Pass-and-Swap Queues

Joint work with Jan-Pieter Dorsman (UvA)

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Eindhoven University of Technology

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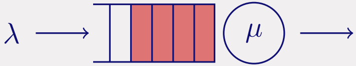
The M/M/1 queue

Model

- Jobs arrive according to a Poisson process with rate λ.
- Service times are i.i.d. exponentially distributed with rate μ.
- A single server.



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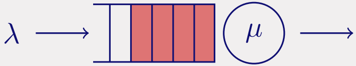

Analysis

- Markov (birth-and-death) process.
- Stationary distribution: $\pi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$.
- Proof: (partial) balance equations + normalization equation.

Order-independent queues

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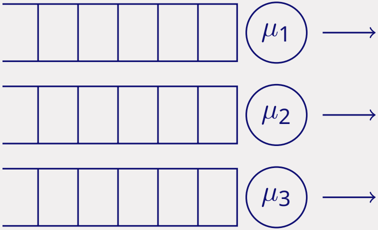
- **Redundancy cancel-on-complete** (Gardner et al., 2016)



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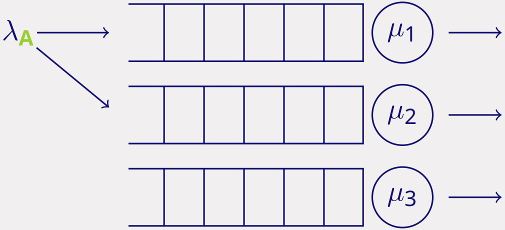

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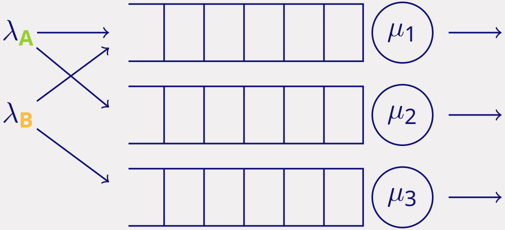

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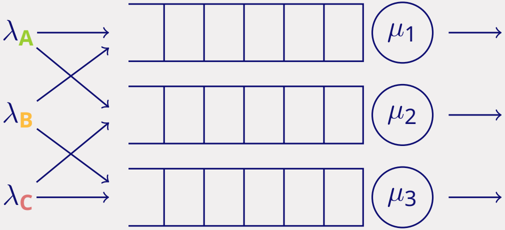

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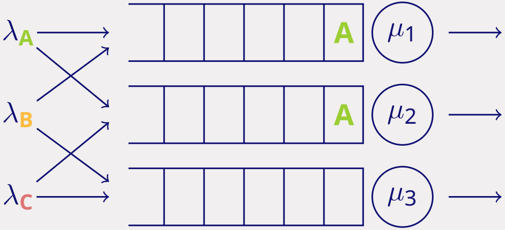

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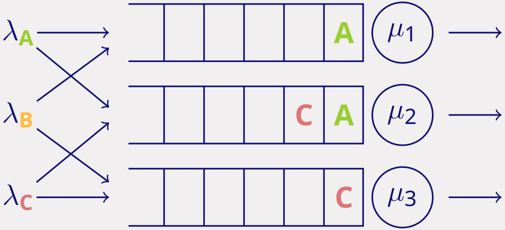

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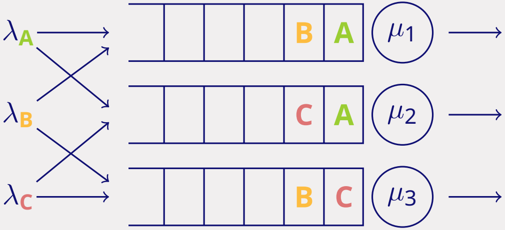

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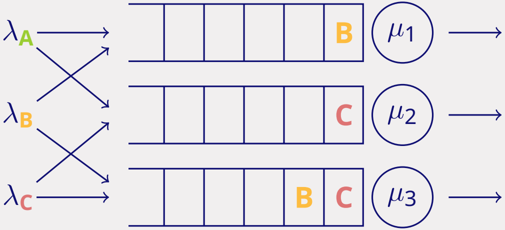

