Filtering Algorithms with Poisson-distributed Observations

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Abstract

Filtering theory basically relates to optimal state estimation in stochastic dynamical systems, particularly when faced with partial and noisy data. This field, closely intertwined with control theory, focuses on designing estimators doing real-time computation while maintaining an acceptable level of accuracy as measured by the mean square error. The necessity for such estimates becomes increasingly critical with the proliferation of network-controlled systems, such as autonomous vehicles and complex industrial processes, where the observation processes are subject to randomness in transmission and this gives rise to varying information patterns under which the estimation must be carried out.

This thesis addresses the important task of state estimation in continuous-time stochastic dynamical systems when the observation process is available only at some discrete time instants governed by a random process. By adapting classical estimation methods, we derive equations for optimal state estimator, explore their properties and practicality, and propose and evaluates sub-optimal alternatives, showcasing parallels to the existing techniques within the classical estimation domain when applied to Poisson-distributed observation processes.

The study covers three classes of mathematical models for the continuous-time dynamical system and the discrete observation process. First, we consider Itô stochastic differential equations with Lipschitz drift terms and constant diffusion coefficient, whereas the lower-dimensional discrete observation process comprises the nonlinear mapping of the state and additive Gaussian noise. We propose easy-to-implement continuous-discrete suboptimal state estimators for this system class. Assuming that a Poisson counter governs discrete times at which the observations are available, we compute the expectation or error covariance process. Analysis is carried out to provide conditions for boundedness of the error covariance process, as well as, the dependence on the mean sampling rate.

Secondly, we study continuous-discrete ensemble filters for Ornstein–Uhlenbeck processes with discrete observations described by linear functions of state and additive Gaussian noise. Ensemble filters have gained a lot of interest for state estimation in large-scale models with noisy measurements where the computation of optimal gain is either computationally expensive or not entirely feasible due to complexity of the dynamics. In this thesis, we propose continuous-discrete McKean–Vlasov type diffusion processes, which serve as the mean-field model for describing the limit dynamics of the ensemble of particles. We study several kinds of mean-field processes depending on how the noise terms are included in mimicking the state process and the observation model. The resulting particles are coupled through empirical covariances which are updated at discrete times with the arrival of new observations. With appropriate analysis of the first and second moments, we show that under certain conditions on system parameters, the performance of the ensemble filters approaches the optimal filter as the number of particles gets larger.

Lastly, we consider the dynamical systems described by continuous-time Markov chains with finite state space, and the observation process is obtained by discretizing a conventional stochastic process driven by a Wiener process. For this case, the L_1-convergence of the derived optimal estimator to the classical (purely continuous) optimal estimator (Wonham filter) is shown with respect to increasing intensity of Poisson processes.

Keywords: nonlinear filtering, linear filtering, Poisson-distributed observations, random observations, ensemble filter, continuous-discrete filter

Résumé

La théorie du filtrage concerne essentiellement l'estimation optimale de l'état dans les systèmes stochastiques, surtout avec des mesures partielles et bruitées. Ce domaine, fortement lié à la théorie du contrôle, se concentre sur la synthèse d'estimateurs effectuant des calculs en temps réel pour minimiser l'erreur quadratique moyenne. La nécessité de telles estimations devient de plus en plus critique avec la prolifération de systèmes contrôlés par réseau, tels que les véhicules autonomes et les processus industriels complexes, où les processus d'observations sont soumis au caractère aléatoire de la transmission, ce qui donne lieu à des modèles d'information variables pour résoudre le problème d'estimation.

Cette thèse aborde la tâche importante de l'estimation d'état dans les systèmes dynamiques stochastiques en temps continu lorsque les mesures sont disponible aux certains instants discrets defini par un processus aléatoire. En adaptant les méthodes d'estimation classiques, nous développons des équations pour un estimateur optimal d'état, explorons leurs propriétés et les aspect pratiques, et proposons et analysons des alternatives sous-optimales, présentant des parallèles avec les techniques existantes dans le domaine d'estimation classique lorsqu'elles sont appliquées aux processus d'observation Poisson-distribués.

L'étude couvre trois classes de modèles mathématiques pour le système dynamique en temps continu et le processus d'observation discret. Tout d'abord, nous considérons des équations différentielles Itôstochastiques avec le champ de vecteur Lipschitz et un coefficient de diffusion constant, alors que le processus d'observation discrète de dimension inférieure comprend la fonction nonlinéaire de l'état et un bruit Gaussien additif. Nous proposons des estimateurs d'état sous-optimaux continus-discrets, qui sont faciles à implémenter pour cette classe de systèmes. En supposant qu'un compteur de Poisson décrit les instants discrets auxquels les observations sont disponibles, nous calculons le processus de covariance d'erreur d'estimation. L'analyse est effectuée pour fournir les conditions de limitation du processus de covariance d'erreur, ainsi que la dépendance au taux d'échantillonnage moyen.

Deuxièmement, nous étudions les filtres d'ensemble continus-discrets pour les processus d'Ornstein-Uhlenbeck avec des observations discrètes décrites par des fonctions d'état linéaires et un bruit Gaussien additif. Les filtres d'ensemble ont gagné beaucoup d'intérêt pour l'estimation d'état dans les modèles à grande échelle avec des mesures bruitées où le calcul du gain optimal est soit coûteux en calcul, soit pas entièrement réalisable en raison de la complexité de la dynamique. Dans cette thèse, nous proposons des processus de diffusion de type McKean–Vlasov continus-discrets, qui servent de modèle de champ moyen pour décrire la dynamique limite de l'ensemble des particules. Nous étudions plusieurs types de processus de champ moyen en fonction de la manière dont les termes de bruit sont inclus pour l'imitation du processus d'état et du modèle d'observation. Les particulaires résultantes sont couplées via des covariances empiriques qui sont mises à jour en temps discrets avec l'arrivée de nouvelles observations. Avec une analyse appropriée des premier et deuxième instants, nous montrons que sous certaines conditions sur les paramètres du système, les performances des filtres d'ensemble se rapprochent du filtre optimal quand le nombre de particulaires augmente.

Enfin, nous considérons les systèmes dynamiques décrits par des chaînes de Markov en temps continu avec un espace d'état fini, et le processus d'observation est obtenu en discrétisant un processus stochastique conventionnel piloté par un processus de Wiener. Dans ce cas, nous montrons la convergence L_1 de l'estimateur optimal vers l'estimateur optimal classique (purement continu) (filtre de Wonham) quand l'intensité des processus de Poisson augmente.

Les mots cles: filtrage nonlinéaire, filtrage linéaire, observations distribuées par Poisson, observations aléatoires, filtre d'ensemble, filtre continu-discret

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Notation and Abbreviations

Number sets

- $\mathbb{N}$: set of nonnegative integers $\{0, 1, 2, \ldots\}$
- $\mathbb{N}^+$: set of positive integers $\{1, 2, 3 \ldots\}$
- $\mathbb{Z}$: set of integer numbers
- $\mathbb{R}$: set of real numbers
- $\mathbb{C}$: set of complex numbers
- $[a, b]$: closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $]a, b[$: open interval $\{x \in \mathbb{R} \mid a < x < b\}$
- $[a, b[$: interval $\{x \in \mathbb{R} \mid a \leq x < b\}$
- $]a, b]$: interval $\{x \in \mathbb{R} \mid a < x \leq b\}$

Probability

- Law: law of a random variable
- $\mathcal{L}$: generator
- $\mathcal{N}(\mu, \Sigma)$: multivariate Gaussian distribution with mean μ and covariance Σ
- N: Poisson process
- $\mathsf{P}[\cdot]$: probability of an event
- $\mathsf{E}[\cdot]$: expectation
- $\sigma(\cdot)$: the smallest sigma-algebra containing the argument

Symbols

- X_{t-}: $= \lim_{s \uparrow t} X_s$
- ΔX_t: $= X_t - X_{t-}$
- $\mu_j(A)$: j-th eigenvalue of a matrix A
- $\Re$: real part of a complex number
- $[a]_i$: i-th element of a vector a
- $[A]_{i,j}$: i, j-th element of a matrix A
- $A^\top$: transpose of a matrix A
- $\det A$: determinant of a matrix A
- $\operatorname{tr} A$: trace of a matrix A
- $\otimes$: Kronecker product

Abbreviations

a.s. almost surely

càdlàg right continuous with left limits

FV finite variation

Lem. Lemma

ODE Ordinary differential equation

PDE Partial differential equation

Prop. Proposition

resp. respectively

s.t. such that

Th. Theorem

ucp uniformly on compacts in probability

wrt with respect to

Chapter 1

Introduction

Modern engineering applications continue to grow in complexity, and very often they involve analysis of large volume of data. In filtering theory, the analysis of data corresponds to accurately inferring the true state of a dynamical system amidst noisy observations. Due to the dynamic nature of the variables which are modeled by differential equations with uncertainties, the problem is intrinsically quite challenging and emerges as a cornerstone discipline for many applications ranging including cryptography, tracking and guidance, speech recognition, image and video processing, genetics, financial modeling, etc. At its essence, filtering theory endeavors to distill signal from noise, enabling practitioners to extract meaningful information from imperfect and often incomplete observations.

In this thesis, we basically study the filtering problem for a certain class of dynamical systems which are modeled by continuous-time stochastic differential equations where the dynamics may be linear or nonlinear, and the underlying state-space may be finite (Markov Chains), or infinite (Itô differential equations). The common element that appears in all the problems is that the observation process is available only at some discrete time instants, which is partly motivated by the fact that, in many applications, the measurements are often not available at all times, and one needs to properly analyze the filtering algorithm taking into consideration this loss of information due to sampling of the observation process. In the sequel, we provide an overview of some developments in the field of filtering theory which are relevant for the topics studied in subsequent chapters, and also some details about the discrete observation process and its implication in filtering problems.

1.1 Overview of the filtering results

The theory of signal filtering first emerged in the first half of the 20th century in the pioneering works of Norbert Wiener [WH31; Wie49] and Andrey Kolmogorov [Kol41], and gained invaluable contribution in 1960's thanks to the publication of Rudolf Kalman's classic work [Kal60] and subsequent joint work [KB61] on the derivation of the Kalman–Bucy filter. The Kalman–Bucy filter is currently applied in numerous engineering and scientific fields, including machine learning, medicine, finance, and robotics (see reviews [CR11; Che03] and relevant references). The filtering theory finds relevance in problems of reconstruction, prediction, approximation, control, and stabilization of partially observed (and noisy) processes or signals.

To define the filtering problem in the context of stochastic dynamical systems, we consider a state process $(x_t)_{t\geq 0}$ described by a non-deterministic evolution rule and an observation process $(y_t)_{t\geq 0}$, which provides partial noisy measurements of the state. The problem is to compute the *best* estimator $(\widehat{x}_t)_{t\geq 0}$ using the observations and the system model. To quantify *best*, one has to specify some constraints and criteria which must be optimized when computing the estimator. In this regard, we take the following elements into consideration:

- the estimator is $\mathcal{Y}_t$-measurable, where $\mathcal{Y}_t$ is the sigma-algebra generated by observation process y on $[0, t]$;
- the estimator minimizes the mean square of the error $x_t - \widehat{x}_t$, i.e., $\widehat{x}_t$ is $\mathcal{Y}_t$-measurable integrable with the second moment stochastic processes that solves the following

$$\min_{\widehat{x}_t} \quad \mathsf{E}\|x_t - \widehat{x}_t\|^2.$$

A generic solution to this problem is provided by choosing $\widehat{x}_t$ as the expectation of the state process conditioned upon the observation process (up to time t). Depending upon the system dynamics, this conditional distribution, often called the *posterior distribution*, may or not be readily available in the closed-form. Research over the years in the area of filtering theory goes in the direction of computing this conditional distribution for different system classes, and providing different numerical tractable approximations when it is not readily computable. Here, we provide an overview of some techniques that are relevant for the developments in subsequent chapters.

Markov chains: For Markov chains, the state process evolves over a finite set and the transition from one value to another occurs at a random time instant. A probability distribution supported over this finite state space represents the evolution of the underlying state. The observation process is described by a nonlinear function of the state plus a diffusion term driven by Wiener process. The objective is to compute the conditional distribution of the state process given the observation. Because of the finite state space, this conditional distribution at each time instant is represented by a finite dimensional vector. For continuous-time Markov chains, Wonham filters proposed in [Won64] provide the evolution of conditional distribution in the form of observation-driven finite-dimensional stochastic differential equations.

Linear Gaussian case: Probably the most commonly studied class of systems in filtering theory is where the state process and the observation process are both linear in state and driven by standard Wiener process. The state space is the Euclidean space, and hence not necessarily finite. However, under the assumption that the initial condition is Gaussian, the state process and the observation process remain Gaussian at all times. This allows us to parameterize these processes entirely by the mean and covariance. Moreover, the conditional posterior distribution, which we need for computing the mean square estimate, is also Gaussian. Hence, as shown in [KB61] and several books by now [AM79; KS72], filtering problem boils down to computing the mean and the variance of the conditional distribution. The variance is obtained from a completely deterministic, observation independent, quadratic differential equation, also called the *Riccati* differential equation. The evolution of mean, on the other hand, is described by a linear stochastic differential equation where the randomness is due to the innovation process of the observations, and it involves an injection gain described by the time-varying covariance matrix. One can then analyze the properties of such filters under system theoretic assumptions like controllability (with respect to diffusion term) and the observability.

Nonlinear case: In case of dynamical systems described by nonlinear differential equations, the state process is not necessarily Gaussian and the conditional distribution is no longer characterized by the first and second moments. The exact solution of the filtering problem requires us to compute the conditional distribution at all times, and under certain hypothesis on system data, one can derive partial differential equations (PDEs) for describing the evolution of the conditional density. Several works have been pursued in this direction, and one may refer to [Str60; Kus67b; Kus67a; Zak69; FKK72] for development of the theory and the relevance of such results in different applications. While the problem is relatively well-studied for linear dynamical systems with closed-form solutions, the analysis and implementation of nonlinear filters have proven to be a rather challenging problem. For this reason, many research works have focused on providing relaxed versions of optimal filters for nonlinear systems, which are more amenable to implementation. However, from an analysis perspective, relatively fewer works exist that rigorously quantify the performance of nonlinear filters. For these reasons, nonlinear filtering continues to be an active area of research where the search for effective implementation algorithms and analysis methods is of interest for many applications [DJ11; GMP02; Han10; BS15; Tag+17; YMM13; SGK16; KSY19; Kar+20].

Ensemble Filters: The closed form expression for optimal filters is very often difficult to implement. In linear case, this may be due to higher order Riccati equations where the number of differential equations gets very large as the system dimension increases, and in the nonlinear case, we have to very often look for solutions to stochastic PDEs for solving the problem. Ensemble filters are similar in spirit to Monte-Carlo methods and are designed to get an approximation of the optimal filter with much lighter computational burden. The use of Monte-Carlo integration methods for approximating the optimal distribution have gained significant interest in the literature [DFG01; RAG03]. In the same spirit, [Eve94] introduced the technique of ensemble Kalman filters to develop filtering methods for large scale applications related to geophysical sciences, so that the error covariance is computed from a *collection* of state estimators rather than from a single (Riccati) differential equation. Since then, the use of ensemble Kalman filters have had a notable impact where estimation with noisy data is required in large-scale models. This approach is based on simulating the evolution of different particles through differential equations that are coupled through the associated empirical mean and the empirical error covariance. Several review articles [Che03; ESS22] and the books [Eve09] provide an overview of developments in that area. In most of these works, we do not find much details about the theoretical analysis of the proposed filtering techniques, and this area of mathematical analysis of the ensemble Kalman filters has gathered attention only very recently. Also, over the past decade, we see that the ensemble filtering methods can be viewed from the lens of mean-field models described by stochastic differential equations, and the particles are simply the approximations of these mean field models. Recent review articles which elaborate on this viewpoint are [BD23; TM23].

1.2 Random observation process

In many engineering applications and control systems in particular, an important aspect to consider in implementation of filtering and control algorithms is the loss of information from the measurements as the observations are transmitted between the plant and the estimator/controller. This may be in the form packet dropouts at certain time instants, or due to mapping of the information over an alphabet used for communication over channels with finite capacity. Over the past two decades, researchers in control theory have tried to study the effect of such loss in information due to communication networks [HNX07; Sch+07]. In this thesis, we address the filtering problem in continuous-time stochastic systems when the observations cannot be transmitted over the communication channel exactly and it is stipulated that the observation process is randomly time-sampled. Such an abstraction for the observation process is motivated by the use of certain communication protocols as pointed out in [TT17].

Let us now recall some references that have addressed filtering problems in the presence of measurement errors or communication uncertainties. For continuous-time linear systems, filtering problems with randomly sampled measurements have been studied in [Mic01] and [MJ02], where the authors discretize both the observation and state processes, compute a conditional density function, and use it to get the desired expectation. The papers [MS03; Sin+04; HD07; Sch08; YFX11; DCS14; PSD19] consider a discrete-time linear dynamical system and analyze the performance of filtering algorithms using the observations that have been transmitted over a communication channel. A different toolset, based on relative entropy, is adopted in [MY18] to study the stability and convergence of filters under relaxed assumptions on observation channels. For continuous-time dynamical system driven by white noise, one can adopt the structure of continuous-discrete observer proposed in [Jaz07]. However, there are relatively fewer works for filtering in continuous-time stochastic systems using irregular sampling [HMR11], and such setup is further explored in this thesis.

Other than the filtering algorithms, there have been other control-theoretic problems which have been studied when the measurements are randomly time-sampled, or the dynamical system is subject to resets at random time instants. In particular, [ACM00] addresses the optimal control in linear Itô stochastic differential equations with different distributions on the sampling process of the measurements. The reader may also refer to [HT06; AHS12] for results on stability with resets at random times, and stabilization with randomly sampled measurements. The book chapter [TCL18] provides an overview of such results and some recent developments.

More often than not, in our work we will assume that the sampling process (for the observations) is a Poisson counter. Consequently, this allows us to consider differential equations driven by Poisson processes. One may find an introduction to such classes of dynamical systems in [Bro09]. Based on the same idea and notation, applications are provided in [BGG99; MGT99]. Another advantage of considering the Poisson distribution for sampling process is that the mean sampling rate is characterized by a scalar parameter. When analyzing the performance of our filters subject to random sampling, this allows us to analyze study the suboptimality and stability related estimates in terms of the mean sampling rate.

1.3 Linear filters in closed form

We now revisit some of the basic results stated above, and motivate the modifications that appear when taking randomly sampled discrete observations into consideration. We start with the case of linear Itô differential equations in Gaussian setting. For this case, the closed form solution of the optimal filter, the so-called Kalman–Bucy filter, can be presented directly. Moreover, when dealing with discrete observations one can draw parallels rather easily with the classical solution.

1.3.1 Case of continuous observations

In a linear conventional model, the processes are given as Itô differential equations:

$$\begin{aligned} \text{state } x_t \in \mathbb{R}^n : \qquad & \mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \\ \text{continuous observations } z_t \in \mathbb{R}^p : \qquad & \mathrm{d}z_t = Cx_t\mathrm{d}t + \mathrm{d}\eta_t, \end{aligned}$$

where $(x_t)_{t\geq 0}$ is an $\mathbb{R}^n$-valued diffusion process describing the state, and $(z_t)_{t\geq 0}$ is an $\mathbb{R}^p$-valued observation process. It is assumed that, for each $t \geq 0$, $(w_t)_{t\geq 0}$ is an $\mathbb{R}^m$-valued, and $(w_t)_{t\geq 0}$ is an $\mathbb{R}^p$-valued, standard Wiener process with the property that $\mathsf{E}[\mathrm{d}w_t\,\mathrm{d}w_t^\top] = I_m\mathrm{d}t$, and $\mathsf{E}[\mathrm{d}\eta_t\,\mathrm{d}\eta_t^\top] = V\mathrm{d}t$, for each $t \geq 0$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are taken as constant, and the processes $(w_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ are independent and do not depend on the state. The solutions of the stochastic differential equations are interpreted in the sense of Itô stochastic integral. It follows that, if x_0 is Gaussian normally distributed then the process $(x_t)_{t\geq 0}$ is also Gaussian.

Using the classical results, see for example [Øks03, Th. 6.3.1], the optimal estimator minimizing the mean square estimation error is given by the conditional distribution $\mathsf{E}[x_t \,|\, (\mathrm{d}z_s)_{0\leq s\leq t}]$ which is Gaussian normal $\mathcal{N}(\widehat{x}_t, P_t)$, where $P_t = \mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top]$ denotes the covariance of the estimation error. The mean $\widehat{x}_t$ and the covariance P_t of this conditional distribution are described by the following equations:

$$\mathrm{d}\widehat{x}_t = \underbrace{A\widehat{x}_t\mathrm{d}t}_{\text{drift}} + \underbrace{P_tC^\top V^{-1}(\mathrm{d}z_t - C\widehat{x}_t\mathrm{d}t)}_{\text{gain}\times\text{innovation}},$$

where P_t satisfies the following matrix-valued ordinary differential equation:

$$\mathrm{d}P_t = (AP_t + P_tA^\top + GG^\top - P_tC^\top V^{-1}CP_t)\mathrm{d}t.$$

1.3.2 Case of Poisson-distributed observations

Our work is primarily aimed at studying filtering problem for continuous-time systems where the amount of information provided by the output measurements is limited not only by continuous white noise but also by time sampling. In particular, the output measurement is stipulated to arrive randomly at some discrete time instants only.

For the continuous observation model introduced in the previous section, let us introduce the identifications $y_t \sim \frac{\mathrm{d}z_t}{\mathrm{d}t}$ and $v_t \sim \frac{\mathrm{d}\eta_t}{\mathrm{d}t}$, so that v_t is a white Gaussian process, $v_t \sim N(0, V_t)$; see [Jaz07, Chapter 4] for further details. In other words the observation model is formally equivalent to $y_t = Cx_t + v_t$. Due to this observation, a natural approach to model observations at discrete time instants $\{\tau_k\}_{k\in\mathbb{N}}$ would be as follows:

$$y_{\tau_k} = Cx_{\tau_k} + v_{\tau_k}$$

where we additionally assume that the the instances τ_k are randomly distributed. We next show that this framework facilitates the handling of the optimal estimator for each realization of the observation process.

Model description

In the light of above discussion, we can now write down the system with state process in continuous-time, and observations in discrete-time in a compact form using the following equations:

$$\begin{aligned} \text{state } x_t \in \mathbb{R}^n : \qquad & \mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \\ \text{discrete observations } y_{\tau_k} \in \mathbb{R}^p : \qquad & y_{\tau_k} = Cx_{\tau_k} + v_{\tau_k}, \end{aligned}$$

where $\{v_{\tau_k}\}_{k\in\mathbb{N}}$ is a sequence of independent $\mathbb{R}^p$-valued normal random variables distributed as $\mathcal{N}(0, V)$. The nondecreasing sequence $\{\tau_k\}_{k\in\mathbb{N}}$ is assumed to be random. The particular case of our interest is when the time between two consecutive sampling instants is exponentially distributed, i.e., for each $k \in \mathbb{N}$,

$$\mathbb{P}((\tau_{k+1} - \tau_k) > h) = e^{-\lambda h}. \tag{1.3.1}$$

For each realization of $\{\tau_k\}_{k\in\mathbb{N}}$, one can construct the optimal filter for the aforementioned continuous-discrete system by using the model of the system for propagation of the estimate between two sampling instants, and updating the estimate at sampling times using the available measurements. This approach has appeared in [Jaz07, Th. 7.1]. It is also provided in Section 2.6 by Th. 2.21.

Let us notice that (1.3.1) implies that a counting process defined as

$$N_t = \sum_{k\in\mathbb{N}} \delta_{(\tau_k \le t)}$$

is a Poisson process with intensity λ. Consequently, observations can be called Poisson-sampled or Poisson-distributed; the times of observations are the arrival times of a Poisson process. Moreover, for $t > 0$, one may use the increments $\mathrm{d}N_t = \lim_{\delta\downarrow 0} N_t - N_{t-\delta}$, as it equals 1 only at the observation arrival time, and otherwise it equals zero. The rigorous definition and more details can be found in Chapter 2.

Filtering equations

In the proposed notation, the continuous-discrete filtering system might be rewritten in another form:

$$\begin{aligned} \text{state } x_t \in \mathbb{R}^n: \qquad & \mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \\ \text{discretely updated observations } y_t \in \mathbb{R}^p: \qquad & y_t = Cx_{\tau_{N_t}} + v_{\tau_{N_t}}, \end{aligned}$$

where $v_{\tau_{N_t}}$ on $t \in [0,\infty[$ coincides with v_{τ_k} for $k \in \mathbb{N}$ since the range of values of N_t is the set of nonnegative integers. Integration with respect to Poisson processes is discussed in Chapter 2, but here one can observe that the observation process $(y_t)_{t\ge 0}$ is piecewise constant and it updates only at times $(\tau_k)_{k\in\mathbb{N}}$. The optimal estimator $(\widehat{x}_t)_{t\ge 0}$ can be written in a compact form as follows:

$$\mathrm{d}\widehat{x}_t = \underbrace{A\widehat{x}_t\mathrm{d}t}_{\text{drift}} + \underbrace{P_tC^\top M_{P_t}^{-1}(y_t - C\widehat{x}_t)\mathrm{d}N_t}_{\text{gain}\times\text{innovation}},$$

where $P_t := \mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top \mid (y_s)_{0\le s\le t}]$ and satisfies a matrix-valued ordinary differential equation

$$\mathrm{d}P_t = (AP_t + P_tA^\top + GG^\top)\mathrm{d}t - P_tC^\top M_{P_t}^{-1}CP_t\,\mathrm{d}N_t$$

for $M_{P_t} := CP_tC^\top + V$.

One should notice that the definitions of covariances are not the same. Both are independent of the values of the observations, but for the continuous-discrete process, the covariance is conditioned upon the observation times. The study of the covariance matrix is important because the trace of the covariance matrix gives some indicator of the filter's performance. When the sampling process is random, the error covariance is conditioned upon the sampling process and the evolution of $(P_t)_{t\geq 0}$ is no longer deterministic. Nevertheless, we can study the expected performance of the filter with respect to the sampling rate of the observation process. In addition to the covariance being non-deterministic, we observe that the expression for the injection gain is now changed; instead of the inverse of the noise variance described by V^{-1}, we see that now $M_{P_t}^{-1}$ appears which also contains the error covariance P_t and it complicates the analysis. This gives a glimpse of some of the issues that arise in studying filters with discrete observations. In Chapter 3, we will address them in detail while generalizing the system class to include certain nonlinearities as well.

1.4 Ensemble filters

The exact closed-form solutions that one could obtain for linear Gaussian systems are not so easy to obtain for other system classes, or even when the process is not Gaussian. The generic solution given by the conditional expectation of the state process given the observations is applicable for broader settings, but this conditional distribution is not readily computable. It is therefore interesting to develop techniques which allow us to approximate the conditional distribution using simpler calculations. Also, for very high-dimensional linear systems, the optimal solution requires solving matrix-valued Riccati differential equation which can be computationally expensive.

The basic idea behind the design of ensemble filters is based on simulating several particles through differential equations that are coupled with each other through the empirical mean and the empirical error covariance. More recently, several research papers study estimation techniques based on ensembles using the viewpoint of mean-field process described by stochastic differential equations. The review articles [BD23; TM23] advocate this approach. The limiting behavior of these particles is described by a McKean–Vlasov type diffusion process also called the mean-field process. In the literature, this mean-field process is seen to be chosen in different ways, e.g., by adding noise in the prediction term and the correction term of the Kalman–Bucy process [DT18], or as a non-diffusion equation that is optimal in the measure transportation sense [TM20].

In contrast to these aforementioned works, our focus is on developing online ensemble filters suitable for Poisson-distributed observations. For this purpose, we propose a continuous-discrete limiting mean-field process and show that the distribution of this process conditioned upon the discrete observations coincides with the optimal distribution of the optimal filter. We use this process equation to describe a system of interacting particles driven by empirical mean and covariance. These particles are shown to be consistent with the proposed limiting process. This kind of analysis sets up the groundwork for studying the performance of these filters as a function of the mean sampling rate.

1.4.1 Working conjecture for vanilla ensemble filter

Let us consider the ensemble filter approach as another instance of the conceptual similarities between the purely continuous case and those with Poisson-distributed observations. In the literature on ensemble filters, we find different methodologies for describing the equations that govern the motion of particles, depending upon whether the particle equations are purely deterministic or they comprise identical copies of the noise processes that appear in state and observation processes.

As a first instance, we consider the case where the particle dynamics involve a copy of state process noise and the injection term involves a copy of observation noise process. The generalization of this particular design to the case of discrete observations can be carried out along the same lines as described earlier for the Kalman–Bucy filter. However, the resulting implementation is much different due to the interactions involved in ensemble filters. We provide an overview of this approach and later in Chapter 4, we provide a complete analysis of convergence to the optimal solution with the increase in the number of particles.

Let us recall the linear Itô system with continuous observations, described as follows:

$$\begin{aligned} \text{state } x_t \in \mathbb{R}^n : \qquad & \mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \\ \text{continuous observations } z_t \in \mathbb{R}^p : \qquad & \mathrm{d}z_t = Cx_t\mathrm{d}t + \mathrm{d}\eta_t, \end{aligned}$$

with usual assumptions on the system data as described earlier. The so-called vanilla ensemble filter, also known as ensemble Kalman filter (EnKF), introduced in [Eve94], is described as,

$$\mathrm{d}x_t^i = \underbrace{A_t x_t^i \mathrm{d}t + G\mathrm{d}w_t^i}_{\text{drift}} + \underbrace{P_t^{\mathrm{m}} C^\top V^{-1}\Big(\mathrm{d}z_t - (Cx_t^i\mathrm{d}t + \mathrm{d}\eta_t^i)\Big)}_{\text{gain}\times\text{innovation}}, \tag{1.4.1}$$

where (w_t^i, η_t^i) are independent copies of (w_t, η_t) and

$$P_t^{\mathrm{m}} := \frac{1}{\mathrm{m}-1}\sum_{i=1}^{\mathrm{m}}(x_t^i - \widehat{x}_t^{\mathrm{m}})(x_t^i - \widehat{x}_t^{\mathrm{m}})^\top, \quad \text{and} \quad \widehat{x}_t^{\mathrm{m}} := \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} x_t^i.$$

Note that the injection gain in (1.4.1) does not depend on the exact error covariance equation that is obtained from solving the Riccati differential equation. Instead, in (1.4.1), we use the empirical covariance which in turn is obtained from the empirical mean of different particles. Thus, the particles are coupled to each other through empirical covariance.

The structure of (1.4.1) can be replicated for the case of Poisson-distributed observations. Indeed, the innovation part, which is a zero-mean process, should be the subtraction of the observation dynamics. Hence, it can be written as $\Big(y_t - Cx_t^i + v_{N_t}^i\Big)\mathrm{d}N_t$. Let us denote particles by s^i instead of x^i to highlight the transition to the Poisson case. Thus, one may anticipate that

Vanilla filter in Poisson case is

$$\mathrm{d}s_t^i = \underbrace{A_t s_t^i \mathrm{d}t + G\mathrm{d}w_t^i}_{\text{drift}} + \underbrace{P_t^{\mathrm{m}} C^\top M_{P_t^{\mathrm{m}}}^{-1}\Big(y_t - (Cs_t^i + v_{N_t}^i)\Big)\mathrm{d}N_t}_{\text{gain}\times\text{innovation}}, \tag{1.4.2}$$

where (w_t^i, η_t^i) are independent copies of (w_t, η_t) and

$$\widehat{s}_t^{\mathrm{m}} = \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} s_t^i, \quad \text{and} \quad P_t^{\mathrm{m}} = \frac{1}{\mathrm{m}-1}\sum_{i=1}^{\mathrm{m}}(s_t^i - \widehat{s}_t^{\mathrm{m}})(s_t^i - \widehat{s}_t^{\mathrm{m}})^\top.$$

The notation $M_{P_t^{\mathrm{m}}}$ is for $CP_t^{\mathrm{m}}C^\top + V$ similarly to the linear optimal case. It will be shown in Section 4.5 of Chapter 4 that this intuitive modification provides consistent results, in the sense that, the expected covariance converges to optimal expected covariance as the number of particles gets large.

