Stochastic Dynamic Matching in Graphs

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Outline

Model and notation

Performance under the first-come-first-matched policy
Comte, Stochastic Models (2022)

Matching rates under an arbitrary policy
Comte, Mathieu, and Bušić, arXiv:2112.14457 (2022)

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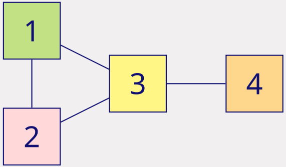
Compatibility graph

Graph $G = (V, E)$ undirected, connected, without loop



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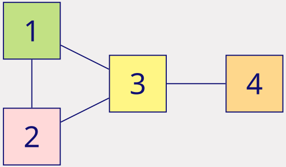

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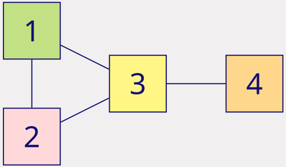

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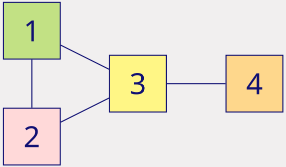

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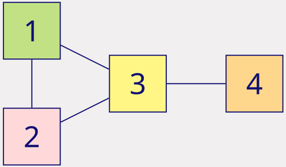

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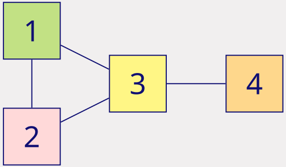

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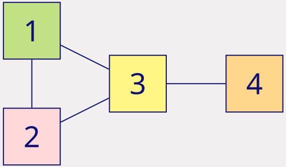

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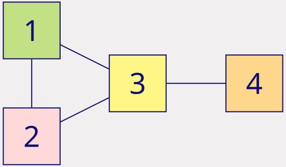
- Independent sets
  $\mathbb{I} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}\}$
  $\mathbb{I}_0 = \mathbb{I} \cup \{\emptyset\}$

Random dynamics



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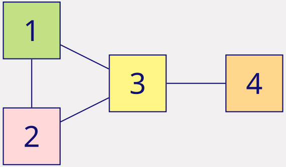

Class-i items arrive as a Poisson process with rate μ_i

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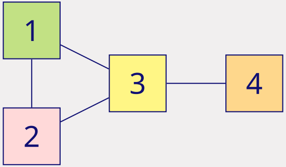

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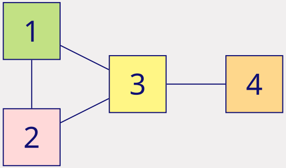

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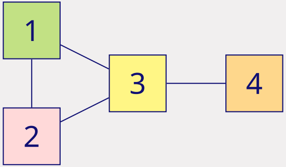

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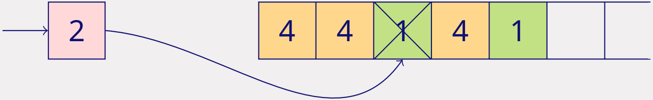

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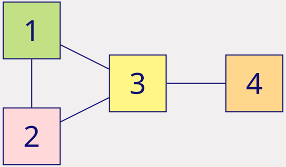

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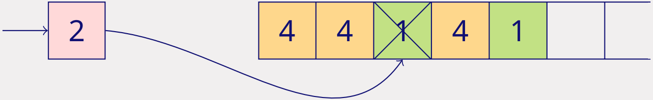

The system dynamics depend on:

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- the matching policy.

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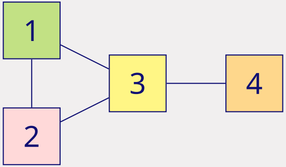

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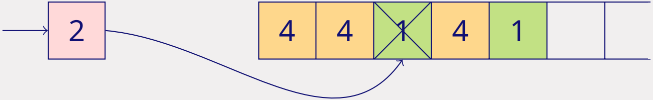

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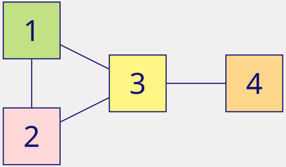
Additional notation:

- Arrival rate $\mu(U) = \sum_{i \in U} \mu_i$, $U \subseteq V$
- Load $\rho(I) = \frac{\mu(I)}{\mu(V(I))}$, $I \in \mathbb{I}$



“Stabilizability”

(Bušić, Gupta, and Mairesse, 2013) (Mairesse and Moyal, 2016)

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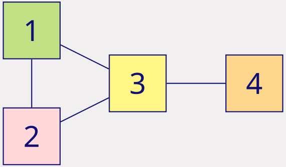

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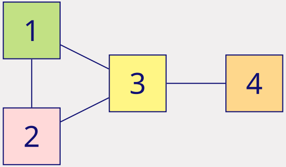

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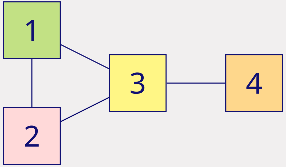

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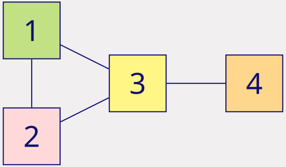

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- The compatibility graph G is stabilizable if and only if G is non-bipartite.

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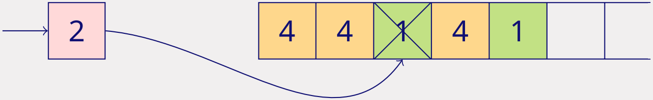

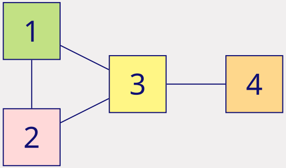

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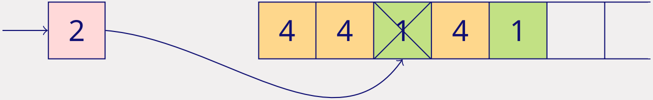

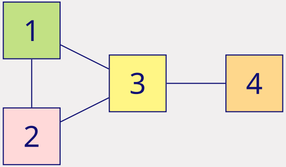

- Perceived as “fair”, greedy, easy to implement, easy to analyze.

