Foundations of Computing

Module Introduction

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Outline

1. Constraint Programming

2. Clause Learning in CP

Constraint Programming

- Constraint Satisfaction Problems are generalization of Boolean satisfiability to non-Boolean domains
- Standard constraint programming solvers are similar to DPLL
  - No clause learning (Clause-learning CSP solvers existed before CDCL but were not that successful)
  - But stronger propagation

Constraint Propagation

Given a constraint $c = (R(c), S(c))$, a *propagator* is an algorithm that reduce the domains so that the constraint is *arc consistent*.

- A constraint solver is a library of constraints, each with its dedicated propagator

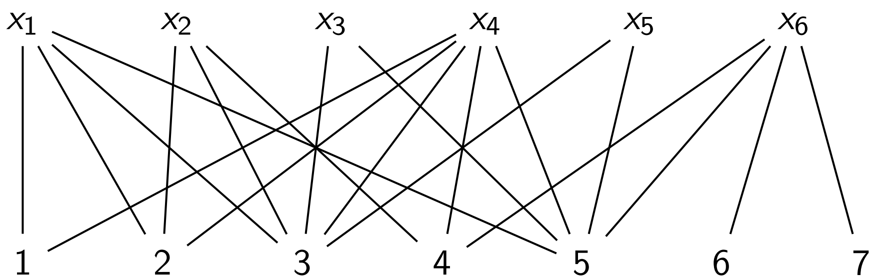
Arc Consistency

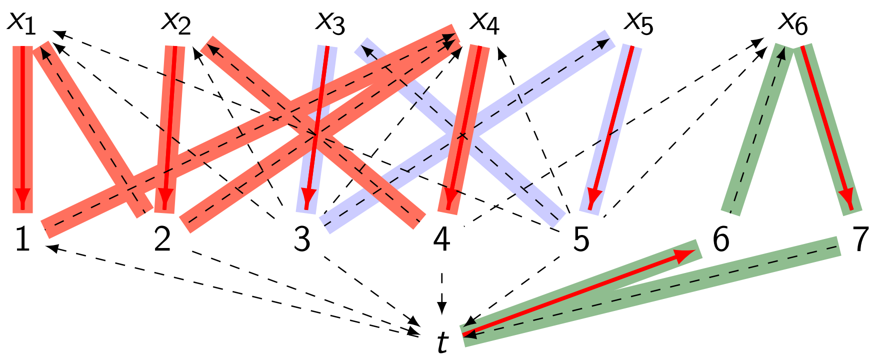
A constraint c is *Arc Consistent* on domain $\mathcal{D}$ if and only if for every $x \in S(c)$ and for every $j \in \mathcal{D}(x)$, there exists a tuple $\sigma \in R(c) \cap \prod_{x \in \mathcal{X}} \mathcal{D}(x)$ such that $\sigma(x) = j$.

- The constraint can be a clause: arc consistency corresponds to unit propagation
- The constraint can be a primitive relation (e.g., '$\leq$') and arc consistency is easy and efficient
  - Propagation of $x \leq y$:
    - Event *lower bound of x ($\min(x)$) has changed:* update $\min(y)$ to $\min(x)$
    - Event *upper bound of y has changed:* update $\max(x)$ to $\max(y)$
    - Do not wake up on other events
- Can be a much larger and more complex relation, even an $\mathbf{NP}$-hard relation
  - E.g., "the graph given by the incidence matrix $\mathbf{x}$ is a clique of size greater than or equal to y"
  - Arc consistency is not required for correctness (and is $\mathbf{NP}$-hard when the constraint relation is $\mathbf{NP}$-hard)

- $\textsc{AllDifferent}(x_1, \ldots, x_n) \Leftrightarrow \forall 1 \leq i < j \leq n,\ x_i \neq x_j$
- For instance: $\textsc{AllDifferent}(x_1, x_2, x_3, x_4)$
  - $\mathcal{D}(x_1) = \{1\}$
  - $\mathcal{D}(x_2) = \{1, 2, 3\}\mathcal{D}(x_2) = \{2, 3\}$
  - $\mathcal{D}(x_3) = \{1, 2, 3\}\mathcal{D}(x_3) = \{2, 3\}$
  - $\mathcal{D}(x_4) = \{1, 2, 3, 4\}\mathcal{D}(x_4) = \{4\}$
- Only two solutions: $(1, 2, 3, 4)$ and $(1, 3, 2, 4)$, therefore:
  - $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_4 = 2$, $x_4 = 3$ are not *viable*
- How can we compute that *efficiently*?
  - Generating and testing the validity all permutations would take exponential time

