Window Flow Control

- Communication network S
- Window size $w \rightarrow \gamma_w$
- Limit the amount of data



Goal

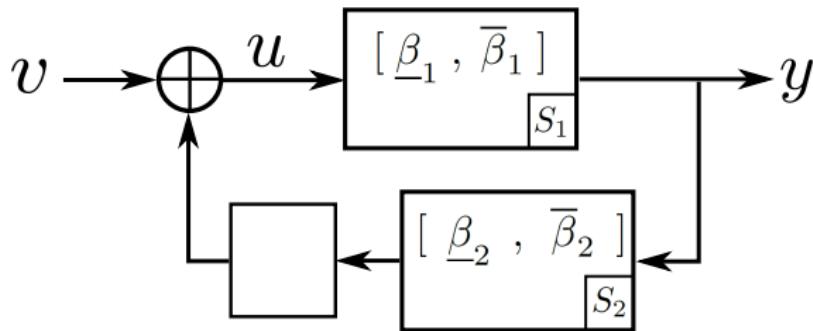
- Data stream S_1 and acknowledgments stream S_2 ¹⁰



¹⁰R Agrawal et al.: Performance bounds... Transactions on Networking, 1999.

Goal

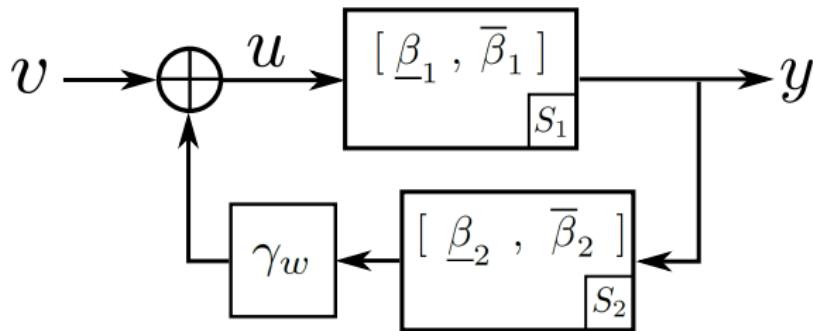
- Data stream S_1 and acknowledgments stream S_2 ¹⁰
- Interval of service $[\underline{\beta}, \bar{\beta}]$



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Goal

- Data stream S_1 and acknowledgments stream S_2 ¹⁰
- Interval of service $[\underline{\beta}, \bar{\beta}]$
- Computation of the minimal window size $\hat{\gamma}_w$ s.t.
closed-loop behavior (S_1 and S_2) = open-loop behavior (S_1)



¹⁰R Agrawal et al.: Performance bounds... Transactions on Networking, 1999.

Interval of service [maximum service , minimum service]

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- Open-loop system (S_1)

$$[\underline{\beta}_1 , \overline{\beta}_1]$$

Interval of service [maximum service , minimum service]

- Open-loop system (S_1)

$$[\underline{\beta}_1 , \overline{\beta}_1]$$

- Closed-loop system (S_1 and S_2)

$$[\underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^*, \overline{\beta}_1(\gamma_w \overline{\beta}_2 \overline{\beta}_1)^*]$$

Proposition 3

$$\hat{\gamma_w} = \bigoplus \{ \gamma_w \mid \underline{\beta}_1 (\gamma_w \underline{\beta}_2 \underline{\beta}_1)^* \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \bar{\beta}_1 (\gamma_w \bar{\beta}_2 \bar{\beta}_1)^* \preccurlyeq \bar{\beta}_1 \}$$

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Proposition 3

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Sketch of proof (lower bound of the interval) ¹¹

Because of properties of deconvolution \setminus and subadditive closure $*$

$$\underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^* \preccurlyeq \underline{\beta}_1$$

¹¹ B Cottenceau et al.: Model reference control... Automatica, 2001.

Proposition 3

$$\begin{aligned}\hat{\gamma}_w &= \bigoplus \{\gamma_w \mid \underline{\beta}_1(\gamma_w \underline{\beta}_2 \underline{\beta}_1)^* \preccurlyeq \underline{\beta}_1 \quad \text{and} \quad \bar{\beta}_1(\gamma_w \bar{\beta}_2 \bar{\beta}_1)^* \preccurlyeq \bar{\beta}_1\} \\ &= (\underline{\beta}_1 \setminus \underline{\beta}_1 \phi(\underline{\beta}_2 \underline{\beta}_1)) \wedge (\bar{\beta}_1 \setminus \bar{\beta}_1 \phi(\bar{\beta}_2 \bar{\beta}_1))\end{aligned}$$

Sketch of proof (lower bound of the interval)¹¹Because of properties of deconvolution \setminus and subadditive closure $*$

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Outlines

1 (Min,+) algebra and Network Calculus

- (Min,+) algebra
- Network Calculus modelling
- Performance characteristics

2 Flow control

- Arrival curve computation
- Window flow control

3 Conclusions

What have we done?

Traffic regulation and performance guarantee: flow control

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Traffic regulation and performance guarantee: flow control

- Optimal arrival curve $\hat{\alpha}^*$
→ respect of delay or backlog constraints
- Optimal window size $\hat{\gamma}_w$
→ window flow control

Thank you for your attention ...

Questions ?

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Why $u \succcurlyeq \alpha u \Leftrightarrow u = \alpha^* u$?

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and

α (and α^*) is an arrival curve for u iff $u = \alpha^* u$

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