1.4.2 False conjecture for transport-inspired ensemble filter

In the literature on ensemble filters, we have more recently seen algorithms where equations of particles are developed from the principle of optimal transport, and are purely deterministic (other than the stochastic observation process entering the innovation part), see for example [TM20; BD23]. E.g., one may consider the transport-inspired filter, proposed in [TM20], that takes the following form:

$$\mathrm{d}x_t^i = A_t x_t^i \mathrm{d}t + \frac{1}{2}GG^\top (P_t^{\mathrm{m}})^{-1}(x_t^i - \widehat{x}_t)\mathrm{d}t + P_t^{\mathrm{m}} C^\top V^{-1}\Big(\mathrm{d}z_t - C\frac{x_t^i + \widehat{x}_t}{2}\mathrm{d}t\Big),$$

where once again, we take

$$P_t^{\mathrm{m}} := \frac{1}{\mathrm{m}-1}\sum_{i=1}^{\mathrm{m}}(x_t^i - \widehat{x}_t^{\mathrm{m}})(x_t^i - \widehat{x}_t^{\mathrm{m}})^\top, \quad \text{and} \quad \widehat{x}_t^{\mathrm{m}} := \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} x_t^i.$$

The foregoing particle equation contains the innovation part $\mathrm{d}z_t - C\frac{x_t^i + \widehat{x}_t}{2}\mathrm{d}t$. For the case of Poisson-distributed observations, it may be tempting to take the corresponding innovation term as $\Big(y_t - C\frac{s_t^i + \widehat{s}_t}{2}\Big)\mathrm{d}N_t$. Hence, a conjecture for adapting transport-inspired filter to the case of Poisson-distributed observations is

$$\mathrm{d}s_t^i = A_t s_t^i \mathrm{d}t + \frac{1}{2}GG^\top (P_t^{\mathrm{m}})^{-1}(s_t^i - \widehat{s}_t)\mathrm{d}t + P_t^{\mathrm{m}} C^\top M_{P_t^{\mathrm{m}}}^{-1}\Big(y_t - C\frac{s_t^i + \widehat{s}_t}{2}\Big)\mathrm{d}N_t.$$

where once again we take

$$P_t^{\mathrm{m}} := \frac{1}{\mathrm{m}-1}\sum_{i=1}^{\mathrm{m}}(s_t^i - \hat{s}_t^{\mathrm{m}})(s_t^i - \hat{s}_t^{\mathrm{m}})^\top, \quad \text{and} \quad \widehat{s}_t^{\mathrm{m}} := \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} s_t^i.$$

Indeed, one can compute the dynamics of the empirical moments using the chain rule, derived in Chapter 2. Surprisingly, as the number of particles tends to infinity, the empirical covariance does not converge to the optimal one. Instead, we propose to change the average $\frac{s_t^i + \widehat{s}_t}{2}$ in the innovation part to a parameterized sum. Namely, we prove that the following filter serves our purposes:

Proposed transport-inspired filter for Poisson case:

$$\begin{aligned}\mathrm{d}s_t^i = A s_t^i \mathrm{d}t + \frac{1}{2}GG^\top (P_t^{\mathrm{m}})^{-1}(s_t^i - \widehat{s}_t^{\,\mathrm{m}})\mathrm{d}t + \\ \Big(P_t^{\mathrm{m}} C^\top M_{P_t^{\mathrm{m}}}^{-1}(y_t - C\widehat{s}_t^{\mathrm{m}}) - \Xi_t^{\mathrm{m}}(s_t^i - \widehat{s}_t)\Big)\mathrm{d}N_t,\end{aligned}$$

where Ξ_t satisfies the following for $t = \tau_{N_t}$:

$$(\Xi_t - I)P_t^{\mathrm{m}}(\Xi_t^\top - I) = P_t^{\mathrm{m}} - P_t^{\mathrm{m}} C^\top (C P_t^{\mathrm{m}} C^\top + V_t)^{-1} C P_t^{\mathrm{m}}.$$

More details about this design appear in Chapter 4.

1.5 Nonlinear filtering in Markov chains

So far we discussed filtering problem for linear systems with Gaussian processes. The natural question is how can one address similar questions for the nonlinear setting. In the literature on filtering problems, probably the first instances of nonlinear filtering appeared in the setting of Markov chains. Because of the finiteness of the state-space, the conditional distribution, which solves the optimal filtering problem, is described by finitely many stochastic differential equations. This is relatively easier to analyze than the stochastic partial differential equations describing the optimal conditional distribution for generic nonlinear Itô differential equations.

The pioneering work [Won64] introduces the optimal filter (called now the Wonham filter). Over the past twenty-five years, we have seen a certain interest in better understanding some system-theoretic properties of these filters. Earlier papers in this direction studied stability with respect to initial conditions [OP96; BCL04; BD17] under different assumptions on the underlying Markov chain, and tools such as Lyapunov exponents were developed to characterize the convergence rates for the decay in error due to mismatched initial conditions. The PhD dissertation [Han07] revisits some of these results and, in addition, provides results on robustness with respect to unknown parameters in the models [Han07, Chapter 3]. Such connections between system-theoretic tools and nonlinear filtering continue to develop in more recent works [KMM19; KM21], where the authors develop a dual of the filtering problem in the form of a backward stochastic differential equation and reformulate the stability of nonlinear filter in terms of stabilizability properties of the dual system. The stability analysis of nonlinear filters continues to attract the attention of different communities [KSY19; Kar+20]. In the presence of communication channels, the stability of filters is treated in [MY18], with connections to observability and information rates.

On a conceptual level, Wonham filter provides the optimal solution to the filtering problem and the aforementioned works analyze how the solution deviates from the optimal solution if the initial condition, or some system parameters, are changed. In our work, we also consider the deviation of a finite-state filter from the optimal solution. However, the source of this deviation is considerably different; that is, we stipulate that the continuous-time observation process is not available for measurement, but instead, the realizations of this observation process are available at some randomly drawn time instants. In other words, the information used to compute the conditional distribution is different in our setup. We are primarily interested in defining an optimal filter for the continuous-time Markov chain subject to this discrete information structure.

1.6 Contribution of the thesis

1.6.1 Overview and comparisons

So far, we reviewed some known filtering techniques from the literature and showed how those algorithms can be adapted for the case when the observation process is randomly time-sampled. The later chapters will provide a more formal development of these filtering algorithms with discrete observations. In addition to the algorithm description, our focus is also on analyzing the performance of these filters. For the Itô differential equations, when studying the generalization of Kalman–Bucy filters, we study the suboptimality of this algorithm in the nonlinear setting. We also compute the evolution of expected error covariance and the conditions under which it remains bounded. In the process, we can also analyze the dependence of the expected error covariance on the mean sampling rate of the observations.

For the case of ensemble filters, we draw connections with continuous-discrete McKean–Vlasov type diffusion processes which are used to describe the evolution of the particles. Since the individual particles do not provide the exact optimal solution, it is important to carry out the suboptimality analysis by looking at the difference between the empirical covariance (respectively, empirical mean) and the optimal covariance (resp., optimal mean) of the conditional distribution. We will also discuss tools that allow us to study the convergence of ensemble filters to the optimal filters as the number of particles gets large.

For the case of Markov chains, the discrete process that we consider is different than the ones consider Itô differential equations. Here, our focus is on studying the convergence of the discretized optimal filter to continuous-time optimal filter as the mean sampling rate gets large.

An overview of our results, which appear in this thesis, is given in the following table in the form of comparisons with some standard references.

Conventional case	**Poisson-distributed** y_{τ_k}
(optimal) Kalman–Bucy filter [KB61; Jaz07]	the linear optimal filter[TY20]
(optimal) Wonham filter [Won64]	the optimal filter on Markov chains[YT22]
vanilla, or ensemble Kalman filter[Eve94]	the ensemble filter[TY24]
optimal-transport ensemble filter [TM20; BD23]	the ensemble filter[YT23]

1.6.2 List of publications

The thesis is based on the following publications of the author:

Peer-reviewed international journals:

[TY20] Error covariance bounds for suboptimal filters with Lipschitzian drift and Poisson-sampled measurements. A. Tanwani, O. Yufereva. Automatica (2020)

[TY24] Convergence analysis of ensemble filters for linear stochastic systems with Poisson-sampled observations. A. Tanwani, O. Yufereva. To appear in IFAC-PapersOnLine (2024).

Peer-reviewed international conferences:

[YT22] Approximation of Nonlinear Filters for Continuous-Time Markov Chains under Randomly-Sampled Observations. O. Yufereva, A. Tanwani. Proceedings of IEEE Conf. Decision & Control (2022).

[YT23] Transport Inspired Particle Filters with Poisson-Sampled Observations in Gaussian Setting. O. Yufereva, A. Tanwani. Proceedings of IEEE Conf. Decision & Control (2023).

1.7 Structure of the thesis

Motivated by the need to model and process filtering problems with Poisson-distributed observations, an introduction to stochastic differential equations driven by a Poisson process is provided in Chapter 2. This chapter proposes an approach for stochastic integration in a particular form and provides its properties and comparison to known approaches. The main results are the corresponding form of the chain rule (Prop. 2.14), the derivation of infinitesimal generator (Prop. 2.19), and the implementation of analog of Dynkin's formula (Th. 2.20).

Chapter 3 studies a nonlinear case with a structural assumption on the drifts. Similar assumptions can be found in works [GR73; Pic91]. In order to have a computation-friendly estimator, a suboptimal filter is proposed, and its performance under Poisson-sampled observations is analyzed. Moreover, the proposed filter can be treated by computing a dynamical optimal gain matrix, assuming the gain matrix is constant. The main results cover the following points. Prop. 3.3 and 3.4 estimate the covariance of the proposed filter for an arbitrary and the optimal gain matrices resp. Prop. 3.5 presents the expectation of the covariance with respect to the sampling rate (intensity of the Poisson process). Convergence to a finite steady state with sufficient condition is shown in Th. 3.12, and reformulation of the sufficient conditions as an eigenvalue test is done in Prop. 3.14. For the linear case, the lower bound on the sampling rate is given in Prop.3.17 and Prop.3.18 states the monotonicity of expected covariances with respect to the sampling rate.

Chapter 4 investigates the ensemble filters. The sufficient conditions for a continuous-discrete mean-field process to be an exact filter are stated in Prop 4.2. These conditions can be met in different ways, and we propose two types of ensemble filters satisfying these conditions. The consistency of the transport-inspired ensemble filter is provided by Th. 4.4; notice that it implies that the convergence with respect to the number of particles is not needed. The consistency of the vanilla ensemble filter is shown by Th. 4.5, while the stability follows from Prop. 4.8.

In Chapter 5, we restrict our attention to a system with a finite state Markov chain state process, which also makes the optimal filter a Markov chain. We are then interested in analyzing how the continuous-discrete filter, derived for Poisson-distributed observations, differs from the continuous-time filter. In particular, we assign Poisson distribution to the randomly drawn time instants and look at the expectation of the conditional probability with respect to the distribution of the sampling process. The main result, Th. 5.4, shows that as Poisson intensity tends to infinity, the derived filter converges to the classical one in L_1. In addition, an optimal filter for uniform discrete observations and an auxiliary process play a role in convergences proved by intermediate statements (Lem. 5.3.5, Prop. 5.7).

The conclusion is provided in Chapter 6. First it recalls the problem frames and the contribution of this work. Secondly it describes the various future directions.

Chapter 2

Basic Tools

This chapter presents some basic tools for analysis of stochastic dynamical systems that arise in studying filtering problem with discrete observations. In particular, our attention is on Poisson-distributed observations which give rise to differential equations driven by Poisson processes. Such a setup allows us to investigate and pave the way for new methods. The problem setting considered here presents a simple instance of studying discrete observations and avoids several complexities which may not be necessary for the basic understanding of the problem.

The primary object of the analysis is a Poisson process. We provide its definition and comparison with generalized forms. Integration and the corresponding chain rule are introduced as self-contained statements. They can not be seen as new results as they are particular cases of more general theories. However, the proposed methods are sufficient to deal with the cases considered in this manuscript and fairly easy to comprehend, contributing to the analysis.

2.1 Poisson process

An excellent introduction and numerous examples of applications for the Poisson processes can be found in [SK08]. In order to study integration with respect to the Poisson processes, it is beneficial to revisit the properties of counting processes first. Such an approach is close to [Pro05].

Poisson process as a counting process

Assume that a filtered[1] probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ satisfies the usual hypotheses, i.e. every $\mathcal{F}_t$ contains all sets of probability 0 and $\mathcal{F}$ is complete. All functions are assumed to be càdlàg, that means right continuous with left limits (from French: continue à droite, limite à gauche).

Definition 2.1. *The process $(N_t)_{0\leq t\leq\infty}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{\{t\geq\tau_n\}}$$

with values in $\mathbb{N}\cup\{\infty\}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is called the counting process associated to the sequence $(\tau_n)_{n\geq 1}$.

The increment $N_t - N_s$ counts the number of random time instances τ_n that occur between the fixed times s and t. The Poisson process is one of the counting processes. Moreover, note that if an adapted[2] counting process has stationary independent increments, it is a Poisson process, as shown below.

The random variable $T = \sup_n \tau_n$ is the explosion time of $(N)_{t\geq 0}$. If $T = \infty$ a.s., then $(N)_{t\geq 0}$ is a counting process without explosions. For $T = \infty$, note that for $0 \leq s < t < \infty$, we have

$$N_t - N_s = \sum_{n\geq 1} 1_{\{s<\tau_n\leq t\}}.$$

As we have defined, a counting process is not necessarily adapted to the filtration $\mathbb{F}$. Indeed, we have the following.

Theorem 2.2 ([Pro05, Ch.I, Th.22])**.** *A counting process $(N)_{t\geq 0}$ is adapted if and only if the associated random variables $(\tau_n)_{n\geq 1}$ are stopping times[3].*

Note that a counting process without explosions has right continuous paths with left limits; hence, a counting process without explosions is càdlàg.

Definition 2.3. *An adapted counting process $(N)_{t\geq 0}$ is a Poisson process if*

1. ***(increments are independent of the past)*** *$N_t - N_s$ is independent of $\mathcal{F}_s$ for any $s, t, 0 \leq s < t < \infty$;*

2. ***(stationary increments)*** *the distribution of $N_t - N_s$ is the same as that of $N_v - N_u$ for any non-negative s, t, u, v such that $t - s = v - u > 0$.*

Therefore we consider only adapted processes.

---

[1] Here filtration $\{\mathcal{F}_t\}_{t\geq 0}$ means non-decreasing and right-continuous family of sub-σ-fields of $\mathcal{F}$.

[2] A process $\alpha(t,\omega) : [0,\infty[\times\Omega \to \mathbf{R}^n$ is called $\mathcal{F}_t$-adapted if for each $t \geq 0$ the function $\omega \to \alpha(t,\omega)$ is $\mathcal{F}_t$-measurable.

[3] A random variable $T : \Omega \to [0,\infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$, every $t, 0 \leq t \leq \infty$.

Properties of Poisson processes

Further we recall the properties of Poisson processes related to its distribution.

Theorem 2.4 ([Pro05, Ch.I, Th.23]). *Let $(N_t)_{t\geq 0}$ be a Poisson process. Then*

$$\mathsf{P}\left(N_t = n\right) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$n = 0, 1, 2, \ldots$, for some $\lambda > 0$. Such λ is called the intensity of the Poisson process. That is, N_t has the Poisson distribution with parameter λt. Moreover, $(N_t)_{t\geq 0}$ is continuous in probability[4] *and does not have explosions.*

Notice that Theorem 2.4 immediately implies the following facts on a Poisson process $(N_t)_{t\geq 0}$ with intensity λ.

1. The expectation $\mathsf{E}[N_t] = \lambda t$;
2. $(N_t - \lambda t)_{t\geq 0}$ is a martingale;
3. Almost all sample path of $(N)_{t\geq 0}$ has a finite number of jumps on a finite interval;
4. (Markov property) the distribution of $(N_{t+s} - N_t)$ coincides with the distribution of N_s for $s \geq 0$;
5. (Strong Markov property [SK08, Th. 2.3.5]) the distribution of $(N_{\tau_k+s} - k)$ coincides with the distribution of N_s for any $s \geq 0$ and any arrival time τ_k.

As an equivalent definition one may consider the following corollary.

Corollary 2.5 ([SK08, Th. 2.3.2]). *If $(N)_{t\geq 0}$ is a Poisson process of intensity λ, then*

1. $N_0 = 0$;
2. *its increments are independent, i.e. $(N_{t_1} - N_{t_0}), \ldots, (N_{t_n} - N_{t_{n-1}})$ are independent for all $0 = t_1 < \ldots < t_n$;*
3. *for all $t \geq 0$, the following infinitesimal property holds*

$$\begin{aligned}
&\mathsf{P}(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta),\\
&\mathsf{P}(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta),\\
&\mathsf{P}(N_{t+\delta} - N_t \geq 2) = o(\delta).
\end{aligned}$$

[4] I.e. at an arbitrary time t almost all sample paths are continuous, while each sample path is discontinuous on $[0, \infty[$.

Hence, Corollary 2.5 provides us with infinitesimal behavior of a Poisson process. If a time interval is fixed and the number of arrivals in the interval is given, their distribution is described as follows.

Theorem 2.6 ([SK08, Th. 2.3.7])**.** *If $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, then for all $s, t > 0$ and $m = 1, 2, \ldots$, conditional on $N_{t+s} - N_s = m$, the jump points $J_1 = J_1(s,t), \ldots, J_m = J_m(s,t)$ in $]s, s+t[$ exhibit the joint probability density function*

$$\begin{aligned}
&f_{J_1,\ldots,J_m}\left(x_1, \ldots, x_m \mid m \text{ jumps in total in }]s, s+t[\right) \\
&= \left(\frac{m!}{t^m}\right) \mathbf{1}\left(s < x_1 < \cdots < x_m < t+s\right).
\end{aligned}$$

That is, conditional random variables

$$(J_1, \cdots, J_m \mid m \text{ jumps in interval }]s, s+t[)$$

are obtained by throwing m uniform independent identically distributed points on the interval $]s, s+t[$ and listing them in the increasing order.

In particular, conditional on $m = 1$ (a single jump), the point J_1 is distributed by the uniform distribution on $]s, s+t[$.

2.2 Integration with respect to Poisson process

For the details on Itô integration, we refer to [Øks03]. Integration with respect to a Poisson process is a particular case of the integration with respect to semimartingales, studied in [Pro05]. Employing the properties of Poisson processes one concludes from [Pro05, Ch. II] the following

Definition 2.7. *For an adapted càdlàg process $(\gamma_t)_{t\geq 0}$ the stochastic integral on interval $]t_0, t_1[$ of $(\gamma_t)_{t\geq 0}$ with respect to a Poisson process $(N_t)_{t\geq 0}$ with arrival times $\{\tau_k\}_{k\in\mathbb{N}}$ is*

$$\int_{t_0}^{t_1} \gamma_s \mathrm{d}N_s = \sum_{k:\tau_k \in]t_0,t_1]} \gamma_{\tau_k}.$$

Notice that

1. $\int_{t_0}^{t_1} \gamma_s \mathrm{d}N_s = \int_{t_0}^{t} \gamma_s \mathrm{d}N_s + \int_{t}^{t_1} \gamma_s \mathrm{d}N_s$ for $t \in]t_0, t_1[$,

2. $\int_{t_0}^{t_1}\Big(\gamma_s+\zeta_s\Big)\mathrm{d}N_s=\int_{t_0}^{t_1}\gamma_s\mathrm{d}N_s+\int_{t_0}^{t_1}\zeta_s\mathrm{d}N_s$,

for adapted càdlàg processes $(\gamma_t)_{t\geq 0}, (\zeta_t)_{t\geq 0}$. This makes it natural to consider the more general stochastic integrals of the form

$$e_{t_1}=e_{t_0}+\int_{t_0}^{t_1}\alpha_s\mathrm{d}s+\int_{t_0}^{t_1}\sigma_s\mathrm{d}w_s+\int_{t_0}^{t_1}\gamma_s\mathrm{d}N_s,$$

where $(w_t)_{t\geq 0}$ is a Wiener process and α, σ, γ are càdlàg processes bounded on compacts.

2.2.1 SDEs with a Poisson term

A fruitful technique that permits a unified treatment of time varying and nonlinear problems was introduced in [Bro09]. It combines deterministic integration with Poisson integration. Such framework provides sound intuition for stochastic differential equations and their uses without allowing the technicalities to dominate. Building upon this study, we combine Poisson integration with Itô integration.

A sample path of $(N_t)_{t\geq 0}$ has only a finite number of jumps on a finite interval. Hence, for a sample path of a Poisson process $(N_t)_{t\geq 0}$, an integral with $\mathrm{d}N_t$ can be defined as a Lebesgue–Stieltjes integral. Moreover, one can consider the following stochastic integral equation

$$e_{t_1}=e_{t_0}+\int_{t_0}^{t_1}f(s,e_s)\mathrm{d}s+\int_{t_0}^{t_1}G(s,e_s)\mathrm{d}w_s+\int_{t_0}^{t_1}g(s,e_s)\mathrm{d}N_s \tag{2.2.1}$$

that is in differential form

$$\mathrm{d}e_s=f(s,e_s)\mathrm{d}s+G(s,e_s)\mathrm{d}w_s+g(s,e_s)\mathrm{d}N_s, \tag{2.2.2}$$

where $(w_t)_{t\geq 0}$ is a Wiener process, $(N)_{t\geq 0}$ is a Poisson process and functions f, G, g satisfy Assumption 2.8.

Assumption 2.8 ([Gra92, hypothesis of Th. 1.2])**.** *Let a triplet of integrable càdlàg functions*

$$\begin{aligned} f\colon&\quad [0,\infty[\times\mathbb{R}^n\to\mathbb{R}^n,\\ G\colon&\quad [0,\infty[\times\mathbb{R}^n\to\mathbb{R}^{n\times m},\\ g\colon&\quad [0,\infty[\times\mathbb{R}^n\to\mathbb{R}^n \end{aligned}$$

be such that for each compact set B in $\mathbb{R}^+$ there exists a constant K and

$$\|G(t,x)-G(t,y)\|^2+\|f(t,x)-f(t,y)\|^2\leq K\|x-y\|^2; \tag{2.2.3a}$$

$$\|g(t,x)-g(t,y)\|\leq K\|x-y\| \tag{2.2.3b}$$

for $x, y\in\mathbb{R}^n$, $t\in B$.

We define a solution of (2.2.2) in a particular way, allowed by the features of the choice of differentials.

Definition 2.9. *Let $(w)_{t\geq 0}$ be a standard m-dimensional Wiener process and $(N)_{t\geq 0}$ be a Poisson process. Let functions f, G, g satisfy Assumption 2.8. An $\mathbb{R}^n$-valued càdlàg process $(e_t)_{t\geq 0}$ is called a solution of*

$$\mathrm{d}e_t = f(t, e_t)\mathrm{d}t + G(t, e_t)\mathrm{d}w_t + g(t, e_t)\mathrm{d}N_t, \qquad e_0 = \bar{e} \tag{2.2.4}$$

if, for almost all sample paths of Poisson process $(N_t)_{t\geq 0}$, it satisfies

$$\mathrm{d}e_t = f(t, e_t)\mathrm{d}t + G(t, e_t)\mathrm{d}w_t, \qquad \text{for } t \in]\tau_k, \tau_{k+1}[,\ k \in \mathbb{N}, \tag{2.2.5a}$$

$$e_{\tau_k} = \lim_{s\uparrow\tau_k}\Big(e_s + g(s, e_s)\Big), \qquad \text{for } k \in \mathbb{N}^+, \tag{2.2.5b}$$

$$e_{\tau_0} = e_0 = \bar{e}. \tag{2.2.5c}$$

This definition is easy to implement. In addition, the definition is consistent with the general forms that are discussed in Section 2.2.2.

In a formal manner, the last term of equation (2.2.4) can be written with the left limits: $g(t_-, e_{t_-})\mathrm{d}N_t$. For the sake of brevity, this formal notation is used only in ambiguous cases.

2.2.2 Comparison

There are methods of stochastic integration with respect to 1) martingales, 2) semimartingales, and 3) random measures. They all can be found in [JS13]. The use of Poisson random measures is rather frequent, see e.g. [ØS19; PR16] and references therein. Let us briefly describe approaches that develop more general cases.

Semimartingales

Notice that a Poisson process N_t is a finite variation semimartingale, and Definition 2.9 is consistent with integration with respect to semimartingales given in [Pro05, Ch. II, Section 4]. This can be easily checked due to the fact that N_t is a simple predictable process with decomposition $N_t = \sum_i \delta_{\tau_i \leq t}$. In this approach stochastic integration with respect to semimartingales can be thought of as an extension of path-by-path Stieltjes integration thanks to the following theorem.

Theorem 2.10 ([Pro05, Ch. II, Th. 17])**.** *Let all paths of a semimartingale $(H_t)_{t\geq 0}$ have finite variation on compacts. If $(g_t)_{t\geq 0}$ is a càdlàg adapted process, then $\int_0^{t'} g_t \mathrm{d}H_t$ is indistinguishable from the Lebesgue–Stieltjes integral computed pathwise for $t' \in]0, \infty[$.*

Jump diffusion

It is worth mentioning the book [ØS19] on stochastic differential equations driven by Lévy processes[5]. Every càdlàg Lévy process is a semimartingale, so this work focuses on a more nuanced case than [Pro05]. Moreover, a Lévy process can be split as follows.

Theorem 2.11 ([ØS19, Th. 1.7]). *Let $(\eta_t)_{t\geq 0}$ be a Lévy process. Then η_t has the decomposition*

$$\eta_t = \alpha t + \beta B(t) + \int_{|z|<R} z\tilde{N}(t, dz) + \int_{|z|\geq R} zN(t, \mathrm{d}z), \tag{2.2.6}$$

for some constants $\alpha \in \mathbb{R}, \beta \in \mathbb{R}, R \in [0, \infty]$. Here

$$\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$$

is the compensated Poisson random measure of $\eta(\cdot)$, $\nu(\cdot)$ is the corresponding Lévy measure, and $(B)_{t\geq 0}$ is an independent Brownian motion.

Equation (2.2.6) can be transformed into SDE with respect to $\mathrm{d}\eta_t$. Its time homogeneous is called jump diffusion. It is the main object of [ØS19]. Note that instead of a Poisson process, equation (2.2.6) includes a Poisson random measure. It has a Borel set as the second argument, which can be considered as a spatial density of jumps. In our case, this argument is implicitly assumed to be constant and so is always omitted. Our case is less general and thus it inherits properties, including the conditions on existence and uniqueness of SDEs.

Itô–Skorohod equations

Another similar case is presented in [Gra92], which calls the corresponding objects as Itô–Skorohod equations. These equations can be recognized as (2.2.6), however, the assumption $\lambda = 1$ appears in [Gra92] for the sake of simplicity. Indeed, it can be easily changed to an arbitrary $\lambda > 0$ by rescaling time.

In contrast to this possible rescaling, filtering problems impose constrains on the system dynamics and using an appropriate intensity of Poisson processes is crucial. Nonetheless, the existence and uniqueness conditions provided in [Gra92] for SDEs can be used in our case as well. Moreover, the work [Gra92] also studies McKean–Vlasov equations with Poisson term, that one will find in Chapter 4.

[5] An $\mathbb{F}$-adapted process $(\eta_t)_{t\geq 0}$ with $\eta_0 = 0$ a.s. is called a Lévy process if η_t is continuous in probability and has stationary, independent increments.

2.2.3 Existence and uniqueness

Exploiting the similarity, we obtain that [Gra92, Th.1.2] provides suitable existence and uniqueness conditions. For our case they can be reformulated as follows.

Theorem 2.12 (Reformulation of [Gra92, Th.1.2])**.** *If functions f, G, g satisfy Assumption 2.8, then there exists a unique strong solution for the equation* (2.2.4) *starting at $e_0 = \bar{e}$.*

In addition, the following SDE is of specific interest to us:

$$\mathrm{d}e_t = A_t e_t \mathrm{d}t + \Gamma_t \mathrm{d}w_t + [\Xi_t e_t + \Omega_t]\mathrm{d}N_t, \qquad e_0 = \bar{e} \tag{2.2.7}$$

which is easy to treat as far as a realization of the Poisson process is given.

Corollary 2.13. *Let A, Γ, Ξ, Ω are càdlàg adapted processes, bounded on compacts. There exists a unique strong solution for the equation* (2.2.7)*.*

Such equations are discussed in Section 2.4.

2.3 Chain rule

The chain rule involving Poisson processes is often used in this manuscript. Our assumptions allows us to prove the chain rule in a direct way. Nonetheless, it can be considered as a corollary of [Pro05, Ch. II, Th. 31–32] on the chain rule for semimartingales.

Proposition 2.14 (Chain rule)**.** *Let a càdlàg n-dimensional process $(e_t)_{t\geq 0}$ satisfy*

$$\mathrm{d}e_t = \alpha_t \mathrm{d}t + \sigma_t \mathrm{d}w_t + \gamma_t \mathrm{d}N_t,$$

where $(w_t)_{t\geq 0}$ is an m-dimensional Wiener process and α, σ, γ are càdlàg processes bounded on compacts. If $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function, then[6]

$$\begin{aligned} \mathrm{d}\psi(e_t) = \sum_i \frac{\partial \psi(e_t)}{\partial [e_t]_i} [\alpha_t]_i \mathrm{d}t + \frac{1}{2} \sum_{i,j} \frac{\partial \psi(e_t)}{\partial [e_t]_i \partial [e_t]_j} [\sigma_t \sigma_t^\top]_{i,j} \mathrm{d}t \\ + \sum_i \frac{\partial \psi(e_t)}{\partial [e_t]_i} [\sigma_t \mathrm{d}w_t]_i + \Big(\psi\Big(e_t + \gamma_t\Big) - \psi(e_t)\Big) \mathrm{d}N_t. \end{aligned} \tag{2.3.1}$$

[6] $[\cdot]_i$ denotes the i-th element of the vector and $[\cdot]_{i,j}$ denotes the (i,j)-th element of the matrix.

Proof. Recall that for almost all sample paths of the Poisson process $(N_t)_{t\geq 0}$ the arrival times $\{\tau_k\}_{k\in\mathbb{N}}$ form an increasing sequence with $\lim_k \tau_k = \infty$. All $\{\tau_k\}_{k\in\mathbb{N}}$ are stopping times. For an arbitrary time $T > 0$ let us consider the integrals form of the processes $(e_t)_{t\geq 0}$ and $(\psi(e_t))_{t\geq 0}$ on $[0, T]$.

Let $\theta = \sup\limits_{k\in\mathbb{N}}\{\tau_k \leq T\}$. Firstly, if $T \neq \theta$ and say θ equals to a τ_n, then

$$e_T = e_{\tau_n} + \int\limits_{\tau_n}^{T} \alpha_t \mathrm{d}t + \int\limits_{\tau_n}^{T} \sigma_t \mathrm{d}w_t$$

since there are no jumps on $]\tau_n, T]$. Hence we can apply the classical chain rule [Øks03, Th. 4.2.1] and obtain

$$\psi(e_T) = \psi(e_{\tau_n}) + \int\limits_{\tau_n}^{T} \sum_i \frac{\partial\psi(e_t)}{\partial[e_t]_i}[\alpha_t]_i \mathrm{d}t \\ + \int\limits_{\tau_n}^{T} \frac{1}{2}\sum_{i,j} \frac{\partial\psi(e_t)}{\partial[e_t]_i\partial[e_t]_j}[\sigma_t\sigma_t^\top]_{i,j}\mathrm{d}t + \int\limits_{\tau_n}^{T} \sum_i \frac{\partial\psi(e_t)}{\partial[e_t]_i}[\sigma_t \mathrm{d}w_t]_i.$$

The integral $\int\limits_{\tau_n}^{T} \Big(\psi\Big(e_t + \gamma_t\Big) - \psi(e_t)\Big)\,\mathrm{d}N_t$ equals zero as $N_T - N_{\tau_n} = 0$ and can be added to obtain integral form of (2.3.1) formally.

Thus, it remains to consider the second case which is $T = \theta$; it means $T = \tau_n$ for a certain $n \in \mathbb{N}$. In contrast to the classical proof, it suffices to consider the following decomposition

$$\psi(e_{\tau_n}) - \psi(e_0) = \sum_{\tau_k<T} [\psi(e_{\tau_{k+1}}) - \psi(e_{\max\{\tau_{k+1}-\varepsilon,\tau_k\}})] \\ + \sum_{\tau_k<T} [\psi(e_{\max\{\tau_{k+1}-\varepsilon,\tau_k\}}) - \psi(e_{\tau_k})]$$

where $\varepsilon > 0$ will tend to zero. The second sum excludes jumps and so can be written in the form of classical Itô integrals. The first sum tends to $\sum_{\tau_k<T}[\psi(e_{\tau_k}) - \psi(e_{\tau_k^-})]$ that is

$$\sum_{\tau_k<T} [\psi(e_{\tau_k^-} + \gamma_{\tau_k^-}) - \psi(e_{\tau_k^-})] = \int\limits_0^{\tau_n} \psi(e_t + \gamma_t) - \psi(e_t)]\mathrm{d}N_t$$

that implies the differential form (2.3.1).

□

2.4 Particular cases

Let us now consider a few basic elements which are described using the notion of Poisson process. These examples will be used later in the context of filtering problem as well.

Piecewise constant zero-mean process

Assume that $\{v_k\}_{k\in\mathbb{N}}$ is a sequence of independent random variables where $v_k \sim \mathcal{N}(0, V_k)$. Consider a process $(b_t)_{t\geq 0}$ defined as

$$b_t := v_{N_t},$$

for a Poisson process $(N_t)_{t\geq 0}$. One should notice that the entities v_{N_t} are defined for all $t \geq 0$ since $(N_t)_{t\geq 0}$ is $\mathbb{N}$-valued process. In addition, each v_k for $k \in \mathbb{N}$ is used, since $N_t \to \infty$ as $t \to \infty$ and there is no explosion.

Indeed, the obtained process $(b_t)_{t\geq 0}$ is piece-wise constant, càdlàg and jumps at the arrival times τ_k of the Poisson process only. Moreover,

$$b_\theta = v_k, \qquad \text{for } \theta \in [\tau_k, \tau_{k+1}[.$$

Since all τ_k are stopping times and due to the symmetry of the density of each v_k, one obtains then $\mathsf{E}[b_t] = 0$ for each $t \geq 0$.

In what follows, instead of defining such a process $(b_t)_{t\geq 0}$ we define only the sequence $\{v_k\}_{k\in\mathbb{N}}$ and then use explicitly the process $(v_{N_t})_{t\geq 0}$.

Jumps in linear Gaussian dynamics

In the linear Gaussian setting, a filtering problem with Poisson-distributed observations produces equations of the form (2.4.1) below. It is crucial that presence of jumps keeps the distribution normal.

Lemma 2.15. *Assume that $\widehat{x}_0 \in \mathbb{R}^n$ is given and the stochastic process $(\widehat{x}_t)_{t\geq 0}$ satisfies*

$$\mathrm{d}\widehat{x}_t = A_t\widehat{x}_t\mathrm{d}t + G_t\mathrm{d}w_t + (B_t\widehat{x}_t + \beta_{N_t})\mathrm{d}N_t, \tag{2.4.1}$$

where $(w_t)_{t\geq 0}$ is m-dimensional standard Wiener process, $\{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of independent Gaussian random variables, A, G and B are functions of finite variation on compacts. Then $\widehat{x}_t$ is a Gaussian random variable for each $t \geq 0$.

