Nonlinear systems of equations and parallel asynchronous iterative algorithms

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Abstract

This paper deals with the solution of nonlinear systems of equations via asynchronous iterative algorithms. New conditions underlying convergence of asynchronous relaxation algorithms are given. Conditions of convergence of asynchronous versions of a generalization of Richardson's method for linear systems of equations are also presented. Application to nonlinear network flow problems is considered and computational experiences using a distributed memory multiprocessor are presented.

1. INTRODUCTION

In this paper we concentrate on the solution of nonlinear systems of equations via asynchronous iterative methods whereby iterations are carried out in parallel by several processors in arbitrary order and without any synchronization (see in particular [1] and [2]). The restrictions imposed on asynchronous iterative algorithms are very weak: no component of the iterate vector is abandoned forever and more and more recent values of the components of the iterate vector have to be used as the computation progresses.

In [3], we have considered nonlinear systems of equations, $F(x) = z$, and we have given a convergence result for point asynchronous relaxation when F is continuous, off-diagonally antitone, and strictly diagonally isotone. In this paper we present weaker conditions underlying convergence of asynchronous relaxation (AR). We propose also and prove convergence of asynchronous versions of a generalization of Richardson's method for linear systems of equations (AGR). These results apply to the discretization of certain boundary value problems and economic problems (see [4]), they generalize convergence results for network flows (see [5] and [6]). Finally, we present computational experiences for network flows using a distributed memory multiprocessor. The reader is also referred to [7-8] for convergence results under partial ordering for block asynchronous relaxation.

2. PARALLEL ASYNCHRONOUS RELAXATION ALGORITHMS

Consider the nonlinear systems of equations :

$$F(x) \equiv \begin{pmatrix} F_1(x_1, ..., x_n) \\ ... \\ F_n(x_1, ..., x_n) \end{pmatrix} = \begin{pmatrix} z_1 \\ ... \\ z_n \end{pmatrix}, \tag{1}$$

where $x_1, ..., x_n$ denote the components of vector x element of the n-dimensional real linear space R^n. We define the natural partial ordering on R^n by: for $x, y \in R^n$, $x \le y$ if and only if $x_i \le y_i$, $i = 1, ..., n$. We operate under the following weak assumptions.

Assumption 2.1. $F : D \subset R^n \to R^n$ is continuous, off-diagonally antitone, and diagonally isotone, i.e. for any $x \in R^n$, the functions $\phi_{ij} : \{t \in R^1 / x + t.e^j \in D\} \to R^1, \phi_{ij}(t) = F_i(x + t.e^j), j \neq i, i, j = 1, ..., n$, and $\phi_{ii} : \{t \in R^1 / x + t.e^i \in D\} \to R^1, \phi_{ii}(t) = F_i(x + t.e^i), i = 1, ..., n$, where $e^i, i = 1, ..., n$, are the unit basis vectors with i-th component one and all others zero, are antitone and isotone, respectively.

For study of off-diagonally antitone and diagonally isotone mappings, reference is made to [4, Section 13.5] and [9, Section 2].

Assumption 2.2. For some $z \in R^n$ there exist points $x^0, y^0 \in D$ such that

$$x^0 \leq y^0, D' = \{x \in R^n / x^0 \leq x \leq y^0\} \subset D, F(x^0) \leq z \leq F(y^0). \tag{2}$$

We consider the point-to-set mapping G, defined on $D' \subset R^n$, with components G_i, $i = 1, ..., n$, given by : $G_i(x) = \{\hat{x}_i / F_i(x_1, ..., \hat{x}_i, ..., x_n) = z_i, x_i^0 \leq \hat{x}_i \leq y_i^0\}$.

Proposition 2.1. Let Assumptions 2.1 and 2.2 hold. For all $i \in \{1, ..., n\}$ and $x \in D'$, $G_i(x)$ is a nonempty and compact interval and the point-to-point mappings $\underline{G}$ and $\overline{G}$, with components: $\underline{G}_i(x) = \min_{\hat{x}_i \in G_i(x)} \hat{x}_i, i = 1, ..., n$, and $\overline{G}_i(x) = \max_{\hat{x}_i \in G_i(x)} \hat{x}_i, i = 1, ..., n$, respectively, are well defined and isotone on D'. Moreover, we have:

$$x^0 \leq \underline{G}(x^0) \leq \underline{G}(y^0) \leq y^0, x^0 \leq \overline{G}(x^0) \leq \overline{G}(y^0) \leq y^0. \tag{3}$$

Proof: see the Appendix.