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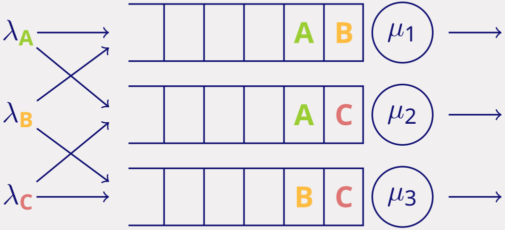

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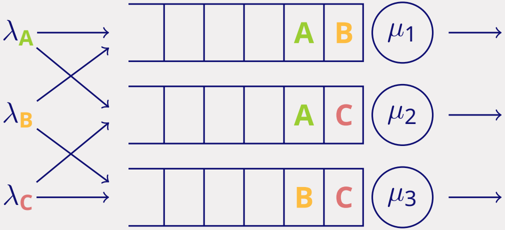
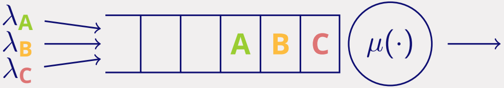


- **Order-independent (OI) queue**
  (Berezner et al., 1995)
  (Bonald and Comte, 2017)



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- Why study product-form queues?
  - Rich in applications.
  - Exact performance analysis is not completely hopeless.

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- A: But, still, new product-form queues keep appearing, not captured by these frameworks, such as the pass-and-swap queue.

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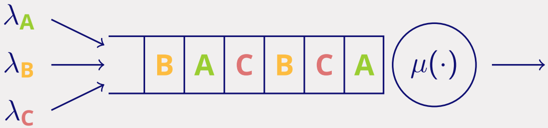

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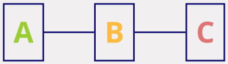

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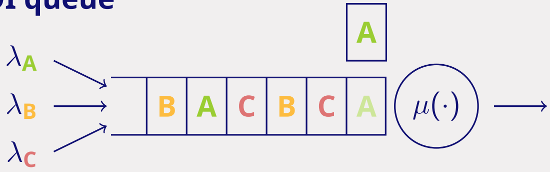

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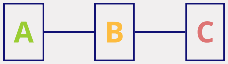

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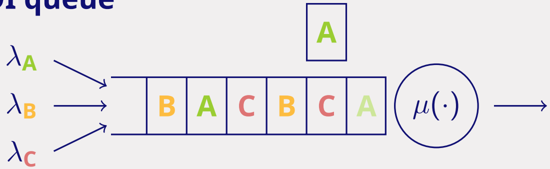

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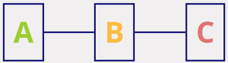

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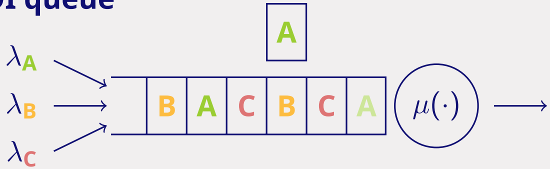

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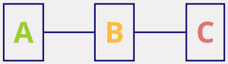

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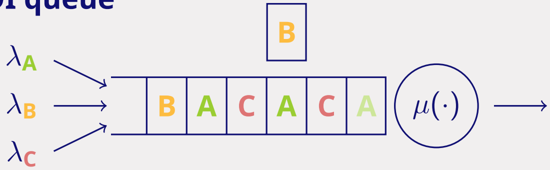

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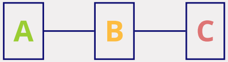

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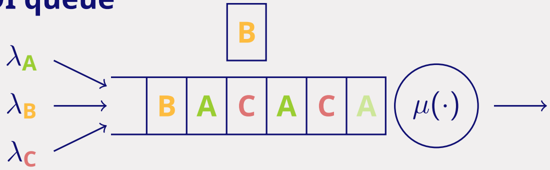

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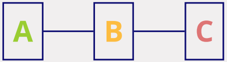

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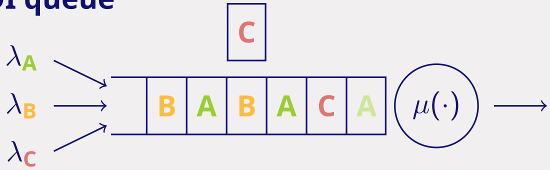