Proof. Notice that all $\tau_1, \tau_2, \ldots$ are stopping times and each $t > 0$ belongs to only one interval $]\tau_k, \tau_{k+1}]$ for $k \in \mathbb{N}$. For $t \in]0, \tau_1[$ we have that $(\widehat{x}_t)_{t\geq 0}$ is an Itô diffusion process and its density satisfies the corresponding Fokker–Plank equation. Due to linearity it remains Gaussian and the left limit at τ_k is Gaussian as well. At τ_1, $\widehat{x}$ is defined as $\lim_{t\uparrow\tau_1}(\widehat{x}_t + B\widehat{x}_t + \beta_{N_t})$, i.e. as a sum of two independent Gaussian random variables. Thus, the jump preserves the normality of the distribution. For all the further intervals $]\tau_k, \tau_{k+1}]$ the same arguments are applicable. □

Evolution of product of random variables

Consider an example of application of the chain rule. For a given stochastic process $(e_t)_{t\geq 0}$ it might be useful to compute the matrix $e_t e_t^\top$, which will be later used for computing the covariance matrices. Proposition 2.14 allows to do it element-wise, and the following remark provides a matrix-valued form.

Remark 2.16. *Let $\mathbb{R}^n$-valued càdlàg process $(e_t)_{t\geq 0}$ be defined as*

$$\mathrm{d}e_t = A_t e_t \mathrm{d}t + \Gamma_t \mathrm{d}w_t + [\Xi_t e_t + \Omega_t]\,\mathrm{d}N_t, \quad e_0 = \bar{e}$$

where w_t is a standard Wiener process of dimension m, càdlàg processes A_t, Γ_t, Ξ_t, and Ω_t are bounded on compacts. For $n \times n$-dimensional process $e_t e_t^\top$ one obtains that $e_0 e_0^\top = \bar{e}\bar{e}^\top$ and

$$\begin{aligned}
\mathrm{d}e_t e_t^\top &= (A_t e_t e_t^\top + e_t e_t^\top A_t^\top + \Gamma_t \Gamma_t^\top)\mathrm{d}t + \Gamma_t \mathrm{d}w_t e_t^\top + e_t \mathrm{d}w_t^\top \Gamma_t^\top \\
&\quad + \Big[(I + \Xi_t)e_t e_t^\top (I + \Xi_t)^\top + (I + \Xi_t)e_t \Omega_t^\top + \Omega_t e_t^\top (I + \Xi_t)^\top \\
&\quad + \Omega_t \Omega_t^\top - e_t e_t^\top\Big]\mathrm{d}N_t.
\end{aligned} \tag{2.4.2}$$

Proof. By Proposition 2.14 one has element-wise dynamics of $(e_t e_t^\top)_{t\geq 0}$

$$\begin{aligned}
\mathrm{d}[e_t e_t^\top]_{ij} = \mathrm{d}[e_t]_i [e_t]_j &= [e_t]_i [A e_t \mathrm{d}t + \Gamma_t \mathrm{d}w_t]_j + [e_t]_j [A e_t \mathrm{d}t + \Gamma_t \mathrm{d}w_t]_i \\
&\quad + \frac{1}{2}\sum_k [\Gamma_t]_{ik}[\Gamma_t]_{jk}\mathrm{d}t \\
&\quad + \Big[\Big[e_t + \Xi_t e_t + \Omega_t\Big]_i \Big[e_t + \Xi_t e_t + \Omega_t\Big]_j - [e_t]_i [e_t]_j\Big]\mathrm{d}N_t
\end{aligned}$$

that can be rewritten as (2.4.2). □

2.5 Expected value

Notice that randomness in the process of our interest (2.2.4) has two independent sources: a Wiener process and a Poisson process. While the intensity of the Wiener process is always fixed, the intensity of the Poisson process plays the role of a parameter that can be controlled. Nonetheless, taking an expectation, one should assume that both intensities are fixed somehow.

Note that the defined process (2.2.4) is Markovian. Given an initial value $e_0 = \bar{e}$, an expectation at time t is taken with respect to the law of e_t generated by the two stochastic processes. However, the initial value will be usually omitted or provided as a condition, so $\mathsf{E}[e_t]$ or $\mathsf{E}[e_t \mid e_0 = \bar{e}]$ can be used depending on the context.

2.5.1 Infinitesimal generator

In this section the infinitesimal generator is derived for the processes driven by a Poisson process. It allows to obtain an analog of Dynkin's formula, but the statements avoid using stopping times since their implementation does not require this feature.

Definition 2.17. *Given a real-valued function $\psi : \mathbb{R}^n \to \mathbb{R}$, the* (infinitesimal) generator *of a random process $(x_t)_{t\geq 0}$ is the linear operator $\psi \mapsto \mathcal{L}\psi$ defined by*

$$\begin{gathered} \mathbb{R}^n \ni z \mapsto \mathcal{L}\psi(z) \in \mathbb{R} \\ \mathcal{L}\psi(z) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\, x(t) = z\Big] - \psi(z)\Big). \end{gathered} \tag{2.5.1}$$

Remark 2.18. *We obtain the expected value of ψ by integrating the generator, which can be seen as a generalization of the classical Dynkin's formula:*

$$\mathsf{E}\Big[\psi(x(t))\Big] = \mathsf{E}\Big[\psi(e(0))\Big] + \mathsf{E}\bigg[\int_0^t \mathcal{L}\psi(x(s))\,\mathrm{d}s\bigg]. \tag{2.5.2}$$

For our purposes, it is useful to compute an explicit expression of the generator which can then be analyzed for studying the qualitative behavior of $\mathsf{E}[\psi]$. Here, we provide a statement specifically tailored for the following process

$$\mathrm{d}x_t = f(x_t)\mathrm{d}t + G(x_t)\mathrm{d}\omega + g(x_t, \nu_{N_t})\mathrm{d}N_t \tag{2.5.3}$$

where N_t is a Poisson process with intensity $\lambda > 0$, ω is a standard Wiener process, and a sequence $\{\nu_k\}_{k\in\mathbb{N}}$ consists of i.i.d. random variables with density $\nu(\cdot)$.

Proposition 2.19. *If the sampling process $(N_t)_{t\geq 0}$ is Poisson with intensity $\lambda > 0$, then the process $(x(t))_{t\geq 0}$ described in* (2.5.3) *is Markovian. Moreover,* *for any function* $\mathbb{R}^n \ni z \mapsto \psi(z) \in \mathbb{R}$ *with at most polynomial growth as* $\|z\| \to +\infty$, *we have*

$$\mathcal{L}\psi(z) = \langle \nabla\psi(z), f(z)\rangle + \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2\psi(z)}{\partial z^2}G(z)G(z)^\top\Big) \\ + \lambda\Big[\int \psi\Big(z + g(z,v)\Big)\mathrm{d}\nu(v) - \psi(z)\Big]. \quad (2.5.4)$$

Proof. The fact that $(x(t))_{t\geq 0}$ is Markovian follows from the observation that the future of $x(t)$ depends on $x(\tau_{N_t})$.

Since the system under consideration is well-posed, we have, for $\varepsilon > 0$ small,

$$\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,x(t) = z\Big] \\ = \mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\Big(1_{\{N_{t+\varepsilon}=N_t\}} + 1_{\{N_{t+\varepsilon}=1+N_t\}} + 1_{\{N_{t+\varepsilon}-N_t\geq 2\}}\Big)\,\Big|\,x(t)\Big]. \quad (2.5.5)$$

We now compute the conditional probability distribution of $\Big(x(t+\varepsilon)\Big)$ for small $\varepsilon > 0$ given $x(t)$. Since the sampling process is independent of the state joint, by definition of the sampling (Poisson) process we have, for $\varepsilon \downarrow 0$,

$$\begin{cases} \mathsf{P}\Big(N_{t+\varepsilon} - N_t = 0\,\Big|\,N_t, x(t)\Big) = 1 - \lambda\varepsilon + o(\varepsilon), \\ \mathsf{P}\Big(N_{t+\varepsilon} - N_t = 1\,\Big|\,N_t, x(t)\Big) = \lambda\varepsilon + o(\varepsilon), \\ \mathsf{P}\Big(N_{t+\varepsilon} - N_t \geq 2\,\Big|\,N_t, x(t)\Big) = o(\varepsilon). \end{cases}$$

Using these expressions, we develop (2.5.5) further for $\varepsilon \downarrow 0$ as

$$\begin{aligned} &\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,x(t) = z\Big] \\ &= \mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\Big(1_{\{N_{t+\varepsilon}=N_t\}} + 1_{\{N_{t+\varepsilon}=1+N_t\}}\Big)\,\Big|\,x(t)\Big] + o(\varepsilon) \\ &= \mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,x(t), N_{t+\varepsilon} = N_t\Big]\cdot\Big(1 - \lambda\varepsilon + o(\varepsilon)\Big) \\ &\quad + \mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,x(t), N_{t+\varepsilon} = 1 + N_t\Big]\Big(\lambda\varepsilon\Big) + o(\varepsilon). \end{aligned} \quad (2.5.6)$$

The two significant terms on the right-hand side of (2.5.6) are now computed separately. First, for $\varepsilon \downarrow 0$, we have

$$\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,N_{t+\varepsilon} = N_t, x(t) = z\Big] \\ = \psi(z) + \varepsilon\Big\langle\nabla\psi\Big(z\Big), f\Big(z\Big)\Big\rangle + 0.5\varepsilon\bar{G}(z) + o(\varepsilon),$$

leading to the first term on the right-hand side of (2.5.6) having the estimate

$$\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,N_{t+\varepsilon} = N_t, x(t) = z\Big]\cdot\Big(1 - \lambda\varepsilon + o(\varepsilon)\Big) \\ = \psi\Big(z\Big) + \varepsilon\Big\langle\nabla\psi\Big(z\Big), f(z)\Big\rangle + \varepsilon\bar{G}(z) - (\lambda\varepsilon)\psi\Big(z\Big) + o(\varepsilon),$$

where $\bar{G}(z) := 0.5\mathrm{tr}\Big((\partial^2\psi(z)/\partial z)G(z)G(z)^\top\Big)$. Concerning the second term on the right-hand side of (2.5.6), we observe that conditional on $N_{t+\varepsilon} = 1 + N_t$, the probability distribution of $\tau_{N_{t+\varepsilon}}$ is uniform over $[t, t+\varepsilon[$ by Th. 2.6, i.e.,

$$\mathsf{P}\Big(\tau_{N_{t+\varepsilon}} \in [s, s+s'[\,\Big|\, N_{t+\varepsilon} = 1 + N_t\Big) = \frac{1}{\varepsilon}s'$$

for $[s, s+s'[\subset [t, t+\varepsilon[$. Since the sampling process is independent of the state process, the preceding conditional probability is equal to

$$\mathsf{P}\Big(\tau_{N_{t+\varepsilon}} \in [s, s+s'[\,\Big|\, N_{t+\varepsilon} = 1 + N_t, x(t) = z\Big).$$

We define $\theta \in [0,1[$ such that $\tau_{N_{t+\varepsilon}} = t + \theta\varepsilon, x(t) = z$; then θ is uniformly distributed on $[0,1[$ given $N_{t+\varepsilon} = 1 + N_t$. We also have, conditioned on the same event,

$$x(\tau_{N_{t+\varepsilon}}) = x(t+\theta\varepsilon) = x(t+\theta\varepsilon^-) + g(x(t+\theta\varepsilon^-), \nu_{N_{t+\theta\varepsilon^-}}),$$

and

$$x(t+\theta\varepsilon^-) = x(t) + \theta\varepsilon\bar{f}\Big(x(t)\Big) + o(\varepsilon),$$

where $\bar{f}(z)$ denotes the linear interpolation of the solution of (2.5.3) over an interval of length ε starting from z. The above expressions then lead to, conditioned on the event $N_{t+\varepsilon} = 1 + N_t, x(t) = z$ and for $\varepsilon \downarrow 0$,

$$\begin{aligned}
x(t+\varepsilon) &= x(t+\theta\varepsilon) + (1-\theta)\varepsilon\bar{f}\Big(x(t+\theta\varepsilon))\Big) + o(\varepsilon)\\
&= x(t) + \theta\varepsilon\bar{f}\Big(x(t)\Big) + g(x(t) + \theta\varepsilon\bar{f}\Big(x(t)\Big), \nu_{N_t})\\
&\quad + (1-\theta)\varepsilon\bar{f}\Big(x(t+\theta\varepsilon)\Big) + o(h)\\
&= x(t) + \theta\varepsilon\bar{f}\Big(x(t)\Big) + g(x(t), \nu_t) + (1-\theta)\varepsilon\bar{f}\Big(x(t+\theta\varepsilon)\Big)\\
&\quad + \theta\varepsilon\nabla_x g(x(t), \nu_t)\cdot\bar{f}(x(t)) + o(\varepsilon).
\end{aligned}$$

Therefore, for $\varepsilon \downarrow 0$,

$$
\begin{aligned}
&\mathsf{E}\Big[\psi\Big(x(t+\varepsilon)\Big)\,\Big|\,x(t)=z, N_{t+\varepsilon}=1+N_t\Big]\cdot(\lambda\varepsilon)\\
&=\int_0^1 \psi\Big(z+\theta\varepsilon\bar{f}(z)+g(z,v)+(1-\theta)\varepsilon\bar{f}\Big(x(t+\theta\varepsilon)\Big)\\
&\qquad\qquad +\theta\varepsilon\nabla_z g(z,v)\cdot\bar{f}(z)+o(\varepsilon)\Big)\,\mathrm{d}\nu(v)\mathrm{d}\theta\cdot(\lambda\varepsilon)\\
&=\int_0^1\Bigg(\psi\Big(z+g(z,v)\Big)+\varepsilon\langle\nabla\psi\Big(z+g(z,v)\Big),o(\varepsilon)+\theta\bar{f}(z)\\
&\quad+(1-\theta)\bar{f}(x(t+\theta\varepsilon))+\theta\nabla_z g(z,v)\cdot\bar{f}(z)\rangle\Bigg)\,\mathrm{d}\nu(v)\mathrm{d}\theta\cdot(\lambda\varepsilon)\\
&=\Big(\int\psi\Big(z+g(z,v)\Big)\mathrm{d}\nu(v)+O(\varepsilon)\Big)\cdot(\lambda\varepsilon)\\
&=(\lambda\varepsilon)\int\psi\Big(z+g(z,v)\Big)\mathrm{d}\nu(v)+o(\varepsilon).
\end{aligned}
$$

Putting everything together, we arrive at

$$
\begin{aligned}
&\mathsf{E}\Big[\psi\Big(x(t+h)\Big)\,\Big|\,x(t)=z\Big]\\
&=\psi(z)+\varepsilon\Big(\langle\nabla\psi(z),f(z)\rangle+\frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2\psi(z)}{\partial z}G(z)G(z)^\top\Big)\Big)\\
&\qquad-(\lambda\varepsilon)\Big(\psi(z)-\int\psi(z+g(z,v))\mathrm{d}\nu(v)\Big)+o(\varepsilon).
\end{aligned}
$$

Substituting these expressions in (2.5.1), we see that for each $z \in \mathbb{R}^n$, we get the expression (2.5.4).

□

2.5.2 Expectation with nonlinear Poisson-driven term

The following theorem takes place due to the nature of a Poisson process and makes a high impact for continuous–discrete filters analysis.

Proposition 2.20. *Let a $\mathbb{R}^n$-valued càdlàg process $(e_t)_{t\geq 0}$ is defined as*

$$
\mathrm{d}e_t = A_t e_t \mathrm{d}t + (G_t)\mathrm{d}w_t + (g(e_t)+v_{N_t})\mathrm{d}N_t \tag{2.5.7}
$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, the sequences $\{b_k\}_{k\in\mathbb{N}}$, $\{v_k\}_{k\in\mathbb{N}}$ consist of $\mathcal{N}(0,V)$-distributed independent random variables, A, G are càdlàg adapted processes, bounded on compacts, the function g is Lipschitz continuous in the second argument and has finite variation on compacts in the first argument. One obtains that

$$
\frac{\mathrm{d}\widehat{e}_t}{\mathrm{d}t} = A_t\widehat{e}_t + \lambda g(t,\widehat{e}_t). \tag{2.5.8}
$$

Proof. Notice that

$$\begin{aligned}
\widehat{e}_{t+\varepsilon} &= \mathsf{E}[e_{t+\varepsilon} \mid e_0 = z] \\
&= z + \mathsf{E}\left[\int_0^{t+\varepsilon} 1 \; \mathrm{d}e_s \mid e_0 = z\right] \\
&= \mathsf{E}[e_t^z] + \mathsf{E}\left[\int_t^{t+\varepsilon} 1 \; \mathrm{d}e_s \mid e_t = \mathsf{E}[e_t^z]\right] \\
&= \mathsf{E}[e_t^z] + \mathsf{E}\left[e_{t+\varepsilon} \mid e_t = \mathsf{E}[e_t^z]\right] \\
&= \widehat{e}_t + \mathsf{E}\left[e_{t+\varepsilon} \mid e_t = \mathsf{E}[e_t^z]\right],
\end{aligned}$$

where e_t^z denotes the random variable $z + \int_0^t \mathrm{d}e_s$. Now denote the deterministic vector $\mathsf{E}[e_t^z] =: y$. Hence

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\widehat{e}_{t+\varepsilon} - \widehat{e}_t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathsf{E}\left[e_{t+\varepsilon} \mid e_t = y\right].$$

Using Corollary 2.5 one can split this expectation into three cases:

$$\begin{aligned}
&\mathsf{E}\Big[e(t+\varepsilon) \,\Big|\, e(t) = y\Big] \\
&= \mathsf{E}\Big[e(t+\varepsilon)\Big(1_{\{N_{t+\varepsilon}=N_t\}} + 1_{\{N_{t+\varepsilon}=1+N_t\}} + 1_{\{N_{t+\varepsilon}-N_t \geq 2\}}\Big) \,\Big|\, e(t) = y\Big] \\
&= \mathsf{E}\Big[e(t+\varepsilon)\Big) \,\Big|\, e(t), N_{t+\varepsilon} = N_t\Big] \cdot \Big(1 - \lambda\varepsilon + o(\varepsilon)\Big) \\
&\quad + \mathsf{E}\Big[e(t+\varepsilon) \,\Big|\, e(t), N_{t+\varepsilon} = 1 + N_t\Big]\Big(\lambda\varepsilon\Big) + o(\varepsilon)
\end{aligned}$$

the first case contains no jumps and is treated correspondingly:

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(1 - \lambda\varepsilon + o(\varepsilon)\Big)\mathsf{E}\Big[e(t+\varepsilon)\Big) \,\Big|\, e(t) = y, N_{t+\varepsilon} = N_t\Big] \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(1 - \lambda\varepsilon + o(\varepsilon)\Big)\mathsf{E}\left[\int_t^{t+\varepsilon} A_s e_s \mathrm{d}s + \int_t^{t+\varepsilon} (G_s + b_{N_s})\mathrm{d}s \mid e(t) = y, N_{t+\varepsilon} = N_t\right] \\
&= A_t y = A_t \mathsf{E}[e_t^z] = A_t \widehat{e}_t.
\end{aligned}$$

For the second case we obtain that

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(\lambda\varepsilon + o(\varepsilon)\Big)\mathsf{E}\Big[e(t+\varepsilon)\Big) \,\Big|\, e(t) = y, N_{t+\varepsilon} = 1 + N_t\Big] \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(\lambda\varepsilon + o(\varepsilon)\Big)\mathsf{E}\Bigg[\int_t^{t+\varepsilon} A_s e_s \mathrm{d}s + \int_t^{t+\varepsilon} (G_s + b_{N_s})\mathrm{d}w_s \\
&\qquad + \int_t^{t+\varepsilon} (g(s, e_s) + v_{N_s})\mathrm{d}N_s \mid e(t) = y, N_{t+\varepsilon} = 1 + N_t\Bigg] \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(\lambda\varepsilon + o(\varepsilon)\Big)\mathsf{E}\left[\int_t^{t+\varepsilon} g(s, e_s)\mathrm{d}N_s \mid e(t) = y, N_{t+\varepsilon} = 1 + N_t\right]
\end{aligned}$$

and thanks to the Theorem 2.6, the last limit equals to $\lambda g(t, y)$, that is $\lambda g(t, \widehat{e}_t)$. Thus (2.5.8) holds.

$\square$

2.6 Implementation in the linear filtering problem

Let us turn our focus to the continuous-discrete filtering system and summarize the key findings that can be achieved. Let the filtering system consist of

$$\begin{aligned} \text{state } x_t \in \mathbb{R}^n : \qquad & \mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t && \text{(2.6.1a)} \\ \text{discrete observations } y_{\tau_k} \in \mathbb{R}^p : \qquad & y_{\tau_k} = Cx_{\tau_k} + v_{\tau_k}, && \text{(2.6.1b)} \end{aligned}$$

where $\{v_{\tau_k}\}_{k\in\mathbb{N}}$ is a sequence of independent $\mathbb{R}^p$-valued normal random variables distributed as $\mathcal{N}(0, V)$ and

$$\{\tau_k\}_{k\in\mathbb{N}} \quad \text{are the arrival times of a Poisson process } N_t. \qquad \text{(2.6.1c)}$$

We let $(\Omega, \mathcal{F}, \mathsf{P})$ denote the underlying probability space.

First, for a given sequence $\{\tau_k\}_{k\in\mathbb{N}}$, we recall the filter design based on computing the covariance matrix exactly via the differential equations.

Theorem 2.21 ([Jaz07, Theorem 7.1])**.** *Consider the state process* (2.6.1a) *and the observation process* (2.6.1b) *with a strictly increasing real-valued sequence* $\{\tau_k\}_{k\in\mathbb{N}}$. *The optimal filter is a Gaussian process, its mean* $\widehat{x}_t$ *described as*[7]

$$\begin{aligned} \mathrm{d}\widehat{x}_t/\mathrm{d}t &= A\widehat{x}_t, && \tau_k \le t < \tau_{k+1} && \text{(2.6.2a)} \\ \widehat{x}_{\tau_k} &= \widehat{x}^-_{\tau_k} + K_{\tau_k}(y_{\tau_k} - C\widehat{x}^-_{\tau_k}) && k \in \mathbb{N} && \text{(2.6.2b)} \end{aligned}$$

and the variance $P_t = \mathsf{E}\left[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top\right]$ *that satisfies*

$$\begin{aligned} \mathrm{d}P_t/\mathrm{d}t &= AP_t + P_tA^\top + GG^\top, && \tau_k \le t < \tau_{k+1} && \text{(2.6.3a)} \\ P_{\tau_k} &= P^-_{\tau_k} - K_{\tau_k}CP^-_{\tau_k} && k \in \mathbb{N} && \text{(2.6.3b)} \end{aligned}$$

where $K_{\tau_k} = P^-_{\tau_k}C^\top(CP^-_{\tau_k}C^\top + V_{\tau_k})^{-1}$.

[7]We use the notation $\widehat{x}^-_{\tau_k} := \lim_{s\uparrow\tau_k} \widehat{x}_s$ and $P^-_{\tau_k} = \lim_{s\uparrow\tau_k} P_s$.

Second, if we assume that a sequence $(\tau_k)_{k\in\mathbb{N}}$ that corresponds to the arrival times of a Poisson process N_t, we can reformulate the mean and covariance equations from Theorem 2.21 as differential equations driven by this Poisson process.

Theorem 2.22 (Reformulation of [Jaz07, Th. 7.1])**.** *The optimal filter for the system* (2.6.1a)–(2.6.1c) *is a Gaussian process, its conditional mean* $\widehat{x}_t = \mathsf{E}[x_t \mid \sigma\{y_{[0,t]}\}]$ *satisfies*

$$\mathrm{d}\widehat{x}_t = A\widehat{x}_t \mathrm{d}t + K_t(y_t - C\widehat{x}_t)\mathrm{d}N_t, \tag{2.6.4}$$

and the covariance matrix $P_t = \mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top \mid \sigma\{y_{[0,t]}\}]$ *satisfies*

$$\mathrm{d}P_t = (AP_t + P_tA^\top + GG^\top)\mathrm{d}t - K_tCP_t\mathrm{d}N_t \tag{2.6.5}$$

where $K_t = P_tC^\top(CP_tC^\top + V_t)^{-1}$.

In the above result, one may notice that by Lemma 2.15 the optimal conditional distribution is Gaussian despite its moments are discontinuous for each sample path.

Let us note that the definition of moments are not the same in the two theorems above. Both are independent of the values of the observations, but Theorem 2.22 involves the covariance matrix conditional on the random arrival times of observations. It crucially differs because the trace of the covariance matrix measures the performance of the filter; now the covariance matrix is not deterministic anymore. Nonetheless, study extends to the expected performance of the filter with respect to the sampling rate of the observation process. By applying Proposition 2.20 to the process (2.6.5), we get

$$\dot{\mathcal{P}}_t = A\mathcal{P}_t + \mathcal{P}_tA^\top + GG^\top - \lambda\mathcal{P}_tC^\top(C\mathcal{P}_tC^\top + V)^{-1}C\mathcal{P}_t \tag{2.6.6}$$

for the process $\mathcal{P}_t := \mathsf{E}[P_t \mid P_0]$. In addition, equation (2.6.6) with

$$\mathcal{K} = \mathcal{P}_tC^\top(C\mathcal{P}_tC^\top + V)^{-1}$$

can be rewritten as

$$\dot{\mathcal{P}}_t = (A - \lambda\mathcal{K}_tC)\mathcal{P}_t + \mathcal{P}_t(A - \lambda\mathcal{K}_tC)^\top + GG^\top + \lambda\mathcal{K}_t(C\mathcal{P}_tC + V)\mathcal{K}_t^\top.$$

Chapter 3

Suboptimal Filter with Lipschitz Drift

We design suboptimal filters for a class of continuous-time nonlinear stochastic systems when the measurements are assumed to arrive randomly at discrete times under a Poisson distribution. The proposed filter is a dynamical system with a differential equation and a reset map which updates the estimate whenever a new measurement is received. We analyze the performance of the proposed filter by computing the expected value of the error covariance which is described by a differential equation. We study throughly the conditions under which the error covariance remains bounded, which depend on the system data and the mean sampling rate associated with the measurement process. We also study the particular cases when the error covariance is seen to decrease with the increase in the sampling rate. For the particular case of linear filters, we can also compare the error covariance bounds with the case when the measurements are continuously available.

3.1 Problem Formulation

3.1.1 System Class

Let us consider stochastic nonlinear systems of the $\mathbb{R}^n$-valued state process

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + G\mathrm{d}w_t \tag{3.1.1a}$$

and the piecewise constant càdlàg measurements y, taking values in $\mathbb{R}^p$, satisfying

$$y_{\tau_k} = h(x_{\tau_k}) + v_{\tau_k} \tag{3.1.1b}$$

for the arrival times τ_k of a Poisson process N_t.

The process noise w and the sensor noise v are assumed to be independent Brownian motions. The drift term $f : \mathbb{R}^n \to \mathbb{R}^n$, and the output map $h : \mathbb{R}^n \to \mathbb{R}^p$ are both assumed to be continuous. For the sake of simplicity, we will assume that the covariances of the process noise and sensor noise remain constants over time and are denoted by W and V respectively.

3.1.2 Structural assumption on system nonlinearities

To analyze the performance of the nonlinear filters, we introduce a structural assumption on the nonlinearities in the model (3.1.1).

(A1) There exist matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, the constants a, c such that for all $\delta \in \mathbb{R}^n$

$$|f(x+\delta) - f(x) - A\delta| \le a\,|\delta|, \tag{3.1.2a}$$

$$|h(x+\delta) - h(x) - C\delta| \le c|\delta|, \tag{3.1.2b}$$

and the pair (A, C) is detectable. Moreover, the diffusion term is constant, that is, $G(x) = G \in \mathbb{R}^{n\times m}$ for every $x \in \mathbb{R}^n$, with (A, G) stabilizable.

Our goal is to work out the filtering equations in the previous section for this particular class of nonlinear systems, and analyze the covariance of the estimation error as a function of the parameters in the model and the observation process.

Remark 3.1. *In other words, the nonlinearities, that we consider, are to be seen as the perturbations in the linear terms and the size of these perturbations is handled by the constants a and c. If we write $f(x) = Ax + f_{\mathrm{nl}}(x)$, and $h(x) = Cx + h_{\mathrm{nl}}(x)$, then Assumption* **(A1)** *requires that f_{nl} is Lipschitz continuous with modulus $a \ge 0$, and that h_{nl} is Lipschitz continuous with modulus $c \ge 0$. This allows us to cover drift terms with at most linear growth, and output measurements (from sensors) with bias and saturation. Assumption* **(A1)***, however, rules out nonlinearities in diffusion coefficient matrix G, and considering such nonlinearities is a topic of further investigations.*

The conventional filtering problem deals with an $\mathbb{R}^p$-valued continuous-time observation process $(z_t)_{t\ge 0}$ described as

$$\mathrm{d}z_t = h(x_t)\,\mathrm{d}t + \mathrm{d}\eta_t, \tag{3.1.3}$$

where η_t is an $\mathcal{F}_t$-adapted standard Wiener process, taking values in $\mathbb{R}^p$, and $\mathsf{E}[\mathrm{d}\eta_t\,\mathrm{d}\eta_t^\top] = V_t\mathrm{d}t$, with $V_t \in \mathbb{R}^{p\times p}$ assumed to be positive definite.

Remark 3.2. *It is possible to consider time-dependence in nonlinearities f, g, h and the covariances of the noise. As a result, the matrices A, C, G, the constants a, c, and the covariance matrices $\mathsf{E}[\mathrm{d}\omega_t \mathrm{d}\omega_t^\top]$, $\mathsf{E}[\mathrm{d}\eta_t \mathrm{d}\eta_t^\top]$ can also be time-dependent. Such generalities can be addressed within the framework of this chapter. In particular, the derivation of bounds in Proposition 3.3, Proposition 3.4, and Proposition 3.5 could be easily generalized without much differences. However, the results of Section 3.5, Section 3.6, and Section 3.6.3, where we analyze the stability of the differential equations associated with the bounds of the error covariance, require more careful analysis for time-varying systems.*

3.2 Suboptimal filtering

In what follows, we consider the dynamical system (3.1.1) under the assumption **(A1)**. As already mentioned, the exact filtering equations are not easy to analyse. Thus, we propose a filter structure, which is clearly a suboptimal solution and analyze the performance of this particular filter.

3.2.1 Filtering equations

For given values of randomly sampled time instants $\{\tau_{N_t}\}_{t\geq 0}$, or in other words, a fixed sample path of $(N_t)_{t\geq 0}$, we define the estimate $\widehat{x}_t$ using the filter (3.2.1), whose sample paths are càdlàg functions; a function $\phi : [0, \infty[\to \mathbb{R}^n$ is a càdlàg function if $\lim_{s\searrow t} \phi_s = \phi_t$ for each $t \in [0, \infty[$, and $\phi_{t^-} := \lim_{s\nearrow t} \phi_s$ exists for each $t \in]0, \infty[$.

$$\mathrm{d}\widehat{x}_t = f(\widehat{x}_t)\,\mathrm{d}t \tag{3.2.1a}$$
$$\widehat{x}_{\tau_{N_t}^+} = \widehat{x}_{\tau_{N_t}^-} + K_{\tau_{N_t}}(y_{\tau_{N_t}} - h(\widehat{x}_{\tau_{N_t}^-})). \tag{3.2.1b}$$

where $\widehat{x}_t$ is the state estimate, and $K_{\tau_k} \in \mathbb{R}^{n\times p}$, $k \in \mathbb{N}$, are the injection gains that need to be designed appropriately. The estimate obtained from (3.2.1) is, in general, not optimal for minimizing the mean square estimation error. However, by analyzing the performance of the suboptimal filter in (3.2.1), we can get an upper bound on the minimal mean square estimation error.

According to the notation of Chapter 2, one can rewrite (3.2.1) as

$$\mathrm{d}\widehat{x}_t = f(\widehat{x}_t)\mathrm{d}t + K_t(y_t - h(\widehat{x}_t))\mathrm{d}N_t.$$

This form explicitly shows dependence on the sampling process N; however, in this chapter, it is more convenient to estimate the filter performance of the form (3.2.1).

Hence, for the particular choice of filter in (3.2.1), we are interested in computing bounds on the expectation (with respect to the sampling process) of the error covariance and find conditions (involving the design parameter K_{τ_k}, $k \in \mathbb{N}$) so that the resulting covariance is bounded. The dependency of the error covariance on the sampling rate is also analyzed for specific cases.

3.2.2 Error covariance bounds with arbitrary injection gains

The following statement describes a bound on the covariance of estimation error resulting from (3.2.1) for a given value of sampling times.