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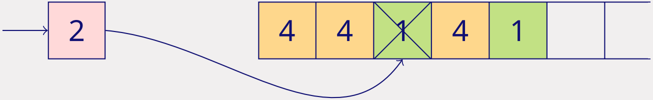

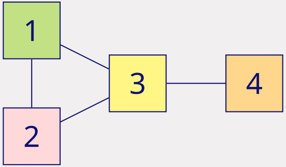

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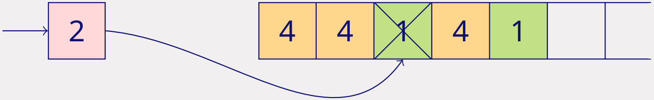

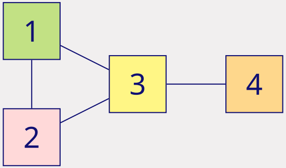

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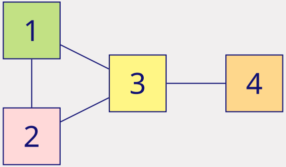
What is the long-term performance under first-come-first-matched?

Calculate long-term performance metrics



- This is an order-independent loss queue!

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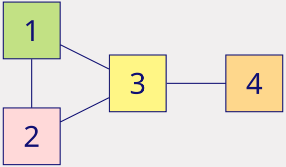

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- Stationary distribution of the set of unmatched classes:

$$\pi(I) = \frac{\rho(I)}{1 - \rho(I)} \left(\sum_{i \in I} \frac{\mu_i}{\mu(I)} \pi(I \setminus \{i\}) \right), \quad I \in \mathbb{I}.$$

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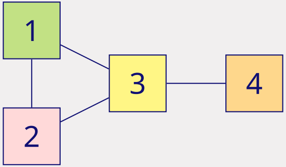

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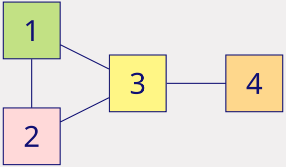

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$$\omega_i = \sum_{\substack{I \in \mathbb{I}_0: \\ i \notin V(I)}} \pi(I),$$

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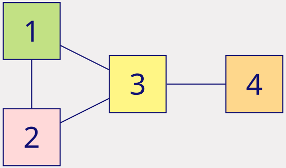

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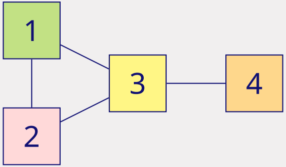
$$\omega_i = \sum_{\substack{I \in \mathbb{I}_0: \\ i \notin V(I)}} \pi(I), \quad \text{which implies} \quad \frac{\sum_{i \in V} \mu_i \omega_i}{\sum_{i \in V} \mu_i} = \frac{1}{2}.$$

TU/e

Calculate long-term performance metrics



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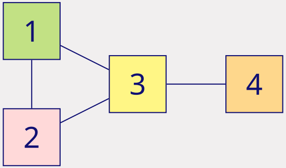

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The mean waiting time of an item follows using Little's law.

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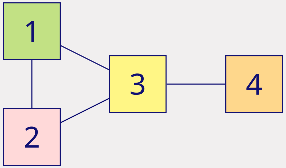

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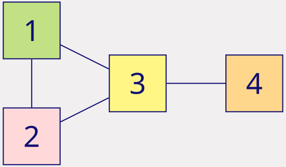

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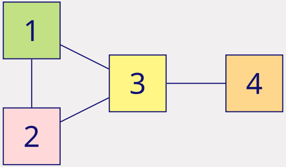
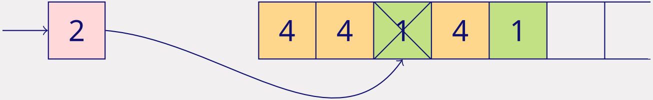
- Similar results for stochastic bipartite matching model (Comte and Dorsman, ASMTA, 2021).

Calculate long-term performance metrics



- Matching rate along edge $k = \{i, j\}$: mean number of matches per time unit between classes i and j.



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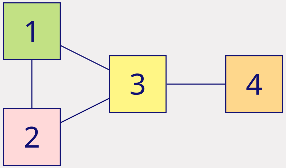

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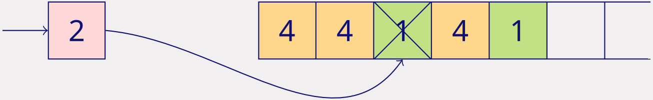

- Closed-form expression: consider a finer partition of the state space.

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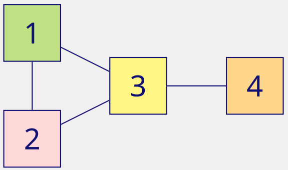

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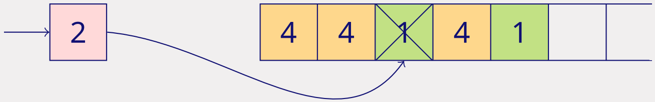
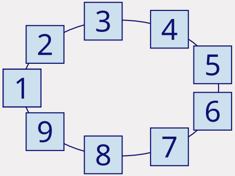


- Closed-form expression: consider a finer partition of the state space.
- More in a few slides...