We call $\underline{G}$ and $\overline{G}$, the minimal and the maximal relaxation mapping, respectively. An asynchronous iterative algorithm relative to the mapping $\underline{G}$ ($\overline{G}$), the starting point x^0, the sequence of delays $\{k^p = (k_1^p, ..., k_n^p)\}$, and the sequence $\{h^p\}$ of nonempty subsets of $\{1, ..., n\}$ is a sequence of points $\{x^p\}$ defined recursively by:

$$x_i^{p+1} = \underline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)}), (x_i^{p+1} = \overline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)})), \forall i \in h^p, x_i^{p+1} = x_i^p, \forall i \notin h^p, \tag{4}$$

where for all $i \in \{1, ..., n\}$: i occurs infinitely often in the sequence $\{h^p\}$, and for all $j \in \{1, ..., n\}$: the function $d_j(p)$ is isotone, $\lim_{p \to \infty} d_j(p) = +\infty$, and $0 \leq d_j(p) = p - k_j^p \leq p, p = 0, 1, ...$

The following proposition is the main result of this section.

Proposition 2.2. Let Assumptions 2.1 and 2.2 hold and assume that the mapping $\underline{G}$ ($\overline{G}$) is continuous on D'. The asynchronous relaxation algorithms $\{x^p\}$ and $\{y^p\}$ defined by (4), relative to the same sequences $\{h^p\}$, $\{k^p\}$, and starting from x^0 and y^0, respectively, are uniquely defined and satisfy:

$$x^0 \leq ... \leq x^p \leq x^{p+1} \leq y^{p+1} \leq y^p \leq ... \leq y^0, p = 0, 1, ... \tag{5}$$

$$\lim_{p \to \infty} x^p = x^* \leq y^* = \lim_{p \to \infty} y^p, \tag{6}$$

$$\underline{G}(x^*) = x^*, \underline{G}(y^*) = y^*, (\overline{G}(x^*) = x^*, \overline{G}(y^*) = y^*), F(x^*) = z, F(y^*) = z. \tag{7}$$

Proof: The proof is given in the Appendix.

3. PARALLEL ASYNCHRONOUS GENERALIZED RICHARDSON'S METHOD

We turn now to a different approach to constructing iterative processes which converge monotonically to a solution of nonlinear systems of equations:

$$F(x) \equiv \begin{pmatrix} F_1(x_1,...,x_n) \\ ... \\ F_n(x_1,...,x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ ... \\ 0 \end{pmatrix}. \tag{8}$$

Assumption 3.1. $F : D \subset R^n \to R^n$ is Lipschitz continuous on D, i.e. there exists a constant α such that

$$\|F(x) - F(y)\|_2 \leq \alpha . \|x - y\|_2 , \forall x, y \in D, \tag{9}$$

where $\|.\|_2$ denotes the Euclidean norm on R^n.

Proposition 3.1. Let Assumptions 2.1, 2.2 (with $z = 0$), and 3.1 hold. The mapping $H : D \subset R^n \to R^n$, defined by $H(x) = x - \frac{1}{\alpha}.F(x)$, is continuous and isotone on D. Moreover: $x^0 \leq H(x^0) \leq H(y^0) \leq y^0$.

Proof: The proof is detailed in the Appendix.

We consider now asynchronous algorithms relative to the mapping H. These algorithms can be viewed as asynchronous versions of a generalization of Richardson's method for linear systems of equations (see [1] and [10]).

Proposition 3.2. Let Assumptions 2.1, 2.2 (with $z = 0$), and 3.1 hold. The asynchronous algorithms $\{x^p\}$ and $\{y^p\}$ relative to the mapping H, the same sequences $\{h^p\}$, $\{k^p\}$, and starting from x^0 and y^0, respectively, are uniquely defined and satisfy:

$$x^0 \leq ... \leq x^p \leq x^{p+1} \leq y^{p+1} \leq y^p \leq ... \leq y^0, p = 0, 1, ... \tag{10}$$

$$\lim_{p\to\infty} x^p = x^* \leq y^* = \lim_{p\to\infty} y^p, \tag{11}$$

$$H(x^*) = x^*, H(y^*) = y^*, F(x^*) = F(y^*) = 0. \tag{12}$$

Proof: The proof is analogous with the proof of Proposition 2.2 and is omitted.

