Mathematical study of perturbed asynchronous iterations designed for distributed termination

D. El Baz[(1)], **P. Spiteri**[(2)], **J.C. Miellou**[(3)], **M. Jarraya**[(1)]

(1) LAAS du CNRS, 7 avenue du Colonel Roche,
F-31077 Toulouse Cedex 4 France

(2) ENSEEIHT-IRIT/LIMA, 2 rue Camichel,
B.P. 7122, F-31071 Toulouse-Cedex -France

(3) Département de Mathématiques, UFR Sciences et Techniques,
16 route de Gray, F-25030 Besançon-Cedex - France

Abstract

This paper deals with the mathematical study of perturbed fixed point asynchronous iterations designed for a distributed termination. The distributed termination of asynchronous iterations is considered by using a perturbed fixed point mapping, which is an approximation of an exact fixed point mapping. In the general framework of φ-approximate contraction, it is shown that the perturbed asynchronous iteration converges in finite time and that the limit of the perturbed asynchronous iteration belongs to a ball of finite radius and center $\tilde{u}^\star$, solution of the exact problem. The value of the radius is given in the case of linear and quadratic convergence, respectively.

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1. Introduction

Asynchronous iterative methods whereby iterations are carried out in parallel by several processors in arbitrary order and without synchronization are an original class of iterative schemes which is directly derived from the concept of parallelism. For more details on asynchronous iterations reference is made in particular to: [7], [17], [18], [1], [13], [2], [21], [3], [4], [9], [16], [14], [10], and [20]. The reader is also referred to [15] for a recent survey on asynchronous iterations.

The distributed termination of asynchronous iterative methods is one of the most complex problems related to the study of this class of algorithms. The complexity of this problem is due to the fact that processors go at their own pace (each processor may have its own clock) and are not synchronized. In a pioneering paper (see [5], see also Section 8.1 of [4]), Bertsekas and Tsitsiklis have presented an original distributed method in order to terminate asynchronous iterations, this method is the first one for which a formal proof of validity has been given. Other distributed termination methods for which there exists formal proof of validity have been proposed recently (see in particular [11] and [12], see also [6]).

In this paper, we concentrate on the distributed termination method of asynchronous iterations proposed by Bertsekas and Tsitsiklis (see [5]). We present this method by using a new mathematical formalism. Owing to the fact that the distributed method proposed by Bertsekas and Tsitsiklis relies on a perturbation, say an approximation, of the exact fixed point mapping, we derive an original result which shows that the perturbed asynchronous iteration converges in finite time and that the limit of the perturbed asynchronous iteration belongs to a ball of finite radius and center $\tilde{u}^\star$, solution of the exact problem. The value of the radius is given in the case of linear and quadratic convergence, respectively. This result holds for a large number of iterative methods including Newton's method and methods with linear convergence such as iterations associated with contraction in the usual sense. This result is established by making use of the concept of φ-approximate contraction. We note that φ-approximately contractant mappings have been introduced by Ortega and Rheinboldt in Section 12.2 of [22] for sequential iterative methods. In [19], this concept was adapted by Miellou et al. to parallel iterative methods in the context of perturbation of fixed point mappings.

Section 2 deals with successive approximation methods and more particularly asynchronous iterations in an abstract topological context. The concepts of φ-contraction and φ-approximate contraction are also presented in Section 2. Perturbed asynchronous iterations designed for distributed termination are presented in Section 3. The results concerning the convergence in finite time and the localization of the limit of a perturbed asynchronous iteration are also given in Section 3.

2. Successive approximation methods and topological context

2.1. General topological context

Let E be a topological space and consider an embedded sequence $\{E^n\}_{n\in N}$ of closed subsets of E, such that $E^{n+1} \subset E^n$ for all $n \in N$, the set of natural numbers. We denote by H the intersection of the subsets E^n; note that H is also closed in E.