Numerical results: Cycle



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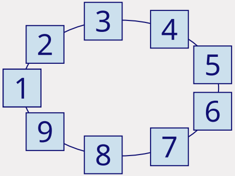

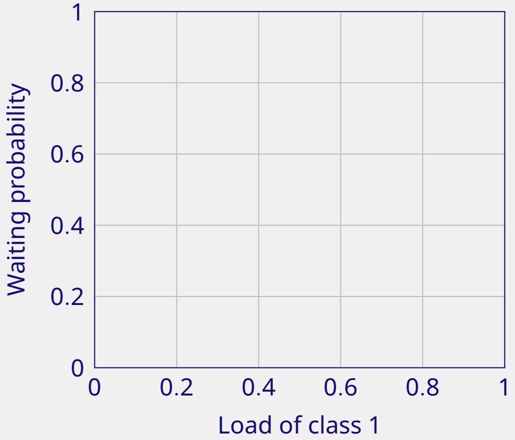

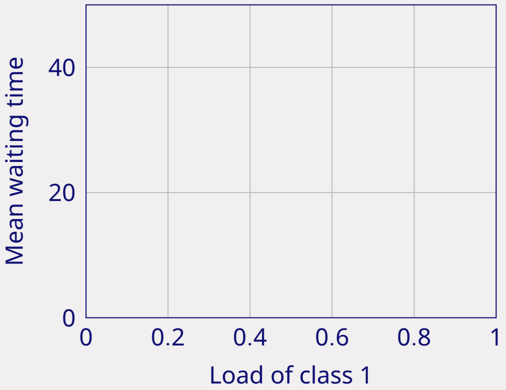

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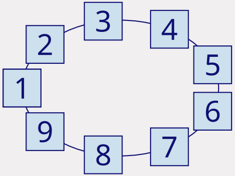

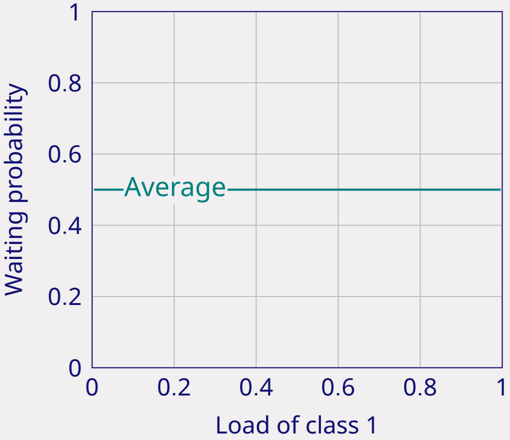

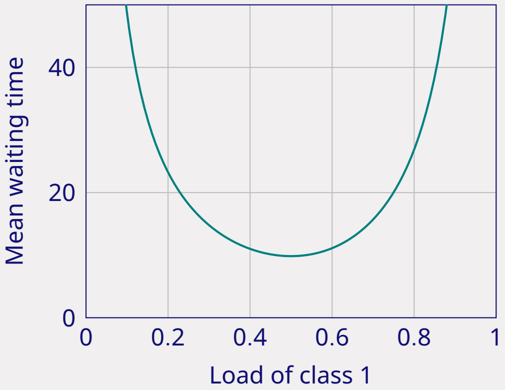

TU/e

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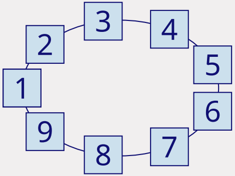

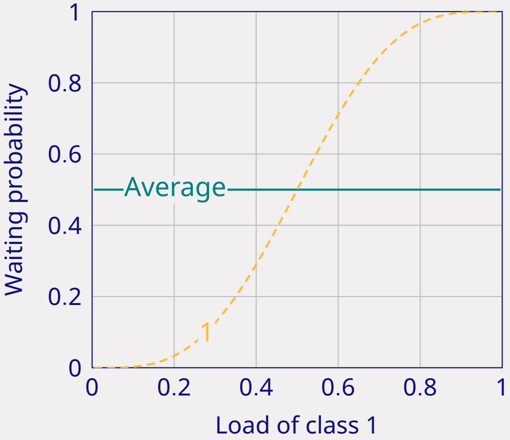

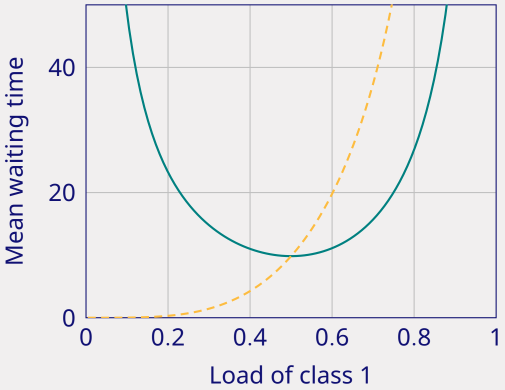

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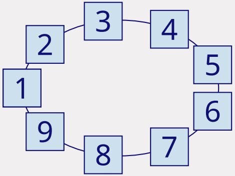