4. APPLICATION TO NETWORK FLOWS

Let $G = (N, A)$ be a directed graph. N and A are referred to as the set of nodes and the set of arcs, respectively. Let $c_{ij} : R \to (-\infty, +\infty]$ be the convex cost function associated with each arc $(i,j) \in A$, c_{ij} is a function of the flow of the arc (i,j), denoted by f_{ij}. Let b_i the supply or demand at node i. The problem is to minimize total cost subject to a conservation of flow constraint at each node:

$$min \sum_{(i,j)\in A} c_{ij}(f_{ij}), \tag{13}$$

subject to $\sum_{(i,j)\in A} f_{ij} - \sum_{(m,i)\in A} f_{mi} = b_i, \forall i \in N$.

We assume that problem (13) has a feasible solution. We also make the following assumptions: c_{ij} is strictly convex, and lower semicontinuous, the conjugate convex function of c_{ij}, defined by: $c^*_{ij}(t_{ij}) = \sup_{f_{ij}} \{t_{ij}.f_{ij} - c_{ij}(f_{ij})\}$, is real valued.

A dual problem for (13) is given by:

$$\min_{p\in R^n} q(p), \tag{14}$$

subject to no constraints on the vector $p = \{p_i/i \in N\}$, where q is the dual functional given by: $q(p) = \sum_{(i,j)\in A} c^*_{ij}(p_i - p_j) - \sum_{i\in N} b_i.p_i$. We refer to p as a price vector and its components as prices. Price p_i is the Lagrange multiplier associated with conservation of flow constraint at node i. Existence of an optimal solution of the dual problem can be guaranteed under the following additional regular feasibility assumption (see [11, p. 360 and p. 329]): there exists a feasible flow vector, $f = \{f_{ij}/(i,j) \in A\}$, such that $c'_{ij-}(f_{ij}) < +\infty$ and $c'_{ij+}(f_{ij}) > -\infty$, for all $(i,j) \in A$, where c'_{ij-} and c'_{ij+} denote the left and right derivatives of c_{ij}, respectively.

The following nonlinear system of equations is derived from the convex, unconstrained, and differentiable dual problem.

$$\left.\frac{\partial q}{\partial p_i}\right|_p = \sum_{(i,j)\in A} g^*_{ij}(p_i - p_j) - \sum_{(m,i)\in A} g^*_{mi}(p_m - p_i) - b_i = 0, i = 1, ..., n, \tag{15}$$

where g^*_{ij} denotes the gradient of c^*_{ij}. We note that g^*_{ij} is isotone, since c^*_{ij} is convex. Thus the mapping of components $\frac{\partial q}{\partial p_i}$ is off-diagonally antitone and diagonally isotone, moreover it is continuous. For many practical problems, it is easy to find a subsolution x^0 and a supersolution y^0 satisfying (2) and to show that the mapping H defined in Section 3 is isotone and continuous (see for example [6]). For network flow problems, the generalization of Richardson's method is in fact a gradient method which will be denoted by G.

We present now computational experiences carried out on a transputer based, distributed memory multiprocessor, T-node 16-32. We have implemented synchronous and asynchronous relaxation and gradient methods on 2, 4, 8, and 16 processors, they are denoted by SRx, ARx, SGx, and AGx, respectively, where x is the number of processors. We have considered grid network flow problems. For each problem there is only one nonzero traffic imput, say $b_1 = 1$ and the arc costs are: $c_{ij}(f_{ij}) = (\frac{1}{1-f_{ij}} + 0.5).f_{ij}$, if $f_{ij} \geq 0$, and $c_{ij}(f_{ij}) = +\infty$, if $f_{ij} < 0$. For the gradient methods we have chosen a stepsize $\alpha = 2$. We have chosen the same starting point for the different problems and methods: the subsolution $p_i = 0, \forall i \in N$. Computations are stopped when $\frac{\partial q}{\partial p_d} \leq 0.1$ where d is the destination node. A detailed description of a preliminary implementation on a multitransputer system can be found in [12].