Let $T : D(T) \subset E \to E$ be a mapping such that:

$$E^0 \subset D(T) \text{ and } T(E^n) \subset E^{n+1}, \forall n \in N.$$

Let $\{u^n\}$ be a sequence of E. We denote by $a(\{u^n\})$, the set, which is possibly empty, of the limits of subsequences of $\{u^n\}$. We denote by $R(T)$ the range of T, $R(T) = \{v \in E \mid \exists u \in D(T) \text{ with } v = Tu\}$, and assume that if $\{u^n\}$ is a sequence of elements belonging to the closure of $E^0 \cap R(T)$, then we can extract a subsequence which converges in E and whose limit will be denoted by $u^\star$. Miellou et al. have shown in [19] that under the above assumptions, the successive approximation method:

$$u^{n+1} = T(u^n), n = 0, 1, ..., \text{ with } u^0 \in E^0, \tag{1}$$

satisfies:

$$a(\{u^n\}) \neq \emptyset \text{ and } a(\{u^n\}) \subset H.$$

In the particular case where $H = \cap_{n\in N} E^n = \{u^\star\}$, we have: $a(\{u^n\}) = \{u^\star\}$.

2.2. φ-contractant mappings

Let E be a metric space endowed with the metric $\mid u, v \mid_E$ for all u, v elements of E. Consider the mapping $T : D(T) \subset E \to E$ such that the interior of $D(T)$, denoted by $\dot{D}(T)$, is nonempty. Let $u^\star \in \dot{D}(T)$ and δ a positive real number such that the closed ball of center $u^\star$ and radius δ in E, denoted by $B_E(u^\star; \delta)$, satisfy: $B_E(u^\star; \delta) \subset \dot{D}(T)$.

Definition 1: The mapping T is φ-contractant in $B_E(u^\star; \delta)$, if there exists a continuous isotone function $\varphi : R^+ \to R^+$ such that : $\varphi(0) = 0$, $\varphi(\delta) < \delta$ and

$$\mid Tu, Tv \mid_E \leq \varphi(\mid u, v \mid_E), \ \forall u, v \in B_E(u^\star; \delta). \tag{2}$$

Definition 2: Let $u^\star$ be a fixed point of T. The mapping T is φ-contractant with respect to $u^\star$ in $B_E(u^\star; \delta)$, if there exists a continuous isotone function $\varphi : R^+ \to R^+$ such that : $\varphi(0) = 0$, $\varphi(\delta) < \delta$ and

$$| u^\star, Tv |_E \leq \varphi(| u^\star, v |_E), \ \forall v \in B_E(u^\star; \delta). \tag{3}$$

Remark 1: The function φ is attached to the fixed point mapping T. In the case of linear convergence we have : $\varphi(x) = l \cdot x$ with $0 \leq l < 1$. In the case of quadratic convergence we have : $\varphi(x) = c \cdot x^2$ with $0 < c$. In the sequel, we shall consider particular cases where we have : $\varphi(x) < x$ for all $x \in (0, \delta]$ i.e. $0 < x \leq \delta$.

Remark 2: In the particular case where $H = \{u^\star\}$ and T is a contractant fixed point mapping defined in a complete metric space, the sets E^n can be naturally chosen for the closed balls of center $u^\star$ and radius $l^n \cdot \mid u^\star, u^0 \mid_E$ where l is the constant of contraction of T and $| u^\star, u^0 |_E$ denotes the distance between $u^\star$ and u^0 in the metric space E.

Remark 3: In the particular case where T is a φ-contractant fixed point mapping in $B_E(u^\star; \delta)$, with $u^\star \in H$ and $\varphi(x) < x$, for all $x \in (0, \delta]$, it follows from $\varphi(0) = 0$, that $x^\star = 0$ is the only solution in $[0, \delta]$ of the fixed point equation: $\varphi(x^\star) = x^\star$. The sets E^n can be naturally chosen as follows: $E^0 = B_E(u^\star; | u^\star, u^0 |_E)$ with $| u^\star, u^0 |_E \leq \delta$ and $E^n = B_E(u^\star; \varphi^n(| u^\star, u^0 |_E))$, where φ^n denotes the n-th power of φ. In this particular case, we note that we have also: $H = \{u^\star\}$ since T is strictly nonexpansive (see Section 5 of [22]).

2.3. φ-approximately contractant mappings

We present now the important concept of φ-approximate contraction which will be very useful in the sequel.

