Algorithms for Computational Logic

Introduction

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Outline

1. Extensions

Outline

Motivation

Facility Location Problem

Suppose that a company has to decide where to install new factories from n potential locations in order to be able to serve m clients.

Let c_i denote the cost for opening a factory at location i and let d_{ij} denote the cost of serving client j from location i.

Provide a formulation that helps the administration to decide where to open the factories such that the overall costs (factory open and serving clients) are minimized.

Facility Location Problem

- Problem variables
  - x_i : denotes if a factory is to be open at location i
  - y_{ij} : denotes if client j is served from location i

$$
\begin{array}{lll}
\text{Minimize} & \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} y_{ij} & \\
\text{Subject to} & \sum_{i=1}^{n} y_{ij} = 1 & \forall j \in \{1 \dots m\} \\
 & x_i - y_{ij} \geq 0 & \forall i \in \{1 \dots n\}, j \in \{1 \dots m\} \\
 & x_i \in \{0,1\}, y_{ij} \in \{0,1\} &
\end{array}
$$

Formulation

$$
\begin{array}{lll}
\text{Minimize} & \sum_{j=1}^{n} c_j x_j & \\
\text{Subject to} & & \\
 & \sum_{j=1}^{n} a_{ij} x_j \quad \{\geq, =, \leq\} & b_i \\
 & x_j \in \{0,1\} & \forall j \in \{1, 2, \dots, n\}
\end{array}
$$

- 0-1 Integer Linear Programming (0-1 ILP)

Pseudo-Boolean Optimization (PBO)

- If we identify $\{\textbf{false}, \textbf{true}\}$ to $\{0, 1\}$, a clause $(x \vee y \vee z)$ is equivalent to $x + y + z \geq 1$
  - $(x \vee \bar{y} \vee z)$ is $x + (1 - y) + z \geq 1$
- Not quite *Integer Programming* because the domain is Boolean
  - Particular case

Pseudo-Boolean Optimization (PBO)

Algorithmic Solutions

- Integer Programming solvers are very powerful
  - We are not going to discuss Integer Programming
- When there is a linear objective, MaxSAT can be a good approach (we will see MaxSAT)
- In some case, a CDCL-like algorithm can be better than IP
  - **Replace clauses by cutting planes**

Combination of two constraints

$$\frac{\begin{array}{c}\delta(\sum_{j=1}^{n} a_j x_j \leq b) \\ \delta'(\sum_{j=1}^{n} a'_j x_j \leq b')\end{array}}{\delta \sum_{j=1}^{n} a_j x_j + \delta' \sum_{j=1}^{n} a'_j x_j \leq \delta b + \delta' b'}$$

Rounding can also be applied

$$\frac{\sum_{j=1}^{n} a_j x_j \leq b}{\sum_{j=1}^{n} \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor}$$

- The correctness of the rounding operation follows from $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$
- Hence, δ coefficients in cutting plane operations do not need to be integer. Rounding can be safely applied afterwards

Rounding Example

$$\frac{0.5(3x_1 + 2x_2 + x_3 + 2x_4 + x_5 \quad \leq 5)}{1.5x_1 + x_2 + 0.5x_3 + x_4 + 0.5x_5 \leq 2.5}$$

After rounding: $x_1 + x_2 + x_4 \leq 2$

- Cutting Planes generalize (p-simulate) CNF clause resolution

Example

$$\frac{\begin{array}{c}(\bar{x}_1 \vee x_2 \vee x_3)\\(x_2 \vee x_4 \vee \bar{x}_3)\end{array}}{\bar{x}_1 \vee x_2 \vee x_4}$$

$$\begin{array}{rll}
(1 - x_1) + x_2 + x_3 & \geq 1 & \\
x_2 + x_4 + (1 - x_3) & \geq 1 & \\
(1 - x_1) & \geq 0 & \\
x_4 & \geq 0 & \\
\hline
2(1 - x_1) + 2x_2 + 2x_4 & \geq 1 & \text{addition} \\
\hline
(1 - x_1) + x_2 + x_4 & \geq 1 & \text{division}
\end{array}$$