Table 1 gives the solution times, in seconds, for R and G on one processor in function of the number of dual variables (i.e. the number of nodes in the network). Table 1 points out that for medium scale problems, G is faster than R. The nonlinear dual functional cannot be minimized analytically with respect to each price. Parallel relaxation algorithms lead to indeterministic load unbalancing, since line search is made by an iterative procedure. Table 2 shows that asynchronous implementation can speedup efficiently a relaxation method. Table 2 points out that synchronous relaxation methods are slower than asynchronous relaxation methods. There is generally deterministic load balancing in the particular case of parallel gradient algorithms since we compute essentially a gradient at each updating and try to balance the computational load. Table 3 points out that synchronous and asynchronous implementations speed up very efficiently the gradient method. We note that AG methods are faster than SG methods. We note also that AG algorithms are faster than AR algorithms.

Table 1

Times of R, G

	R	G
48	99.60	74.37
72	279.10	235.85
96	588.60	527.73
120	1054.00	968.60
144	1696.00	1562.30

Table 2

Speedups of SR2, AR2, SR4, AR4, SR8, AR8, SR16, and AR16

	SR2	AR2	SR4	AR4	SR8	AR8	SR16	AR16
48	1.47	1.65	2.53	2.92	3.89	4.86	-	-
72	1.46	1.67	2.57	3.04	4.44	5.38	-	-
96	1.45	1.67	2.54	3.09	4.61	5.68	7.50	9.40
120	1.44	1.66	2.51	3.10	4.57	5.67	7.26	9.55
144	1.44	1.64	2.49	3.09	4.57	5.85	8.21	10.30

Table 3

Speedups of SG2, AG2, SG4, AG4, SG8, AG8, SG16, and AG16

	SG2	AG2	SG4	AG4	SG8	AG8	SG16	AG16
48	1.83	1.84	3.35	3.47	6.49	6.50	-	-
72	1.84	1.87	3.39	3.54	6.37	6.73	-	-
96	1.84	1.88	3.37	3.55	6.22	6.81	11.78	12.78
120	1.83	1.88	3.32	3.54	6.09	6.78	9.87	12.50
144	1.83	1.87	3.22	3.53	5.89	6.79	11.02	12.90

5. CONCLUSIONS

In this paper we have considered the solution of nonlinear systems of equations and we have given new conditions underlying convergence of asynchronous relaxation algorithms and asynchronous versions of a generalization of Richardson's method for linear systems of equations. We note that the conditions underlying convergence of AGR are more restrictive than the conditions given for AR. The class of problems considered in this paper is broad. Off-diagonally antitone and diagonally isotone mappings occur in the discretization of certain boundary value problems, in economic problems, and in the study of nonlinear network flows. Computational experiences for network flow problems using a distributed memory multiprocessor have mainly shown that asynchronous methods are faster than synchronous methods. Moreover AGR is faster than AR when there is no special structure that makes price relaxation particularly easy.

6. APPENDIX

6.1. Proof of Proposition 2.1

Suppose that for $x \in D'$ and $i \in \{1, ..., n\}$, $F_i(x) < z_i$. From the off-diagonal antitonicity of F and (2) it follows that

$$F_i(x) < z_i \leq F_i(y^0) \leq F_i(x_1, ..., y_i^0, ..., x_n). \tag{16}$$

By the continuity and diagonal isotonicity of F, (16) implies that $G_i(x)$ is nonempty and compact. If $z_i \leq F_i(x)$, the proof is very similar. Hence, for all $x \in D'$, and $i \in \{1, ..., n\}$, $G_i(x)$ is nonempty and compact.

Consider now $x, y \in D'$, such that $x \leq y$. By the definition of $\underline{G}$ we have:

$$F_i(x_1, ..., \underline{G}_i(x), ..., x_n) = F_i(y_1, ..., \underline{G}_i(y), ..., y_n) = z_i, i = 1, ..., n. \tag{17}$$

Moreover we have: $F_i(y_1, ..., \underline{G}_i(y), ..., y_n) \leq F_i(x_1, ..., \underline{G}_i(y), ..., x_n), i = 1, ..., n$, since F is off-diagonally antitone. Suppose in particular that for $i \in \{1, ..., n\}$:

$$F_i(y_1, ..., \underline{G}_i(y), ..., y_n) < F_i(x_1, ..., \underline{G}_i(y), ..., x_n). \tag{18}$$

By the diagonal isotonicity of F, (17) and (18) imply: $\underline{G}_i(x) < \underline{G}_i(y)$. Suppose now that for $i \in \{1, ..., n\}$:

$$F_i(y_1, ..., \underline{G}_i(y), ..., y_n) = F_i(x_1, ..., \underline{G}_i(y), ..., x_n). \tag{19}$$

By the definition of $\underline{G}$, (17) and (19) imply $\underline{G}_i(x) \leq \underline{G}_i(y)$. Thus $\underline{G}$ is isotone on D'. The proof of isotonicity of $\overline{G}$ is very similar.