Definition 3: The mapping T is φ-approximately contractant (in brief φ-a-contractant) with respect to $u^\star$ in $B_E(u^\star; \delta)$, if there exist a nonnegative real number θ, and a continuous isotone function $\varphi : R^+ \to R^+$ such that

$$\varphi(0) = 0, \tag{4}$$

$$\theta \leq \delta - \varphi(\delta), \tag{5}$$

and

$$| u^\star, Tv |_E \leq \varphi(| u^\star, v |_E) + \theta, \forall v \in B_E(u^\star; \delta). \tag{6}$$

Remark 4: In the sequel, the constant of approximation θ is related to the perturbation of the fixed point mapping.

It follows from the continuity and the isotonicity of the function φ that the function $f : R^+ \to R^+$ defined by $f(x) = \varphi(x) + \theta$ is continuous and isotone. It follows from (4) and (5) respectively, that 0 and δ are a subsolution and a supersolution, respectively, of the fixed point equation:

$$f(x^\star) = x^\star. \tag{7}$$

We shall denote by $\delta^\star$ the largest fixed point of (7) that is smaller than δ.

Remark 5: In the particular case where φ is linear and T is approximately contractant with respect to $u^\star$ in $B_E(u^\star;\delta)$, i.e. $\exists\theta, \in R^+$ and $l \in [0,1)$ such that $\theta \leq \delta - l \cdot \delta$ and

$$\mid u^\star, Tv \mid_E \leq l \cdot \mid u^\star, v \mid_E + \theta,\ \forall v \in B_E(u^\star;\delta), \tag{8}$$

we have: $\delta^\star = \frac{\theta}{(1-l)}$ by definition of $\delta^\star$.

Remark 6: Consider now the quadratic case, where for example T is related to the Newton mapping and φ-a-contractant with respect to $u^\star$ in $B_E(u^\star;\delta)$, with $\varphi(x) = c \cdot x^2$, $0 < c$ and the value of θ is small enough so that $c\theta \leq \frac{1}{4}$. The real roots of $c \cdot x^2 - x + \theta = 0$ are:

$$\delta^\star = \frac{1 - (1 - 4c\theta)^{\frac{1}{2}}}{2c},\ \delta^{\star\star} = \frac{1 + (1 - 4c\theta)^{\frac{1}{2}}}{2c}. \tag{9}$$

For $\delta \in (\delta^\star, \delta^{\star\star})$ we have:

$$c\delta^2 + \theta < \delta. \tag{10}$$

We note that for θ sufficiently small we have: $(1 - 4c\theta)^{\frac{1}{2}} \simeq 1 - 2c\theta$ and $\delta^\star \simeq \theta$ with $\delta^\star > \theta$.

2.4. Asynchronous iterations

Let us consider now α metric spaces $E_i,\ i = 1, ..., \alpha$, endowed respectively with the metric $\mid u_i, v_i \mid_{E_i}$, for $u_i, v_i \in E_i$. Let us consider the product space: $E = \prod_{i=1}^{\alpha} E_i$. We write $u \in E$ as follows: $u = (u_1, ..., u_\alpha)$, with $u_i \in E_i$, $i = 1, ..., \alpha$. The space E is endowed with the metric:

$$\mid u, v \mid_E = \max_i \frac{1}{\gamma_i} \mid u_i, v_i \mid_{E_i},\ \forall u, v \in E, \tag{11}$$

where $\gamma_i > 0$, for $i = 1, ..., \alpha$.
Let $T : D(T) \subset E \to E$ be the fixed point mapping defined by:

$$T(u) = (T_1(u), ..., T_\alpha(u)),\ \text{for all } u \in D(T),\ \text{with } T_i(u) \in E_i,\ i = 1, ..., \alpha.$$

We define asynchronous iterations as follows:

Definition 4: An asynchronous iteration associated with the fixed point mapping T and the initial guess $u^0 \in D(T)$, is the sequence $\{u^n\}$ of vectors of E defined recursively as follows for all $i \in \{1, ..., \alpha\}$:

$$u_i^{n+1} = \begin{cases} T_i(...., u_j^{s_j(n)}, ...), \text{ if } i \in J(n), \\ u_i^n, \text{ if } i \notin J(n), \end{cases} \tag{12}$$

where the strategy $J = \{J(n)\}_{n \in N}$ is a sequence of nonempty subsets of $\{1, ..., \alpha\}$, $J(n)$ is the subset of the indices of the blocks of components updated at the n-th iteration and $S = \{(s_1(n), ..., s_\alpha(n))\}_{n \in N}$ is a sequence of elements of N^α, which corresponds to delayed iteration numbers. For all $i \in \{1, ..., \alpha\}$:

$$\text{the set } \{n \in N \mid i \in J(n)\} \text{ is infinite,} \tag{13}$$

$$0 \leq s_i(n) \leq n, \ \forall n \in N, \tag{14}$$

$$s_i(n) = n, \forall i \in J(n) \text{ and } n \in N, \tag{15}$$

$$\lim_{n \to \infty} s_i(n) = +\infty. \tag{16}$$

Remark 7: We note that hypothesis (15) is an extra hypothesis as compared to the standard asynchronous iterations model (see for example [1]); however, this assumption is fulfilled in all current computational models for which there is static allocation of tasks to the processors.

We have the following result (see [19]).

Theorem 1: *Assume that the closure of $B_E(u^\star; \delta) \cap R(T)$ is compact and T is φ-a-contractant with respect to $u^\star$ in $B_E(u^\star; \delta)$. Then, the asynchronous iteration $\{u^n\}$ with initial guess $u^0 \in B_E(u^\star; \delta)$ is well defined and we have:*

$$a(\{u^n\}) \neq \emptyset \text{ and } a(\{u^n\}) \subset B_E(u^\star; \delta^\star). \tag{17}$$

This result was established by using the concept of φ-approximately contractant mappings and the abstract result related to the localization of the limits of subsequences of successive approximation methods.

3. Perturbed asynchronous iterations

We consider now $\tilde{T}$, a given fixed point mapping from $D(\tilde{T}) \subset E \to E$ which is associated with a fixed point algorithm. We can quote for example: methods with quadratic convergence such as Newton's method or methods with linear convergence i.e. methods associated with contraction in the usual sense such as relaxation and multisplitting.

According to Definition 4, an asynchronous iteration associated with the fixed point mapping $\tilde{T}$ and the initial guess $\tilde{u}^0 \in D(\tilde{T})$, is the sequence $\{\tilde{u}^n\}$ of vectors of E defined recursively as follows for all $i \in \{1, ..., \alpha\}$:

$$\tilde{u}_i^{n+1} = \begin{cases} \tilde{T}_i(...., \tilde{u}_j^{s_j(n)}, ...), \text{ if } i \in J(n), \\ \tilde{u}_i^n, \text{ if } i \notin J(n), \end{cases} \tag{18}$$

where the strategy J and the delayed iteration numbers sequence S verify conditions (13) to (16).

In general, asynchronous iterations given by the above model do not converge in finite time and thus never terminate.

We present now the perturbed fixed point mapping $T : D(T) \subset E \to E$, associated with $\tilde{T}$. The perturbed fixed point mapping is introduced in order to derive a perturbed asynchronous iterative scheme which converges in finite time and whose termination is detected in a distributed way.

Definition 5: Let θ' be a given positive real number which is related to the perturbation of the fixed point mapping. The perturbed fixed point mapping $T : D(T) \subset E \to E$ associated with $\tilde{T}$ is such that for all $u \in E$ and $i \in \{1, ..., \alpha\}$, we have:

$$T_i(u) = \tilde{T}_i(u) \text{ if } \mid \tilde{T}_i(u), u_i \mid_{E_i} > \theta',$$

$$T_i(u) = u_i \text{ if } \mid \tilde{T}_i(u), u_i \mid_{E_i} \leq \theta'.$$

Remark 8: We have $D(T) = D(\tilde{T})$. We note that according to Definition 5, the perturbed fixed point mapping T is such that for all $i \in \{1, ..., \alpha\}$, if applying $\tilde{T}_i$ to vector u does not lead to a significant improvement of u_i, then u_i remains unchanged when applying T_i. The mapping T is introduced in order to design a perturbed asynchronous iteration which will eventually reach an inactive state where none of the block components u_i changes and for which we can detect termination in a distributed way.