Proposition 3.3. *Consider system* (3.1.1) *under assumption* **(A1)** *and the filter* (3.2.1) *with* $\{\tau_k\}_{k\in\mathbb{N}}$ *fixed. Let the process* P_t*, with* $P_0 := \mathsf{E}[(x_0 - \widehat{x}_0)(x_0 - \widehat{x}_0)^\top]$*, be defined by*

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = AP_t + P_t A^\top + aP_t + a\,\mathrm{tr}(P_t) I_{n\times n} + GG^\top, \tag{3.2.2a}$$

for $\tau_{N_t} \le t < \tau_{N_t+1}$*, and for* $k = N_t$*, let*

$$\begin{aligned} P_{\tau_k^+} &= P_{\tau_k^-} + cP_{\tau_k^-} - K_{\tau_k} C P_{\tau_k^-} - P_{\tau_k^-} C^\top K_{\tau_k}^\top \\ &\quad + K_{\tau_k}\Big((1+c) C P_{\tau_k^-} C^\top + (2c + c^2)\,\mathrm{tr}\,(P_{\tau_k^-}) I_{p\times p} + V\Big) K_{\tau_k}^\top. \end{aligned} \tag{3.2.2b}$$

Then, it holds that, for each $t \ge 0$,

$$\mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top \,|\, \mathcal{Y}_t] \le P_t. \tag{3.2.3}$$

Proof. We denote the state estimation error by $e_t := x_t - \widehat{x}_t$, so that the resulting error dynamics are given by

$$\mathrm{d}e_t = (f(x_t) - f(\widehat{x}_t))\mathrm{d}t + G\mathrm{d}w_t =: \widetilde{f}(x_t, \widehat{x}_t)\mathrm{d}t + G\mathrm{d}w_t \tag{3.2.4a}$$

for $t \in [\tau_{N_t}, \tau_{N_t+1}[$, and for each $k = N_t$,

$$\begin{aligned} e_{\tau_k^+} &= x_{\tau_k^+} - \widehat{x}_{\tau_k^+} = x_{\tau_k^-} - \widehat{x}_{\tau_k^-} - K_{\tau_k}(y_{\tau_k} - h(\widehat{x}_{\tau_k^-})) \\ &= e_{\tau_k^-} - K_{\tau_k}(y_{\tau_k} - h(\widehat{x}_{\tau_k^-})). \end{aligned} \tag{3.2.4b}$$

Under the small nonlinearity assumption **(A1)**, we can write $f(x) = Ax + f_{\mathrm{nl}}(x)$ for some Lipschitz function $f_{\mathrm{nl}}(\cdot)$ with modulus a, so that

$$\widetilde{f}(x_t, \widehat{x}_t) = f(x_t) - f(\widehat{x}_t) = Ae + f_{\mathrm{nl}}(x_t) - f_{\mathrm{nl}}(\widehat{x}_t),$$

where $f_{\mathrm{nl}}(\cdot)$ satisfies the bound, $|f_{\mathrm{nl}}(x_t)-f_{\mathrm{nl}}(\widehat{x}_t)| \leq a|e_t|$ for every $x_t, \widehat{x}_t \in \mathbb{R}^n$. In what follows, we denote $\mathsf{E}[(x_t-\widehat{x}_t)(x_t-\widehat{x}_t)^\top \,|\, \mathcal{Y}_t]$ by $\mathsf{E}[e_t e_t^\top]$, and show that, for each $t \in [\tau_{N_t}, \tau_{N_t+1}[$,

$$\frac{\mathrm{d}\mathsf{E}[e_t e_t^\top]}{\mathrm{d}t} \leq A\mathsf{E}[e_t e_t^\top] + \mathsf{E}[e_t e_t^\top]A^\top + a\,\mathsf{E}[e_t e_t^\top] + a\,\mathrm{tr}(\mathsf{E}[e_t e_t^\top])\, I_{n\times n} + GG^\top$$

and that, at the sampling instants

$$\mathsf{E}[e_{\tau_k^+} e_{\tau_k^+}^\top] - \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] \leq P_{\tau_k^+} - P_{\tau_k^-}.$$

The bound in (3.2.3) then holds by observing that P_t satisfies (3.2.2a) over $[\tau_{N_t}, \tau_{N_t+1}[$, and (3.2.2b) at $t=\tau_{N_t}$, with $P_0 = \mathsf{E}[e_0 e_0^\top]$.

Continuous part Let us first show that for every $t \in [\tau_{N_t}, \tau_{N_t+1}[$ we have the inequality $\frac{\mathrm{d}\mathsf{E}[e_t e_t^\top]}{\mathrm{d}t} \leq \frac{\mathrm{d}P_t}{\mathrm{d}t}$. On this time interval, let us consider the Itô process in (3.2.4a):

$$\mathrm{d}e_t = \widetilde{f}(x_t, \widehat{x}_t)\mathrm{d}t + G\mathrm{d}w_t,$$

and the function $v: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ given by, $\mathbb{R}^n \ni e \mapsto v(e) = ee^\top \mathbb{R}^{n\times n}$, where an element of the matrix $v(e)$ is denoted by $v_{k,l}(e) = e^k e^l$. Consequently, $(v \circ e_t)_{t\geq 0}$ is an Itô process, and applying Itô's differential chain rule, we obtain

$$\mathrm{d}v_{k,l}(e_t) = \sum_{i=1}^n \frac{\partial v_{k,l}(e_t)}{\partial e_t^i}\mathrm{d}e_t^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 v_{k,l}(e_t)}{\partial e_t^i \partial e_t^j}\mathrm{d}e_t^i \mathrm{d}e_t^j.$$

Substituting $\mathrm{d}e_t^i = \widetilde{f}_i(x_t,\widehat{x}_t)\mathrm{d}t + \sum_{k=1}^m G_{i,k}(\mathrm{d}w_t)_k$ and using Itô's multiplication table [KS98, Page 154], we have $(\mathrm{d}w_t)_i \cdot (\mathrm{d}w_t)_j = \delta_{ij}\mathrm{d}t$, $\mathrm{d}t\cdot\mathrm{d}t = (\mathrm{d}w_t)_i \cdot \mathrm{d}t = \mathrm{d}t \cdot (\mathrm{d}w_t)_i = 0$, and hence

$$\begin{aligned}
\mathrm{d}v_{k,l}(e_t) &= \sum_{i=1}^n \frac{\partial v_{k,l}(e_t)}{\partial e_t^i}\widetilde{f}_i(x_t,\widehat{x}_t)\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 v_{k,l}(e_t)}{\partial e_t^i \partial e_t^j} \sum_{k_i,k_j=1}^m \delta_{k_i,k_j} G_{i,k_i} G_{j,k_j}\mathrm{d}t \\
&\quad + \sum_{i=1}^n \sum_{j=1}^m \frac{\partial v_{k,l}(t,e_t)}{\partial e_t^i} G_{i,j}(\mathrm{d}w_t)_j \\
&= \sum_{i=1}^n \frac{\partial v_{k,l}(e_t)}{\partial e_t^i}\widetilde{f}_i(x_t,\widehat{x}_t)\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 v_{k,l}(e_t)}{\partial e_t^i \partial e_t^j}\langle G_i, G_j\rangle \mathrm{d}t \\
&\quad + \sum_{i=1}^n \sum_{j=1}^m \frac{\partial v_{k,l}(t,e_t)}{\partial e_t^i} G_{i,j}(\mathrm{d}w_t)_j,
\end{aligned}$$

where $\langle G_i, G_j \rangle$ is the inner product of i-th and j-th rows of the matrix G. We are interested in computing $\frac{\mathrm{d}\mathsf{E}[v(e_t)]}{\mathrm{d}t}$, and we have the following expression for its coordinates:

$$\frac{\mathrm{d}\mathsf{E}[v_{k,l}(e_t)]}{\mathrm{d}t} = \lim_{\varepsilon \to 0} \frac{\mathsf{E}[v_{k,l}(e_{t+\varepsilon})] - v_{k,l}(e_t)}{\varepsilon} = \mathsf{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathrm{d}v(e_s)\right]$$

Substituting the expression for $\mathrm{d}v(e_s)$, and using the fact that w_s is a Wiener process, we get

$$\frac{\mathrm{d}\mathsf{E}[v_{k,l}(e_t)]}{\mathrm{d}t} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathsf{E}\left[\sum_{i=1}^{n} \frac{\partial v_{k,l}(e_s)}{\partial e_s^i} \widetilde{f}_i(x_s, \widehat{x}_s) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 v_{k,l}(e_s)}{\partial e_s^i \partial e_s^j} \langle G_i, G_j \rangle\right] \mathrm{d}s$$

and hence, recalling that, $v_{k,l}(e_t) = e_t^k e_t^l$,

$$\frac{\mathrm{d}\mathsf{E}[v_{k,l}(e_t)]}{\mathrm{d}t} = \mathsf{E}\Big[(e_l \widetilde{f}_k(x_t, \widehat{x}_t) + e_k \widetilde{f}_l(x_t, \widehat{x}_t))\Big] + \frac{1}{2}\left(\langle G_k, G_l \rangle + \langle G_l, G_k \rangle\right).$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}\mathsf{E}[v(e_t)]}{\mathrm{d}t} &= \mathsf{E}\Big[\widetilde{f}(x_t, \widehat{x}_t)e^\top + e\widetilde{f}(x_t, \widehat{x}_t)^\top\Big] + GG^\top \\
&= \mathsf{E}\Big[Ae_te_t^\top + \widetilde{f}_{\mathrm{nl}}(x_t, \widehat{x}_t)e_t^\top + e_t(Ae_t)^\top\Big] \\
&\quad + \mathsf{E}\Big[e_t\widetilde{f}_{\mathrm{nl}}(x_t, \widehat{x}_t)^\top\Big] + GG^\top \\
&\le A\mathsf{E}\Big[e_te_t^\top\Big] + \mathsf{E}\Big[e_te_t^\top\Big]A^\top + a\,\mathsf{E}\Big[e_te_t^\top\Big] \\
&\quad + a\mathsf{E}\Big[|e_t|^2\Big]I_{n\times n} + GG^\top \\
&= A\mathsf{E}\Big[e_te_t^\top\Big] + \mathsf{E}\Big[e_te_t^\top\Big]A^\top + a\,\mathsf{E}\Big[e_te_t^\top\Big] + \\
&\quad a\,\mathrm{tr}(\mathsf{E}\Big[e_te_t^\top\Big])\,I_{n\times n} + GG^\top,
\end{aligned}$$

where we used the fact that $\widetilde{f}(x_t, \widehat{x}_t) = Ae_t + \widetilde{f}_{\mathrm{nl}}(x_t, \widehat{x}_t)$, with $\widetilde{f}_{\mathrm{nl}}(x_t, \widehat{x}_t) := f_{\mathrm{nl}}(x_t) - f_{\mathrm{nl}}(\widehat{x}_t)$, and the last inequality was obtained using the bound[1]

$$e\widetilde{f}_{\mathrm{nl}}^\top + \widetilde{f}_{\mathrm{nl}}e^\top \le a\,ee^\top + a|e|^2 I_{n\times n}$$

and the fact that $|e|^2 = \mathrm{tr}\,(ee^\top)$.

[1] It follows from the fact that, for each $z, e \in \mathbb{R}^n$, and $f_{\mathrm{nl}} \in \mathbb{R}^n$ satisfying $|f_{\mathrm{nl}}| \le a|e|$, we have that, $z^\top(e\widetilde{f}_{\mathrm{nl}}^\top + \widetilde{f}_{\mathrm{nl}}e^\top)z \le 2\,|z^\top e| \cdot |z^\top \widetilde{f}_{\mathrm{nl}}| \le 2\,a\,|z^\top e| \cdot |z|\,|e|$
$\le a\,z^\top ee^\top z + a|e|^2 z^\top z$.

Jump part To analyze the jump in error dynamics at time instants when the new measurements are received, we first observe that, under **(A1)**, $h(x) = Cx + h_{\mathrm{nl}}(x)$, where $h_{\mathrm{nl}}(x)$ satisfies the bound, $|h_{\mathrm{nl}}(x-e) - h_{\mathrm{nl}}(x)| \le c|e|$, for all $x, e \in \mathbb{R}^n$. Since x_t is continuous, recalling that $e_t = x_t - \widehat{x}_t$ and (3.2.4b), it follows that, for $k = N_t$,

$$\begin{aligned} e_{\tau_k^+} - e_{\tau_k^-} &= -(\widehat{x}_{\tau_k^+} - \widehat{x}_{\tau_k^-}) = -K_{\tau_k}(y_{\tau_k} - h(\widehat{x}_{\tau_k^-})) \\ &= -K_{\tau_k}(Cx_{\tau_k} + h_{\mathrm{nl}}(x_{\tau_k^-}) + v_{\tau_k} - C\widehat{x}_{\tau_k^-} - h_{\mathrm{nl}}(\widehat{x}_{\tau_k^-})) \\ &= -K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k}) \end{aligned}$$

where $\widetilde{y}_{\tau_k^-} := h_{\mathrm{nl}}(x_{\tau_k^-}) - h_{\mathrm{nl}}(\widehat{x}_{\tau_k^-})$. With this relation for jump in error value at sampling instants, we can compute the change in the error covariance at the sampling instants. We do so by noting that

$$\begin{aligned} \mathsf{E}[e_{\tau_k^+} e_{\tau_k^+}^\top] = \mathsf{E}\Big[&\Big(e_{\tau_k^-} - K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})\Big) \cdot \\ &\Big(e_{\tau_k^-} - K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})\Big)^\top \Big] \\ = \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] &- \mathsf{E}[e_{\tau_k^-}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})^\top K_{\tau_k}^\top \\ &+ K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k}) e_{\tau_k^-}^\top] \\ &+ \mathsf{E}[K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})^\top K_{\tau_k}^\top]. \end{aligned} \tag{3.2.5}$$

We will now get a bound on the second and third terms on the right-hand side. To do so, we observe that, for each $K \in \mathbb{R}^{n\times p}$, $\tilde{y} \in \mathbb{R}^p$, $e \in \mathbb{R}^n$, such that $|\tilde{y}| \le c|e|$, we have

$$-K\widetilde{y}e^\top - e\widetilde{y}^\top K^\top \le cee^\top + c|e|^2 KK^\top. \tag{3.2.6}$$

This indeed follows from the fact that, for each $z \in \mathbb{R}^n$, $z^\top(-K\widetilde{y}e^\top - e\widetilde{y}^\top K^\top)z \le 2\,|z^\top e| \cdot |z^\top K\widetilde{y}| \le 2\,c\,|z^\top e| \cdot |K^\top z|\,|e| \le c\,z^\top ee^\top z + c|e|^2 z^\top KK^\top z$. Applying the inequality (3.2.6), the second term on the right-hand side of (3.2.5) can be bounded as follows:

$$\begin{aligned} &- \mathsf{E}[e_{\tau_k^-}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})^\top K_{\tau_k}^\top - K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k}) e_{\tau_k^-}^\top] \\ &\le -\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] C^\top K_{\tau_k}^\top - K_{\tau_k} C \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] + c\,\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] \\ &\quad + c\,|e|^2 K_{\tau_k} K_{\tau_k}^\top \\ &= -\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] C^\top K_{\tau_k}^\top - K_{\tau_k} C \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] + c\,\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] \\ &\quad + c\,\mathrm{tr}\,(\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]) K_{\tau_k} K_{\tau_k}^\top, \end{aligned}$$

where we used the fact that $|e_{\tau_k^-}|^2 = \mathrm{tr}\,(\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top])$.

For the last term on the right-hand side of (3.2.5), we observe that, for each $C \in \mathbb{R}^{p\times n}$, $e \in \mathbb{R}^n$, and $\widetilde{y} \in \mathbb{R}^p$ satisfying $|\widetilde{y}| \le c|e|$, we have

$$C e \widetilde{y}^\top + \widetilde{y} e^\top C^\top \le c\, C e e^\top C^\top + c|e|^2 I_{p\times p}. \tag{3.2.7}$$

This, in turn, leads to

$$\begin{aligned}
&\mathsf{E}[K_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-} + v_{\tau_k})^\top K_{\tau_k}^\top] \\
&= \mathsf{E}[K_{\tau_k}(Ce_{\tau_k^-} e_{\tau_k^-}^\top C^\top + Ce_{\tau_k^-}\widetilde{y}_{\tau_k^-}^\top + \widetilde{y}_{\tau_k^-} e_{\tau_k^-}^\top C^\top + \widetilde{y}_{\tau_k^-}\widetilde{y}_{\tau_k^-}^\top \\
&\quad + v_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-})^\top + (Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-})v_{\tau_k}^\top + v_{\tau_k}v_{\tau_k}^\top)K_{\tau_k}^\top] \\
&= K_{\tau_k}\mathsf{E}[Ce_{\tau_k^-} e_{\tau_k^-}^\top C^\top + Ce_{\tau_k^-}\widetilde{y}_{\tau_k^-}^\top + \widetilde{y}_{\tau_k^-} e_{\tau_k^-}^\top C^\top + \widetilde{y}_{\tau_k^-}\widetilde{y}_{\tau_k^-}^\top \\
&\quad + v_{\tau_k}(Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-})^\top + (Ce_{\tau_k^-} + \widetilde{y}_{\tau_k^-})v_{\tau_k}^\top + v_{\tau_k}v_{\tau_k}^\top]K_{\tau_k}^\top \\
&\le K_{\tau_k}\Big(C\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]C^\top + \mathsf{E}[c\, Ce_{\tau_k^-} e_{\tau_k^-}^\top C^\top \\
&\qquad + c|e_{\tau_k^-}|^2 I_{p\times p} + c^2|e_{\tau_k^-}|^2] + V\Big)K_{\tau_k}^\top \\
&= K_{\tau_k}\Big(C\mathsf{E}[\tilde{x}_{\tau_k^-}\tilde{x}_{\tau_k^-}^\top]C^\top + c\, C\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]C^\top \\
&\qquad + (c + c^2)\ \mathrm{tr}\,(\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top])I_{p\times p} + V\Big)K_{\tau_k}^\top.
\end{aligned}$$

Plugging this last bound in (3.2.5), we get

$$\begin{aligned}
\mathsf{E}[e_{\tau_k^+} e_{\tau_k^+}^\top] &\le \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] - \mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]C^\top K_{\tau_k}^\top - K_{\tau_k}C\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] \\
&\quad + c\,\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top] + K_{\tau_k}\Big((1+c)C\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]C^\top \\
&\qquad + (2c + c^2)\,\mathrm{tr}\,(\mathsf{E}[e_{\tau_k^-} e_{\tau_k^-}^\top]) + V\Big)K_{\tau_k}^\top
\end{aligned}$$

which is the desired statement. □

3.2.3 Minimizing error covariance with suboptimal filters

In the result of Proposition 3.3, we computed bounds on the estimation error covariance for arbitrary choice of gains. However, by choosing the gains in a certain dynamic manner such that K in (3.2.1b) depends on P_t obtained from equations of form (3.2.2a)-(3.2.2b), we can minimize P_t and $\mathsf{E}[e_t e_t^\top]$. For the chosen class of filters, this optimal choice of dynamic gain and the resulting covariance bound are indicated in the next result.

Proposition 3.4. *Consider system* (3.1.1) *under assumption* **(A1)** *and the filter* (3.2.1) *with* $\{\tau_k\}_{k\in\mathbb{N}}$ *given. If the output injection gains* K_{τ_k}*,* $k \in \mathbb{N}$*, in* (3.2.1b) *are chosen as*

$$K_{\tau_k} = P^\star_{\tau_k^-} C^\top M^{-1}_{P_{\tau_k}} \tag{3.2.8}$$

where $P^\star$ *is obtained from*

$$\frac{\mathrm{d}P^\star_t}{\mathrm{d}t} = AP^\star_t + P^\star_t A^\top + aP^\star_t + a\,\mathrm{tr}(P^\star_t)I + GG^\top, \tag{3.2.9a}$$

$$P^\star_{\tau_k^+} = (1+c)P^\star_{\tau_k^-} - P^\star_{\tau_k^-} C^\top M^{-1}_{P_{\tau_k}} C P^\star_{\tau_k^-} \tag{3.2.9b}$$

and $M_{P_{\tau_k}}$ *is defined as*[2]

$$M_{P_{\tau_k}} := (1+c)CP^\star_{\tau_k^-}C^\top + (2c+c^2)\,\mathrm{tr}\,(P^\star_{\tau_k^-})I_{p\times p} + V, \tag{3.2.10}$$

then, for each $t \geq 0$*, it holds that*

$$\mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top] \leq P^\star_t \leq P_t \tag{3.2.11}$$

where P_t *is obtained as a solution of* (3.2.2)*.*

The proof of this statement follows from Proposition 3.3. It is noted that K_{τ_k} only affects the bound P_t at the jump times via (3.2.2b). The right-hand side of (3.2.2b) is a quadratic convex function of K_{τ_k} which is minimized by choosing K_{τ_k} as in (3.2.8).

3.3 Expectation of error covariance

In the previous section, we computed a bound on the error covariance for a fixed sequence of time instants at which measurements are received. That is, P_t is an upper bound on $\mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top \,|\, \mathcal{Y}_t]$ along one particular sample path as the realization of P_t depends on the observed sampling times. It was initially stipulated that the random sampling instants are governed by a Poisson process. It is thus of interest to compute the expected value of P_t along all possible sample paths generated by Poisson sampling process.

Toward this end, it is noted that the evolution of P_t in (3.2.2) is governed by a piecewise deterministic process, where P_t is obtained from the differential equation (3.2.2a) between two sampling instants, and then at the sampling times, P_t is reset according to the equation (3.2.2b). By Proposition 2.20 one immediately obtains the following important result.

[2] For a given matrix P, the notation M_P used in (3.2.10) will be used extensively in the remainder of this article where we will change the matrix appearing in the subscript. With V assumed to be symmetric positive definite, M_P is a symmetric positive definite matrix, whenever P is symmetric positive semidefinite.

Proposition 3.5. *Consider the process $(P_t)_{t\geq 0}$ given by (3.2.2) with $K_{\tau_k} = K \in \mathbb{R}^{n\times p}$ for each $k \in \mathbb{N}$, and assume that the sampling process $(N_t)_{t\geq 0}$ has intensity $\lambda > 0$. Let $\mathcal{P}_t := \mathsf{E}[P_t \,|\, P_0]$ denote the expected value of the covariance process at time $t \geq 0$. Then, it holds that*

$$\dot{\mathcal{P}}_t = (A-\lambda KC)\mathcal{P}_t+\mathcal{P}_t(A-\lambda KC)^\top+\Pi_a(\mathcal{P}_t)+GG^\top+\lambda\, c\mathcal{P}_t+\lambda K M_{\mathcal{P}_t} K^\top, \tag{3.3.1}$$

where the linear operator $\Pi_a : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is defined as

$$\Pi_a(P) := a\,P + a\,\mathrm{tr}(P)I, \tag{3.3.2}$$

and the matrix $M_{\mathcal{P}_t} \in \mathbb{R}^{p\times p}$ is given by

$$M_{\mathcal{P}_t} := (1+c)C\mathcal{P}_tC^\top + (2c+c^2)\,\mathrm{tr}\,(\mathcal{P}_t)I_{p\times p} + V. \tag{3.3.3}$$

In particular, if we choose $K_{\tau_k} = P^\star_{\tau_k^-} C^\top M^{-1}_{P_{\tau_k}}$, then the expected value of the error covariance process governed by (3.2.9) is

$$\dot{\mathcal{P}}^\star_t = A\mathcal{P}^\star_t + \mathcal{P}^\star_t A^\top + \Pi_a(\mathcal{P}^\star_t) + GG^\top + \lambda\, c\mathcal{P}^\star_t - \lambda\mathcal{P}^\star_t C^\top M^{-1}_{\mathcal{P}^\star_t} C\mathcal{P}^\star_t. \tag{3.3.4}$$

Remark 3.6. *In (3.3.4), if we introduce the variable $\mathcal{K}_t := \mathcal{P}^\star_t C^\top M^{-1}_{\mathcal{P}^\star_t}$, then (3.3.4) is equivalently written as*

$$\begin{aligned}\dot{\mathcal{P}}^\star_t = (A-\lambda\mathcal{K}_tC)\mathcal{P}^\star_t + \mathcal{P}^\star_t(A-\lambda\mathcal{K}_tC)^\top \\ + \Pi_a(\mathcal{P}^\star_t) + GG^\top + \lambda\, c\mathcal{P}^\star_t + \lambda\mathcal{K}_t M_{\mathcal{P}^\star_t}\mathcal{K}_t^\top.\end{aligned}$$

This last equation resembles (3.3.1), the only difference being that a constant K is replaced by a time-varying term $\mathcal{K}_t$.

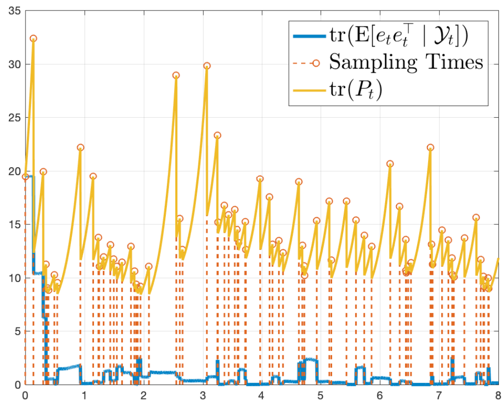
Thus, in linear case, an upper bound on the expectation (with respect to sampling process) of error covariance is obtained by solving a nonlinear differential equations. The boundedness of the solutions of equations in Proposition 3.5 are analyzed in later sections.

Let us provide two examples for an illustration of the results presented so far.

Example 3.7. *Consider the nonlinear system described by the equations*

$$\mathrm{d}x_{1,t} = \mathrm{sat}(x_{2,t})\mathrm{d}t + \mathrm{d}w_t, \quad \mathrm{d}x_{2,t} = (x_{1,t} - x_{2,t})\mathrm{d}t + \mathrm{d}w_t$$

with the measurement process $\mathrm{d}z_t = (1 + x_{1,t})\mathrm{d}t + \mathrm{d}\eta_t$. The noise processes are normalized as $\mathrm{d}w_t \sim \mathcal{N}(0,1)$ and $\mathrm{d}\eta_t \sim \mathcal{N}(0,1)$. The function $\mathrm{sat}(\cdot)$



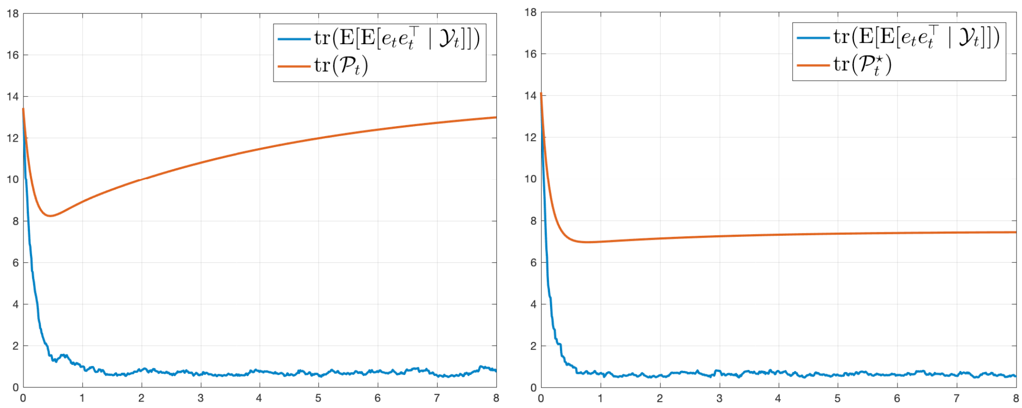
(a) Constant gain $K = [0.5\ 0.5]^\top$. (b) Dynamic gain $K_t = P_t^\star C^\top M_{P_t^\star}^{-1}$.

Figure 3.1: Plot of $\mathrm{tr}(e_t e_t^\top)$ (blue curve) and $\mathrm{tr}(P_t)$ (yellow curve) along one sample path (of sampling times) for two choices of K_{τ_k} with $\lambda = 10$.

represents the standard saturation function, that is, for each $z \in \mathbb{R}$, $\mathrm{sat}(z) = z - f_{\mathrm{nl}}(z)$, *where* f_{nl} *is a Lipschitz function described as,* $f_{\mathrm{nl}}(z) = z - 1$, *if* $z \geq 1$, $f_{nl}(z) = 0$ *if* $z \in [-1, 1]$, *and* $f_{\mathrm{nl}}(z) = z + 1$, *if* $z \leq -1$. *Hence, by letting* $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, $G = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]^\top$, *it is seen that Assumption* **(A1)** *holds with* $a = 1$, *and* $c = 0$. *Associating a Poisson process of intensity* λ, *the corresponding measurement equations are given by,* $y_{\tau_{N_t}} = 1 + x_{1,\tau_{N_t}} + v_{\tau_{N_t}}$, *where we take* $v_{\tau_{N_t}} \sim \mathcal{N}(0, 1)$ *for each* $t \geq 0$. *For appropriately chosen* $k_{1,\tau_{N_t}}, k_{2,\tau_{N_t}} \in \mathbb{R}$, *the proposed filter is then described by*

$$\begin{aligned} \mathrm{d}\widehat{x}_{1,t} &= \mathrm{sat}(\widehat{x}_{2,t})\mathrm{d}t, & \widehat{x}_{1,\tau_{N_t}^+} &= \widehat{x}_{1,\tau_{N_t}^-} + k_{1,\tau_{N_t}}(y_{\tau_{N_t}} - \widehat{x}_{1,\tau_{N_t}^-}) \\ \mathrm{d}\widehat{x}_{2,t} &= (\widehat{x}_{1,t} - \widehat{x}_{2,t})\mathrm{d}t, & \widehat{x}_{2,\tau_{N_t}^+} &= \widehat{x}_{2,\tau_{N_t}^-} + k_{2,\tau_{N_t}}(y_{\tau_{N_t}} - \widehat{x}_{2,\tau_{N_t}^-}). \end{aligned}$$

The simulation results for this example with two different choices of $K_{\tau_{N_t}} := [k_{1,\tau_{N_t}}\ k_{2,\tau_{N_t}}]^\top$ *are reported in Figure 3.1 and Figure 3.2. For one particular choice of randomly determined sampling times, Figure 3.1a provides an illustration of* (3.2.3) *in Proposition 3.3, where we plot* $\mathrm{tr}(\mathsf{E}[e_t e_t^\top \,|\, \mathcal{Y}_t])$ *and the bound* $\mathrm{tr}(P_t)$ *obtained from solving* (3.2.2), *with constant gain* $K_{\tau_{N_t}} = [0.5\ 0.5]^\top$ *for each* $t \geq 0$. *Similarly, Figure 3.1b is obtained by choosing* $K_{\tau_{N_t}} = P^\star_{\tau_{N_t}} C^\top M^{-1}_{P^\star_{\tau_{N_t}}}$ *and is an illustration of inequality* (3.2.11) *stated in Proposition 3.4. In Figure 3.2, we compute expectation with respect to the sampling process. Figure 3.2a plots* $\mathrm{tr}(\mathsf{E}[\mathsf{E}[ee^\top \,|\, \mathcal{Y}_t]])$, *and its upper bound* $\mathrm{tr}(\mathcal{P}_t)$ *obtained from* (3.3.1) *with constant gain* $K_{\tau_{N_t}} = [0.5\ 0.5]^\top$ *for each* $t \geq 0$, *and choosing the intensity of Poisson sampling process* $\lambda = 10$. *In Figure 3.2b, we choose* $K_{\tau_{N_t}} = P^\star_{\tau_{N_t}} C^\top M^{-1}_{P^\star_{\tau_{N_t}}}$, *with the the theoretical upper*



(a) Constant gain $K = [0.5\ 0.5]^\top$. (b) Dynamic gain $K_t = P_t^\star C^\top M_{P_t^\star}^{-1}$.

Figure 3.2: Experimental value (blue curve) and theoretical upper bound (red curve) for expectation (with respect to sampling process) of trace of error covariance with $\lambda = 10$ for two choices of K.

bound $\mathrm{tr}(\mathcal{P}_t^\star)$ *obtained from* (3.3.4). *We indeed observe that* $\mathrm{tr}(\mathcal{P}_t^\star)$ *has a smaller value than* $\mathrm{tr}(\mathcal{P}_t)$, *for each* $t \geq 0$. *It is not obvious, for what choices of gains* $K_{\tau_{N_t}}$, *we should get finite values of our bounds; such questions are addressed in subsequent sections.*

Example 3.8. *Consider the linear stochastic system*

$$\mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \tag{3.3.5a}$$
$$\mathrm{d}z_t = Cx_t\mathrm{d}t + \mathrm{d}\eta_t. \tag{3.3.5b}$$

where $\mathrm{d}z_t$ *is the measurement process, and the processes* w_t, η_t *satisfy the same hypothesis as in system* (3.1.1). *If* (A, C) *is detectable and* (A, G) *is stabilizable, then Assumption* **(A1)** *holds with* $a = c = 0$. *The output observed at randomly sampled time instants is described by* $y_{\tau_{N_t}} = Cx_{\tau_{N_t}} + v_{\tau_{N_t}}$, *where for each* $t \geq 0$, $v_{\tau_{N_t}} \sim \mathcal{N}(0, V)$. *The filter* (3.2.1) *takes the form*

$$\mathrm{d}\widehat{x}_t = A\widehat{x}_t\,\mathrm{d}t, \qquad \widehat{x}_{\tau_{N_t}^+} = \widehat{x}_{\tau_{N_t}^+} + K_{\tau_{N_t}}(y_{\tau_{N_t}} - h(\widehat{x}_{\tau_{N_t}^-})).$$

Inequality (3.2.3) *in Proposition 3.3 holds, where* P_t *satisfies*

$$\dot{P}_t = AP_t + P_tA^\top + GG^\top,$$

for $\tau_{N_t} \leq t < \tau_{N_t+1}$, *and for* $k = N_t$,

$$P_{\tau_k^+} = P_{\tau_k^-} - K_{\tau_k}CP_{\tau_k^-} - P_{\tau_k^-}C^\top K_{\tau_k}^\top + K_{\tau_k}\left(CP_{\tau_k^-}C^\top + V\right)K_{\tau_k}^\top.$$

By choosing, $K_{\tau_k} = P_{\tau_k^-} C^\top M_{P_{\tau_k}}^{-1}$*, for each* $k \in \mathbb{N}$*, the covariance bound is minimized and the resulting equations are given by* (3.2.9) *by setting* $a = c = 0$.

Assuming that the sampling process is Poisson of intensity λ*, and the gain* K *is constant, the expected value of* P_t *is*

$$\dot{\mathcal{P}}_t = (A - \lambda KC)\mathcal{P}_t + \mathcal{P}_t(A - \lambda KC)^\top + GG^\top + \lambda K \left(C\mathcal{P}_t C^\top + V \right) K^\top.$$

However, by choosing $K_{\tau_k} = P^\star_{\tau_k^-} C^\top M_{P_{\tau_k}}^{-1}$*, the expected value of the error covariance process governed by* (3.2.9) *in linear case is*

$$\dot{\mathcal{P}}^\star_t = A\mathcal{P}^\star_t + \mathcal{P}^\star_t A^\top + GG^\top - \lambda \mathcal{P}^\star_t C^\top \left(C\mathcal{P}_t C^\top + V \right)^{-1} C\mathcal{P}^\star_t.$$

3.4 Evolution of error covariance

We are interested in studying the solutions of the equation (3.3.1) and (3.3.4), which characterize the expected value of the error covariance resulting from the suboptimal nonlinear filter when the available measurements are Poisson distributed. In particular, we want to study conditions on the system dynamics and the sampling rates which guarantee boundedness of the solution. In doing so, the following algebraic equation also plays an important role:

$$0 = A\overline{\mathcal{P}} + \overline{\mathcal{P}}A^\top + \Pi_a(\overline{\mathcal{P}}) + \lambda c\overline{\mathcal{P}} + GG^\top - \lambda\overline{\mathcal{P}}C^\top M_{\overline{\mathcal{P}}}^{-1} C\overline{\mathcal{P}}. \tag{3.4.1}$$

In the remainder of this section, we provide a statement of the existence of solution to the differential equation (3.3.4), and then study boundedness of its solution and connections to steady state algebraic equations in the next section.