Consider now x^0, by the definition of $\underline{G}$, we have

$$F_i(x_1^0, ..., \underline{G}_i(x^0), ..., x_n^0) = z_i, i = 1, ..., n. \tag{20}$$

Moreover we have $F(x^0) \leq z$. Suppose that for $i \in \{1, ..., n\}$:

$$F_i(x^0) < z_i. \tag{21}$$

By the diagonal isotonicity of F, (20) and (21) imply $x_i^0 < \underline{G}_i(x^0)$. Suppose now that for $i \in \{1, ..., n\}$:

$$F_i(x^0) = z_i. \tag{22}$$

By the definition of $\underline{G}$, (20) and (22) imply $\underline{G}_i(x^0) = x_i^0$. Hence, $x^0 \leq \underline{G}(x^0)$. We can show analogously that: $\underline{G}(y^0) \leq y^0$. Moreover by the isotonicity of $\underline{G}$, (2) implies $\underline{G}(x^0) \leq \underline{G}(y^0)$. We can show analogously that: $x^0 \leq \overline{G}(x^0) \leq \overline{G}(y^0) \leq y^0$. Q.E.D.

6.2. Proof of proposition 2.2

It follows from (3) and (4) that $x^0 \leq x^1 \leq y^1 \leq y^0$. Suppose now that there exists $p \geq 1$ such that:

$$x^0 \leq ... \leq x^{p-1} \leq x^p \leq y^p \leq y^{p-1} \leq ... \leq y^0. \tag{23}$$

If $i \notin h^p$, then from (4) and (23) it follows that

$$x_i^p = x_i^{p+1} \leq y_i^{p+1} = y_i^p. \tag{24}$$

Consider now the following sets $S_i^p = \{m/i \in h^m, 0 \leq m < p\}, i = 1, ..., n, p = 0, 1, ...$
If $i \in h^p$ and S_i^p is empty, then from (4) it follows that

$$x_i^p = x_i^0, y_i^p = y_i^0. \tag{25}$$

For all $j \in \{1, ..., n\}, d_j(p) \geq 0, p = 0, 1, ...$ By the isotonicity of $\underline{G}$, (4), (23), (3) and (25) imply

$$x_i^{p+1} = \underline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)}) \geq \underline{G}_i(x^0) \geq x_i^0 = x_i^p, \tag{26}$$

$$y_i^{p+1} = \underline{G}_i(y_1^{d_1(p)}, ..., y_n^{d_n(p)}) \leq \underline{G}_i(y^0) \leq y_i^0 = y_i^p. \tag{27}$$

For all $j \in \{1, ..., n\}, d_j(p) \leq p, p = 0, 1, ...$ Hence, by the isotonicity of $\underline{G}$, (23), (26), and (27) imply

$$x_i^p \leq x_i^{p+1} = \underline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)}) \leq \underline{G}_i(y_1^{d_1(p)}, ..., y_n^{d_n(p)}) = y_i^{p+1} \leq y_i^p. \tag{28}$$

Suppose now that $i \in h^p$ and S_i^p is nonempty, consider $\overline{m} = \max_{m \in S_i^p} m$, from (4) we have:

$$x_i^p = x_i^{\overline{m}+1}, y_i^p = y_i^{\overline{m}+1}. \tag{29}$$

By the isotonicity of $\underline{G}$ and $d_j(p), j = 1, ..., n$, (4), (23), and (29) imply

$$x_i^{p+1} = \underline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)}) \geq \underline{G}_i(x_1^{d_1(\overline{m})}, ..., x_n^{d_n(\overline{m})}) = x_i^{\overline{m}+1} = x_i^p, \tag{30}$$

$$y_i^{p+1} = \underline{G}_i(y_1^{d_1(p)}, ..., y_n^{d_n(p)}) \leq \underline{G}_i(y_1^{d_1(\overline{m})}, ..., y_n^{d_n(\overline{m})}) = y_i^{\overline{m}+1} = y_i^p. \tag{31}$$