- $\textsc{AllDifferent}(x_1, \ldots, x_n) \Leftrightarrow \forall 1 \leq i < j \leq n,\ x_i \neq x_j$
- For instance: $\textsc{AllDifferent}(x_1, x_2, x_3, x_4, x_5, x_6)$
  - $\mathcal{D}(x_1) = \{1, 2, 3, 5\}$
  - $\mathcal{D}(x_2) = \{2, 3, 4\}$
  - $\mathcal{D}(x_3) = \{3, 5\}$
  - $\mathcal{D}(x_4) = \{1, 2, 3, 4, 5\}$
  - $\mathcal{D}(x_5) = \{3, 5\}$
  - $\mathcal{D}(x_6) = \{4, 5, 6, 7\}$





- A solution of the AllDifferent constraint is a *maximal matching* of the graph
- We can compute a maximal matching in $O(n^{\frac{3}{2}}m)$ (Hopcroft Karp)
- Cycle: alternative matching. *Strongly Connected Components* are set of vertices all pairwise connected by a cycle. Tarjan's Algorithm finds them all in $O(nm)$
- An edge (x, v) belongs to a strongly connected component iff the value v is viable for $x \Rightarrow$ pruning!

AllDifferent

- $\text{AllDifferent}(x_1, \ldots, x_n) \Leftrightarrow \forall 1 \leq i < j \leq n,\ x_i \neq x_j$
- For instance: $\text{AllDifferent}(x_1, x_2, x_3, x_4, x_5, x_6)$
  - $\mathcal{D}(x_1) = \{1, 2, 3, 5\}$
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  - $\mathcal{D}(x_5) = \{3, 5\}$
  - $\mathcal{D}(x_6) = \{4, 5, 6, 7\}$

  - $\mathcal{D}(x_1) = \{1, 2\}$
  - $\mathcal{D}(x_2) = \{2, 4\}$
  - $\mathcal{D}(x_3) = \{3, 5\}$
  - $\mathcal{D}(x_4) = \{1, 2, 4\}$
  - $\mathcal{D}(x_5) = \{3, 5\}$
  - $\mathcal{D}(x_6) = \{6, 7\}$

Constraint Propagation

- When and how propagators are called?
- Typically via a *Constraint Queue* and an *Event Stack*
- The event stack contains events corresponding to domain *reduction*
  - Variable x is assigned a value v
  - The lower (resp. upper) bound of variable x has increased (resp. decreased)
  - The domain of variable x has lost at least one value
  - The domain of variable x has lost at value v
- Every propagator *watches* some events

Algorithm 0: Constraint Propagation

```
repeat
    while Event-Stack ≠ ∅ do
        e ← Event-Stack.pop-back();
        foreach c ∈ Watchers(e) do
            Constraint-Queue.add(c);
    if Constraint-Queue ≠ ∅ then
        c ← Constraint-Queue.pop-priority();
        c.propagate(e);
        /* might push events on the event stack */
until Event-Stack = ∅;
```