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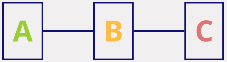

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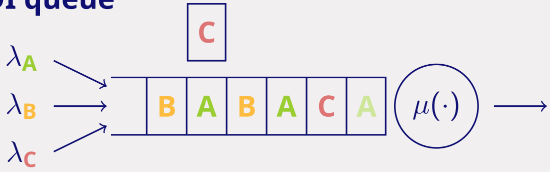

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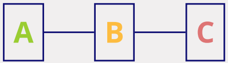

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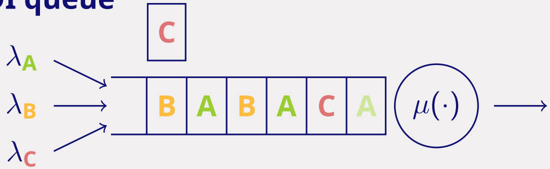

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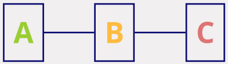

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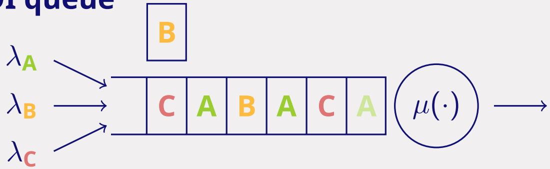

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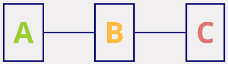

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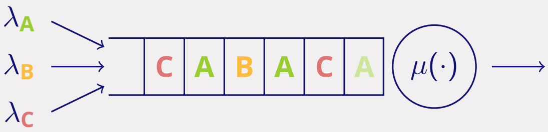

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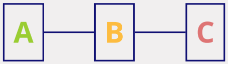

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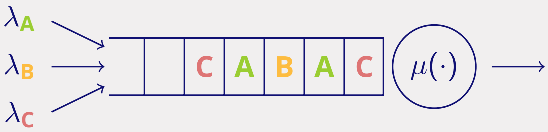

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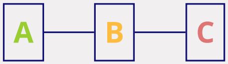

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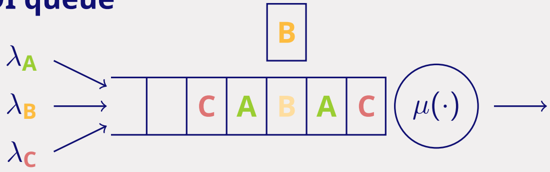

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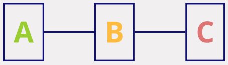

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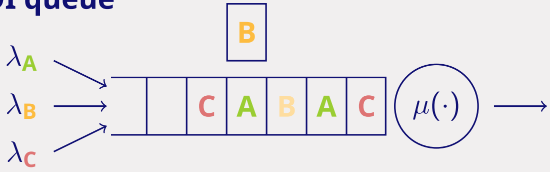

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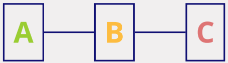

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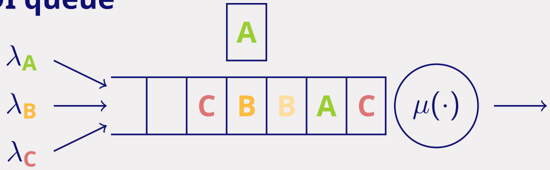

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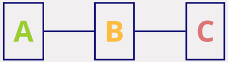

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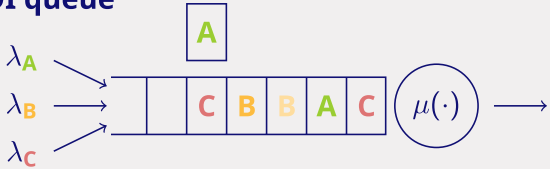

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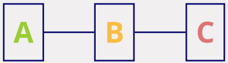

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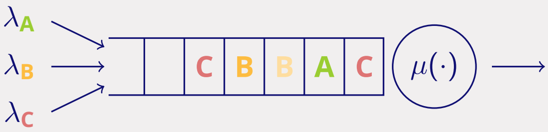

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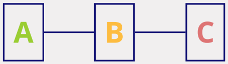

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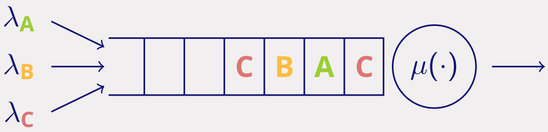

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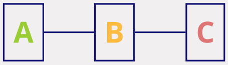

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- Stationary distribution: *exactly* the same as OI queues!