By the isotonicity of $\underline{G}$, (23), (30) and (31) imply

$$x_i^p \leq x_i^{p+1} = \underline{G}_i(x_1^{d_1(p)}, ..., x_n^{d_n(p)}) \leq \underline{G}_i(y_1^{d_1(p)}, ..., y_n^{d_n(p)}) = y_i^{p+1} \leq y_i^p. \tag{32}$$

It follows from (24), (28) and (32) that $x^p \leq x^{p+1} \leq y^{p+1} \leq y^p$. Hence, the monotone sequences $\{x^p\}$ and $\{y^p\}$ have limits x^* and y^*, with $x^* \leq y^*$. Since for all $i \in \{1,...,n\}$, i occurs infinitely often in the sequence $\{h^p\}$, there exist infinite sequences $\{p_i^q\} = \{p / i \in h^p\}, i = 1,...,n$. It follows from (4) that

$$\lim_{q\to\infty} \underline{G}_i(x_1^{d_1(p_i^q)},...,x_n^{d_n(p_i^q)}) = \lim_{q\to\infty} x_i^{p_i^q+1} = x_i^*, i = 1,...,n, \tag{33}$$

$$\lim_{q\to\infty} \underline{G}_i(y_1^{d_1(p_i^q)},...,y_n^{d_n(p_i^q)}) = \lim_{q\to\infty} y_i^{p_i^q+1} = y_i^*, i = 1,...,n. \tag{34}$$

Since for all $j \in \{1,...,n\}$, $\lim_{p\to\infty} d_j(p) = +\infty$, we have

$$\lim_{q\to\infty} x_j^{d_j(p_i^q)} = x_j^*, \lim_{q\to\infty} y_j^{d_j(p_i^q)} = y_j^*, j = 1,...,n, i = 1,...,n. \tag{35}$$

By the continuity of $\underline{G}_i$, (35) implies

$$\lim_{q\to\infty} \underline{G}_i(x_1^{d_1(p_i^q)},...,x_n^{d_n(p_i^q)}) = \underline{G}_i(x^*), i = 1,...,n, \tag{36}$$

$$\lim_{q\to\infty} \underline{G}_i(y_1^{d_1(p_i^q)},...,y_n^{d_n(p_i^q)}) = \underline{G}_i(y^*), i = 1,...,n. \tag{37}$$

It follows from (33), (34), (36), and (37) that $x^* = G(x^*), y^* = G(y^*)$. The proof of convergence for the fixed point mapping $\overline{G}$ is analogous with the one just given. Q.E.D.

6.3. Proof of Proposition 3.1

Since F is continuous on D, H is clearly continuous on D. From (9) it follows that for all $i \in \{1,...,n\}$, and for all $x, x' \in D$, such that $x_i' \leq x_i$ and $x_j' = x_j$ for $j \neq i$, we have

$$F_i(x) - F_i(x') \leq \|F(x) - F(x')\|_2 \leq \alpha. \|x - x'\|_2 = \alpha.(x_i - x_i'). \tag{38}$$

By the off-diagonal antitonicity of F, for all $i \in \{1,...,n\}$, and for all $x', x'' \in D$ such that $x'' \leq x'$ and $x_i'' = x_i'$, we have

$$F_i(x') \leq F_i(x''). \tag{39}$$

It follows from (38) and (39) that for all $i \in \{1,...,n\}$ and for all $x, x', x'' \in D$, such that $x'' \leq x', x_i'' = x_i' \leq x_i$ and $x_j' = x_j$ for $j \neq i$, we have $F_i(x) - F_i(x'') \leq F_i(x) - F_i(x') \leq \alpha.(x_i - x_i') \leq \alpha.(x_i - x_i'')$. Hence, H is isotone on D and (2) with $z = 0$, implies $x^0 \leq H(x^0) = x^0 - \frac{1}{\alpha}.F(x^0) \leq H(y^0) = y^0 - \frac{1}{\alpha}.F(y^0) \leq y^0$. Q.E.D.

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