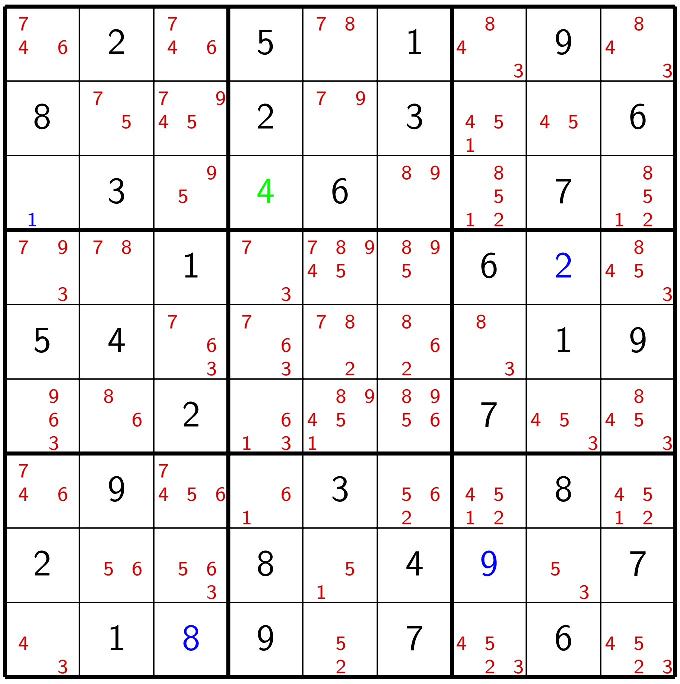
Sudoku

7 8 9 4 5 6 1 2 3	2	7 8 9 4 5 6 1 2 3	5	7 8 9 4 5 6 1 2 3	1	7 8 9 4 5 6 1 2 3	9	7 8 9 4 5 6 1 2 3
8	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	2	7 8 9 4 5 6 1 2 3	3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	6
7 8 9 4 5 6 1 2 3	3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	6	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7	7 8 9 4 5 6 1 2 3
7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	1	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	6	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3
5	4	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	1	9
7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	2	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3
7 8 9 4 5 6 1 2 3	9	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	3	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	8	7 8 9 4 5 6 1 2 3
2	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	8	7 8 9 4 5 6 1 2 3	4	7 8 9 4 5 6 1 2 3	7 8 9 4 5 6 1 2 3	7
7 8 9 4 5 6 1 2 3	1	7 8 9 4 5 6 1 2 3	9	7 8 9 4 5 6 1 2 3	7	7 8 9 4 5 6 1 2 3	6	7 8 9 4 5 6 1 2 3

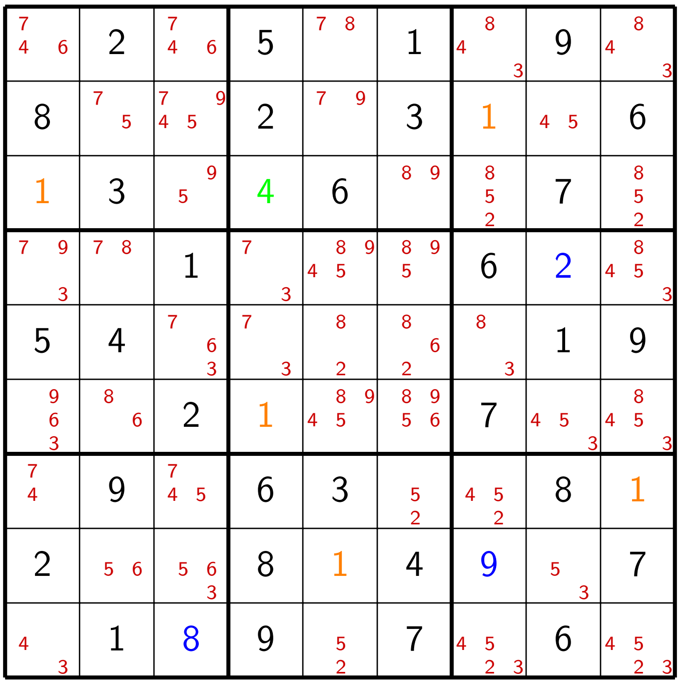
Sudoku $AC(\neq)$

7 4 6	2	7 4 6	5	7 8	1	8 4 3	9	8 4 3
8	7 5	7 9 4 5	2	7 9	3	4 5 1	4 5	6
9 1	3	9 5	4	6	8 9	8 5 1 2	7	8 5 1 2
7 9 3	7 8	1	7 3	7 8 9 4 5 2	8 9 5 2	6	4 5 2 3	8 4 5 2 3
5	4	7 8 6 3	7 6 3	7 8 2	8 6 2	8 2 3	1	9
9 6 3	8 6	2	6 1 3	8 9 4 5 1	8 9 5 6	7	4 5 3	8 4 5 3
7 4 6	9	7 4 5 6	6 1	3	5 6 2	4 5 1 2	8	4 5 1 2
2	5 6	5 6 3	8	5 1	4	9 5 1 3	5 3	7
4 3	1	8 4 5 3	9	5 2	7	4 5 2 3	6	4 5 2 3