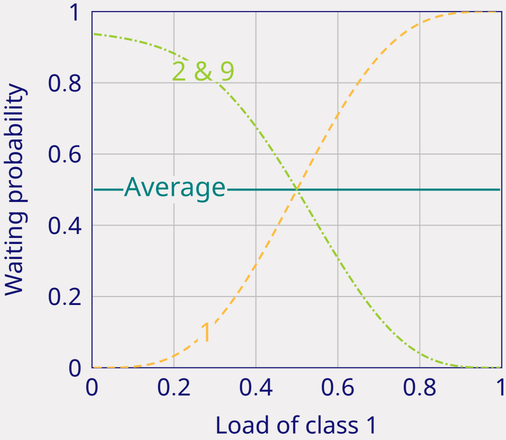

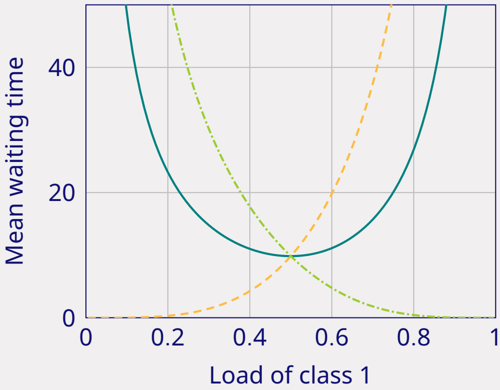

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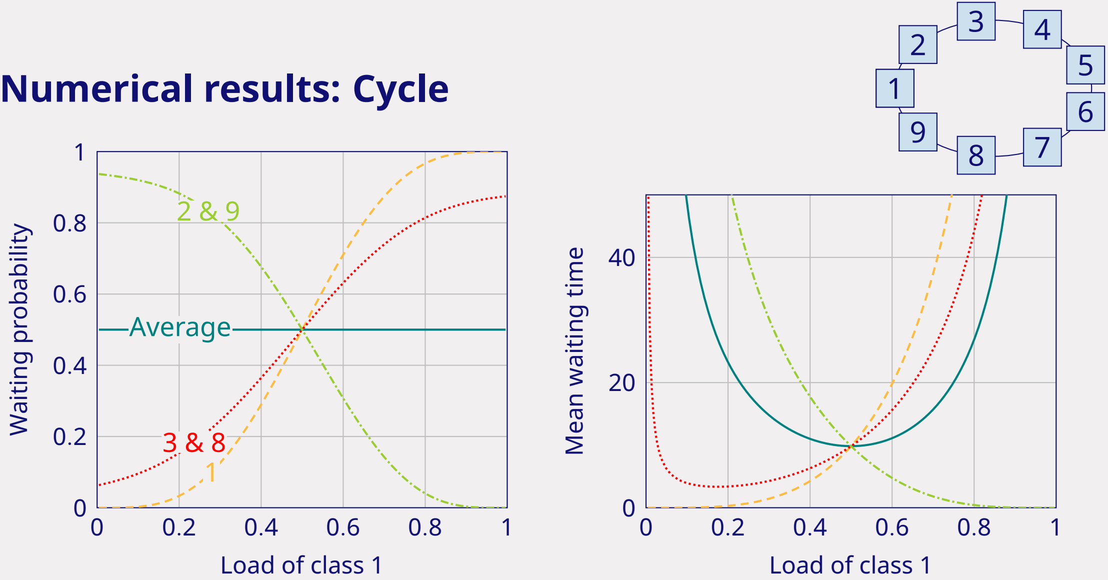

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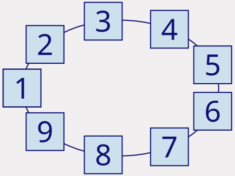

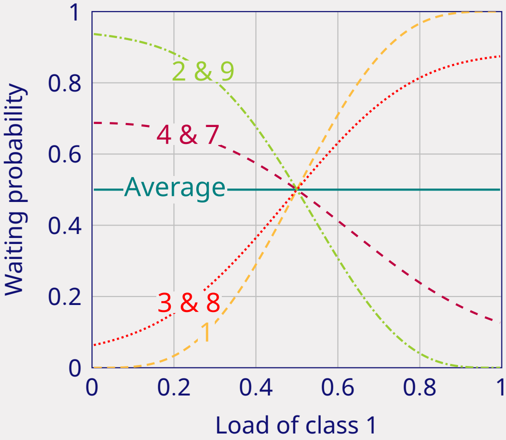

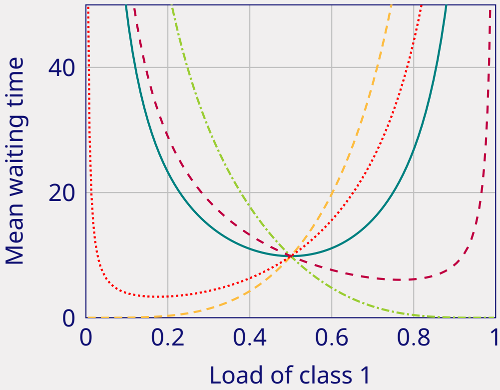

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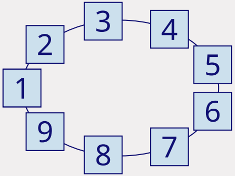
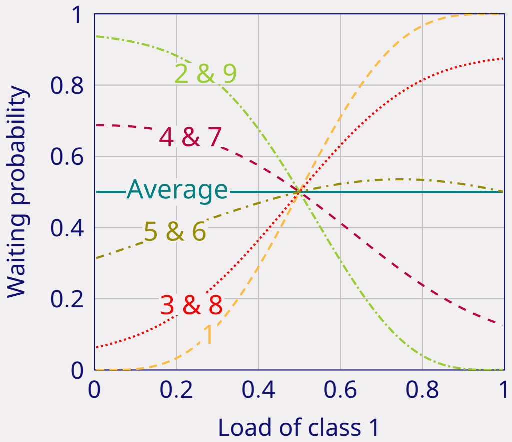
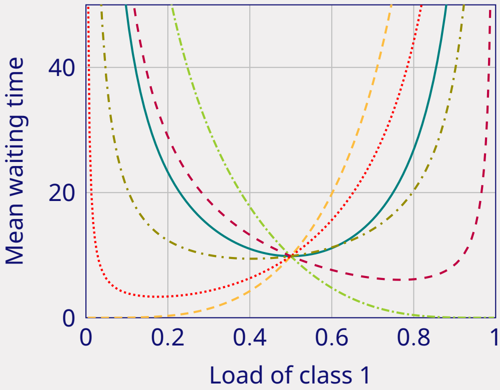
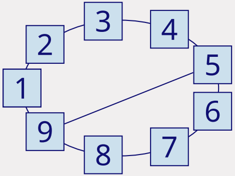






Numerical results: Cycle with a chord



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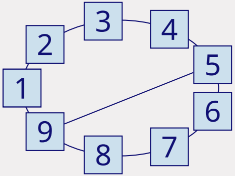
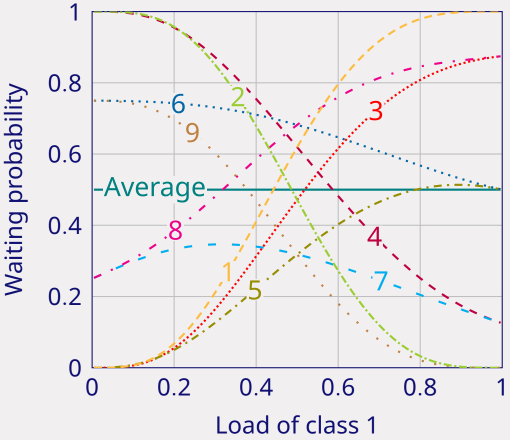
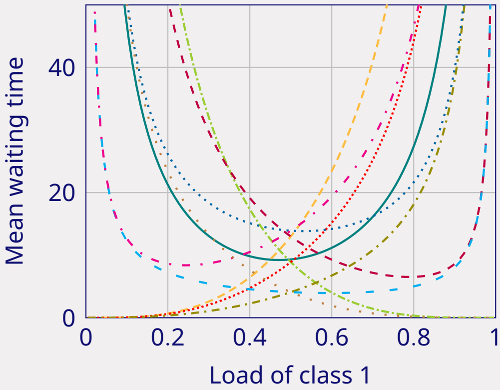