Proposition 1: *Let $\tilde{u}^\star$ be a fixed point of $\tilde{T}$. If the mapping $\tilde{T}$ is φ-*

contractant with respect to $\tilde{u}^\star$ in $B_E(\tilde{u}^\star;\delta)$, then the perturbed fixed point mapping T is φ-a-contractant with respect to $\tilde{u}^\star$ in $B_E(\tilde{u}^\star;\delta)$.

Proof: it follows from Definition 5 that for all $u \in E$ and $i \in \{1,...,\alpha\}$, we have:

$$\mid \tilde{T}_i(u), T_i(u) \mid_{E_i} = 0, \text{ if } \mid \tilde{T}_i(u), u_i \mid_{E_i} > \theta',$$

$$\mid \tilde{T}_i(u), T_i(u) \mid_{E_i} \leq \theta', \text{ if } \mid \tilde{T}_i(u), u_i \mid_{E_i} \leq \theta'.$$

Thus,

$$\mid \tilde{T}_i(u), T_i(u) \mid_{E_i} \leq \theta', \ \forall i \in \{1,...,\alpha\} \text{ and } u \in E. \tag{19}$$

Moreover, we have:

$$\mid \tilde{u}^\star, T(u) \mid_E \leq \mid \tilde{u}^\star, \tilde{T}(u) \mid_E + \mid \tilde{T}(u), T(u) \mid_E, \forall u \in E. \tag{20}$$

From the φ-contractant property of the mapping $\tilde{T}$ with respect to $\tilde{u}^\star$ in $B_E(\tilde{u}^\star;\delta)$ and equations (19) and (20) it follows that:

$$\mid \tilde{u}^\star, T(u) \mid_E \leq \varphi(\mid \tilde{u}^\star, u \mid_E) + \theta, \forall u \in B_E(\tilde{u}^\star;\delta), \tag{21}$$

where $\theta = \theta' \cdot \max_i \frac{1}{\gamma_i}$.

We define now the perturbed asynchronous iteration.

Definition 6: Let us consider an initial guess $u^0 \in D(T)$. A perturbed asynchronous iteration is the sequence $\{u^n\}$ of vectors of E defined recursively as follows for all $i \in \{1,...,\alpha\}$:

$$u_i^{n+1} = \begin{cases} T_i(...., u_j^{s_j(n)}, ...), \text{ if } i \in J(n), \\ u_i^n, \text{ if } i \notin J(n), \end{cases} \tag{22}$$

where T is given in Definition 5, the strategy J and the delayed iteration numbers sequence S satisfy assumptions (13) to (16).

We have the following important result.

Proposition 2: *Let $\tilde{u}^\star$ be a fixed point of $\tilde{T}$. If the mapping $\tilde{T}$ is φ-contractant with respect to $\tilde{u}^\star$ in $B_E(\tilde{u}^\star;\delta)$ and the closure of $B_E(\tilde{u}^\star;\delta) \cap R(T)$*

is compact, then the perturbed asynchronous iteration $\{u^n\}$ *associated with* T *and* $u^0 \in B_E(u^\star;\delta)$ *is well defined and we have:*

$$a(\{u^n\}) \neq \emptyset \text{ and } a(\{u^n\}) \subset B_E(\tilde{u}^\star;\delta^\star). \tag{23}$$

Proof : The proof follows directly from Proposition 1 and Theorem 1 by substituting $\tilde{u}^\star$ for $u^\star$ and perturbed asynchronous iteration for asynchronous iteration in Theorem 1.

Remark 9: If θ is sufficiently small and $1-l$ is small, then we note that the case of linear convergence, where $\delta^\star = \frac{\theta}{(1-l)}$, is worse in terms of accuracy, than the case of quadratic convergence, where $\delta^\star \simeq \theta$ (see Remarks 5 and 6).

In the sequel, we will show that the perturbed asynchronous iteration $\{u^n\}$ converges in finite time to a fixed point $u^\star$ by using a mathematical formalism which is new; we note that this formalism is different from the formalism used in [4] and [5]. With respect to this last remark, we introduce a new strategy $J' = \{J'(n)\}$.