Cutting Planes as a Proof System

- Cutting planes is a stronger proof system than resolution

Cutting Planes

Use of Cutting Planes

- Used in branch and bound algorithms for PBO
  - And in the more general case of Integer Linear Programming (ILP)
- Very common at preprocessing (i.e., at the root node of the search tree)
- Algorithms that use cutting plane techniques during the search process are also known as branch and cut algorithms
- Other types of cutting planes exist (e.g., clique cuts)

Backtrack search with Cutting Plane learning

- DPLL-like algorithms for PBO can perform cutting plane learning instead of clause learning
- Replace clause resolution with cutting planes in implication graph analysis
- Important note: It is not guaranteed that the new constraint will be assertive

Backtrack search with Cutting Plane learning

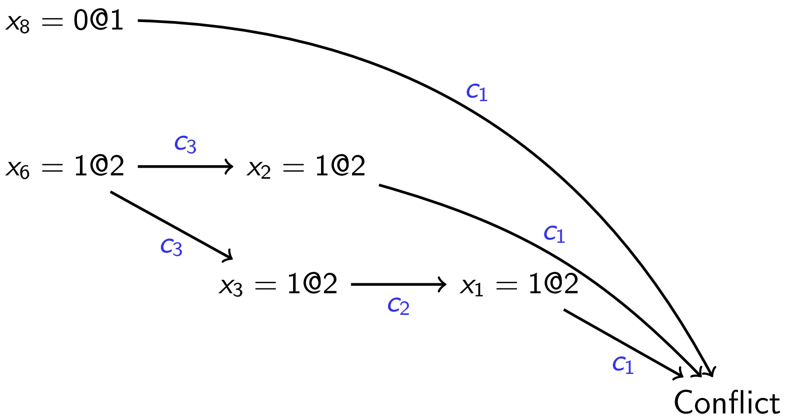
Consider the following constraints:

$$
\begin{aligned}
c_1 :&\quad 3x_1 + x_2 + x_7 - 2x_8 && \leq 3 \\
c_2 :&\quad -3x_1 + x_3 + 2x_7 + x_9 && \leq 0 \\
c_3 :&\quad -x_2 - x_3 + x_6 && \leq -1 - 1
\end{aligned}
$$

- Suppose you start with assignment $x_8 = 0$ at first decision level
- Next, you decide to assign $x_6 = 1$. What happens?
- Constraint propagation on c_3 sets $x_2 = 1, x_3 = 1$
- Constraint propagation on c_2 sets $x_1 = 1$
- Constraint c_1 is violated

Backtrack search with Cutting Plane learning

$$\begin{array}{lll} c_1: & 3x_1 + x_2 + x_7 - 2x_8 & \leq 3 \\ c_2: & -3x_1 + x_3 + 2x_7 + x_9 & \leq 0 \\ c_3: & -x_2 - x_3 + x_6 & \leq -1 \end{array}$$



- Conflict in constraint c_1
- Start backward traversal of graph

Cutting plane between c_1 and c_2 to remove x_1

$$\begin{array}{ll} 1(3x_1 + x_2 + x_7 - 2x_8 & \leq 3) \\ 1(-3x_1 + x_3 + 2x_7 + x_9 & \leq 0) \\ \hline x_2 + x_3 + 3x_7 - 2x_8 + x_9 \leq 3 & \end{array}$$

Cutting plane with c_3 to remove x_3

$$\begin{array}{ll} 1(x_2 + x_3 + 3x_7 - 2x_8 + x_9 & \leq 3) \\ 1(-x_2 + -x_3 + x_6 & \leq -1) \\ \hline x_6 + 3x_7 - 2x_8 + x_9 \leq 2 & \end{array}$$

- Backward traversal to the decision variable x_6
- Learned constraint: $x_6 + 3x_7 - 2x_8 + x_9 \leq 2$
- Backtrack to level 1 and assert $x_7 = 0$