Sudoku *BC*(AllDifferent)



Sudoku *AC*(AllDifferent)



Sudoku (Solution)

4	2	6	5	7	1	3	9	8
8	5	7	2	9	3	1	4	6
1	3	9	4	6	8	2	7	5
9	7	1	3	8	5	6	2	4
5	4	3	7	2	6	8	1	9
6	8	2	1	4	9	7	5	3
7	9	4	6	3	2	5	8	1
2	6	5	8	1	4	9	3	7
3	1	8	9	5	7	4	6	2

Sum

$$\sum_{i=1}^{n} a_i x_i = K$$

- Subset Sum: given a set of integers and an integer K, does there exist a subset whose sum is equal to K
  - A variable with domain $\{0, 1\}$ for each integer, coefficients are the inetegers
- Finding a support is NP-hard
  - Therefore, achieving AC is NP-hard
  - Achieving BC is NP-hard too, since on $\{0, 1\}$ domains, a bounds support is a support
- However, one can enforce BC on each conjunct of:

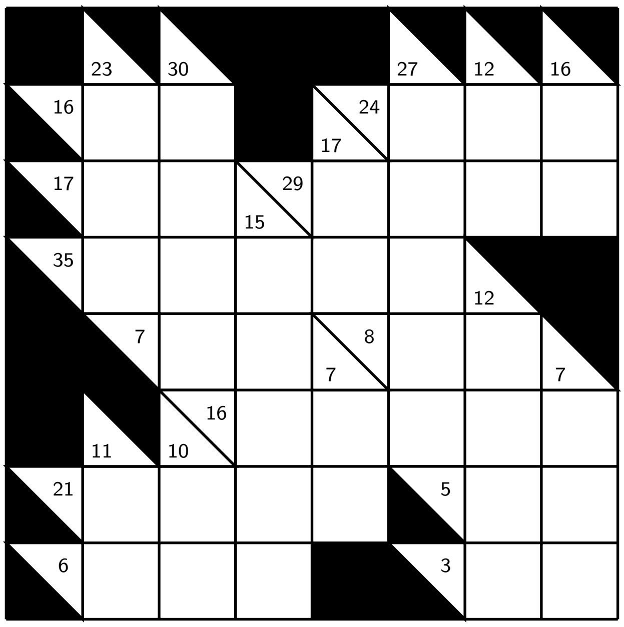
$$\sum_{i=1}^{n} a_i x_i \leq K \text{ and } \sum_{i=1}^{n} a_i x_i \geq K$$

Sum

$$\sum_{i=1}^{n} a_i x_i \leq K$$

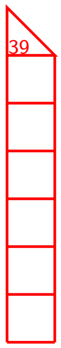
- Assume that all coefficients are positive
- $max(x_i) + \sum_{j=1}^{n} a_j \min(x_j) - min(x_i) \leq K$
  - $x_i \leq K - \sum_{j=1}^{n} a_j \min(x_j) - min(x_i)$
- $min(x_i) + \sum_{j=1}^{n} a_j \max(x_j) - min(x_i) \geq K$
  - $x_i \geq K - \sum_{j=1}^{n} a_j \max(x_j) - min(x_i)$

Kakuro



- $\sum_{i=1}^{6} x_i = 39$
- $\textsc{AllDifferent}(\{x_1, \ldots, x_6\}, \{1, \ldots, 9\})$

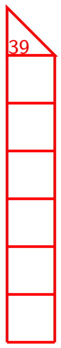
$$
\begin{array}{lllllllllll}
x_1 : & \{ & & & & & & & 8 & 9\} \\
x_2 : & \{1 & 2 & & & 6 & 7 & 8 & 9\} \\
x_3 : & \{ & & & & & & 8 & 9\} \\
x_4 : & \{1 & & & 5 & 6 & & 8 & 9\} \\
x_5 : & \{1 & 2 & & & 6 & 7 & 8 & 9\} \\
x_6 : & \{ & & 4 & 5 & & & 8 & 9\}
\end{array}
$$