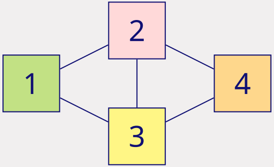
Outline

Model and notation

Performance under the first-come-first-matched policy
Comte, Stochastic Models (2022)

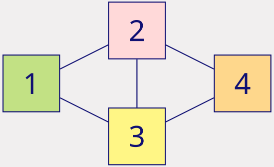
Matching rates under an arbitrary policy
Comte, Mathieu, and Bušić, arXiv:2112.14457 (2022)

Matching rates



- Matching rate λ_k along edge $k = \{i, j\}$:
  mean number of matches per time unit between classes i and j.

Matching rates



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  - They give intuition about the long-term impact of the matching policy.

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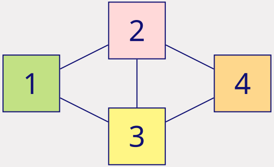

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- Matching rates are particularly interesting:
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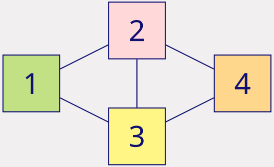
Given a graph $G = (V, E)$ and a vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of arrival rates, what is the set of "feasible" vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of matching rates?

Conservation equation



The matching rates satisfy the conservation equation

$$\sum_{k \in E_i} \lambda_k = \mu_i, \quad i \in \{1, 2, \dots, n\}.$$

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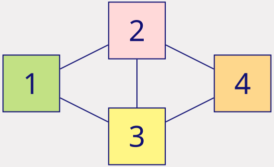

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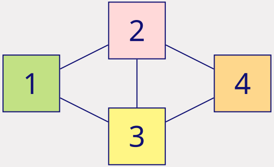

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that is, in matrix form,

$$A\lambda = \mu,$$

where $A = (a_{i,k})$ is the incidence matrix of the compatibility graph.

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

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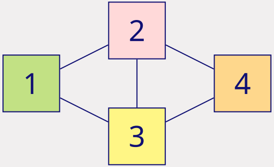

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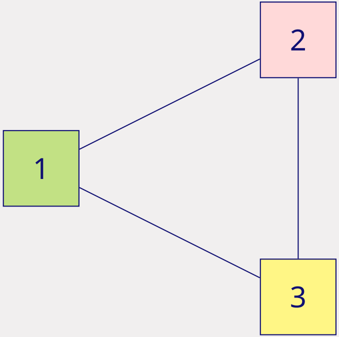
where $A = (a_{i,k})$ is the incidence matrix of the compatibility graph.

Example: Triangle graph



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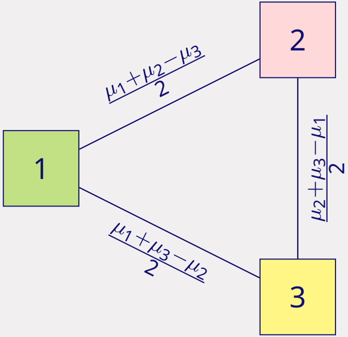

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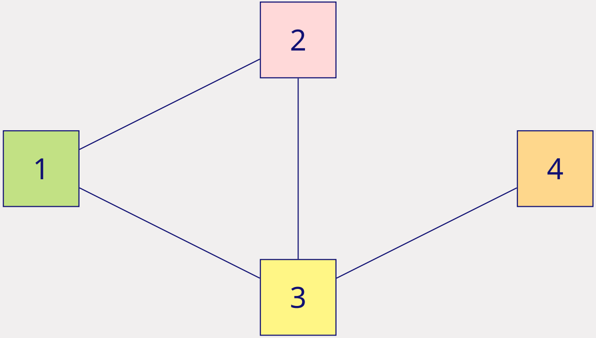
Example: Paw graph



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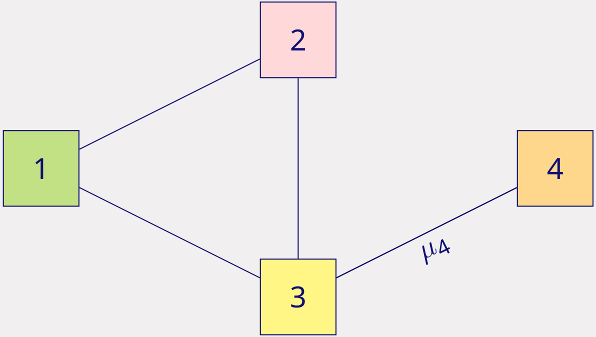

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$$

$$
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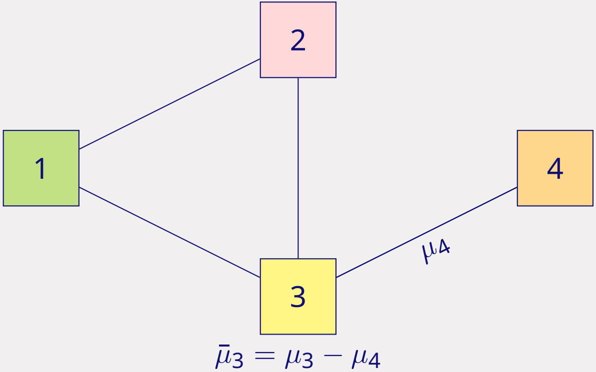