To study Carathédory solutions to the differential equation (3.3.1), we introduce the notation,

$$\Psi(\mathcal{P}, M_{\mathcal{P}}, K) := A_{\lambda K}\mathcal{P} + \mathcal{P}A_{\lambda K}^\top + \Pi(\mathcal{P}) + GG^\top + \lambda K^\top M_{\mathcal{P}} K. \tag{3.4.2}$$

where $M_{\mathcal{P}}$ is defined as in (3.3.3), and we take

$$A_{\lambda K} := (A - \lambda KC) \tag{3.4.3a}$$
$$\Pi(\mathcal{P}) := \Pi_a(\mathcal{P}) + \lambda\, c\, \mathcal{P} = a\mathcal{P} + a\,\mathrm{tr}(\mathcal{P})I + \lambda c\mathcal{P}. \tag{3.4.3b}$$

It is readily seen that the particular choice of $K = \mathcal{K} = \mathcal{P}C^\top M_{\mathcal{P}}^{-1}$ minimizes $\Psi(\mathcal{P}, M_{\mathcal{P}}, K)$ with respect to the ordering in the cone of positive semidefinite matrices. Indeed,

$$
\begin{aligned}
&\Psi(\mathcal{P}, M_{\mathcal{P}}, \mathcal{K}) \\
&\quad = (A - \lambda \mathcal{K} C)\mathcal{P} + \mathcal{P}(A - \lambda \mathcal{K} C)^\top + \Pi(\mathcal{P}) + GG^\top \\
&\qquad + \lambda \mathcal{K} M_{\mathcal{P}} \mathcal{K}^\top \\
&\quad = (A - \lambda K C)\mathcal{P} + \mathcal{P}(A - \lambda K C)^\top + \Pi(\mathcal{P}) + GG^\top \\
&\qquad + \lambda K M_{\mathcal{P}} K^\top - \lambda (K - \mathcal{K}) M_{\mathcal{P}} (K - \mathcal{K})^\top \\
&\quad = \Psi(\mathcal{P}, M_{\mathcal{P}}, K) - \lambda (K - \mathcal{K}) M_{\mathcal{P}} (K - \mathcal{K})^\top \\
&\quad \le \Psi(\mathcal{P}, M_{\mathcal{P}}, K).
\end{aligned} \tag{3.4.4}
$$

We can apply similar arguments to prove the following lemma which compares the solutions of equations (3.3.1) and (3.3.4).

Lemma 3.9. *Let $\mathcal{P}_t^\star$ be the solution of*

$$\dot{\mathcal{P}}_t^\star = \Psi(\mathcal{P}_t^\star, M_{\mathcal{P}_t^\star}, \mathcal{P}_t^\star C^\top M_{\mathcal{P}_t^\star}^{-1}), \quad \mathcal{P}_0^\star = \mathcal{P}_0,$$

and let $\mathcal{P}_t$ be the solution of

$$\dot{\mathcal{P}}_t = \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K), \quad \mathcal{P}_0 \ge 0 \text{ given},$$

where $M_{\mathcal{P}_t^\star}$ and $M_{\mathcal{P}_t}$ are defined as in (3.2.10)*. Then, it holds that*

$$\mathcal{P}_t^\star \le \mathcal{P}_t, \quad \forall\, t \ge 0.$$

Proof. Notice that for $\widetilde{\mathcal{P}}(t) = \mathcal{P}_t - \mathcal{P}_t^\star$ the minimum property (3.4.4) implies

$$
\begin{aligned}
\dot{\widetilde{\mathcal{P}}} &= \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K) - \Psi(\mathcal{P}_t^\star, M_{\mathcal{P}_t^\star}, \mathcal{P}_t^\star C^\top M_{\mathcal{P}_t^\star}^{-1}) \\
&\ge \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K) - \Psi(\mathcal{P}_t^\star, M_{\mathcal{P}_t^\star}, K) \\
&= A_{\lambda K} \widetilde{\mathcal{P}}_t + \widetilde{\mathcal{P}}_t A_{\lambda K}^\top + \Pi(\widetilde{\mathcal{P}}_t) + \lambda K M_{\widetilde{\mathcal{P}}_t} K^\top,
\end{aligned}
$$

where $M_{\widetilde{\mathcal{P}}} = M_{\mathcal{P}_t} - M_{\mathcal{P}_t^\star}$ is consistent and depends linearly on $\widetilde{\mathcal{P}}_t$. So, we have

$$\dot{\widetilde{\mathcal{P}}} = A_{\lambda K} \widetilde{\mathcal{P}} + \widetilde{\mathcal{P}} A_{\lambda K}^\top + \Pi(\widetilde{\mathcal{P}}) + \lambda K M_{\widetilde{\mathcal{P}}} K^\top + R_t,$$

for some $R_t \ge 0$. Since $\widetilde{\mathcal{P}}(0) = 0$, for $t \ge 0$, $\widetilde{\mathcal{P}}_t$ is a solution of the following Volterra equation

$$\widetilde{\mathcal{P}}_t = \int_0^t \mathrm{e}^{s\, A_{\lambda K}} (\Pi(\widetilde{\mathcal{P}}_s) + \lambda K^\top M_{\widetilde{\mathcal{P}}_s} K) \mathrm{e}^{s A_{\lambda K}^\top} \mathrm{d}s.$$

The matrix on the right-hand side is positive semidefinite, and hence $\mathcal{P}_t \ge \mathcal{P}_t^\star$, for each $t \ge 0$. $\square$

A direct consequence of this observation is that the right-hand side of (3.3.1) is minimized by choosing K_{τ_k} as in (3.2.8). Even though the perturbation Π is linear in $\mathcal{P}$, the aforementioned choice of K makes the right-hand side of (3.3.1) a nonlinear function in $\mathcal{P}$, which is different from the quadratic nonlinearity seen in Riccati differential equations. We are thus interested in knowing whether there exists a solution to the resulting equation. To this end, we will focus our attention on the initial value problem (3.3.4) with $\mathcal{P}(0) = \mathcal{P}_0$, where $\mathcal{P}_0$ is a positive semidefinite matrix.

Proposition 3.10. *For each symmetric positive semidefinite matrix $\mathcal{P}_0 \in \mathbb{R}^{n\times n}$ and a given $T > 0$, there exists a unique absolutely continuous solution $\mathcal{P} : [0, T] \to \mathbb{R}^{n\times n}$ in the class of symmetric positive definite matrices such that $\mathcal{P}(0) = \mathcal{P}_0$ and the differential equation* (3.3.4) *holds for almost every $t \in [0, T]$.*

The proof of this result is given in Appendix B.1. In the infinite-time horizon case we have a unique continuous solution $\mathcal{P} : [0, +\infty[\to \mathbb{R}^{n\times n}$ in the class of symmetric positive semidefinite matrices satisfying (3.3.4).

3.5 Conditions for bounded covariance

In this section, we study the conditions under which the error covariance $\mathcal{P}_t$ stays bounded for all $t \geq 0$. In doing so, we recall that the error covariance can be compactly represented as $\dot{\mathcal{P}}_t = \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K)$ where Ψ is defined in (3.4.2). For the purposes of this section, it is convenient to write Ψ in an alternate form. To do so, we let $\Pi_c(P) := c\,P + c\,KCPC^\top K^\top + (2c + c^2)\,\mathrm{tr}(P)KK^\top$, and let

$$\Pi_\lambda^K(P) := \Pi_a(P) + \lambda \Pi_c(P),$$

then we can write

$$\begin{aligned}
&\Psi(\mathcal{P}, M_{\mathcal{P}}, K) \\
&= A\mathcal{P} + \mathcal{P}A^\top + \Pi_\lambda^K(\mathcal{P}) + GG^\top + \lambda KVK^\top \\
&\quad + \lambda\Big[KC\mathcal{P}C^\top K^\top - KC\mathcal{P} - \mathcal{P}C^\top K^\top\Big] \\
&= \left(A - \frac{\lambda}{2}I\right)\mathcal{P} + \mathcal{P}\left(A - \frac{\lambda}{2}I\right)^\top + \Pi_\lambda^K(\mathcal{P}) + GG^\top \\
&\quad + \lambda KVK^\top + \lambda\Big[\mathcal{P} - \mathcal{P}C^\top K^\top - KC\mathcal{P} + KC\mathcal{P}C^\top K^\top\Big] \\
&= \left(A - \frac{\lambda}{2}I\right)\mathcal{P} + \mathcal{P}\left(A - \frac{\lambda}{2}I\right)^\top + \Pi_\lambda^K(\mathcal{P}) + GG^\top \\
&\quad + \lambda KVK^\top + \lambda\Big[(I - KC)\mathcal{P}(I - KC)^\top\Big].
\end{aligned}$$

More compactly, by letting $A_\lambda := \left(A - \frac{\lambda}{2}I\right)$, $J_K := (I - KC)$, we get

$$\Psi(\mathcal{P}, M_{\mathcal{P}}, K) := A_\lambda\mathcal{P} + \mathcal{P}A_\lambda^\top + \Pi_\lambda^K(\mathcal{P}) + GG^\top + \lambda KVK^\top + \lambda J_K\mathcal{P}J_K^\top. \tag{3.5.1}$$

Based on this representation, we can write the solution $\mathcal{P}_t$ to equation (3.3.1) as follows:

$$\begin{aligned}
\mathcal{P}_t = \mathrm{e}^{A_\lambda t}\mathcal{P}_0\mathrm{e}^{A_\lambda^\top t} + \int_0^t \mathrm{e}^{A_\lambda(t-s)}\Big[\Pi_\lambda^K(\mathcal{P}_s) + \lambda J_K\mathcal{P}_s J_K^\top \\
+ GG^\top + \lambda KVK^\top\Big]\mathrm{e}^{A_\lambda^\top(t-s)}\,\mathrm{d}s.
\end{aligned}$$

Naturally, to study the bounds on $\mathcal{P}_t$ for large t, we consider the operator $\mathcal{F}_\lambda^K : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined as:

$$\mathcal{F}_\lambda^K(P) := \int_0^\infty \mathrm{e}^{A_\lambda t}\left[\Pi_\lambda^K(P) + \lambda J_K P J_K^\top\right]\mathrm{e}^{A_\lambda^\top t}\,\mathrm{d}t. \tag{3.5.2}$$

This dependence is stated in the following result:

Theorem 3.11. *Consider the matrix valued function $\mathcal{P}_t$ described by* (3.3.1)*, and the following statements:*

(S1) *All the eigenvalues of the operator $\mathcal{F}_\lambda^K$, defined in* (3.5.2)*, lie inside the unit disk of the complex plane, that is,*

$$\rho(\mathcal{F}_\lambda^K) < 1$$

where $\rho(\cdot)$ denotes the spectral radius of its argument.

(S2) *There exists a positive semidefinite solution $\mathcal{Q}$ to the algebraic equation*

$$0 = A_\lambda \mathcal{Q} + \mathcal{Q} A_\lambda^\top + \Pi_\lambda^K(\mathcal{Q}) + GG^\top + \lambda K V K^\top + \lambda (I - KC)\mathcal{Q}(I - KC)^\top. \quad (3.5.3)$$

Statement **(S1)** *implies that* **(S2)** *holds. If (A, G) is controllable, then* **(S2)** *implies that*

$$\lim_{t\to\infty} \mathcal{P}_t = \mathcal{Q}, \quad (3.5.4)$$

where $\mathcal{P}_t$ is the solution of (3.3.1) *and $\mathcal{Q}$ is obtained from* **(S2)**.

Proof. The implication that **(S2)** follows from **(S1)** is a consequence of [FCS98, Theorem 3.1: (d) $\Rightarrow$ (b)].

The implication that **(S2)** implies (3.5.4) can be obtained as a result of Lemma A.2 given in Appendix A. To apply this result, we need to show that $(A_\lambda, \overline{G})$ is controllable, where $\overline{G}$ is such that $\lambda K V K^\top + GG^\top = \overline{G}\,\overline{G}^\top$. This indeed follows from the fact that (A, G) is controllable. To see this, we use Lemma A.1 and show that (A_λ, G) is controllable. To prove the later statement, let us proceed by contradiction. If (A_λ, G) is not controllable, there exists $v \in \mathbb{R}^n$ such that $v^\top(A - 0.5\lambda I) = (\mu - 0.5\lambda)v^\top$ and $v^\top G = 0$, where μ is an eigenvalue of A. This implies that $v^\top A = \mu v^\top$ and $v^\top G = 0$, meaning (A, G) is not controllable, which is a contradiction. $\square$

We now prove a similar result for the nonlinear differential equation (3.3.4).

Theorem 3.12. *Consider the matrix valued function $\mathcal{P}_t$ described by the nonlinear equation differential equation* (3.3.4)*, and the following statement:*

(S3) *There exists a positive semidefinite solution $\mathcal{Q}$ to the algebraic equation*

$$0 = A\mathcal{Q} + \mathcal{Q}A^\top + \Pi_a(\mathcal{Q}) + \lambda\, c\, \mathcal{Q} - \lambda \mathcal{Q} C^\top M_{\mathcal{Q}}^{-1} C \mathcal{Q} + GG^\top \quad (3.5.5)$$

1. *If (A, G) is stabilizable and there exists K such that* **(S1)** *holds, then* **(S3)** *holds and $(A - \lambda \mathcal{Q} C^\top M_{\mathcal{Q}}^{-1} C)$ is a Hurwitz matrix.*

2. *If (A, G) is controllable,* **(S3)** *implies that,*

$$\lim_{t\to\infty} \mathcal{P}_t = \mathcal{Q}, \quad (3.5.6)$$

where $\mathcal{P}_t$ solves (3.3.4) *and $\mathcal{Q}$ is obtained from* **(S3)**.

Proof. To prove the first item, we first choose K_1 such that **(S1)** holds. Therefore, there exists a positive semidefinite matrix $\mathcal{Q}_1$ that solves the matrix equation (3.5.3), and $\mathcal{Q}_1$ satisfies the fixed point equation, $\mathcal{Q}_1 = \mathcal{F}_\lambda^{K_1}(\mathcal{Q}_1) + \mathfrak{G}_{K_1}$, where $\mathcal{F}_\lambda^{K_1}(\cdot)$ is defined as in (3.5.2), and

$$\mathfrak{G}_{K_1} = \int_0^\infty \mathrm{e}^{A_\lambda t}(GG^\top + \lambda K_1 V K_1^\top)\mathrm{e}^{A_\lambda^\top t}\mathrm{d}t.$$

Due to linearity of $\mathcal{F}_\lambda^{K_1}$, it readily follows that $\mathcal{Q}_1 = \mathcal{F}_\lambda^{K_1}(\mathcal{Q}_1) + \mathfrak{G}_{K_1} = (\mathcal{F}_\lambda^{K_1})^i(\mathcal{Q}_1) + \sum_{j=0}^{i-1}(\mathcal{F}_\lambda^{K_1})^j(\mathfrak{G}_{K_1})$, for each $i \in \mathbb{N}$. Further, since the spectral radius of $\mathcal{F}_\lambda^{K_1}$ is strictly less than one, there exist $\gamma \geq 0$ and $0 < \delta < 1$ such that $\|(\mathcal{F}_\lambda^{K_1})^i\| \leq \gamma\delta^i$ for each $i \in \mathbb{N}$. Consequently, let

$$\mathcal{G}_{K_1,i_1}(\mathcal{Q}) := (\mathcal{F}_\lambda^{K_1})^{i_1}(\mathcal{Q}) + \sum_{j=0}^{i_1-1}(\mathcal{F}_\lambda^{K_1})^j(\mathfrak{G}_{K_1})$$

where i_1 is chosen such that $\gamma\delta^{i_1} < 1$. Since $\mathcal{G}_{K_1,i_1}$ is a contraction mapping, there is a unique solution of the equation $\mathcal{Q} = \mathcal{G}_{K_1,i_1}(\mathcal{Q})$, which we have denoted by $\mathcal{Q}_1$. Further, let $K_2 := M_{\mathcal{Q}_1}^{-1}C^\top\mathcal{Q}_1$, $\mathcal{Q}_2 = \mathcal{G}_{K_2,i_1}(\mathcal{Q}_2)$, where $\mathcal{Q}_2$ can be obtained by successive approximations:

$$\mathcal{Q}_2^{(1)} = 0, \quad \mathcal{Q}_2^{(k+1)} = \mathcal{F}_\lambda^{K_2}(\mathcal{Q}_2^{(k)}) + \mathfrak{G}_{K_2};$$

Indeed, $\mathcal{G}_{K_2,i_1}(\mathcal{Q}_2)$ is well-defined since A_λ is Hurwitz. Notice that the sequence of $\mathcal{Q}_2^{(k)}$ is nondecreasing and nonnegative.

Recall that $\mathcal{Q}_1$ satisfies $\Psi(\mathcal{Q}_1, M_{\mathcal{Q}_1}, K_1) = 0$, so that, from (3.4.4) and our choice of K_2, we have

$$0 = -\Psi(\mathcal{Q}_1, M_{\mathcal{Q}_1}, K_1) \leq -\Psi(\mathcal{Q}_1, M_{\mathcal{Q}_1}, K_2)$$

which leads to (recall that $J_{K_2} = (I - K_2C)$),

$$\begin{aligned}0 \leq &-\int_0^\infty \mathrm{e}^{A_\lambda t}\Big(A_\lambda\mathcal{Q}_1 + \mathcal{Q}_1A_\lambda^\top + J_{K_2}\mathcal{Q}_1J_{K_2}^\top \\ &\quad + \Pi_\lambda^{K_2}(\mathcal{Q}_1) + GG^\top + \lambda K_2VK_2^\top\Big)\mathrm{e}^{A_\lambda^\top t}\mathrm{d}t \\ \leq &\ \mathcal{Q}_1 - \int_0^\infty \mathrm{e}^{A_\lambda t}\Big(\Pi_\lambda^{K_2}(\mathcal{Q}_1) + J_{K_2}\mathcal{Q}_1J_{K_2}^\top\Big)\mathrm{e}^{A_\lambda^\top t}\mathrm{d}t - \mathfrak{G}_{K_2}.\end{aligned}$$

Let us show by induction that $\mathcal{Q}_2^{(k)} \leq \mathcal{Q}_1$ for each k. Firstly it holds for

$k = 1$, and for $k > 1$,

$$\begin{aligned}\mathcal{Q}_2^{(k+1)} &= \mathcal{F}_\lambda^{K_2}(\mathcal{Q}_2^{(k)}) + \mathfrak{G}_{K_2}\\ &= \int_0^\infty \mathrm{e}^{tA_\lambda}\Big(\Pi_\lambda^{K_2}(\mathcal{Q}_2^{(k)}) + \lambda J_{K_2}(\mathcal{Q}_2^{(k)})J_{K_2}^\top\Big)\mathrm{e}^{tA_\lambda^\top}\mathrm{d}t + \mathfrak{G}_{K_2}\\ &\leq \mathcal{Q}_1 - \int_0^\infty \mathrm{e}^{tA_\lambda}\Big(\Pi_\lambda^{K_2}(\mathcal{Q}_1 - \mathcal{Q}_2^{(k)})\\ &\qquad + \lambda J_{K_2}(\mathcal{Q}_1 - \mathcal{Q}_2^{(k)})J_{K_2}^\top\Big)\mathrm{e}^{tA_\lambda^\top}\mathrm{d}t.\end{aligned}$$

Since the second-term on the right-hand side is negative definite, for each $k \in \mathbb{N}$, $\mathcal{Q}_2^{(k)}$ is no greater than $\mathcal{Q}_1$ and consequently $0 \leq \mathcal{Q}_2 \leq \mathcal{Q}_1$.

This procedure can be repeated to construct the sequences $\{K_i\}$ and $\{\mathcal{Q}_i\}$ similarly. Since $0 \leq \mathcal{Q}_{i+1} \leq \mathcal{Q}_i$ there exists $\overline{\mathcal{Q}} = \lim_{i\to\infty}\mathcal{Q}_i$, and moreover, for $M_{\overline{\mathcal{Q}}} = (1+c)C\overline{\mathcal{Q}}C^\top + (2c+c^2)\,\mathrm{tr}\,(\overline{\mathcal{Q}})I_{n\times n} + V$, we obtain, $\overline{K} = \lim_{i\to\infty} K_i = M_{\overline{\mathcal{Q}}}^{-1}C^\top\overline{\mathcal{Q}}$, and get $\Psi(\overline{\mathcal{Q}}, M_{\overline{\mathcal{Q}}}, \overline{K}) = 0$ where $\overline{\mathcal{Q}} \geq 0$.

To complete the proof of first item, we next show that $A - \lambda\overline{K}C$ is stable. It is seen that $\overline{\mathcal{Q}}$ satisfies the equation

$$\begin{aligned}\overline{\mathcal{Q}} &= \int_0^\infty \mathrm{e}^{sA_{\lambda\overline{K}}}\Big(\Pi(\overline{\mathcal{Q}}) + GG^\top + \lambda\overline{K}M_{\overline{\mathcal{Q}}}K^\top\Big)\mathrm{e}^{sA_{\lambda\overline{K}}^\top}\mathrm{d}s\\ &\geq \int_0^\infty \mathrm{e}^{sA_{\lambda\overline{K}}}GG^\top\mathrm{e}^{sA_{\lambda\overline{K}}^\top}\,\mathrm{d}s.\end{aligned}$$

Existence of positive semidefinite $\overline{\mathcal{Q}}$ implies that the integral on the right-hand side is bounded and since (A, G) is assumed to be stabilizable, it follows that $A_{\lambda\overline{K}}$ is Hurwitz.

For the proof of second item in Theorem 3.12, we show that there are $\overline{\mathcal{P}}_t$ and $\underline{\mathcal{P}}_t$ such that $\underline{\mathcal{P}}_t \leq \mathcal{P}_t \leq \overline{\mathcal{P}}_t$ and both converge to $\mathcal{Q}$. Set $K_\mathcal{Q} = \mathcal{Q}C^\top M_\mathcal{Q}^{-1}$, $A_\mathcal{Q} = A - \lambda K_\mathcal{Q}C$, where $A_\mathcal{Q}$ is Hurwitz stable by the first item. Denote by $\overline{\mathcal{P}}_t$ the solution of $\dot{\overline{\mathcal{P}}}_t = \Psi(\overline{\mathcal{P}}_t, M_{\overline{\mathcal{P}}_t}, K_\mathcal{Q}) = A_\lambda\overline{\mathcal{P}}_t + \overline{\mathcal{P}}_tA_\lambda^\top + \Pi_\lambda^{K_\mathcal{Q}}(\overline{\mathcal{P}}_t) + GG^\top + \lambda K_\mathcal{Q}VK_\mathcal{Q}{}^\top + \lambda(I - K_\mathcal{Q}C)\overline{\mathcal{P}}_t(I - K_\mathcal{Q}C)^\top$, with $\overline{\mathcal{P}}_0 = \mathcal{P}_0 \geq 0$. By the minimum property (Lemma 3.9) $\mathcal{P}_t \leq \overline{\mathcal{P}}_t$. Let $\overline{G}_\mathcal{Q}$ be such that $\lambda K_\mathcal{Q}VK_\mathcal{Q}^\top + GG^\top = \overline{G}_\mathcal{Q}\overline{G}_\mathcal{Q}^\top$. Using the same reasoning as in the proof of Theorem 3.11, it follows from Lemma A.1 that $(A_\lambda, \overline{G}_\mathcal{Q})$ is controllable. Then, applying Lemma A.2, we have $\overline{\mathcal{P}}_t \to \mathcal{Q}$ as $t \to +\infty$.

Next, consider the function $\underline{\mathcal{P}}_t$ obtained by solving the equation, $\dot{\underline{\mathcal{P}}}_t = \Psi(\underline{\mathcal{P}}_t, M_{\underline{\mathcal{P}}_t}, \underline{\mathcal{P}}_tC^\top M_{\underline{\mathcal{P}}_t}^{-1})$, with $\underline{\mathcal{P}}(0) = 0$. Due to zero initial condition, $\underline{\mathcal{P}}_t$ enjoys a monotonicity property which is stated in the following lemma and its proof appears in Appendix B.2.

Lemma 3.13. *The solution $\mathcal{P}_t$ of the following Cauchy problem*

$$\frac{\mathrm{d}\mathcal{P}_t}{\mathrm{d}t} = \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, \mathcal{P}_tC^\top M_{\mathcal{P}_t}^{-1}), \quad \mathcal{P}_0 = 0$$

is monotone nondecreasing for $t \in [0, +\infty[$.

Notice that the limit $\lim_{t\to+\infty} \underline{\mathcal{P}}_t$ exists as $\underline{\mathcal{P}}_t$ is monotone function by Lemma 3.13 and this limit is bounded because of minimal property in Lemma 3.9 and Theorem 3.11. Moreover, this means that $\frac{\mathrm{d}\underline{\mathcal{P}}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\underline{\mathcal{P}}}{\mathrm{d}t^2}$ are bounded too. As $\underline{\mathcal{P}}_t \geq 0$ the equality $\lim_{t\to+\infty} \underline{\mathcal{P}}_t = \int_0^{+\infty} \frac{\mathrm{d}\underline{\mathcal{P}}_t}{\mathrm{d}t}\mathrm{d}t$ implies that $\frac{\mathrm{d}\underline{\mathcal{P}}_t}{\mathrm{d}t} \to 0$ as $t \to \infty$. Since $\mathcal{Q}$ is a unique solution of the algebraic equation, it follows that $\underline{\mathcal{P}}_t \to \mathcal{Q}$ as $t \to \infty$.

It remains to prove that $\underline{\mathcal{P}}_t \leq \mathcal{P}_t$. Let $K_{\mathcal{P}_t} := \mathcal{P}_t C^\top M_{\mathcal{P}_t}^{-1}$ and consider $\widetilde{\mathcal{P}}_t := \mathcal{P}_t - \underline{\mathcal{P}}_t$, so that $\widetilde{\mathcal{P}}_0 = \mathcal{P}_0$, and

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\mathcal{P}}_t}{\mathrm{d}t} &= \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K_{\mathcal{P}_t}) - \Psi(\underline{\mathcal{P}}_t, M_{\underline{\mathcal{P}}_t}, \underline{\mathcal{P}}_t C^\top M_{\underline{\mathcal{P}}_t}^{-1}) \\
&\geq \Psi(\mathcal{P}_t, M_{\mathcal{P}_t}, K_{\mathcal{P}_t}) - \Psi(\underline{\mathcal{P}}_t, M_{\underline{\mathcal{P}}_t}, K_{\mathcal{P}_t}) \\
&= (A - \lambda K_{\mathcal{P}_t} C)\widetilde{\mathcal{P}} + \widetilde{\mathcal{P}}(A - \lambda K_{\mathcal{P}_t} C)^\top + \Pi(\widetilde{\mathcal{P}}_t) \\
&\qquad + \lambda K_{\mathcal{P}_t} M_{\widetilde{\mathcal{P}}_t} K_{\mathcal{P}_t}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\widetilde{\mathcal{P}}_t &= \Phi(t,0)\mathcal{P}_0\Phi^\top(t,0) \\
&\qquad + \int_0^t \Phi(t,s)\Big[\Pi(\widetilde{\mathcal{P}}_s) + \lambda K_{\mathcal{P}_s} M_{\widetilde{\mathcal{P}}_s} K_{\mathcal{P}_s}^\top\Big]\Phi^\top(t,s)\,\mathrm{d}s \\
&\geq \int_0^t \Phi(t,s)\Big[\Pi(\widetilde{\mathcal{P}}_s) + \lambda K_{\mathcal{P}_s} M_{\widetilde{\mathcal{P}}_s} K_{\mathcal{P}_s}^\top\Big]\Phi^\top(t,s)\,\mathrm{d}s,
\end{aligned}
$$

where $\Phi(t,s)$ is the fundamental matrix associated with $A - \lambda K_{\mathcal{P}_t} C$. From this expression[3], we obtain $\widetilde{\mathcal{P}}_t \geq 0$ and hence $\underline{\mathcal{P}}_t \leq \mathcal{P}_t$. $\square$

3.6 Sufficient conditions for boundedness

In this section, we study some particular cases which provide sufficient conditions to guarantee the boundedness of the error covariance matrix, based on the results given in the previous section.

3.6.1 Eigenvalue assignment condition

Since the conventional design of the state estimators provide conditions for bounded estimation error based on the eigenvalues of certain matrix, we rewrite the conditions from previous section as an eigenvalue test.

[3] If $Q_t \geq \int_0^t \Phi(t,s)(\Pi(Q_s) + \lambda K_s M_{Q_s} K_s)\Phi^\top(t,s)\mathrm{d}s$, then $Q_t \geq 0$.

Proposition 3.14. *For a given $\lambda > 0$ and $K \in \mathbb{R}^{n\times p}$, consider the matrices,*[4]

$$\begin{aligned}\mathcal{A}_{\lambda K} &:= (I \otimes (A - \lambda KC)^\top) + ((A - \lambda KC)^\top \otimes I)\\ &\quad + \lambda((KC)^\top \otimes (KC)^\top)\\ \Sigma_a &:= a(I \otimes I + \operatorname{vec}(I)\cdot \operatorname{vec}(I)^\top)\\ \Sigma_c &:= c(I \otimes I + (KC)^\top \otimes (KC)^\top)\\ &\quad + (2c + c^2)\cdot(\operatorname{vec}(I)\cdot\operatorname{vec}(I)^\top)\cdot(I \otimes KK^\top)\end{aligned}$$

and the following statement,

(S4) *The matrix $\mathcal{A}_{\lambda K} + \Sigma_a + \lambda\Sigma_c \in \mathbb{R}^{n^2\times n^2}$ is Hurwitz.*

Then **(S4)** *implies* **(S2)** *stated in Theorem 3.11.*

Proof. To prove this result, we will make use of [FCS98, Theorem 3.1(ii)]. Using this result, we just need to show that, if **(S4)** holds, then the matrix-valued function $\mathcal{R} : [0,\infty) \to \mathbb{R}^{n\times n}$ satisfying the differential equation $\dot{\mathcal{R}} = \Sigma(\mathcal{R})$ has the property that $\|\mathcal{R}_t\|$ converges to zero, as $t \to \infty$, where $\Sigma(\mathcal{R}) = A_\lambda^\top\mathcal{R} + \mathcal{R}A_\lambda + \lambda J_K^\top\mathcal{R}J_K + \Pi^*_{\lambda,K}(\mathcal{R})$ and $\Pi^*_{\lambda,K}$ denotes the adjoint[5] of Π^K_λ. In particular,

$$\Pi^*_{\lambda,K}(\mathcal{R}) = a\mathcal{R} + a\operatorname{tr}(\mathcal{R})I + \lambda c\mathcal{R} + \lambda cC^\top K^\top\mathcal{R}KC \\ + \lambda(2c + c^2)\operatorname{tr}(KK^\top\mathcal{R})I.$$

We now write the matrix differential equation in vectorial form, so that, $\operatorname{vec}(\dot{\mathcal{R}}) = \operatorname{vec}(\Sigma(\mathcal{R}))$, and using the properties of the Kronecker product of the matrices, we observe that[6]

$$\begin{aligned}\operatorname{vec}(\Sigma(R)) &= (I \otimes A^\top)\operatorname{vec}(R) + (A^\top \otimes I)\operatorname{vec}(R)\\ &\quad - \lambda(I \otimes I)\operatorname{vec}(R) + \lambda(J_K^\top \otimes J_K^\top)\operatorname{vec}(R) + \operatorname{vec}(\Pi^*_{\lambda,K}(\mathcal{R})). \quad (3.6.1)\end{aligned}$$

Recalling the definition of Σ_a and Σ_c, the last term in (3.6.1) can be written as, $\operatorname{vec}(\Pi^*_{\lambda,K}(R)) = \Sigma_a\operatorname{vec}(R) + \lambda\Sigma_c\operatorname{vec}(R)$. Also, since $J_K = (I - KC)$, the second to last term in (3.6.1) is rewritten as $(I - KC)^\top \otimes (I - KC)^\top = (I\otimes I) - ((KC)^\top\otimes I) - (I\otimes(KC)^\top) + ((KC)^\top\otimes(KC)^\top)$ so that, $\operatorname{vec}(\Sigma(\mathcal{R})) = (\mathcal{A}_{\lambda K} + \Sigma_a + \lambda\Sigma_c)\operatorname{vec}(\mathcal{R})$. Clearly, if the matrix $(\mathcal{A}_{\lambda K} + \Sigma_a + \lambda\Sigma_c)$ is Hurwitz, then $\operatorname{vec}(\mathcal{R}_t)$ converges to 0, as $t \to \infty$, and, in particular, $\|\mathcal{R}_t\|$ converges to 0 as well. □

[4] The symbol $\otimes$ denotes the matrix Kronecker product.

[5] For a linear operator $\Pi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, its adjoint is a linear operator $\Pi^* : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times}$ that satisfies $\langle \Pi(M_1), M_2\rangle = \langle M_1, \Pi^*(M_2)\rangle$, for all matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$. Here, the inner product over the space of matrices is defined as $\langle M_1, M_2\rangle = \operatorname{tr}(M_1^\top M_2)$.