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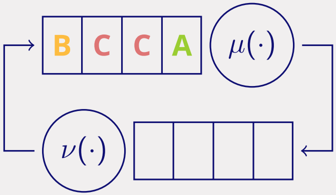
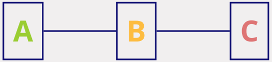
- We also prove a simple stability condition (also valid for OI queues).

Closed network of P&S queues

- Tandem network of two P&S queues with the same swapping graph:





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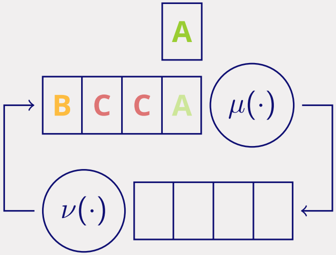

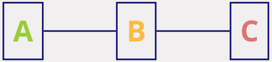

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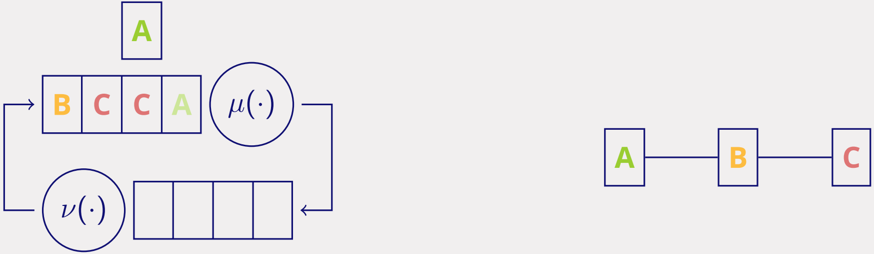

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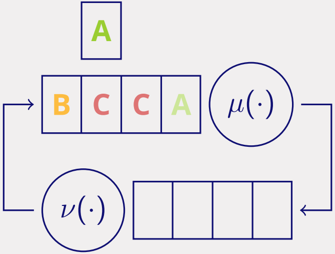

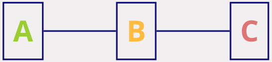

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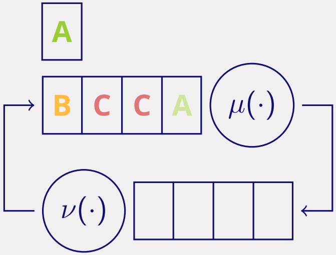

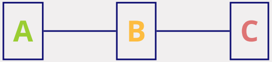

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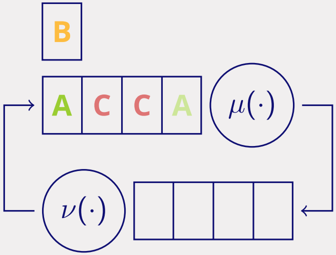

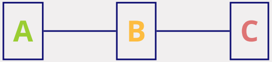

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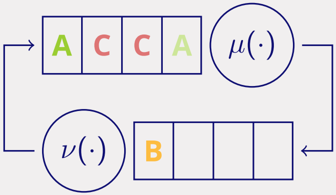

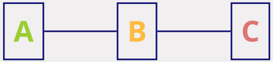

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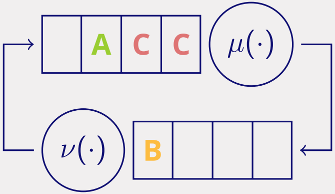

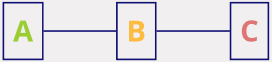

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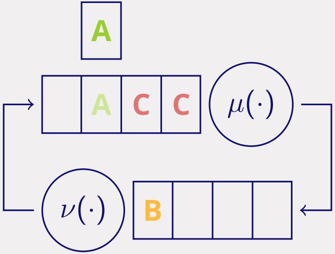