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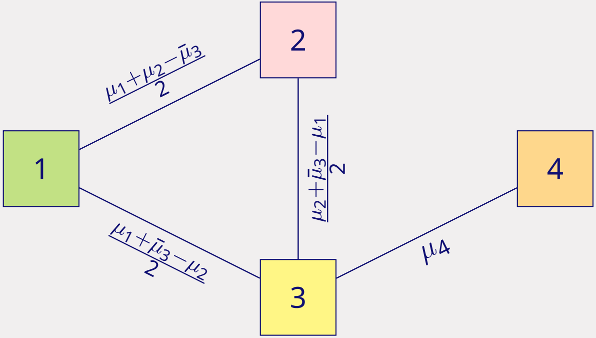

$$\bar{\mu}_3 = \mu_3 - \mu_4$$

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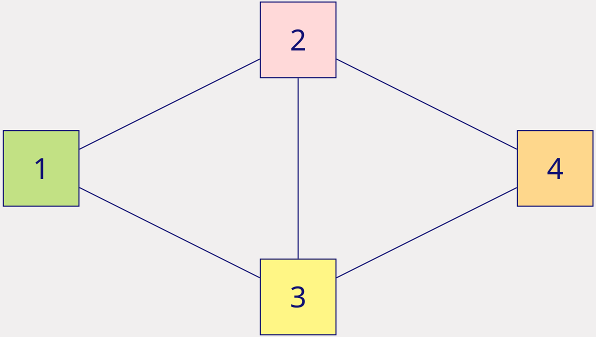
Example: Diamond graph



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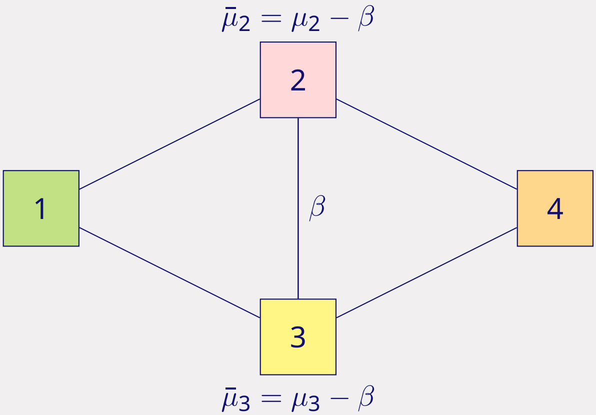

$$\beta = \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$
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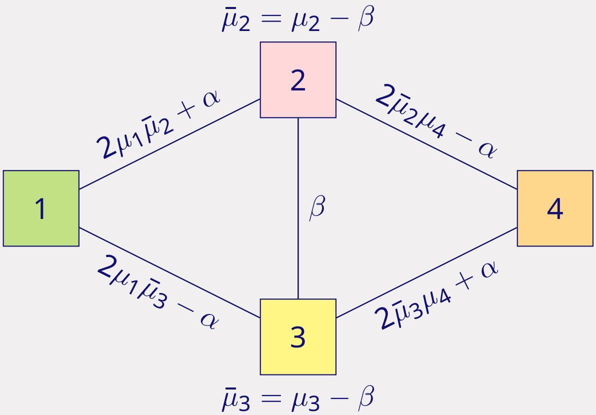

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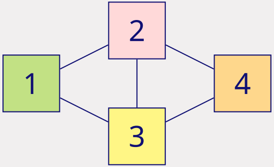
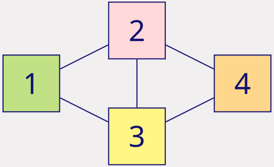
Surjectivity, injectivity, and bijectivity



- The compatibility graph G is surjective if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.

Surjectivity, injectivity, and bijectivity



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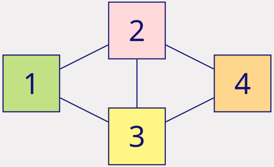
Surjectivity, injectivity, and bijectivity



- The compatibility graph G is surjective if
  - The linear application $\lambda \in \mathbb{R}^m \mapsto A\lambda \in \mathbb{R}^n$ is surjective.

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- The compatibility graph G is bijective if G is surjective and injective.

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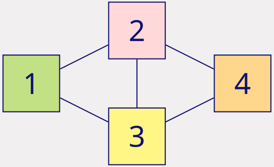

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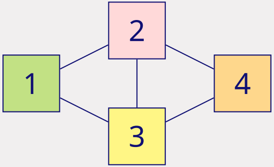

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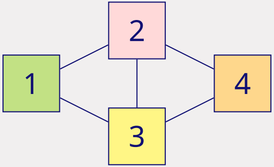

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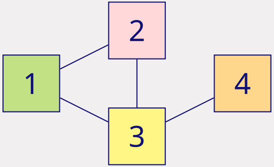
Surjectivity, injectivity, and bijectivity



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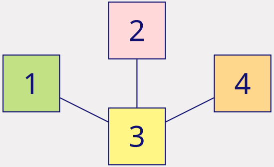
Surjectivity, injectivity, and bijectivity



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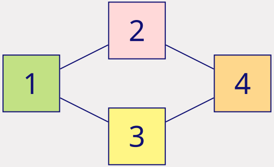

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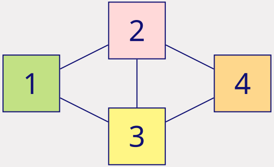

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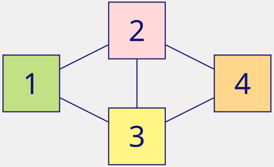
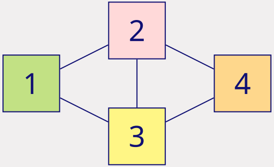
"Stabilizability"



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

"Stabilizability"



- A matching problem (G, μ) is stabilizable if and only if $\rho(I) < 1$ for each $I \in \mathbb{I}$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

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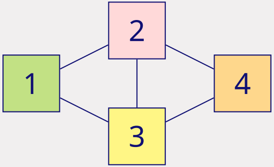

- A matching problem (G, μ) is stabilizable
  if and only if the conservation equation $A\lambda = \mu$ has a solution $\lambda > 0$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

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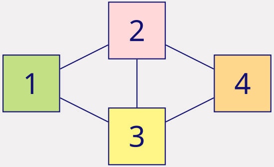

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  ☺ The time complexity to verify this condition is polynomial in n and m.