[6] For given matrices A, B, C, D of appropriate dimensions, we have: 1) $(A\otimes B)(C\otimes D) = AC \otimes BD$, and 2) If $ABC = D$, then $\operatorname{vec}(D) = (C^\top \otimes A)\operatorname{vec}(B)$.

3.6.2 A sufficient condition with c square and invertible

In the particular case, when C is square and invertible, one can provide certain conditions under which a particular choice of the gain matrix K bounds the error covariance. This is captured by the following result:

Proposition 3.15. *Consider the system* (3.1.1) *and the filter* (3.2.1). *Assume that $C \in \mathbb{R}^{n\times n}$ is invertible and take $K = C^{-1}$. Let*

$$\alpha := \int_0^\infty \mathrm{e}^{(A-0.5\lambda I)t}\mathrm{e}^{(A-0.5\lambda I)^\top t}\mathrm{d}t \tag{3.6.2}$$

Then, the error covariance is bounded if

$$a(n+1) + 2\lambda c + \lambda n(2c+c^2)\,\|C^{-2}\| < \frac{1}{\alpha}. \tag{3.6.3}$$

Proof. Based on the result of Theorem 3.11, it suffices to show that (3.6.3) leads to the spectral radius of $\mathcal{F}_\lambda^K$ being strictly less than 1. By letting $K = C^{-1}$, it is seen that $\mathcal{F}_\lambda^K(P) = \int_0^\infty \mathrm{e}^{A_\lambda t}\Pi_\lambda^K(P)\mathrm{e}^{A_\lambda t}\,\mathrm{d}t$. We can now introduce the bound

$$-\|P\|\,\Pi_\lambda^K(I) \leq \Pi_\lambda^K(P) \leq \Pi_\lambda^K(I)\,\|P\|$$

so that, using the definition of α in (3.6.2), $\|\mathcal{F}_\lambda^K(P)\| \leq \alpha\,\|\Pi_\lambda^K(I)\|\,\|P\|$. Condition (3.6.3) ensures that $\alpha\|\Pi_\lambda^K\| < 1$ because

$$\Pi_\lambda^K(I) = a(1+\mathrm{tr}(I))I + 2\lambda cI + \lambda(2c+c^2)\,\mathrm{tr}(I)C^{-2}.$$

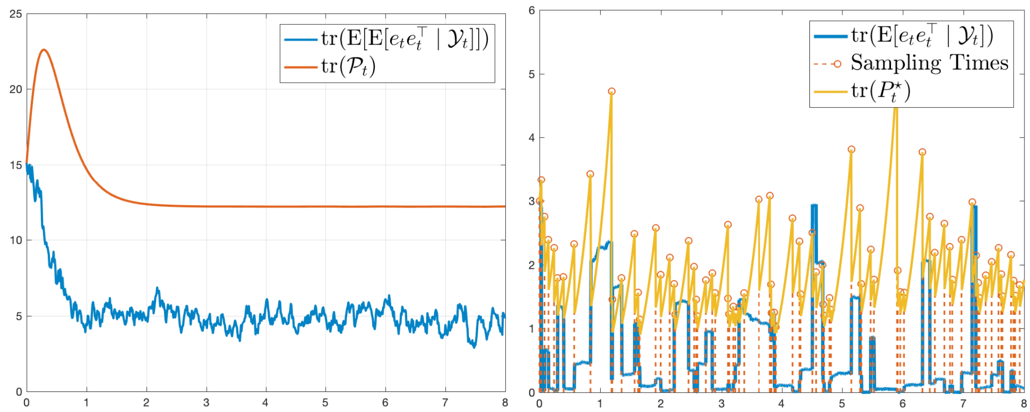
As a result, $\|\mathcal{F}_\lambda^K\| < 1$ and in particular, the spectral radius of $\mathcal{F}_\lambda^K$ is less than 1. $\square$

In the proof of Proposition 3.15, we basically showed that, if C is invertible and the nonlinearities are small enough, then for certain choice of K and λ such that

$$\|\mathcal{F}_\lambda^K\| < 1 \tag{3.6.4}$$

it follows that the spectrum of $\mathcal{F}_\lambda^K$ lies within the unit disk of the complex plane, that is, the statement **(S1)** holds. However, if C is not invertible, then even for linear systems ($a = 0$ and $c = 0$), there may not exist λ and K such that (3.6.4) holds. This is shown by the following example.

Example 3.16. *Consider the dynamical system* (3.1.1) *with $f(t,x) = Ax$ and $h(x) = Cx$, $G = I_{2\times 2}$, where $A = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$; $C = \left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]$. With this example, even if there are no nonlinearities, that is, $a, c = 0$, it is seen that, for any $\lambda > 0$, there does not exist a $K \in \mathbb{R}^2$ such that* (3.6.4) *holds for this*



(a) Expectation with respect to sampling times with constant gain $K = [1\ 2]^\top$.

(b) One sample path with bound minimizing gain.

Figure 3.3: Trace of error covariance and their theoretical upper bounds for Example 3.16 with $\lambda = 10$.

system. To see this, observe that (3.6.4) *translates to finding a matrix K such that, $\lambda \left| \int_0^\infty e^{t(A-0.5\lambda I)} J_K J_K^\top e^{t(A-0.5\lambda I)^\top} \mathrm{d}t \right| < 1$. We let $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$, for some $k_1, k_2 \in \mathbb{R}$. However, with straightforward computations involving matrix exponential, we see that, the integral is finite for each $\lambda > 2$, and*

$$e^{t(A-0.5\lambda I)} J_K J_K^\top e^{t(A-0.5\lambda I)^\top} = \begin{bmatrix} * & * \\ * & e^{(1-0.5\lambda)t}(1+k_2^2)e^{(1-0.5\lambda)t} \end{bmatrix}.$$

Clearly, with $\lambda > 2$, $\lambda \int_0^\infty e^{(1-0.5\lambda)t}(1+k_2^2)e^{(1-0.5\lambda)t} > 1$, and hence the condition (3.6.4) *does not hold true.*

However, it is possible to check that for each $\lambda > 2$, we can find K such that **(S4)** *holds true. For simulation purposes, we choose $\lambda = 10$ and see that $K = [1\ 2]^\top$ satisfies the condition* **(S4)**, *where we take $\Sigma_a = \Sigma_c = 0$. The simulation results are reported in Figure 3.3. Moreover, since eigenvalues of a matrix are continuous functions of its entries, the matrix $\mathcal{A}_{\lambda K} + \Sigma_a + \lambda \Sigma_c$ also has eigenvalues in open left-half of the complex plane, for small enough values of a and c.*

3.6.3 Dependence on sampling rate

It is interesting to investigate the dependence of the filter efficiency with respect to the sampling rate λ of the Poisson measurement process. This question has not been straightforward for the class of systems studied in

Chapter 3, that is, the systems with (f, h) Lipschitz continuous. However, some constructive statements can be derived for the linear system

$$\mathrm{d}x_t = Ax_t\,\mathrm{d}t + G\mathrm{d}w_t \tag{3.6.5a}$$
$$y_{\tau_k} = Cx_{\tau_k} + v_{\tau_k} \tag{3.6.5b}$$

for the arrival times τ_k of a Poisson process N_t. That is the system (3.1.1) under **(A1)** with $a = 0$, $c = 0$.

A lower bound on sampling rate

We first provide a necessary lower bound on the sampling rate which ensures that the error covariance remains bounded for all times. In what follows, we denote an eigenvalue of A by $\mu_j(A)$ and its real part by $\Re\mu_j(A)$, $j = 1, \dots, n$.

Proposition 3.17. *Consider system* (3.6.5) *with* (A, C) *detectable and* (A, G) *is controllable. Then, for each* $\lambda < 2\max\{\Re\mu_j(A) \mid j = 1, \dots, n\}$*, the solution to* (3.3.1) *is such that* $\lim_{t\to\infty} \mathcal{P}(t) = \infty$.

Proof. We first look at the differential equation

$$\dot{Q}_t = A_\lambda Q_t + Q_t A_\lambda^\top + GG^\top$$

with initial condition $Q_0 = \mathcal{P}_0$. It is clear that Q_t stays bounded if and only if

$$0 = A_\lambda \overline{Q} + \overline{Q} A_\lambda^\top + GG^\top$$

has a symmetric positive definite solution $\overline{Q}$. With (A, G) controllable, this is the case if and only if A_λ is Hurwitz, that is,

$$\lambda > 2\max\{\Re\mu_j(A) \mid j = 1, \dots, n\}.$$

We claim that $Q_t \le \mathcal{P}_t$, for every $t \ge 0$. To see this, let $Q_0 = \mathcal{P}_0$. Let $\widetilde{Q}_t = \mathcal{P}_t - Q_t$, then

$$\dot{\widetilde{Q}}_t = \psi_1(\widetilde{Q}_t) + \lambda\Big[(I - KC)\mathcal{P}_t(I - KC)^\top + KVK^\top\Big]$$

where

$$\psi_1(\widetilde{Q}_t) = A_\lambda \widetilde{\mathcal{Q}}_t + \widetilde{\mathcal{Q}}_t A_\lambda^\top + GG^\top.$$

Because of the zero initial condition, this equation has the solution given by

$$\widetilde{Q}_t = \int_0^t \mathrm{e}^{A_\lambda(t-s)}\Big[\lambda(I-KC)\mathcal{P}_s(I-KC)^\top + GG^\top - \lambda KVK^\top\Big]\mathrm{e}^{A_\lambda^\top(t-s)}\,\mathrm{d}s \ge 0$$

and hence $\widetilde{Q}_t \ge 0$, that is, $\mathcal{P}_t \ge Q_t$ for each $t \ge 0$. As a result, $\mathcal{P}_t$ goes to $+\infty$ when Q_t grows unbounded. $\square$

Monotonicity relation

After deriving a lower bound that ensures boundedness, then next question is to study how this steady state upper bound on the error covariance changes as a function of the sampling rate.

Proposition 3.18. *Consider system* (3.6.5). *Let* $\lambda_1, \lambda_2 \geq 0$ *describe the rate of Poisson measurement process and* $\mathcal{P}_{\lambda_1}$, $\mathcal{P}_{\lambda_2}$ *denote the corresponding solutions to* (3.3.4) *with* $\mathcal{P}_{\lambda_1}(0) = \mathcal{P}_{\lambda_2}(0)$. *For each* $\lambda_2 \geq \lambda_1$, *we have*

$$\mathcal{P}_{\lambda_2}(t) \leq \mathcal{P}_{\lambda_1}(t), \quad \forall\, t \geq 0. \tag{3.6.6}$$

Proof. For a given λ, the solution to (3.3.4) satisfies

$$\dot{\mathcal{P}}_\lambda = \Psi_\lambda(\mathcal{P}_\lambda, M_{\mathcal{P}_\lambda}, \mathcal{P}_\lambda C^\top M_{\mathcal{P}_\lambda}^{-1})$$

where, due to the linearity of the measurement map

$$\Psi_\lambda(\mathcal{P}_\lambda, M_{\mathcal{P}_\lambda}, \mathcal{P}_\lambda C^\top M_{\mathcal{P}_\lambda}^{-1}) = A\mathcal{P} + \mathcal{P}A^\top + GG^\top - \lambda \mathcal{P}_\lambda C^\top M_{\mathcal{P}_\lambda}^{-1} C \mathcal{P}_\lambda.$$

It therefore holds that

$$\frac{\partial \Psi_\lambda}{\partial \lambda} \leq 0$$

and thus, for each $\lambda_2 \geq \lambda_1$, we have

$$\Psi_{\lambda_2} \leq \Psi_{\lambda_1}, \quad \forall\, \lambda_2 \geq \lambda_1.$$

In particular, $\dot{\mathcal{P}}_{\lambda_2}(t) \leq \dot{\mathcal{P}}_{\lambda_2}(t)$, for each $t \geq 0$ and hence (3.6.6) holds. □

3.6.4 Convergence with large sampling rate

We now compare the error covariance of the sampled system with the error covariance of an estimator with continuous time measurements. Toward this end, let us consider the *optimal* Kalman–Bucy filter for (3.3.5) with continuously available output measurements, which is given by

$$\mathrm{d}\widehat{z}_t = A\widehat{z}_t\,\mathrm{d}t + K_t^{\mathrm{cont}}(\mathrm{d}z_t - C\widehat{z}_t\,\mathrm{d}t), \tag{3.6.7}$$

with $\widehat{z}_0 = \mathsf{E}[x_0]$. To minimize the error covariance, the injection gain K_t^{cont} for this estimator is obtained as

$$K_t^{\mathrm{cont}} = \mathcal{R}_t C^\top V^{-1} \tag{3.6.8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}_t = A\mathcal{R}_t + \mathcal{R}_t A^\top + GG^\top - \mathcal{R}_t C^\top V^{-1} C \mathcal{R}_t. \tag{3.6.8b}$$

When $t \to \infty$, the error covariance $\mathsf{E}[|x_t - \widehat{z}_t|^2 \,|\, \mathcal{Y}_t]$ is given by the matrix $\overline{\mathcal{R}}$ that solves

$$0 = A\overline{\mathcal{R}} + \overline{\mathcal{R}}A^\top + GG^\top - \overline{\mathcal{R}}C^\top V^{-1} C\overline{\mathcal{R}}. \tag{3.6.9}$$

We are interested in knowing how the steady state error covariance for the estimator using sampled measurements compares with $\overline{\mathcal{R}}$ when we increase the sampling rate λ, and in particular, when we let λ go to ∞.

The next result addresses these issues and to formulate it, we consider the operator $\overline{\mathcal{F}}_\lambda^K$ which is obtained from $\mathcal{F}_\lambda^K$ by setting $\Pi_\lambda^K = 0$ in (3.5.2), that is,

$$\overline{\mathcal{F}}_\lambda^K(P) := \lambda \int_0^\infty \mathrm{e}^{tA_\lambda}(I - KC)P(I - KC)^\top \mathrm{e}^{tA_\lambda^\top}\,\mathrm{d}t, \tag{3.6.10}$$

with $\rho\left(\overline{\mathcal{F}}_\lambda^K\right)$ denoting its spectral radius. We denote the Neumann series associated with $\overline{\mathcal{F}}_\lambda^K$ by $\mathcal{N}_\lambda$, so that $\mathcal{N}_\lambda := \left(I - \overline{\mathcal{F}}_\lambda^K\right)^{-1} = \sum_{j=0}^\infty (\overline{\mathcal{F}}_\lambda^K)^j$. Thus, if $\rho\left(\overline{\mathcal{F}}_\lambda^K\right) < 1$, there exists ℓ such that $\left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\| < 1$, and we can write

$$\begin{aligned}\mathcal{N}_\lambda &= \left(I + \overline{\mathcal{F}}_\lambda^K + \cdots + \left(\overline{\mathcal{F}}_\lambda^K\right)^{\ell-1}\right)\left(I + \left(\overline{\mathcal{F}}_\lambda^K\right)^\ell + \left(\overline{\mathcal{F}}_\lambda^K\right)^{2\ell} + \cdots\right)\\ &= \left(I + \overline{\mathcal{F}}_\lambda^K + \cdots + \left(\overline{\mathcal{F}}_\lambda^K\right)^{\ell-1}\right)\left(I - \left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right)^{-1}.\end{aligned}$$

Using triangle inequality[7], it follows that $\|\mathcal{N}_\lambda\| \le \beta_\lambda := \left\|I + \overline{\mathcal{F}}_\lambda^K + \cdots + \left(\overline{\mathcal{F}}_\lambda^K\right)^{\ell-1}\right\| \Big/ \left(1 - \left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\|\right)$.

Theorem 3.19. *Consider system* (3.1.1) *under* **(A1)** *with $a = 0$, $c = 0$, and assume that (A, G) is controllable. Let $\overline{\mathcal{R}}$ denote the steady state minimal covariance of the estimator* (3.6.7) *that satisfies* (3.6.9). *Let $K = K^{\mathrm{cont}} = \overline{\mathcal{R}}C^\top V^{-1}$ and assume that the operator $\overline{\mathcal{F}}_\lambda^K$ in* (3.6.10) *satisfies*

$$\rho\left(\overline{\mathcal{F}}_\lambda^K\right) < 1, \tag{3.6.11}$$

for each $\lambda > 0$. Then the matrix $\overline{\mathcal{P}}_\lambda \ge 0$ that solves (3.5.3), *and provides an upper bound on the error covariance due to filter* (3.2.1) *with Poisson measurement process of intensity λ, satisfies*

$$\overline{\mathcal{P}}_\lambda = \overline{\mathcal{R}} + \mathcal{N}_\lambda\left(\int_0^\infty \mathrm{e}^{A_\lambda t} K\Big((1-\lambda)V + \lambda C\overline{\mathcal{R}}C^\top\Big)K^\top \mathrm{e}^{A_\lambda^\top t}\mathrm{d}t\right). \tag{3.6.12}$$

---

[7] We use the fact that $\left\|\left(I - \left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right)^{-1}\right\| \le \|I\| + \left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\| + \left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\|^2 + \cdots = \sum_{j=0}^\infty \left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\|^j = \left(1 - \left\|\left(\overline{\mathcal{F}}_\lambda^K\right)^\ell\right\|\right)^{-1}$.

In particular, if $\beta_\infty = \limsup_{\lambda\to\infty} \beta_\lambda$, *then*

$$\limsup_{\lambda\to\infty} \|\overline{\mathcal{P}}_\lambda\| \le \|\overline{\mathcal{R}}\| + \beta_\infty \|K(C\overline{\mathcal{R}}C^\top - V)K^\top\|. \tag{3.6.13}$$

Proof. Consider the solution $\mathcal{P}_t^\lambda$ of (3.3.1) with $\Pi_\lambda^K = 0$ and $K := \overline{\mathcal{R}}C^\top V^{-1}$. Let $\mathcal{Q}_t^\lambda = \mathcal{P}_t^\lambda - \overline{\mathcal{R}}$, then $\mathcal{Q}_t^\lambda$ satisfies the following relation

$$\begin{aligned}
\dot{\mathcal{Q}}_t^\lambda &= A_\lambda \mathcal{Q}_t^\lambda + \mathcal{Q}_t^\lambda A_\lambda^\top + (1+\lambda)KVK^\top + \lambda(I-KC)\mathcal{P}_t^\lambda(I-KC)^\top - \lambda\overline{\mathcal{R}} \\
&= A_\lambda \mathcal{Q}_t^\lambda + \mathcal{Q}_t^\lambda A_\lambda^\top + (1+\lambda)KVK^\top - \lambda\overline{\mathcal{R}} + \lambda(I-KC)\mathcal{Q}_t^\lambda(I-KC)^\top \\
&\quad + \lambda(I-KC)\overline{\mathcal{R}}(I-KC)^\top \\
&= A_\lambda \mathcal{Q}_t^\lambda + \mathcal{Q}_t^\lambda A_\lambda^\top + KVK^\top - \lambda KVK^\top + \lambda(I-KC)\mathcal{Q}_t^\lambda(I-KC)^\top \\
&\quad + \lambda KC\overline{\mathcal{R}}C^\top K^\top
\end{aligned}$$

where the last equality followed from the fact that

$$\begin{aligned}
(I-KC)\overline{\mathcal{R}}(I-KC)^\top - \mathcal{R} &= KC\overline{\mathcal{R}}C^\top K^\top - 2\overline{\mathcal{R}}C^\top V^{-1}C\mathcal{R} \\
&= -2KVK^\top + KC\overline{\mathcal{R}}C^\top K^\top.
\end{aligned}$$

Because of condition (3.6.11) and Theorem 3.11, we have $\lim_{t\to\infty} \mathcal{P}_t^\lambda(t) = \overline{\mathcal{P}}_\lambda$. Hence, by letting $\overline{\mathcal{Q}}_\lambda := \overline{\mathcal{P}}_\lambda - \overline{\mathcal{R}}$ (which is not necessarily positive definite), we have that $\lim_{t\to\infty} \mathcal{Q}_t^\lambda(t) = \overline{\mathcal{Q}}_\lambda$. Moreover, $\overline{\mathcal{Q}}_\lambda$ solves the following algebraic equation:

$$0 = A_\lambda\overline{\mathcal{Q}}_\lambda + \overline{\mathcal{Q}}_\lambda A_\lambda^\top + KVK^\top - \lambda KVK^\top + \lambda(I-KC)\overline{\mathcal{Q}}_\lambda(I-KC)^\top + \lambda KC\overline{\mathcal{R}}C^\top K^\top$$

Alternatively, the matrix $\overline{\mathcal{Q}}_\lambda$ satisfies the equation,

$$\overline{\mathcal{Q}}_\lambda = \overline{\mathcal{F}}_\lambda^K(\overline{\mathcal{Q}}_\lambda) + \overline{\mathfrak{G}}_\lambda^K,$$

where $\overline{\mathfrak{G}}_\lambda^K := \int_0^\infty \mathrm{e}^{A_\lambda t} K\Big((1-\lambda)V + \lambda C\overline{\mathcal{R}}C^\top\Big)K^\top \mathrm{e}^{A_\lambda^\top t}\mathrm{d}t$. Iterating the fixed point equation, we get

$$\overline{\mathcal{Q}}_\lambda = \left(\overline{\mathcal{F}}_\lambda^K\right)^k(\overline{\mathfrak{G}}_\lambda^K) + \sum_{j=0}^{k-1}\left(\overline{\mathcal{F}}_\lambda^K\right)^j(\overline{\mathfrak{G}}_\lambda^K) \overset{k\to\infty}{=} (I-\overline{\mathcal{F}}_\lambda^K)^{-1}(\overline{\mathfrak{G}}_\lambda^K),$$

and hence we arrive at (3.6.12): $\overline{\mathcal{P}}_\lambda = \overline{\mathcal{R}} + (I-\overline{\mathcal{F}}_\lambda^K)^{-1}(\overline{\mathfrak{G}}_\lambda^K)$. The inequality in (3.6.13) is obtained by taking the norms and using the bound $\left\|\mathrm{e}^{tA_\lambda}\right\| = \left\|\mathrm{e}^{tA}\cdot\mathrm{e}^{-0.5t\lambda I}\right\| \le \mathrm{e}^{(\|A\|-0.5\lambda)t}$, which results in

$$\limsup_{\lambda\to\infty}\left\|\overline{\mathfrak{G}}_\lambda^K\right\| \le \left\|K(C\overline{\mathcal{R}}C^\top - V)K^\top\right\|,$$

along with the bound $\|\mathcal{N}_\lambda\| \le \beta_\lambda$. □

Chapter 4

Ensemble Filters

Ensemble filters have a notable impact if estimation with noisy data is required in large-scale models. This approach is based on simulating the evolution of different particles through differential equations that are coupled through the associated empirical measure or empirical moments. This chapter presents the derivation of a continuous-discrete McKean–Vlasov type diffusion process with additive Gaussian noise in the observation model, which is used to describe the evolution of the particles. Such a process is proven to be an exact filter on the one hand. Then, such an exact filter is approximated by an ensemble of particles, and the equations for evolution of particles are coupled through the empirical covariance.

Such an approach requires less computations for implementation than the optimal ones based on solving Riccati differential equations. It is shown that if the mean sampling rate of the observation process is sufficiently large, then the empirical mean and the sample covariance of the ensemble filters converge to the mean and covariance of the optimal filter, as the number of particles tends to infinity.

4.1 Setup

4.1.1 Filtering system

Let us consider $\mathbb{R}^n$-valued continuous-time state process

$$\mathrm{d}x_t = Ax_t\mathrm{d}t + G\,\mathrm{d}w_t \tag{4.1.1}$$

with $\mathbb{R}^p$-valued discrete-time observations

$$y_{\tau_k} = C\,x(\tau_k) + v_{\tau_k}, \qquad \tau_{k+1} \geq \tau_k,\ k \in \mathbb{N} \tag{4.1.2}$$

where τ_k are the arrival times of a Poisson process N_t, w_t is an $\mathbb{R}^m$-valued standard Wiener process, $v_{\tau_k} \sim \mathcal{N}(0, V)$ for each $k \in \mathbb{N}$. We let $(\Omega, \mathcal{F}, \mathsf{P})$ denote the underlying probability space. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{n\times m}$ are assumed to be constant.

The intensity $\lambda > 0$ of the Poisson process is counted as a parameter; it affects the properties of the filter such as stability. Notice that N_t is associated with τ_k as a counting process, i.e.

$$N_t = \sum_i \delta_{\tau_i \leq t}$$

and so its range of values is the set of natural numbers Hence the sequence $\{v_{N_t}\}_{t\in\mathbb{R}^+}$ coincides with the sequence $\{v_{\tau_k}\}_{k\in\mathbb{N}}$.

The observations (4.1.2) restricted to an interval $[0, t]$, as $\{y_{\tau_k} \mid \tau_k \in [0, t]\}$ generate the sigma-algebra $\mathcal{Y}_t$.

4.1.2 Optimal filter

To analyze the performance of an ensemble filter, one needs to be aware of the optimal filter behavior. This section recalls the developments of Section 2.6.

Description by two moments

Let us recall that such filtering system has an optimal solution, given earlier in Theorem 2.22, that consists in describing the Gaussian process $\widehat{x}_t = \mathsf{E}[x_t \mid \mathcal{Y}_t]$ as follows:

$$\mathrm{d}\widehat{x}_t = A\widehat{x}_t \mathrm{d}t + K_t(y_t - C\widehat{x}_t)\mathrm{d}N_t, \tag{4.1.3}$$

where $K_t = P_t C^\top (C P_t C^\top + V_t)^{-1}$ and the covariance matrix $P_t = \mathsf{E}[(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)^\top \mid \sigma\{N_t\}]$ satisfies

$$\mathrm{d}P_t = (AP_t + P_t A^\top + GG^\top)\mathrm{d}t - K_t C P_t \mathrm{d}N_t. \tag{4.1.4}$$

Expectation with respect to Poisson process

The expectation of the error covariance P_t with respect to Poisson process can be obtained then by applying Proposition 2.20 to the processes (4.1.4). Thus, we get

$$\dot{\mathcal{P}}_t = A\mathcal{P}_t + \mathcal{P}_t A^\top + GG^\top - \lambda \mathcal{P}_t C^\top (C\mathcal{P}_t C^\top + V)^{-1} C\mathcal{P}_t \tag{4.1.5}$$

for $\mathcal{P}_t := \mathsf{E}[P_t \mid P_0]$. In addition, for the scalar case, it can be further studied using Lemma B.1.

4.1.3 Ensemble filter approach

Due to the linearity, and Gaussian noise, the distribution of the optimal estimator is Gaussian and it suffices to have only its first and second moments. However, the evolution of $\widehat{x}_t := \mathsf{E}[x_t \mid \mathcal{Y}_t]$, depends on the evolution of the error covariance matrix and the computation of P_t is often costly for higher dimensional systems.

On the other hand, using the ensemble approach, it is possible to compute (an approximation) of the conditional distribution without solving the differential equation for the evolution of covariance matrix.

This can be done by finding a process S_t that implicitly describes the optimal filter. Namely, the following condition (4.1.6) is on the consideration.

Definition 4.1. *For a filtering system with state process $(x_t)_{t\geq 0}$ and observation process $(y_t)_{t\geq 0}$, a stochastic process $(s_t)_{t\geq 0}$ satisfying*

$$\mathrm{Law}(s_t \mid \mathcal{Y}_t) = \mathrm{Law}(x_t \mid \mathcal{Y}_t) \qquad \forall t \geq 0 \tag{4.1.6}$$

is called an exact filter.

Different processes might play the role of exact filters. It turns out that there are several ways to construct exact filters that can be approximated by computation-friendly methods and provide sufficiently good efficiency. Such approximations are provided by an ensemble of particles whose limiting behavior (as the number of particles tend to infinity) converges to S_t. The computational advantage of this approach is that the simulation of particles, in general, is more efficient and potentially applicable in nonlinear systems as well (although there are very few instances of formal analysis in nonlinear setting). As an example of approximating a process using the particles, one can take for instance a result from [Szn91, Theorem 1.4, p.172] where it is shown that if a process S_t satisfies a simple version of McKean–Vlasov type equation

$$\mathrm{d}S_t = \mathrm{d}B_t + \Big(\int b(S_t, \bar{S})\mu_t(\mathrm{d}\bar{S})\Big)\mathrm{d}t,$$

where $\mu_t(\mathrm{d}\bar{S})$ is the law of S_t, B_t is a standard Brownian motion, the function b is bounded and Lipschitz continuous and initial distribution is given, then S_t can be approximated by an ensemble of m interacting particles S_t^i, $i = 1, \ldots, \mathrm{m}$ with the dynamics

$$\mathrm{d}S_t^i = \mathrm{d}B_t^i + \frac{1}{\mathrm{m}}\sum_{j=1}^{\mathrm{m}} b(S_t^i, S_t^j)\mathrm{d}t, \quad i = 1, \ldots, \mathrm{m}$$

and with the corresponding initial distributions. In particular, when the number of particles m tends to ∞, each S_t^i approaches a process which is an independent copy of the process S_t. For the filtering problem, with the conventional observation process z_t given in the form[1] $\mathrm{d}z_t = h(x_t)\ \mathrm{d}t + \mathrm{d}\eta_t$, the authors in [TM20] show that the property $\mathrm{Law}(S_t \mid \mathcal{Y}_t) \sim \mathrm{Law}(x_t \mid \mathcal{Y}_t)$ holds if S_t satisfies the following McKean–Vlasov type equation

$$\mathrm{d}S_t = AS_t\mathrm{d}t + K_{1,t}(S_t - \widehat{S}_t)\mathrm{d}t + K_{2,t}\left(\mathrm{d}z_t - \frac{CS_t + C\widehat{S}_t}{2}\mathrm{d}t\right), \tag{4.1.7}$$

where $\widehat{S}_t = \mathsf{E}[S_t \mid Z_{[0,t]}]$, and $K_{1,t}$, $K_{2,t}$ are certain gain matrices. The authors in [TM20] also show that this evolution rule minimizes certain cost associated with the transportation of probability measures. The evolution equations for each individual particle are then obtained from (4.1.7) by replacing $\widehat{S}_t$ with the empirical mean.

The above discussion shows one particular instance of ensemble filters with continuous observations. There are other McKean–Vlasov type equations which serve as exact filters. The basic problem studied in this chapter is to design ensemble filters for the continuous-time system (4.1.1) with the discrete observation process (4.1.2), and this is done in following steps:

- Find a process s_t such that $\mathsf{E}(s_t \mid \mathcal{Y}_t) \sim \mathsf{E}(x_t \mid \mathcal{Y}_t)$.
- Describe an ensemble of particles s_t^i, $i = 1, \ldots, \mathrm{m}$ coupled to each other via empirical mean and empirical variance, such that, each s_t^i represents an independent copy of s_t when $\mathrm{m} \to \infty$.
- Show that the empirical mean of the particles is consistent with the optimal solution to the filtering problem.

4.2 Mean-field type model

To derive an exact filter for the continuous-discrete problem in a general form, let us examine a suitable process s. Its parameters $\Delta_t, \Gamma_t, \Lambda_t, \Xi_t, \Theta_t, \Omega_t$ are be càdlàg $\mathcal{F}$-measurable processes, and the system is defined as follows

$$\mathrm{d}s_t := As_t\mathrm{d}t + \Delta_t(s_t - \widehat{s}_t)\mathrm{d}t + \Gamma_t\mathrm{d}\bar{w}_t + [\Lambda_t y_t + \Xi_t s_t + \Theta_t\widehat{s}_t + \Omega_t\bar{v}_t]\,\mathrm{d}N_t \tag{4.2.1a}$$

$$\widehat{s}_t := \mathsf{E}[s_t \mid \mathcal{Y}_t], \tag{4.2.1b}$$

$$Q_t := \mathsf{E}[(s_t - \widehat{s}_t)(s_t - \widehat{s}_t)^\top \mid \mathcal{Y}_t], \tag{4.2.1c}$$

[1] Assuming $\mathsf{E}[\mathrm{d}\eta_t\,\mathrm{d}\eta_t^\top] = V_t\mathrm{d}t$, for positive definite $V_t \in \mathbb{R}^{p\times p}$.

where $\bar{w}_t \sim \mathcal{N}(0, I), \bar{v}_t \sim \mathcal{N}(0, V_t)$ are independent copies of w_t and v_t. One can find a definition of solution of (4.2.1) and conditions for its existence in Appendix C.