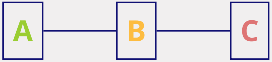

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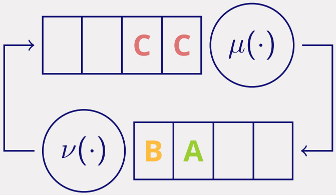

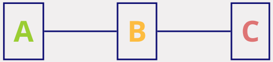

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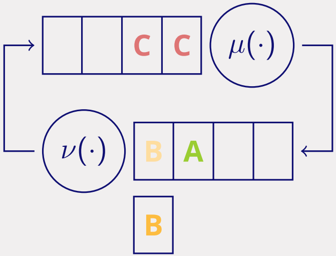

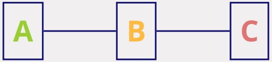

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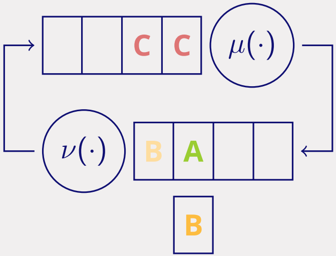

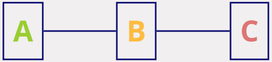

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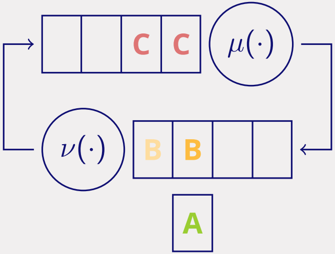

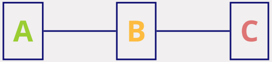

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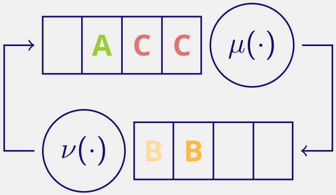

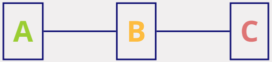

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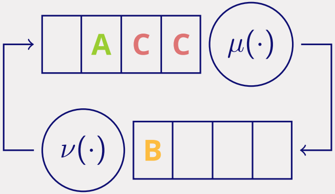

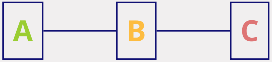

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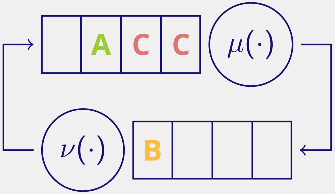

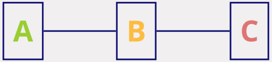

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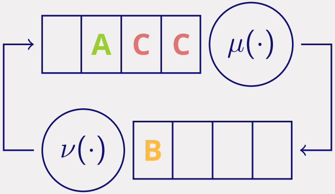
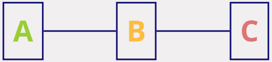
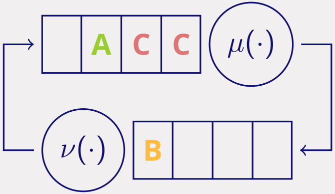




- B always last in the upper queue, first in the lower queue: *placement* order.
- The stationary distribution again has a product form (on a restricted space)!

From two queues to one queue

- The lower queue models “state-dependent arrivals” to the upper queue.



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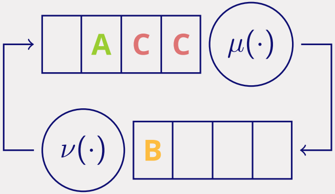

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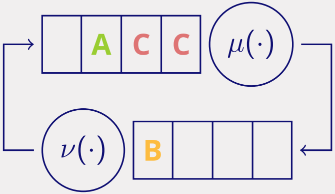
- If the two queues are simple ·/M/1 queues, the upper queue can be seen as an M/M/1 queue with blocking.
- This is very powerful:
  - Redundancy cancel-on-start and cancel-on-commit.
  - Hierarchical load-distribution algorithms.



Conclusion

Take away

- P&S queues broaden the family of product-form queues by allowing for intra-queue routing.
- Networks of P&S queues also have a product form.
- Paves the way for performance analysis of other algorithms.

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Future works

- How big is the family of product-form queues?
- Are there other routing mechanisms that lead to a product form?
- Can we find other applications of P&S queues?