Propagation

- $\sum_{i=1}^{6} x_i = 39$
- $\textsc{AllDifferent}(\{x_1, \ldots, x_6\}, \{1, \ldots, 9\})$

$$
\begin{array}{lllllllllll}
x_1 : & \{ & & & & & & 8 & 9\} \\
x_2 : & \{1 & 2 & & & 6 & 7 & & \} \\
x_3 : & \{ & & & & & & 8 & 9\} \\
x_4 : & \{1 & & & 5 & 6 & & & \} \\
x_5 : & \{1 & 2 & & & 6 & 7 & & \} \\
x_6 : & \{ & & 4 & 5 & & & & \}
\end{array}
$$



Propagation

- $\textsc{AllDifferent}(\{x_1, x_3\}, \{8, 9\})$

Example: Kakuro

- $\sum_{i=1}^{6} x_i = 39$
- $\textsc{AllDifferent}(\{x_1, \dots, x_6\}, \{1, \dots, 9\})$

$$\begin{array}{lllllllllll}
x_1: & \{ & & & & & & & 8 & 9\} \\
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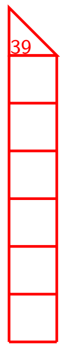
x_5: & \{ & & & & 6 & 7 & & & \} \\
x_6: & \{ & & 4 & 5 & & & & & \}
\end{array}$$



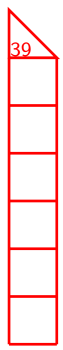
Propagation

- $\sum_{i=1}^{6} = 39$
  - $\Rightarrow \min(x_2) \geq 39 - \sum_{i \neq 2} \max(x_i)$
  - $\Rightarrow \min(x_2) \geq 3, (\& \ \min(x_5) \geq 3 \ \& \ \min(x_4) \geq 2)$

Example: Kakuro

- $\sum_{i=1}^{6} x_i = 39$
- $\textsc{AllDifferent}(\{x_1, \dots, x_6\}, \{1, \dots, 9\})$

$$\begin{array}{lllllllllll}
x_1: & \{ & & & & & & & 8 & 9\} \\
x_2: & \{ & & & & 6 & 7 & & & \} \\
x_3: & \{ & & & & & & & 8 & 9\} \\
x_4: & \{ & & & 5 & & & & & \} \\
x_5: & \{ & & & & 6 & 7 & & & \} \\
x_6: & \{ & & 4 & & & & & & \}
\end{array}$$



Propagation

- $\textsc{AllDifferent}(\{x_2, x_5\}, \{6, 7\})$
- $\textsc{AllDifferent}(\{x_4\}, \{5\})$

Outline

Motivation

- Constraint programming has powerful propagation algorithm
- Example, Kakuro:

Constraint Programming

- One variable $x_{i,j} \in \{1, \ldots, 9\}$ for every cell
- For every clue:
  - One AllDifferent constraint and two Cardinality constraints

SAT Encoding

- One variable $x_{i,j,v}$ for every cell and every $v \in \{1, \ldots, 9\}$ plus a linear number of clauses (somewhat equivalent)
- For every clue of size n:
  - 9(n-1)n/2 binary clauses to encode AllDifferent: unit propagation is not as strong as constraint propagation on AllDifferent
  - SAT encoding of cardinality: unit propagation is not as efficient as constraint propagation on Cardinality

- But no clause learning!
  - Clause learning was developped in CP (even before zChaff and GRASP) but was not as successful

Domains

- There are efficient encoding of domains, e.g., *sequential counters*
  - x_v: variable x takes value v, s_v: variable x lower than or equal to v
- Same space complexity ($O(|\mathcal{D}|)$)
- Domain change slightly less efficient
  - Assignement, value removal and bound change take $O(|\mathcal{D}|)$ time in the SAT encoding
  - They are in constant time in CP
  - However, amortized to the same worst-case down a branch (removing all values one at a time takes $O(|\mathcal{D}|)$ time in both cases)
  - There are many more *read* operations than *write* operations
- Domain events correspond to *domain literals*:
  - Upper bound of x has changed to v: s_v
  - Lower bound of x has changed to v: $\bar{s}_{v-1}$
  - Value v was removed from the domain of x: $\bar{x}_v$
  - Value v has been assigned to variable x: x_v