Namely, we are interested in the case that $\widehat{s} = \widehat{x}$, and since $\widehat{x}_t$ is given in (4.1.3) we need to obtain the following:

$$\mathrm{d}\widehat{s}_t = A\widehat{s}_t\mathrm{d}t + Q_tC^\top(CQ_tC^\top + V_t)^{-1}(y_t - C\widehat{s}_t)\mathrm{d}N_t, \tag{4.2.2a}$$

$$\mathrm{d}Q_t = (AQ_t + Q_tA^\top + GG^\top)\mathrm{d}t - Q_tC^\top(CQ_tC^\top + V_t)^{-1}CQ_t\mathrm{d}N_t \tag{4.2.2b}$$

Let us derive equations for $\widehat{s}$ and Q from (4.2.1) and match them to equations (4.2.2). Straightforwardly, we have

$$\mathrm{d}\widehat{s}_t = A\widehat{s}_t\mathrm{d}t + [\Lambda_t y_t + (\Xi_t + \Theta_t)\widehat{s}_t]\,\mathrm{d}N_t, \tag{4.2.3}$$

Then, for the process $e_t := s_t - \widehat{s}_t$ we get

$$\mathrm{d}e_t = (A + \Delta_t)e_t\mathrm{d}t + \Gamma_t\mathrm{d}\bar{w}_t + [\Xi_t e_t + \Omega_t\bar{v}_t]\,\mathrm{d}N_t,$$

and

$$\begin{aligned}\mathrm{d}[e_te_t^\top]_{ij} = \mathrm{d}[e_t]_i[e_t]_j &= [e_t]_i[(A + \Delta_t)e_t\mathrm{d}t + \Gamma_t\mathrm{d}w_t]_j \\ &+ [e_t]_j[(A + \Delta_t)e_t\mathrm{d}t + \Gamma_t\mathrm{d}w_t]_i + \frac{1}{2}\sum_k[\Gamma_t]_{ik}[\Gamma_t]_{jk}\mathrm{d}t \\ &+ \Big[\Big[e_t + \Xi_te_t + \Omega_t\bar{v}_t\Big]_i\Big[e_t + \Xi_te_t + \Omega_t\bar{v}_t\Big]_j - [e_t]_i[e_t]_j\Big]\mathrm{d}N_t\end{aligned}$$

so

$$\begin{aligned}&\mathrm{d}\mathsf{E}[e_te_t^\top \mid \mathcal{Y}_t] = \mathrm{d}Q_t = (AQ_t + Q_tA^\top + \Delta_tQ_t + Q_t\Delta_t^\top + \Gamma_t\Gamma_t^\top)\mathrm{d}t \\ &+ \mathsf{E}\Big[\Big(e_t + \Xi_te_t + \Omega_t\bar{v}_t\Big)\Big(e_t + \Xi_te_t + \Omega_t\bar{v}_t\Big)^\top - e_te_t^\top \mid \mathcal{Y}_t\Big]\mathrm{d}N_t \\ &= (AQ_t + Q_tA^\top + \Delta_tQ_t + Q_t\Delta_t^\top + \Gamma_t\Gamma_t^\top)\mathrm{d}t \\ &+ \Big[(I + \Xi_t)Q_t(I + \Xi_t)^\top - Q_t + \Omega_tV_t\Omega_t^\top\Big]\mathrm{d}N_t\end{aligned}$$

From (4.2.3) and (4.2.2a) we get the following necessary condition

$$Q_tC^\top(CQ_tC^\top + V_t)^{-1}(y_t - C\widehat{s}_t)\mathrm{d}N_t = (\Lambda_ty_t + (\Xi_t + \Theta_t)\widehat{s}_t)\mathrm{d}N_t. \tag{4.2.4a}$$

To obtain (4.2.2) it is necessary to have

$$\begin{aligned}&\Big(AQ_t + Q_tA^\top + GG^\top\Big)\,\mathrm{d}t \\ &= \Big(AQ_t + Q_tA^\top + \Delta_tQ_t + Q_t\Delta_t^\top + \Gamma_t\Gamma_t^\top\Big)\,\mathrm{d}t, \end{aligned}\tag{4.2.4b}$$

$$\begin{aligned}&\Big(-Q_tC^\top(CQ_tC^\top + V_t)^{-1}CQ_t\Big)\,\mathrm{d}N_t \\ &= \Big((I + \Xi_t)Q_t(I + \Xi_t)^\top - Q_t + \Omega_tV_t\Omega_t^\top\Big)\,\mathrm{d}N_t, \end{aligned}\tag{4.2.4c}$$

Moreover, conditions (4.2.4) are also sufficient. Summing up, we obtain the following results.

Proposition 4.2. *Let $\Delta_t, \Gamma_t, \Lambda_t, \Xi_t, \Theta_t, \Omega_t$ be càdlàg $\mathcal{Y}_t$-measurable processes. The system* (4.2.1) *satisfies* (4.2.2) *if and only if for all $t > 0$*

$$GG^\top = \Delta_t Q_t + Q_t \Delta_t^\top + \Gamma_t \Gamma_t^\top \tag{4.2.5a}$$

and for jump times τ_k

$$\begin{aligned} \Lambda_{\tau_k} &= Q_{\tau_k} C^\top (C Q_{\tau_k} C^\top + V_t)^{-1}, && (4.2.5b)\\ \Theta_{\tau_k} + \Xi_{\tau_k} &= -Q_{\tau_k} C^\top (C Q_{\tau_k} C^\top + V_{\tau_k})^{-1} C, && (4.2.5c)\\ -Q_{\tau_k} C^\top (C Q_{\tau_k} C^\top + V_t)^{-1} C Q_{\tau_k} &= \\ &(I + \Xi_{\tau_k}) Q_{\tau_k} (I + \Xi_{\tau_k})^\top - Q_{\tau_k} + \Omega_{\tau_k} V_{\tau_k} \Omega_{\tau_k}^\top, && (4.2.5d) \end{aligned}$$

To simplify equations below, the gain matrix is denoted by L, i.e.

$$L_t = Q_t C^\top (C Q_t C^\top + V_t)^{-1}.$$

There are different ways to choose the parameters in equation (4.2.1) such that the conditions in Proposition 4.2 are met. Below we provide two specific choices that are inspired by the existing techniques in the literature on ensemble filters.

1. 'Transport-inspired' case assumes $\Omega_t = 0$ and $\Gamma_t = 0$ so one can take $\Delta_t = \frac{1}{2} G G^\top Q_t^{-1}$ and obtain

$$\begin{aligned} \mathrm{d}s_t := {} & A s_t \mathrm{d}t + \frac{1}{2} G G^\top Q_t^{-1} (s_t - \widehat{s}_t) \mathrm{d}t \\ & + \Big(L_t y_t - L_t C \widehat{s}_t + \Xi_t (s_t - \widehat{s}_t) \Big) \mathrm{d}N_t, \end{aligned} \tag{4.2.6}$$

$$\Xi_{\tau_k} : \quad Q_{\tau_k} \Xi_{\tau_k}^\top + \Xi_{\tau_k} Q_{\tau_k} \Xi_{\tau_k}^\top + \Xi_{\tau_k} Q_{\tau_k} = -L_{\tau_k} C Q_{\tau_k}. \tag{4.2.7}$$

2. 'Vanilla' case assumes $\Omega_t \neq 0$, $\Delta_t = 0$, and $\Theta_t = 0$ so $\Gamma_t = G$, $\Xi_t = -L_t C$ and one can take $\Omega_t = -L_t$

$$\mathrm{d}s_t := A s_t \mathrm{d}t + G \mathrm{d}w_t + L_t \Big(y_t - C s_t - v_t \Big) \mathrm{d}N_t, \tag{4.2.8}$$

4.3 Ensemble filters

For particles s^i, $i = 1, \dots, \mathrm{m}$, let the empirical mean and the empirical covariance be

$$\widehat{s}_t^{\mathrm{m}} = \frac{1}{\mathrm{m}} \sum_{i=1}^{\mathrm{m}} s_t^i, \tag{4.3.1a}$$

$$Q_t^{\mathrm{m}} = \frac{1}{\mathrm{m}} \sum_{i=1}^{\mathrm{m}} (s_t^i - \widehat{s}_t^{\mathrm{m}})(s_t^i - \widehat{s}_t^{\mathrm{m}})^\top. \tag{4.3.1b}$$

The gain matrix becomes $L_t^{\mathrm{m}} = Q_t^{\mathrm{m}} C^\top (C Q_t^{\mathrm{m}} C^\top + V_t)^{-1}$. The corresponding ensemble filters have the following forms:

1. 'Transport-inspired' ensemble filter:

$$\begin{aligned} \mathrm{d}s_t^i &:= A s_t^i \mathrm{d}t + \frac{1}{2} G G^\top (Q_t^{\mathrm{m}})^{-1} (s_t^i - \widehat{s}_t^{\mathrm{m}}) \mathrm{d}t \\ &\quad + \Big(L_t^{\mathrm{m}} y_t - L_t^{\mathrm{m}} C \widehat{s}_t + \Xi_t^{\mathrm{m}} (s_t^i - \widehat{s}_t^{\mathrm{m}}) \Big) \mathrm{d}N_t, \end{aligned} \tag{4.3.2}$$

$$\Xi_{\tau_k}^{\mathrm{m}} : \quad Q_{\tau_k}^{\mathrm{m}} (\Xi_{\tau_k}^{\mathrm{m}})^\top + \Xi_{\tau_k}^{\mathrm{m}} Q_{\tau_k}^{\mathrm{m}} (\Xi_{\tau_k}^{\mathrm{m}})^\top + \Xi_{\tau_k}^{\mathrm{m}} Q_{\tau_k}^{\mathrm{m}} = -L_{\tau_k}^{\mathrm{m}} C Q_{\tau_k}^{\mathrm{m}}. \tag{4.3.3}$$

2. 'Vanilla' ensemble filter:

$$\mathrm{d}s_t^i := A s_t^i \mathrm{d}t + G \mathrm{d}w_t^i + L_t^{\mathrm{m}} \Big(y_t - C s_t^i - v_t^i \Big) \mathrm{d}N_t, \tag{4.3.4}$$

The noises $w_t^i \sim \mathcal{N}(0, I), v_t^i \sim \mathcal{N}(0, V_t)$ are independent copies of w_t and v_t respectively, while the Poisson process N_t is common for all the equations.

Interaction of particles

Both proposed ensemble filters use the empirical covariance, that couples the dynamics of particles. In addition, the transport-inspired case involves the empirical mean.

In contrast to it, the vanilla ensemble filter has uncoupled continuous-time part of the dynamics: between observations, each particle s^i follows

$$\mathrm{d}s^i = A s_t^i \mathrm{d}t + G \mathrm{d}w_t^i, \qquad t \in (\tau_k, \tau_{k+1}).$$

Notice that this equation does not contain any interaction between the particles, while the independent noises are added and assures diversity of the dynamics.

Minimum number of particles

Despite the assumption that the number of particles is large enough to approximate the mean-field evolution accurately, it becomes essential to ascertain the minimum number of particles required.

1. For transport-inspired ensemble filters, the minimum number of particles equals n, the dimension of the state process. This necessity arises due to the requirement to compute the inverse of the matrix Q_t^{m} at each time t. The existence of this inverse is guaranteed only if the columns of the matrix Q_t^{m} are linearly independent. Therefore, to ensure this condition the initial choice of at least n linearly independent particles is imperative.

2. For the noise-driven ensemble filters, the minimum number of particles equals 1. The proposed ensemble of particles is such that a single particle evolution can be computed. In some cases, it is common to take the coefficient of the empirical covariance equal to $\frac{1}{m-1}$. Then it is critical to ensure that $m-1 \neq 0$, and at least two particles are necessary.

In the sequel, we will look more closely at the aforementioned two classes of ensemble filters. In particular, we analyze the consistency and approximation with respect to the optimal filter.

4.4 Transport-inspired ensemble filter

We recall that our proposed transport-inspired ensemble filter is described by the following equations:

$$
\begin{aligned}
\mathrm{d}s_t^i &:= A s_t^i \mathrm{d}t + \frac{1}{2} G G^\top (Q_t^{\mathrm{m}})^{-1} (s_t^i - \widehat{s}_t^{\mathrm{m}}) \mathrm{d}t \\
&\quad + \Big(L_t^{\mathrm{m}} y_t - L_t^{\mathrm{m}} C \widehat{s}_t + \Xi_t^{\mathrm{m}} (s_t^i - \widehat{s}_t^{\mathrm{m}}) \Big) \mathrm{d}N_t, \\
\Xi_{\tau_k}^{\mathrm{m}} : &\quad Q_{\tau_k}^{\mathrm{m}} (\Xi_{\tau_k}^{\mathrm{m}})^\top + \Xi_{\tau_k}^{\mathrm{m}} Q_{\tau_k}^{\mathrm{m}} (\Xi_{\tau_k}^{\mathrm{m}})^\top + \Xi_{\tau_k}^{\mathrm{m}} Q_{\tau_k}^{\mathrm{m}} = -L_{\tau_k}^{\mathrm{m}} C Q_{\tau_k}^{\mathrm{m}}.
\end{aligned}
$$

In this section, we show that these filters are indeed consistent with the optimal filters and provide exact reconstruction of the optimal estimate for a large enough number of particles.

The presence of the algebraic equation (4.3.3) requires to study condition on its solvability.

4.4.1 Solvability of the design parameters

The transport-inspired case require computation of the injection gains L_t^{m} and Ξ_t^{m} (in the case of the ensemble filter) or L_t^{m} and Ξ_t^{m} respectively for the mean-field type form. Without loss of generality one can consider the mean-field type case, since these matrices can be viewed as functions of the covariance.

The matrix $L_{\tau_k} = Q_{\tau_k} C^\top (C Q_{\tau_k} C^\top + V)^{-1}$ can be computed directly, while the matrix Ξ_{τ_k} satisfies

$$Q_{\tau_k} \Xi_{\tau_k}^\top + \Xi_{\tau_k} Q_{\tau_k} \Xi_{\tau_k}^\top + \Xi_{\tau_k} Q_{\tau_k} = -L_{\tau_k} C Q_{\tau_k}. \tag{4.4.1}$$

Notice that in all cases the injection gains L_t and Ξ_t only apply at times of observation updates. In these equations, we note that Q_t gets updated due to reset in s_t and $\widehat{s}_t$ with the arrival of new observations. Thus, the key step for the simulation of the particles is to solve the algebraic equation (4.4.1) at renewal times of the Poisson process. Solvability of this equation can be guaranteed and carried out in following steps:

Step 1: We can find an invertible matrix E_t such that $E_t E_t^\top = Q_t$. This is always possible because, for the number of particles m large enough, Q_t is symmetric and positive definite. Using the eigenvalue decomposition, we can choose E_t to be the matrix of orthonormal eigenvectors multiplied by a diagonal matrix with square root of the eigenvalues.

Step 2: We then observe that $(Q_t - L_t C Q_t)$ is positive semi-definite. This follows from the fact that, we can write:

$$(Q_t - L_t C Q_t) = (I - L_t C) Q_t (I - L_t C)^\top + L_t V_t L_t^\top$$

Step 3: Next, we compute the matrix F_t such that $F_t F_t^\top = (Q_t - L_t C Q_t)$. Due to the fact that $(Q_t - L_t C Q_t)$ is symmetric positive semi-definite, such a matrix F_t always exists.

Thus, using eigenvalue decomposition of the matrices Q_t and $(Q_t - L_t C Q_t)$, we have an analytic solution. That is, for each renewal time $t \geq 0$, we let $X_t = \Xi_t - I$, then the quadratic equation becomes

$$X_t E_t E_t^\top X_t^\top = F_t F_t^\top$$

for which the solution is $X_t = F_t E_t^{-1}$, that is, $\Xi_t = I + X_t$.

4.4.2 Filter consistency

We now show that the ensemble filter (4.3.2) with the injection matrix Ξ satisfying (4.3.3) is indeed consistent with the optimal filter.

Proposition 4.3. *Let* $\mathrm{m} \geq \mathrm{n}$. *The evolution of the sample mean* $\widehat{s}_t^{\mathrm{m}}$ *and the sample covariance* Q_t^{m} *satisfies the following equations*

$$\mathrm{d}\widehat{s}_t^{\mathrm{m}} = A\widehat{s}_t^{\mathrm{m}} + L_t^{\mathrm{m}}(y_t - C\widehat{s}_t^{\mathrm{m}})\mathrm{d}N_t, \tag{4.4.2a}$$

$$\mathrm{d}Q_t^{\mathrm{m}} = \left(AQ_t^{\mathrm{m}} + Q_t^{\mathrm{m}}A^\top + GG^\top\right)\mathrm{d}t - L_t^{\mathrm{m}}CQ_t^{\mathrm{m}}\mathrm{d}N_t. \tag{4.4.2b}$$

where $L_t^{\mathrm{m}} = Q_t^{\mathrm{m}}C^\top(CQ_t^{\mathrm{m}}C^\top + V)^{-1}$.

Proof. The differential equation for $\widehat{s}_t^{\mathrm{m}}$ can be obtained by taking the time derivative of the equation $\widehat{s}_t^{\mathrm{m}} = \frac{1}{\mathrm{m}}\sum\limits_{i=1}^{\mathrm{m}} s_t^i$. Next, for an arbitrary i let us consider the error variable $e_t^i := s_t^i - \widehat{s}_t^{\mathrm{m}}$, and observe that

$$\mathrm{d}e_t^i := \mathrm{d}(s_t^i - \widehat{s}_t^{\mathrm{m}}) = Ae_t^i\mathrm{d}t + \frac{1}{2}GG^\top(Q_t^{\mathrm{m}})^{-1}e_t^i\mathrm{d}t - \Xi_t^{\mathrm{m}}e_t^i\mathrm{d}N_t.$$

Using the chain rule from Proposition 2.14, we get

$$\begin{aligned}\mathrm{d}e_t^i(e_t^i)^\top = \Big(&Ae_t^i(e_t^i)^\top + \frac{1}{2}GG^\top(Q_t^{\mathrm{m}})^{-1}e_t^i(e_t^i)^\top \\ &+ e_t^i(e_t^i)^\top A^\top + \frac{1}{2}e_t^i(e_t^i)^\top(GG^\top(Q_t^{\mathrm{m}})^{-1})^\top\Big)\mathrm{d}t \\ &+ \left((e_t^i - \Xi_t^{\mathrm{m}}e_t^i)(e_t^i - \Xi_t^{\mathrm{m}}e_t^i)^\top - e_t^i(e_t^i)^\top\right)\mathrm{d}N_t.\end{aligned}$$

Using $Q_t^{\mathrm{m}} = \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}}(s_t^i - \widehat{s}_t^{\mathrm{m}})(s_t^i - \widehat{s}_t^{\mathrm{m}})^\top = \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} e_t^i(e_t^i)^\top$, we obtain that

$$\begin{aligned}\mathrm{d}Q_t^{\mathrm{m}} = &\left(AQ_t^{\mathrm{m}} + Q_t^{\mathrm{m}}A^\top + GG^\top\right)\mathrm{d}t \\ &+ \left(\Xi_t^{\mathrm{m}}Q_t^{\mathrm{m}}(\Xi_t^{\mathrm{m}})^\top - \Xi_t^{\mathrm{m}}Q_t^{\mathrm{m}} - Q_t^{\mathrm{m}}(\Xi_t^{\mathrm{m}})^\top\right)\mathrm{d}N_t.\end{aligned}$$

Then we notice that assumption on Ξ_t^{m} is equivalent to

$$\Xi_t^{\mathrm{m}}Q_t^{\mathrm{m}}(\Xi_t^{\mathrm{m}})^\top - \Xi_t^{\mathrm{m}}Q_t^{\mathrm{m}} - Q_t^{\mathrm{m}}(\Xi_t^{\mathrm{m}})^\top = L_t^{\mathrm{m}}CQ_t^{\mathrm{m}}$$

which leads to (4.4.2b). $\square$

This result is quite elegant as the moments dynamics coincide with the moments dynamics for the optimal filter (4.1.3)–(4.1.4). This allows us to arrive at the following result which summarizes our findings for the transport-inspired filter.

Theorem 4.4. *Let* $\mathrm{m} \geq \mathrm{n}$*,* P_0 *is positive definite, and* $Q_0^{\mathrm{m}} = P_0$*. For* m-*particle system* (4.3.2) *with the injection matrix* Ξ *satisfying* (4.3.3)*, if an initial distribution of particles* $\{s_0^i\}_{i=1,\dots,\mathrm{m}}$ *is such that*

$$\widehat{s}_0^{\mathrm{m}} = \widehat{x}_0 \quad \textit{and} \quad P_0^{\mathrm{m}} = P_0,$$

then a normal distribution $\mathcal{N}(\widehat{s}_t^{\mathrm{m}}, P_t^{\mathrm{m}})$ *equals the distribution of the optimal filter* $\widehat{x}_t$ *for all* $t \geq 0$*.*

The results of Proposition 4.3 and Theorem 4.4 therefore show that the differential equations for empirical mean $\widehat{s}_t^{\mathrm{m}}$ and covariance Q_t^{m} are the same as those given in Theorem 2.22. To build on this result and show that s_t^{m} and Q_t^{m} are indeed consistent with $\widehat{x}_t$ and P_t, we also need consistency of initial conditions. Assuming that $s_0 \sim x_0$, we see that a realization of s_0^{m} approximates $\mathsf{E}[x_0]$ if the number of particles m is large enough. In the same way, for large enough m, a realization of Q_0 approximates P_0. Formal analysis of this approximation and studying the effect of sampling rate on the asymptotic behavior of the resulting error is a topic of further research.

4.5 Vanilla ensemble filter

In this section, we analyze the second class of ensemble filters, that is, the vanilla filters. We recall that these filters are described by (4.3.4). We will show that these filters actually approximate the optimal estimator as the number of particles gets large. We begin with derivation of the moments dynamics for the vanilla ensemble filter.

4.5.1 Dynamics of the moments

For an l-th particle s^l we have

$$\mathrm{d}s_t^l := As_t^l \mathrm{d}t + G\mathrm{d}\bar{w}_t^l + L_t^{\mathrm{m}}\Big(y_t - Cs_t^l - v_t^l\Big)\mathrm{d}N_t$$

with empirical moments defined by (4.3.1a)–(4.3.1b).

In this case, we straightforwardly obtain that

$$\mathrm{d}\widehat{s}_t^{\mathrm{m}} = A\widehat{s}_t^{\mathrm{m}}\mathrm{d}t + \frac{1}{\sqrt{\mathrm{m}}}G\mathrm{d}\widetilde{w}_t^{\mathrm{m}} + L_t^{\mathrm{m}}\Big(y_t - C\widehat{s}_t^{\mathrm{m}} - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{v}_t^{\mathrm{m}}\Big)\mathrm{d}N_t \tag{4.5.1}$$

where $\widetilde{w}_t^{\mathrm{m}} = \frac{1}{\sqrt{\mathrm{m}}}\sum_{i=1}^{\mathrm{m}} \bar{w}_t^i \sim \mathcal{N}(0, I)$ and $\widetilde{v}_t^{\mathrm{m}} = \frac{1}{\sqrt{\mathrm{m}}}\sum_{i=1}^{\mathrm{m}} \bar{v}_t^i \sim \mathcal{N}(0, V_t)$.

For $l = 1, \ldots, \mathrm{m}$, for the auxiliary process $q_t^l = s_t^l - \widehat{s}_t^{\mathrm{m}}$ we get

$$\mathrm{d}q_t^l = Aq_t^l \mathrm{d}t + G\mathrm{d}(\bar{w}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{w}_t^{\mathrm{m}}) + L_t^{\mathrm{m}}\left[Cq_t^l - \bar{v}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{v}_t^{\mathrm{m}}\right]\mathrm{d}N_t,$$

and the chain rule provides

$$\begin{aligned}\mathrm{d}q_t^l(q_t^l)^\top = &\left(Aq_t^l(q_t^l)^\top + q_t^l(q_t^l)^\top A^\top + (1-\frac{1}{\mathrm{m}})G(G)^\top\right)\mathrm{d}t \\ &+ G\mathrm{d}(\bar{w}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{w}_t^{\mathrm{m}})(q_t^l)^\top + q_t^l\mathrm{d}(\bar{w}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{w}_t^{\mathrm{m}})^\top G \\ &+ \left[\left(q_t^l + L_t^{\mathrm{m}}Cq_t^l - L_t^{\mathrm{m}}(\bar{v}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{v}_t^{\mathrm{m}})\right)\left(q_t^l + L_t^{\mathrm{m}}Cq_t^l - L_t^{\mathrm{m}}(\bar{v}_t^l - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{v}_t^{\mathrm{m}})\right)^\top\right. \\ &\left. - q_t^l(q_t^l)^\top\right]\mathrm{d}N_t,\end{aligned}$$

and, since $\sum_l q_t^l = 0$, the terms like $\sum_l \widetilde{w}_t^{\mathrm{m}}(q_t^l)^\top$ eliminates, so it implies

$$\begin{aligned}&\mathrm{d}Q_t^{\mathrm{m}} = \left(AQ_t^{\mathrm{m}} + Q_t^{\mathrm{m}}A^\top + (1-\frac{1}{\mathrm{m}})GG^\top\right)\mathrm{d}t \\ &+\frac{1}{\mathrm{m}}G\sum_l \mathrm{d}\bar{w}_t^l(q_t^l)^\top + \frac{1}{\mathrm{m}}\left(\sum_l q_t^l\mathrm{d}(\bar{w}_t^l)^\top\right)G \\ &+\Big[(I + L_t^{\mathrm{m}}C)Q_t^{\mathrm{m}}(I + L_t^{\mathrm{m}}C)^\top - Q_t^{\mathrm{m}} \\ &\quad + L_t^{\mathrm{m}}\Big(\frac{1}{\mathrm{m}}\sum_l \bar{v}_t^l(\bar{v}_t^l)^\top - \frac{1}{\mathrm{m}}\sum_l(I + \Xi_t^{\mathrm{m}})q_t^l(\bar{v}_t^l)^\top - \frac{1}{\mathrm{m}}\sum_l \bar{v}_t^l(q_t^l)^\top(I + \Xi_t^{\mathrm{m}})^\top \\ &-\frac{1}{\mathrm{m}\sqrt{\mathrm{m}}}\sum_l \widetilde{v}_t^{\mathrm{m}}(\bar{v}_t^l)^\top - \frac{1}{\mathrm{m}\sqrt{\mathrm{m}}}\sum_l \bar{v}_t^l(\widetilde{v}_t^{\mathrm{m}})^\top + \frac{1}{\mathrm{m}}\widetilde{v}_t^{\mathrm{m}}(\widetilde{v}_t^{\mathrm{m}})^\top\Big)(L_t^{\mathrm{m}})^\top\Big]\mathrm{d}N_t.\end{aligned}$$

4.5.2 Scalar case

Let us consider the scalar case to provide precise estimations. In particular, the one-dimensional equations for empirical moments of the vanilla filters simplify as follows. The empirical mean $\widehat{s}^{\mathrm{m}}$ satisfies

$$\mathrm{d}\widehat{s}_t^{\mathrm{m}} = A\widehat{s}_t^{\mathrm{m}} + \frac{1}{\sqrt{\mathrm{m}}}G\mathrm{d}\widetilde{w}_t^{\mathrm{m}} + L_t^{\mathrm{m}}\left(y_t - C\widehat{s}_t^{\mathrm{m}} - \frac{1}{\sqrt{\mathrm{m}}}\widetilde{v}_t^{\mathrm{m}}\right)\mathrm{d}N_t \tag{4.5.2}$$

and the empirical covariance satisfies

$$
\begin{aligned}
\mathrm{d}Q_t^{\mathrm{m}} = &\left(2AQ_t^{\mathrm{m}} + G^2 - \frac{1}{\mathrm{m}}G^2\right)\mathrm{d}t + 2G\sum_{\ell=1}^{\mathrm{m}} q_t^\ell \left(\mathrm{d}w_t^\ell - \frac{1}{\sqrt{\mathrm{m}}}\mathrm{d}\widetilde{w}_t^{\mathrm{m}}\right) \\
&+ \Big[- L_t^{\mathrm{m}} C Q_t^{\mathrm{m}} - (L_t^{\mathrm{m}})^2 V + \frac{1}{\mathrm{m}}(L_t^{\mathrm{m}})^2 \sum_{\ell=1}^{\mathrm{m}} (v_t^\ell)^2 + \frac{1}{\mathrm{m}}(L_t^{\mathrm{m}})^2 (\widetilde{v}_t^{\mathrm{m}})^2 \\
&- \frac{2}{\mathrm{m}\sqrt{\mathrm{m}}}(L_t^{\mathrm{m}})^2 \sum_{\ell=1}^{\mathrm{m}} v_t^\ell \widetilde{v}_t^{\mathrm{m}} \\
&- \frac{2}{\mathrm{m}}(1 - L_t^{\mathrm{m}} C) L_t^{\mathrm{m}} \sum_{\ell=1}^{\mathrm{m}} q_t^\ell (v_t^\ell - \frac{1}{\sqrt{\mathrm{m}}} L_t^{\mathrm{m}} \widetilde{v}_t^{\mathrm{m}}) \Big] \mathrm{d}N_t.
\end{aligned}
\tag{4.5.3}
$$

Let us compute the expectation $\mathcal{Q}_t^{\mathrm{m}} := \mathsf{E}[Q_t^{\mathrm{m}}]$ with respect to the noise terms and, in addition, the sampling times of the Poisson counter. Using the fact that v_t^ℓ and $\widetilde{v}_t^{\mathrm{m}}$ are mean zero processes, the Proposition 2.20 yields, for the process Q_t^{m} defined in (4.5.3), that

$$
\dot{\mathcal{Q}}_t^{\mathrm{m}} = 2A\mathcal{Q}_t^{\mathrm{m}} + G^2 - \frac{1}{\mathrm{m}}G^2 - \lambda\frac{(\mathcal{Q}_t^{\mathrm{m}})^2 C^2}{\mathcal{Q}_t^{\mathrm{m}} C^2 + V} + \frac{\lambda}{\mathrm{m}}\frac{(\mathcal{Q}_t^{\mathrm{m}})^2 C^2 V}{(\mathcal{Q}_t^{\mathrm{m}} C^2 + V)^2}. \tag{4.5.4}
$$

4.5.3 Convergence of covariances

Consider the scalar version of optimal estimator equations (4.1.3)–(4.1.4) and the scalar vanilla m-particle system (4.5.2)–(4.5.3). Let us state describe the difference in the expected covariances depending on the number of particle, and then prove the stated results consequently.

Theorem 4.5. *Let Poisson intensity $\lambda > 2A$ and number of particles $m \geq 1$. If a number $\Upsilon > 0$ is such that $\mathcal{P}_t, \mathcal{Q}_t^{\mathrm{m}} \geq \Upsilon$ for each $t \geq 0$, then it holds that*

$$
\begin{aligned}
\mathsf{E}|P_t - Q_t^{\mathrm{m}}| \leq & E_0 \exp\{A_{\lambda,\Upsilon} t\} + \frac{1}{\mathrm{m}}\left(G^2 + \lambda V C^{-2}\right)|A_{\lambda,\Upsilon}|^{-1}, \\
\textit{for} \quad A_{\lambda,\Upsilon} = & 2A - \lambda + \lambda\frac{V^2}{(\Upsilon C^2 + V)^2},
\end{aligned}
$$

where $E_0 := |P_0 - Q_0| + \frac{1}{\mathrm{m}}(G^2 + \lambda V C^{-2})|A_{\lambda,\Upsilon}|^{-1}$ is a constant.

Remark 4.6. *If $\lambda > 2A$, $\mathrm{m} \geq 1$ and $\Upsilon := \frac{1}{2}G^2/(\lambda - 2A)$, then $\mathcal{P}_t, \mathcal{Q}_t^{\mathrm{m}} \geq \Upsilon$ for each $t \geq 0$. Indeed, by Lemma B.2 $\mathcal{Q}_t^{\mathrm{m}} \geq (1 - \frac{1}{\mathrm{m}})G^2/(\lambda - 2A)$ and by lemma B.1, $\mathcal{P}_t$ has a uniform lower bound $\min\{\mathcal{P}_0, G^2/(\lambda - 2A)\}$.*

Remark 4.7. *Theorem 4.5 immediately implies that for each time $t > 0$ one has $\mathsf{E}|P_t - Q_t^{\mathrm{m}}| \to 0$ as $\mathrm{m} \to \infty$ and initial conditions are such that $\mathcal{Q}_0^{\mathrm{m}} \to \mathcal{P}_0$.*

Proposition 4.8. *Let* $\mathrm{m} \geq 1$. *If the filtering system parameters satisfy*[2]

$$2(\lambda - 2A)V\Big((1 - \tfrac{1}{\mathrm{m}})G^2C^2 - AV\Big) + (1 - \tfrac{1}{\mathrm{m}})^2G^4C^4 > 0,$$

and $\lambda > 2A + (1 - \frac{1}{\mathrm{m}})G^2/\min\{P_0, Q_0\}$, *then* $A_{\gamma,\Upsilon} = 2A - \lambda + \lambda\frac{V^2}{(\Upsilon C^2+V)^2} < 0$ *for* $\Upsilon = (1 - \frac{1}{\mathrm{m}})G^2/(\lambda - 2A)$. *In particular*

$$\limsup_{t\to\infty} |\mathsf{E}[P_t] - \mathsf{E}[Q^{\mathrm{m}}_t]| \leq |P_0 - Q_0| + \tfrac{2}{\mathrm{m}}\left(G^2 + \lambda VC^{-2}\right)/|A_{\gamma,\Upsilon}|.$$

Proof of Theorem 4.5

Proof. Using the fact that v^ℓ_t and $\widetilde{v}^{\mathrm{m}}_t$ are mean zero processes, the application of Proposition 2.20 to the process Q^{m}_t in (4.5.3) yields that $\mathcal{Q}^{\mathrm{m}}_t := \mathsf{E}[Q^{\mathrm{m}}_t \mid Q_0]$ satisfies

$$\dot{\mathcal{Q}}^{\mathrm{m}}_t = 2A\mathcal{Q}^{\mathrm{m}}_t + G^2 - \frac{1}{\mathrm{m}}G^2 - \lambda\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2}{\mathcal{Q}^{\mathrm{m}}_tC^2 + V} + \frac{\lambda}{\mathrm{m}}\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2V}{(\mathcal{Q}^{\mathrm{m}}_tC^2 + V)^2}. \tag{4.5.5}$$

To analyze the difference between expected optimal and empirical covariance, let us introduce $\mathcal{E}^{\mathrm{m}}_t := \mathsf{E}[P_t - Q^{\mathrm{m}}_t] = \mathcal{P}_t - \mathcal{Q}^{\mathrm{m}}_t$, which satisfies the following equation:

$$\begin{aligned}\dot{\mathcal{E}}^{\mathrm{m}}_t &:= \mathrm{d}(\mathcal{P}_t - \mathcal{Q}^{\mathrm{m}}_t)/\mathrm{d}t\\ &= 2A\mathcal{E}^{\mathrm{m}}_t + \lambda\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2}{\mathcal{Q}^{\mathrm{m}}_tC^2 + V} - \lambda\frac{\mathcal{P}_t^2C^2}{\mathcal{P}_tC^2 + V} + \frac{1}{\mathrm{m}}G^2 - \frac{\lambda}{\mathrm{m}}\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2V}{(C^2\mathcal{Q}^{\mathrm{m}}_t + V)^2}.\end{aligned}$$

Let us note what bounds can be derived. For the last term, the following bounds hold

$$0 < \frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2V}{(C^2\mathcal{Q}^{\mathrm{m}}_t + V)^2} \leq VC^{-2}.$$

The nonlinear part transforms as follows

$$\begin{aligned}\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2}{\mathcal{Q}^{\mathrm{m}}_tC^2 + V} - \frac{\mathcal{P}_t^2C^2}{\mathcal{P}_tC^2 + V} &= \mathcal{Q}(1 - \frac{V}{\mathcal{Q}^{\mathrm{m}}_tC^2 + V}) - \mathcal{P}(1 - \frac{V}{\mathcal{P}_tC^2 + V})\\ &= -\mathcal{E}^{\mathrm{m}}_t + \frac{V^2\mathcal{E}^{\mathrm{m}}_t}{(\mathcal{Q}^{\mathrm{m}}_tC^2 + V)(\mathcal{P}_tC^2 + V)}.\end{aligned}$$

[2]The inequality holds if $AV \leq G^2C^2$, or the measurement noise covariance V is relatively small.