Definition 7: The new strategy $J' = \{J'(n)\}$ is such that:

$$J'(n) = \{i \in \{1,...,\alpha\} \mid i \in J(n) \text{ and } \mid \tilde{T}_i(....,u_j^{s_j(n)},...), u_i^n \mid_{E_i} > \theta'\}, \tag{24}$$

where $J(n)$ and S are defined according to Definition 4.

We note that $J'(n) \subset J(n)$. It follows from (15) and Definitions 5 to 7 that we have the following alternative definition of the perturbed asynchronous iteration $\{u^n\}$. This definition will be useful in the sequel in order to show the convergence in finite time of the perturbed asynchronous iteration $\{u^n\}$. For all $i \in \{1,...,\alpha\}$ we have:

$$u_i^{n+1} = \begin{cases} \tilde{T}_i(....,u_j^{s_j(n)},...), \text{ if } i \in J'(n), \\ u_i^n, \text{ if } i \notin J'(n), \end{cases} \tag{25}$$

where the delayed iteration numbers sequence S satisfies assumptions (14) to (16).

We define the sets $P(i), i \in \{1,...,\alpha\}$ as follows:

$$P(i) = \{n \in N \mid i \in J'(n)\}. \tag{26}$$

We define the set I as follows:

$$I = \{i \in \{1, ..., \alpha\} \mid Card(P(i)) = +\infty\}. \tag{27}$$

In fact, I denotes the subset of $\{1, ..., \alpha\}$ associated with the blocks of components which are updated an infinite number of times by the perturbed asynchronous iteration.

Let $u_i^\star$ be the limiting value of $\{u_i^n\}$, for all $i \in \bar{I}$, where $\bar{I}$ denotes the complementary subset of I in $\{1, ..., \alpha\}$. We introduce now a new fixed point mapping and a new asynchronous iterative sequence $\{\hat{u}^n\}$ which will be useful in order to show the convergence in finite time of the perturbed asynchronous iteration $\{u^n\}$.

Definition 8: The new fixed point mapping $T^I : D(\tilde{T}) \subset E \to E$ is such that for all $u \in E$ and $i \in \{1, ..., \alpha\}$ we have:

$$T_i^I(u) = \tilde{T}_i(u), \text{ if } i \in I, \tag{28}$$

$$T_i^I(u) = u_i^\star, \text{ if } i \in \bar{I}. \tag{29}$$

Definition 9: Let us consider an initial guess $\hat{u}^0 = u^0 \in D(T)$. The new asynchronous iteration $\{\hat{u}^n\}$ is the sequence of vectors of E defined recursively as follows for all $i \in \{1, ..., \alpha\}$:

$$\hat{u}_i^{n+1} = \begin{cases} T_i^I(...., \hat{u}_j^{s_j(n)}, ...), \text{ if } i \in J(n), \\ \hat{u}_i^n, \text{ if } i \notin J(n), \end{cases} \tag{30}$$

where the strategy $J = \{J(n)\}_{n \in N}$ and the delayed iteration numbers sequence S verify conditions (13) to (16).

Proposition 3: *Let $\tilde{u}^\star \in \dot{D}(\tilde{T})$ and assume that the mapping $\tilde{T}$ is φ-contractant in $B_E(\tilde{u}^\star; \delta)$, with $\varphi(x) < x$, for all $x \in (0, \delta]$ and $\tilde{T}(B_E(\tilde{u}^\star; \delta)) \subset B_E(\tilde{u}^\star; \delta)$. Then, the mapping T^I, defined in Definition 8, has a fixed point $\hat{u}^\star$ in $B_E(\tilde{u}^\star; \delta)$ and the new asynchronous iteration $\{\hat{u}^n\}$, defined in Definition 9, with initial guess $\hat{u}^0 \in B_E(\hat{u}^\star; \delta') \subset B_E(\tilde{u}^\star; \delta)$ converges to $\hat{u}^\star$. Moreover, the perturbed asynchronous iteration $\{u^n\}$ defined in Definition 6 with initial guess $u^0 = \hat{u}^0 \in B_E(\hat{u}^\star; \delta') \subset B_E(\tilde{u}^\star; \delta)$ converges in finite time to $u^\star = \hat{u}^\star$.*