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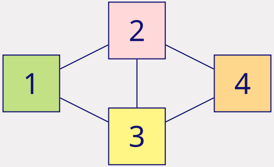

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- A compatibility graph G is stabilizable if and only if G is non-bipartite.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

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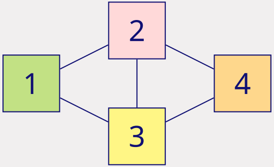

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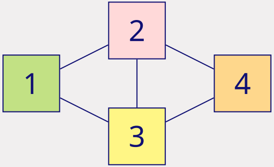

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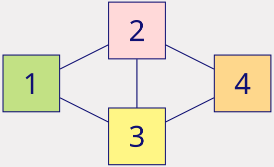
- A compatibility graph G is stabilizable if and only if G is surjective.
  ☺ The rank of matrix A is n.
  The nullity of matrix A is $d = m - n$
  (according to the rank-nullity theorem).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

TU/e

Affine space of solutions

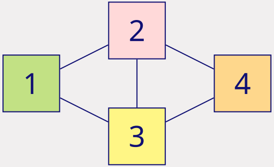


- The solution set of the conservation equation is

$$\Lambda = \left\{ \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d : \alpha \in \mathbb{R}^d \right\}$$

where λ° is a particular solution of the conservation equation and $\{b_1, b_2, \ldots, b_d\}$ is a basis of Ker(A), of cardinality $d = m - n$.

Affine space of solutions



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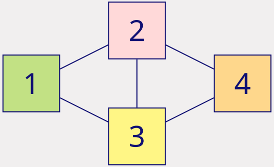
- We borrowed an algorithm from (Doob, 1973) to build a basis of Ker(A).

Affine space of solutions



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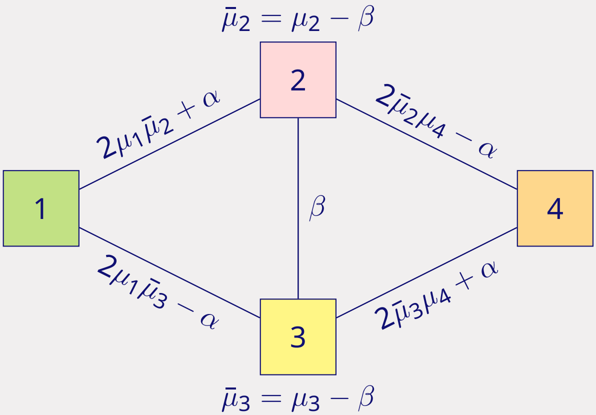
- We borrowed an algorithm from (Doob, 1973) to build a basis of Ker(A).
- We use two coordinate systems:
  - Edge coordinates $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$.
  - Kernel coordinates $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$.

Example: Diamond graph

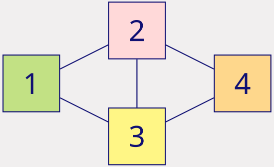


$$\beta = \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - \mu_4)$$
$$\mu_1 + \mu_4 = \bar{\mu}_2 + \bar{\mu}_3 = \frac{1}{2}$$

$$\begin{cases} \lambda_{1,2} + \lambda_{1,3} = \mu_1 \\ \lambda_{1,2} + \lambda_{2,3} + \lambda_{2,4} = \mu_2 \\ \lambda_{1,3} + \lambda_{2,3} + \lambda_{3,4} = \mu_3 \\ \lambda_{2,4} + \lambda_{3,4} = \mu_4 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{1,2} \\ \lambda_{1,3} \\ \lambda_{2,3} \\ \lambda_{2,4} \\ \lambda_{3,4} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

Convex polytope of non-negative solutions

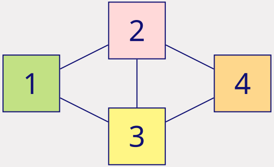


- The set of non-negative solutions of the conservation equation is

$$\Lambda_{\geq 0} = \Lambda \cap \mathbb{R}_+^m \approx \left\{ \alpha \in \mathbb{R}^d : \lambda^\circ + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d \geq 0 \right\}.$$

This is a *d*-dimensional convex polytope.

Convex polytope of non-negative solutions



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- The subgraph restricted to the support of a vertex of $\Lambda_{\geq 0}$ is injective

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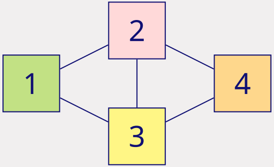

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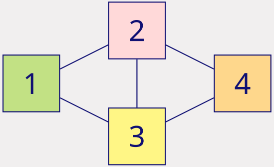

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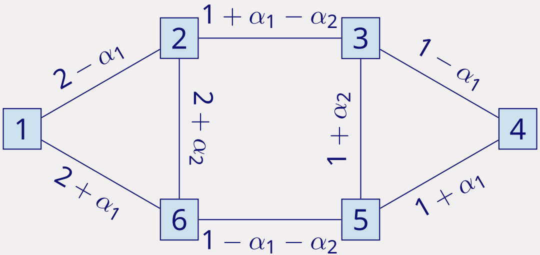
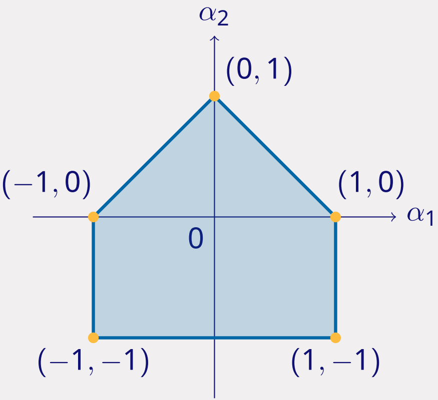
  - If this subgraph is injective but not surjective, it's more complicated...

Example: Codomino graph 1



(a) Solution of the conservation equation $A\lambda = \mu$ with $\mu = (4, 5, 3, 2, 3, 5)$.



(b) Polytope $\Lambda_{\geq 0}$ in kernel coordinates.