Lazy Clause Generation

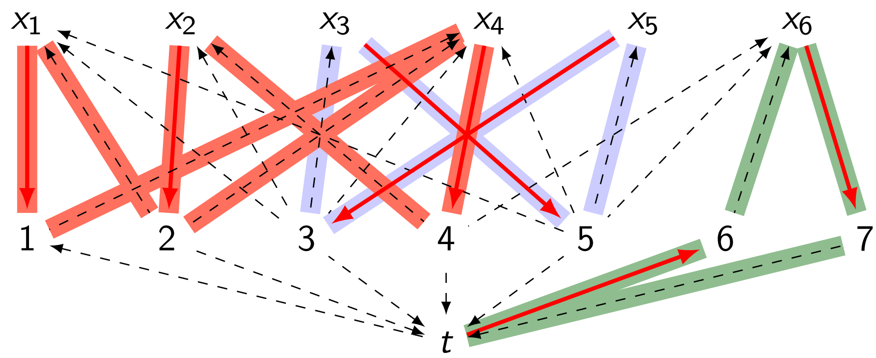
- Initially only *domain clauses*, constraints are propagated as in CP [Katsirelos & Bacchus]
- For every domain reduction l made by propagating a constraint generate an asserting explanation clause $(p_1 \vee p_2 \vee \ldots \vee l)$
  - Used during conflict analysis, but *not* for unit propagation (the propagator already does this pruning)
  - Learn first UIP clauses exactly as CDCL (and unit propagate them)
- Every constraint has a dedicated propagation algorithm *and an explanation algorithm*
  - Explanation clauses can be generated *a posteriori* (during conflict analysis) to avoid unecessary calls to the explanation algorithm

Example: $x \leq y$

- Propagation of $x \leq y$:
  - Event $\bar{x_v}$ (lower bound of x has changed to $v+1$): triggers $\bar{y_v}$
    - Explanation clause $(x_v \vee \bar{y_v})$
  - Event y_v (upper bound of y has changed): triggers x_v
    - Explanation clause $(x_v \vee \bar{y_v})$
- Suited for lazy explanation: the context is irrelevant

Explaining CARDINALITY

Explaining AllDifferent: Hall sets



- Strongly connected components that do not include t have as many variables as values (*Hall sets*)
  - The only way *to* a free value is via t
- Consider any edge $(v \rightarrow x)$ connecting a Hall set to a distinct SCC
  - There cannot be a edge between x and the Hall set of v otherwise the SCCs would not be distinct
- A *Hall set* is a set of variables $\mathcal{X}$ such that $|\bigcup_{x \in \mathcal{X}} \mathcal{D}(x)| = |\mathcal{X}|$
  - An edge $(v \rightarrow x)$ is arc inconsistent if and only if v is in a Hall set and x is not in the same SCC

Explaining AllDifferent

- For instance: $\textsc{AllDifferent}(x_1, x_2, x_3, x_4)$
  - $\mathcal{D}(x_1) = \{1, 2, 3\}$
  - $\mathcal{D}(x_2) = \{1, 2, 3\}$
  - $\mathcal{D}(x_3) = \{1, 2, 3\}$
  - $\mathcal{D}(x_4) = \{1, 2, 3, 4\}\mathcal{D}(x_4) = \{4\}$
- $\{1, 2, 3\}$ is a Hall set, therefore $\{1, 2, 3\}$ are not viable for x_4
- We can use the Hall set as explanation clause:

$$(s_{1,3} \wedge s_{2,3} \wedge s_{3,3}) \implies \neg s_{4,3}$$
$$\iff$$
$$(\neg s_{1,3} \vee \neg s_{2,3} \vee \neg s_{3,3} \vee \neg s_{4,3})$$

(i.e., if $x_1 \leq 3$ and $x_2 \leq 3$ and $x_3 \leq 3$, then $x_4 > 3$)

- Mapping between CSP variables and Boolean variables (can be implicit)
- Propagation of the original constraints is done via propagators (dedicated algorithms)
- Propagators generate explanation clauses, used to encode the *conflict graph*
- Learn First-UIP clauses with this conflict graph
- Propagate the learnt clauses via unit-propagation