Proof: Since $\tilde{T}$ is φ-contractant in $B_E(\tilde{u}^\star; \delta)$, it follows from Definitions 1 and 8 that we have:

$$\mid T^I(u), T^I(v) \mid_E \leq \mid \tilde{T}(u), \tilde{T}(v) \mid_E \leq \varphi(\mid u, v \mid_E), \forall u, v \in B_E(\tilde{u}^\star; \delta).$$

Thus, T^I is φ-contractant in $B_E(\tilde{u}^\star;\delta)$. It follows from $\tilde{T}(B_E(\tilde{u}^\star;\delta)) \subset B_E(\tilde{u}^\star;\delta)$ and Definition 8 that $T^I(B_E(\tilde{u}^\star;\delta)) \subset B_E(\tilde{u}^\star;\delta)$. Thus, the mappings T^I and $\tilde{T}$ have a fixed point in $B_E(\tilde{u}^\star;\delta)$ (see Section 6 of [22]). However, the mappings T^I and $\tilde{T}$ may not have the same fixed point in $B_E(\tilde{u}^\star;\delta)$. Let $\hat{u}^\star$ be a fixed point of T^I and consider the ball: $B_E(\hat{u}^\star;\delta') \subset B_E(\tilde{u}^\star;\delta)$. It follows from Remark 3 and classical results of convergence such as the Theorem of convergence of Bertsekas (see [2] and p. 431 of [4]) that $\{\hat{u}^n\} \to \hat{u}^\star$ which shows the first part of the Proposition.

Consider now the perturbed asynchronous iteration $\{u^n\}$ associated with T and $u^0 = \hat{u}^0$. It follows from the definitions of I and $u_i^\star$ for all $i \in \bar{I}$, that for all ϵ, there exists $n(\epsilon)$ such that for all $n \geq n(\epsilon)$, we have: $\mid u_i^n, u_i^\star \mid_{E_i} \leq \epsilon$, for all $i \in \bar{I}$, Moreover, according to Definitions 5, 6, 8, and 9, the perturbed asynchronous iteration $\{u^n\}$ which is different from the new asynchronous iteration $\{\hat{u}^n\}$ eventually becomes identical with $\{\hat{u}^n\}$ after a certain number of iterations and therefore converges to the same limit point.

We show now that $\{u^n\}$ converges in finite time. It follows from the convergence of $\{u^n\}$ that for all ϵ, there exists $n(\epsilon)$ such that for all $n \geq n(\epsilon)$, we have: $\mid u_i^n, u_i^\star \mid_{E_i} \leq \epsilon$ for all $i \in \{1, ..., \alpha\}$. Thus, for all ϵ, there exists $n(\epsilon)$ such that for all $n, n' \geq n(\epsilon)$, we have: $\mid u_i^n, u_i^{n'} \mid_{E_i} \leq \mid u_i^n, u_i^\star \mid_{E_i} + \mid u_i^{n'}, u_i^\star \mid_{E_i} \leq 2\epsilon$ for all $i \in \{1, ..., \alpha\}$.
For $\epsilon \leq \frac{\theta'}{2}$ and all $n \geq n(\epsilon)$ we have in particular: $\mid u_i^{n+1}, u_i^n \mid_E \leq \theta'$ for all $i \in \{1, ..., \alpha\}$. Thus, it follows from Definition 7 that there exists an $\hat{n}$ such that for all $n \geq \hat{n}$, $J'(n) = \emptyset$, as a consequence $I = \emptyset$. It follows from (25) that the perturbed asynchronous iteration $\{u^n\}$ converges in finite time to $u^\star$.

Remark 10: Since the perturbed asynchronous iteration converges in finite time and the perturbed fixed point mapping T uses the local termination test: $\mid \tilde{T}_i(u), u_i \mid_{E_i} \leq \theta'$, the distributed termination procedure of the perturbed asynchronous iteration $\{u^n\}$ will be derived from the combination of the local termination tests by using for example the distributed termination detection procedure of Dijkstra and Scholten (see [8]), see also [4] and [5]) which is based on message acknowledgment and generation of an activity graph.

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