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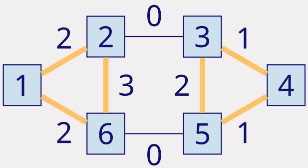
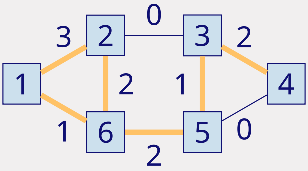
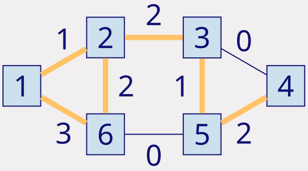


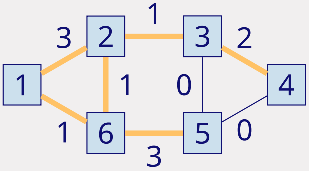
(a) Vertex $(0, 1)$.



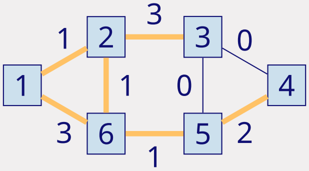
(b) Vertex $(-1, 0)$.



(c) Vertex $(1, 0)$.

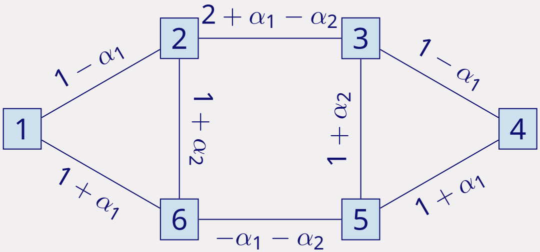


(d) Vertex $(-1, -1)$.

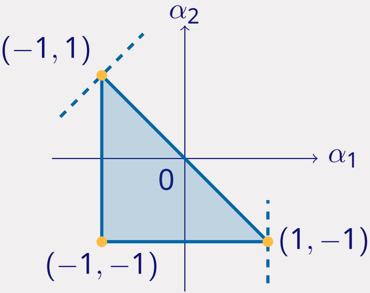


(e) Vertex $(1, -1)$.

Example: Codomino graph 2

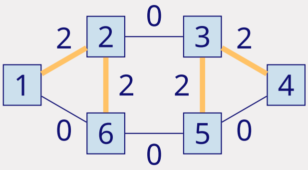


(a) Solution of the conservation equation $A\lambda = \mu$ with $\mu = (2, 4, 4, 2, 2, 2)$.

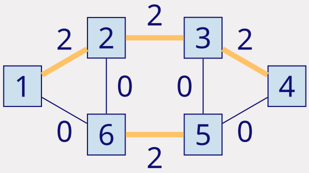


(b) Polytope $\Lambda_{\geq}$ in kernel coordinates.

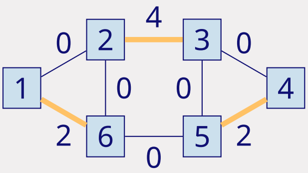
Example: Codomino graph 2



(a) Vertex $(-1, 1)$.

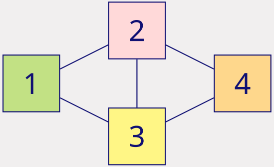


(b) Vertex $(-1, -1)$.



(c) Vertex $(1, -1)$.

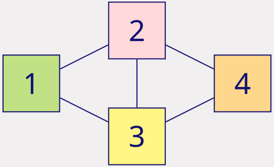
Conclusion



Take-away

- Stochastic dynamic matching problem associated with organ transplant programs and assembly systems.
- Performance evaluation under the first-come-first-matched policy.
- Analysis of the matching rates under an arbitrary matching policy.

Conclusion



Take-away

- Stochastic dynamic matching problem associated with organ transplant programs and assembly systems.
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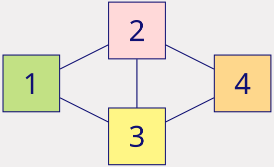
Future works

- More realistic model: hypergraph? state-dependent arrival rates?
- Optimization and learning: graph structure? arrival rates? policy?

References



C. Comte. "Stochastic non-bipartite matching models and order-independent loss queues". *Stochastic Models* 38.1 (Jan. 2022), pp. 1–36

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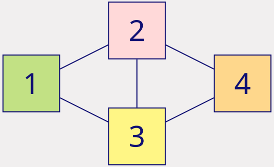
C. Comte, F. Mathieu, and A. Bušić. "Stochastic dynamic matching: A mixed graph-theory and linear-algebra approach". (Jan. 2022). arXiv: 2112.14457

Basis of the kernel of the matrix A



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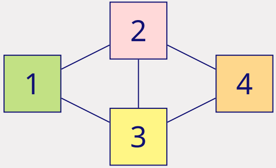

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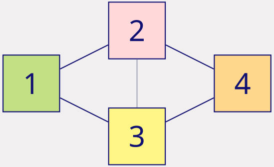

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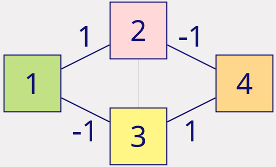

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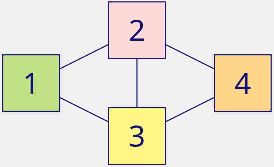

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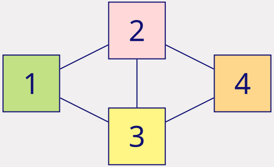

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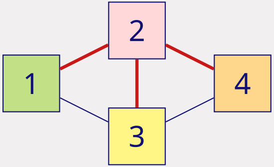

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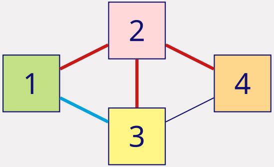

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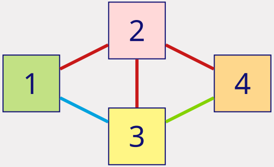

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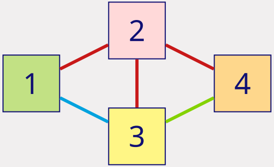

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