By assumption $\mathcal{P}_t, \mathcal{Q}_t^{\mathrm{m}} \geq \Upsilon$ for all $t \geq 0$. To get a bound on the evolution of $\mathcal{E}_t^{\mathrm{m}}$, let us study the derivative of $(\mathcal{E}_t^{\mathrm{m}})^2$, that is,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_t^{\mathrm{m}})^2 &= 2\mathcal{E}_t^{\mathrm{m}}\dot{\mathcal{E}}_t^{\mathrm{m}} \\
&= (4A - 2\lambda + 2\lambda\frac{V^2}{(\mathcal{Q}_t^{\mathrm{m}}C^2+V)(\mathcal{P}_tC^2+V)})(\mathcal{E}_t^{\mathrm{m}})^2 \\
&+ \left(\frac{2}{\mathrm{m}}G^2 - \frac{2\lambda}{\mathrm{m}}\frac{(\mathcal{Q}_t^{\mathrm{m}})^2C^2V}{(C^2\mathcal{Q}_t^{\mathrm{m}}+V)^2}\right)\mathcal{E}_t^{\mathrm{m}} \\
&\leq (4A - 2\lambda + 2\lambda\frac{V^2}{(\Upsilon C^2+V)^2})(\mathcal{E}_t^{\mathrm{m}})^2 + \frac{2}{\mathrm{m}}\left(G^2 + \lambda V C^{-2}\right)|\mathcal{E}_t^{\mathrm{m}}|
\end{aligned}$$

Consider the modified equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\bar{\mathcal{E}}_t^{\mathrm{m}})^2 = 2A_{\lambda,\Upsilon}(\bar{\mathcal{E}}_t^{\mathrm{m}})^2 + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)\bar{\mathcal{E}}_t^{\mathrm{m}}$$

with $\bar{\mathcal{E}}_0^{\mathrm{m}} = |\mathcal{E}_0^{\mathrm{m}}| > 0$, where $A_{\lambda,\Upsilon} = 2A - \lambda + \lambda\frac{V^2}{(\Upsilon C^2+V)^2}$. One gets that $(\mathcal{E}_t^{\mathrm{m}})^2 \leq (\bar{\mathcal{E}}_t^{\mathrm{m}})^2$ and so $|\mathcal{E}_t^{\mathrm{m}}| \leq |\bar{\mathcal{E}}_t^{\mathrm{m}}|$. The modified equation has the exact solution:

$$\bar{\mathcal{E}}_t^{\mathrm{m}} = E_0\exp\{\tfrac{1}{2}A_{\gamma,\Upsilon}t\} - \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)/A_{\gamma,\Upsilon},$$

where $E_0 = \mathcal{E}_0 + \frac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)/A_{\gamma,\Upsilon}$. Thus

$$\begin{aligned}
|\mathcal{E}_t^{\mathrm{m}}| &\leq \left|(\mathcal{E}_0^{\mathrm{m}} + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)/A_{\gamma,\Upsilon})\exp\{\tfrac{1}{2}A_{\gamma,\Upsilon}t\} - \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)/A_{\gamma,\Upsilon}\right| \\
&\leq (|\mathcal{E}_0^{\mathrm{m}}| + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)|A_{\gamma,\Upsilon}^{-1}|)\exp\{\tfrac{1}{2}A_{\gamma,\Upsilon}t\} + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)|A_{\gamma,\Upsilon}^{-1}|,
\end{aligned}$$

that completes the proof. □

Proof of Proposition 4.8

Proof. Recall that by Lemma B.2 one has $\mathcal{Q}_t^{\mathrm{m}} \geq (1-\frac{1}{\mathrm{m}})G^2/(\lambda-2A)$ and by Lemma B.1, $\mathcal{P}_t$ has a uniform lower bound $\min\{\mathcal{P}_0, G^2/(\lambda-2A)\}$. Hence taking $\Upsilon := (1-\frac{1}{\mathrm{m}})G^2/(\lambda-2A)$ the condition $\mathcal{P}_t, \mathcal{Q}_t^{\mathrm{m}} \geq \Upsilon$ for all $t \geq 0$ is met.

Since all the hypothesis of Theorem 4.5 hold true, one has

$$|\mathcal{E}_t| \leq (|\mathcal{E}_0| + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)|A_{\gamma,\Upsilon}^{-1}|)\exp\{\tfrac{1}{2}A_{\gamma,\Upsilon}t\} + \tfrac{2}{\mathrm{m}}\left(G^2+\lambda VC^{-2}\right)|A_{\gamma,\Upsilon}^{-1}|,$$

where $\mathcal{E}_t = \mathsf{E}(P_t - Q_t^{\mathrm{m}})$, $A_{\lambda,\Upsilon} = 2A - \lambda + \lambda\frac{V^2}{(\Upsilon C^2+V)^2}$.

Examine the condition $A_{\lambda,\Upsilon} < 0$, that is $\lambda - 2A > \lambda\frac{V^2}{(\Upsilon C^2+V)^2}$. For $\widetilde{\lambda} := \lambda - 2A$ and the assumed $\Upsilon := (1-\frac{1}{\mathrm{m}})G^2/\widetilde{\lambda}$ it becomes

$$\widetilde{\lambda} > \frac{(\widetilde{\lambda}+2A)V^2\widetilde{\lambda}^2}{((1-\frac{1}{\mathrm{m}})G^2C^2+V\widetilde{\lambda})^2}\ .$$

Using simple algebraic manipulations, the last inequality holds if and only if $\widetilde{\lambda} > 0$ and

$$2\widetilde{\lambda}V((1-\tfrac{1}{\mathrm{m}})G^2C^2 - AV) + (1-\tfrac{1}{\mathrm{m}})^2G^4C^4 > 0,$$

that is assumed. Thus it holds that

$$\begin{aligned}\limsup_{t\to\infty}|\mathcal{E}_t| &\leq |\mathcal{E}_0| + \frac{4}{\mathrm{m}}\left(G^2 + \lambda VC^{-2}\right)|A^{-1}_{\gamma,\Upsilon}| \\ &= |P_0 - Q_0| + \frac{2}{\mathrm{m}}\left(G^2 + \lambda VC^{-2}\right)/(\lambda - \lambda\frac{V^2}{(\Upsilon C^2+V)^2} - 2A) \\ &= |P_0 - Q_0| + \frac{2}{\mathrm{m}}\left(G^2 + \lambda VC^{-2}\right)/|A_{\gamma,\Upsilon}|.\end{aligned}$$

□

4.6 Numerical experiments

We will demonstrate the results of Section 4.4 and Section 4.5 through an academic example. Consider the linear stochastic system

$$\mathrm{d}x_t = Ax_t\mathrm{d}t + G\mathrm{d}w_t \tag{4.6.1a}$$

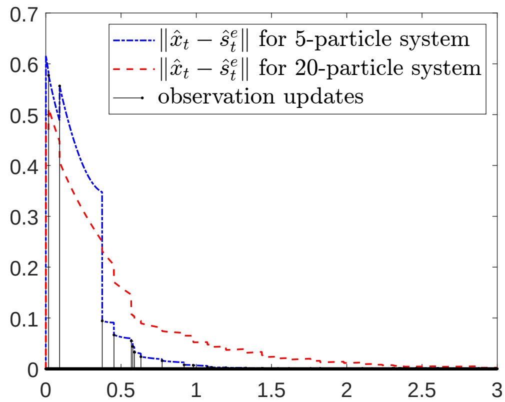
$$y_{\tau_k} = Cx_{\tau_k} + v_{\tau_k}, \tag{4.6.1b}$$

where $A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & -2 & 1 \\ -2 & 1 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}^\top$, and v_t is normally distributed with mean $(0,0)^\top$ and the constant variance $V = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$.

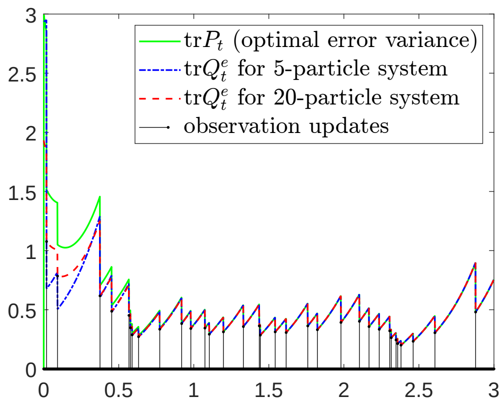
To measure the effectiveness of the proposed ensemble of particles, we compare them with the optimal estimator. We fix a reasonable Poisson intensity $\lambda = 10$. For a simulated and fixed path of the state x_t and the observation noise v_t, we compare the first and second moments.

The comparison can be conducted for a realization of a Poisson process, as presented in Figures 4.1a–4.1b for the case of transport-inspired ensemble filter. For the first moments ($\widehat{x}_t$ and $\widehat{s}^{\mathrm{m}}_t$), we consider the time plot of $\|\widehat{x}_t - \widehat{s}^{\mathrm{m}}_t\|$. While $\widehat{x}_t$ is defined by the differential equation (4.1.3), to compute $\widehat{s}^{\mathrm{m}}_t$ we simulate particles s^i_t and take their empirical mean. Similarly, for the second moments, to compute P_t, we need to solve the matrix differential equation and Q^{m}_t is computed as the empirical covariance of the simulated ensemble of particles.

To get a more general picture we simulate 200 sample paths of the Poisson process, and compare the averaged evolutions. E.g., for the time evolution of $\|\widehat{x}_t - \widehat{s}^{\mathrm{m}}_t\|$ we take its pointwise (in time) average, denoted as $\mathsf{E}\|\widehat{x}_t - \widehat{s}^{\mathrm{m}}_t\|$

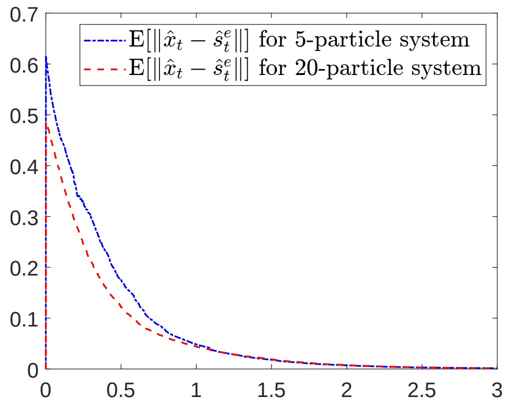


(a) The first moment error.

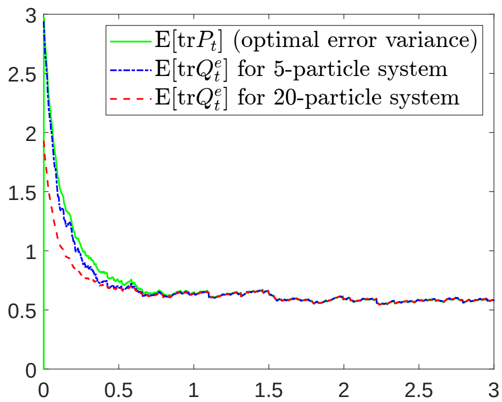


(b) The second moment error.

Figure 4.1: Plot of error moments along one sample path for the deterministic ensemble filter.



(a) The first moment error.

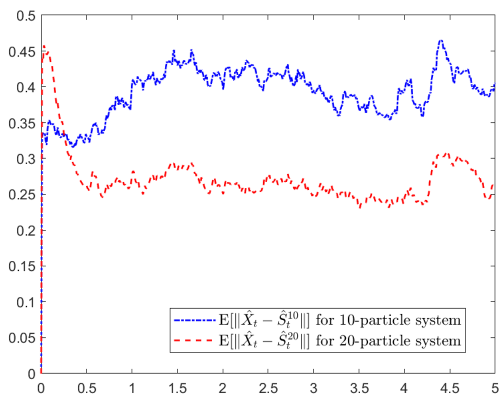


(b) The second moment error.

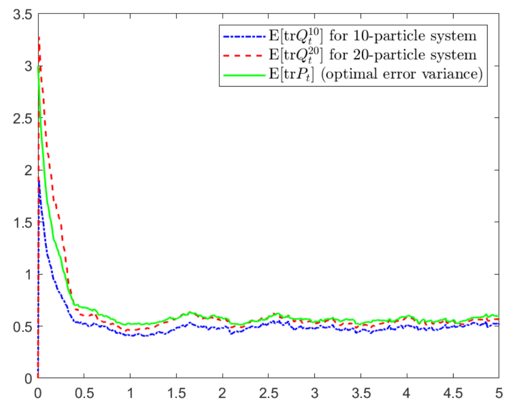
Figure 4.2: Plot of error moments smoothed with respect to the sample paths of Poisson process for the deterministic ensemble filter

and presented on Figure 4.2a. In the similar way, the smoothed evolution of the second moments is provided by Figure 4.2b.

In the case of vanilla ensemble filter, it is interesting to consider different number of particles, so the plots are provided only for the average behavior of the moments; i.e. 200 sample paths of the Poisson process are simulated and then their pointwise average is computed. In this way, Figure 4.3a presents the error between the optimal and empirical means, while Figure 4.3b describes the error between the covariances by presenting the trace of the corresponding matrices.



(a) The first moment error.



(b) The second moment error.

Figure 4.3: Plot of error moments smoothed with respect to the sample paths of Poisson process for the vanilla ensemble filter

Chapter 5

Convergence with Finite State Space

In this chapter, the state process is assumed to be a continuous-time Markov chain, which allows us to work with nonlinear processes. The observation process model differs from the other chapters: observations are defined as sampling from a continuous-time output process. This change allows us to prove the convergence to the classical optimal filter.

The idea of using discrete observations aims to reduce computation complexity at a low cost in its accuracy. It follows that interarrival times should not be too large to prevent significant inaccuracy and should not be too small to prevent overcomputation. In the case of random observations, one cannot obtain a precise upper bound on the interarrival times. This chapter studies the convergence of optimal filters as their expected value of interarrival times tends to zero.

5.1 Setup

For the purpose of analytical tractability, we will consider the filtering problem for systems with finitely many states. In particular, we consider the evolution of states described by a continuous-time Markov chain with an observation process which is a nonlinear function of the state process with additive Gaussian noise. We refer the reader to [MT09] for a standard exposition on Markov chains and related stability notions. The classical solution to the filter design problem for such systems is given by Wonham in [Won64], which is in the form of continuous-time stochastic differential equations evolving over a simplex.

State process

The dynamical systems considered in this chapter are described by a finite-state continuous-time Markov chain $(x_t)_{t\geq 0}$. The notation $\mathcal{S} := \{a_1, \ldots, a_d\}$ is used to denote the state space and the matrix $\Lambda := \{\lambda_{ij}\} \in \mathbb{R}^{d\times d}$, $i, j \in \{1, \ldots, d\}$, denotes the transition intensities. The transition rates are such that, $\lambda_{ij} \geq 0, \quad j \neq i$ and $\sum_{j\in\{1,\ldots,d\}} \lambda_{ij} = 0$, for each $i \in \{1, \ldots, d\}$. The later condition is true only if $\lambda_{ii} \leq 0$, for each $i \in \{1, \ldots, d\}$. For $t \geq 0$, let us denote the probability distribution of x_t by $p(t)$, so that, $p_j(t) = \mathsf{P}\left(x(t) = a_j\right)$. Similarly, the distribution conditioned upon initial condition is denoted by $p_{ij}(t) = \mathsf{P}\left(x(t) = a_j \mid x_0 = a_i\right)$, and we let $P(t) := [p_{ij}(t)] \in \mathbb{R}^{d\times d}$ denote the matrix of transition probabilities. For each $t \geq 0$, the matrix $P(t)$ is obtained from the transition matrix Λ by solving the following Chapman-Kolmogorov equation:

$$\dot{P}(t) = P(t)\Lambda, \quad P(0) = I_{d\times d}.$$

For a given initial distribution of x_0, given by $p_j(0)$, the distribution $p_j(t)$, for each $t \geq 0$, is

$$p(t) = P^\top(t)\, p(0) = p(0) + \int_0^t \left(\Lambda^\top p(\tau)\right) \mathrm{d}\tau, \tag{5.1.1}$$

or, in the differential form, it can be written as $\dot{p}(t) = \Lambda^\top p(t)$, subject to the initial condition $p(0)$.

5.1.1 Continuous-time optimal Wonham filter

Using the model from [Won64], an output process z_t is assumed to be generated by state process x_t and the observation noise η_t, which is assumed to be a Wiener process independent of x_t. It is described by the Itô equation

$$\mathrm{d}z_t = h(x_s)\mathrm{d}s + \mathrm{d}\eta_t \tag{5.1.2}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a measurable function on $\mathcal{S}$. For simplicity, we take our observation process to be one-dimensional. The initial condition z_0 is assumed to be zero. The noise covariance is assumed to be constant and denoted by R, that is, $\mathsf{E}[\mathrm{d}\eta^\top \mathrm{d}\eta] = R\,\mathrm{d}t$. We denote by $\sigma\{z_{[0,t]}\}$ the filtration generated by $(z_s)_{s\leq t}$. We recall that if ϕ is some square integrable function of the signal process x_t, then the 'best estimate' (in mean square sense) of $\phi(x_t)$ given the observations up to time t is

$$\mathsf{E}\left[\phi(x_t) \mid \sigma\{z_{[0,t]}\}\right].$$

To compute this conditional expectation, we are therefore interested in the corresponding conditional probability

$$\rho_t^j := \mathsf{P}(x_t = a_j \,|\, \sigma\{z_{[0,t]}\}), \quad j \in \{1, \ldots, d\}. \tag{5.1.3}$$

The classical Wonham filter provides a recursive expression for the conditional probabilities $\rho_t := [\rho_t^1, \ldots, \rho_t^d]^\top$ in the form of following stochastic differential equation [Han07, Corollary 1.2.1]:

$$\mathrm{d}\rho_t = \Lambda^\top \rho_t \mathrm{d}t + (H - \overline{h})\rho_t(\mathrm{d}z_t - \overline{h}\mathrm{d}t), \qquad \rho_0 = p(0) \tag{5.1.4}$$

where $\overline{h} = \sum_{i=1}^d h(a_i)\rho_t^i$, and H denotes a diagonal matrix,

$$H := \operatorname{diag}\{h(a_1), \cdots, h(a_d)\}.$$

Due to the presence of $\mathrm{d}z_t$ on the right-hand side of (5.1.4), ρ_t is a vector-valued random variable, for each $t \geq 0$. It is noted that the computation of ρ_t requires continuous measurements of the process $\mathrm{d}z_t$.

5.1.2 Randomly-sampled observation process

Assume N_t is a Poisson process. The discretized and noisy observation process is thus defined as

$$\widehat{z}_t = \widehat{z}_{\tau_{N_t}} = z_{\tau_{N_t}}, \quad t \geq 0, \tag{5.1.5}$$

where z satisfies the Itô equation (5.1.2). This type of the observation process will be denoted by $\widehat{z}$ and should not be mistaken for the observation process y, defined in other chapters. To condition the optimal estimator on observations, we consider the filtration generated by $(\widehat{z}_s)_{s \leq t}$, which is $\sigma\{\widehat{z}_{[0,t]}\} = \sigma\{\widehat{z}_{\tau_1}, \ldots, \widehat{z}_{\tau_{N_t}}\}$.

5.1.3 Problem statement

Our primary objective is to study the filtering problem subject to the random sampling of the observation process. Toward this end, we first consider the conditional probabilities

$$\widehat{\rho}_t^j := \mathsf{P}(x_t = a_j \,|\, \sigma\{\widehat{z}_{[0,t]}\}), \quad j \in \{1, \ldots, d\}, \tag{5.1.6}$$

and develop an expression for the vector $\widehat{\rho}_t := [\widehat{\rho}_t^1, \ldots, \widehat{\rho}_t^d]^\top$, for each $t \geq 0$, using $\{\widehat{z}_{\tau_k} \,|\, k \leq N_t\}$. These developments are carried out in Section 5.2.

Next, we are interested in comparing $\widehat{\rho}_t$ with ρ_t. It is noted that $\widehat{\rho}_t$, as defined in (5.1.6) is random not only due to the observation noise $\widehat{z}_t$, but it also depends on the random time instants $(\tau_k)_{k\in\mathbb{N}}$. If we average out the randomness due to sampling times and take the mean sampling rate of the Poisson counter N_t to be large enough, then it is natural to compare the resulting random variable with ρ_t. To formally state this intuition, we look at the expectation of $\widehat{\rho}_t$ with respect to the sampling times, and show in Section 5.3 that, for each $t \geq 0$, the resulting random variable converges to ρ_t in the mean as the sampling rate gets large.

5.2 Optimal filter

In this section, we develop expressions that allow us to compute the conditional probability $\widehat{\rho}_t^j$, for each $j \in \{1, \dots, d\}$ and each $t \geq 0$, for a single realization of the random time instants $\{\tau_k\}_{k\in\mathbb{N}}$. By developing a recursive expression, it is possible to compute $\widehat{\rho}_t^j$ simply by updating the last stored value as a function of the newly received information.

To present the recursive filter, we consider the innovation sequence Δz_{N_t} obtained from taking the difference of two consecutive measurements:

$$\Delta z_{N_t} = \widehat{z}_{\tau_{N_t}} - \widehat{z}_{\tau_{N_t-1}} = \xi_{N_t} + \eta_{\tau_{N_t}} - \eta_{\tau_{N_t-1}},$$

where we used the notation

$$\xi_k := \int_{\tau_{k-1}}^{\tau_k} h(x_s)\,\mathrm{d}s, \quad k \in \mathbb{N}.$$

With probability one, a Poisson process N_t yields finite number of random variables Δz_i and ξ_i over a finite interval $[0, t]$. Notice that

$$\begin{aligned}
p\left(\Delta z_1, \dots, \Delta z_{N_t} \mid x_0, x_t\right) &= \mathsf{E}\left[p\left(\Delta z_1, \dots, \Delta z_{N_t}, \xi_1, \dots, \xi_{N_t}\right) \,\Big|\, x_0, x_t\right] \\
&= \mathsf{E}\left[\prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R\Delta\tau_k}} \exp\left[-\frac{(\Delta z_k - \xi_k)^2}{2R\Delta\tau_k}\right] \,\Big|\, x_0, x_t\right] \\
&= \prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R\Delta\tau_k}} \mathsf{E}\left[\exp\left[-\sum_{k=1}^{N_t} \frac{(\Delta z_k - \xi_k)^2}{2R\Delta\tau_k}\right] \,\Big|\, x_0, x_t\right]
\end{aligned}$$

since $\Delta z_i - \xi_i$ are independent for each i and are Gaussian with mean 0 and variance $R\Delta\tau_i$. Let

$$\Psi_n = \Psi_n(\Delta z, \xi, \Delta\tau) := \exp\left(-\sum_{k=1}^{n} \frac{(\Delta z_k - \xi_k)^2}{2R\Delta\tau_k}\right). \tag{5.2.1}$$

To get the expression for $\widehat{\rho}_t^j$ at $t = \tau_{N_t}$ for $N_t \in \mathbb{N}$, we observe that

$$\widehat{\rho}_t^j = \mathsf{P}(x(t) = a_j \mid \widehat{z}_{\tau_k}, k = 0, \dots, N_t) = \mathsf{P}(x(t) = a_j \mid \Delta z_k, k = 1, \dots, N_t)$$
$$= \frac{\sum\limits_{i=1}^{d} p_i(0) p_{ij}(t) \mathsf{E}\left[\Psi_{N_t} \mid x_0 = a_i, x_t = a_j\right]}{\sum_{l=1}^{d} \sum\limits_{i=1}^{d} p_i(0) p_{il}(t) \mathsf{E}\left[\Psi_{N_t} \mid x_0 = a_i, x_t = a_l\right]}$$

as the term $\prod\limits_{k=1}^{N_t} \frac{1}{\sqrt{2\pi R \Delta\tau_k}}$ cancels out from the numerator and the denominator. Then the numerator is the unnormalized conditional density. Let us denote it as follows.

Definition 5.1. *Let $\bar{z}_{[0,t]}$ be a pure jump process realization with jumps only at $\{\tau_k\}_{k=1}^{n_t}$. For each $t \geq 0$, and for each $j \in \{1, \dots, d\}$, define the function $U^j(t, \bar{z}_{[0,t]})$ as follows*

$$U^j(t, \bar{z}_{[0,t]}) := \sum_{i=1}^{d} p_i(0) p_{ij}(t) \mathsf{E}\left[\Psi_{n_t} \middle| x_0 = a_i, x_t = a_j\right],$$

where Ψ_{n_t} is from (5.2.1) *and x_t is the state process.*

Remark 5.2. *One can consider U_t^j as a function of a pure jump process $\bar{z}_t$ since $\{\tau_k\}$ is also defined by $\bar{z}_t$. The Markov chain x affects the value of U_t^j only by its transitional probabilities, not by realizations.*

We define the evolution of U_t^j with differential equations over the intervals $]\tau_{N_t}, \tau_{N_t+1}[$, and via jumps at τ_{N_t}, for all $t \geq 0$. Firstly, if $t \in]\tau_{N_t}, \tau_{N_t+1}[$, we obtain

$$\begin{aligned}
&U_t^j \\
&= \textstyle\sum_{i=1}^{d} p_i(0) p_{ij}(t) \mathsf{E}\left[\Psi_{N_t} \middle| x_0 = a_i, x_t = a_j\right] \\
&= \textstyle\sum_{\ell=1}^{d} \sum_{i=1}^{d} p_i(0) p_{i\ell}(\tau_{N_t}) p_{\ell j}(t - \tau_{N_t}) \mathsf{E}\left[\Psi_{N_t} \middle| x_0 = a_i, x_{\tau_{N_t}} = a_\ell, x_t = a_j\right] \\
&= \textstyle\sum_{\ell=1}^{d} p_{\ell j}(t - \tau_{N_t}) \sum_{i=1}^{d} p_i(0) p_{i\ell}(\tau_{N_t}) \mathsf{E}\left[\Psi_{N_t} \middle| x_0 = a_i, x_{\tau_{N_t}} = a_\ell\right] \\
&= \textstyle\sum_{\ell=1}^{d} p_{\ell j}(t - \tau_{N_t}) U_{\tau_{N_t}}^\ell.
\end{aligned}$$

It yields $\dot{U} = U\Lambda$ for $t \in]\tau_{N_t}, \tau_{N_t+1}[$. Secondly, for $t = \tau_{N_t}$ with $n = N_t > 1$, the strong Markov property and independence of the different η_k give

$$\begin{aligned}
&\mathsf{E}\left[\Psi_{N_t} \mid x_0 = a_i, x_{\tau_n} = a_j\right] \\
&= \sum_{l=1}^{d} \mathsf{E}\left[\Psi_n \mid x_0 = a_i, x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right] \mathsf{P}(x_{\tau_{n-1}} = a_l \mid x_0 = a_i, x_{\tau_n} = a_j) \\
&= \sum_{l=1}^{d} \mathsf{E}\left[\exp\left(-\tfrac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \,\Big|\, x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right] \\
&\quad \times \mathsf{E}\left[\Psi_{n-1} \mid x_0 = a_i, x_{\tau_{n-1}} = a_l\right] \\
&\quad \times \mathsf{P}(x_{\tau_{n-1}} = a_l \mid x_0 = a_i, x_{\tau_n} = a_j)
\end{aligned}$$

By Bayes rule and strong Markov property

$$\begin{aligned}
&p_{ij}(t)\mathsf{P}(x_{\tau_{n-1}} = a_l \mid x_0 = a_i, x_{\tau_n} = a_j) \\
&= \mathsf{P}(x_{\tau_{n-1}} = a_l \mid x_0 = a_i)\mathsf{P}\left(x_{\tau_n} = a_l \,\middle|\, \begin{array}{l} x_0 = a_i, \\ x_{\tau_n} = a_j \end{array}\right) \\
&= p_{li}(\tau_{n-1})p_{jl}(\Delta\tau_n)
\end{aligned}$$

and then $U^j_{\tau_n}$ equals

$$\begin{aligned}
&\sum_{l=1}^{d}\sum_{i=1}^{d} p_i(0)p_{ij}(t)\mathsf{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \,\middle|\, \begin{array}{l} x_{\tau_n} = a_j, \\ x_{\tau_{n-1}} = a_l \end{array}\right] \\
&\mathsf{E}\left[\Psi_{n-1} \,\middle|\, \begin{array}{l} x_0 = a_i, \\ x_{\tau_{n-1}} = a_l \end{array}\right] \mathsf{P}\left(x_{\tau_{n-1}} = a_l \,\middle|\, \begin{array}{l} x_0 = a_i, \\ x_{\tau_n} = a_j \end{array}\right) \\
&= \sum_{l=1}^{d}\sum_{i=1}^{d} p_i(0)p_{li}(\tau_{n-1})p_{jl}(\Delta\tau_n) \\
&\mathsf{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \,\middle|\, x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right] \\
&\mathsf{E}\left[\Psi_{n-1} \,\middle|\, x_0 = a_i, x_{\tau_{n-1}} = a_l\right] \\
&= \sum_{l=1}^{d} p_{jl}(\Delta\tau_n)U^l_{n-1}\mathsf{E}\left[\exp\left(-\frac{(\Delta z_n - \xi_n)^2}{2R\Delta\tau_n}\right) \,\middle|\, x_{\tau_n} = a_j, x_{\tau_{n-1}} = a_l\right].
\end{aligned}$$

Thus the recursive rule for U is the following.

Proposition 5.3. *For a Markov chain $(x_t)_{t\geq 0}$, a fixed sample path of N_t associated with jump times τ_{N_t}, the observations $\widehat{z}_t$ defined in* (5.1.5)*, the conditional density $\widehat{\rho}_t = (\widehat{\rho}^1_t, \ldots, \widehat{\rho}^d_t)^\top$ is*

$$\widehat{\rho}^j_t = \frac{U^j_t}{\sum_{i=1}^d U^i_t},$$

where the vector $U_t = (U_t^1, \dots, U_t^d)$ satisfies

$$\begin{aligned} \dot{U}_t &= U_t \Lambda & t \in]\tau_{N_t}, \tau_{N_t+1}[\\ U_{\tau_n} &= K_n \lim_{s \nearrow \tau_n} U_s & \forall \tau_n = \tau_{N_t} \end{aligned} \tag{5.2.2}$$

where lj-th component of the matrix K_n is the following

$$\mathsf{E}\left[\exp\left(- \frac{\left(\Delta z_n - \int_{\tau_{n-1}}^{\tau_n} h(x_s)\mathrm{d}s\right)^2}{2R\Delta\tau_n} \right) \,\middle|\, \begin{matrix} x_{\tau_n} = a_j, \\ x_{\tau_{n-1}} = a_l \end{matrix} \right].$$

5.3 Convergence in mean

We now turn to the question of comparing the filter obtained from randomly sampled observations with the continuous Wonham filter. For this purpose, we assume that the state process is a Markov chain. In what follows, we use the notation E^z to denote expectation with respect to noise in z-process, and E^λ to denote expectation with respect to the distribution of the Poisson sampling process with intensity $\lambda > 0$. We also recall that $\sigma(\cdot)$ is used to denote the filtration generated by its argument. Our main result is formulated as follows:

Theorem 5.4. *Let $(x_t)_{t\geq 0}$ be a continuous-time Markov chain with non-zero initial distribution. For each $t > 0$, it holds that*

$$\lim_{\lambda\to\infty} \mathsf{E}^\lambda \mathsf{E}^z \left| \mathsf{E}\left[x_t \mid \sigma\left\{\widehat{z}_{[0,t]}\right\}\right] - \mathsf{E}\left[x_t \mid \sigma\left\{z_{[0,t]}\right\}\right] \right| = 0.$$

The proof of this result is carried in the remainder of this section. An outline of the main steps of the proof is drawn in Figure 5.1.

$$\boxed{\begin{tikzcd} \mathsf{E}\left[x_t \mid \sigma\left\{(H_n^t z)_{[0,t]}\right\}\right] \arrow[r, "\mathrm{L.5.6}", "\substack{\text{in } \mathsf{E}^z \\ n\to\infty}"'] & \mathsf{E}\left[x_t \mid \sigma\left\{z_{[0,t]}\right\}\right] \\ \frac{U\left(t,(H_n^t\widehat{z})_t\right)}{\|U\left(t,(H_n^t\widehat{z})_t\right)\|_1} \arrow[u, "\mathrm{P.\,5.7}", "\substack{\text{in } \mathsf{E}^\lambda\mathsf{E}^z \\ \lambda\to\infty}"'] \arrow[r, "\mathrm{L.\,5.8}", "\substack{\text{in } \mathsf{E}^\lambda\mathsf{E}^z \\ n\to\infty,\forall\lambda>0}"'] & \mathsf{E}\left[x_t \mid \sigma\left\{\widehat{z}_{[0,t]}\right\}\right] \arrow[u, dashed, "\mathrm{Th.5.4}", "\substack{\text{in } \mathsf{E}^\lambda\mathsf{E}^z \\ \lambda\to\infty}"'] \end{tikzcd}}$$

Figure 5.1: Schematic of convergences

5.3.1 Intermediate results

Let us start with the definition of operators θ and H that are used excessively in the sequel.

Definition 5.5. *For $n \in \mathbb{N}, t > 0$, let θ_n^t and H_n^t be the following mappings:*

$$\theta_n^t(s) := \max\left\{\tfrac{mt}{n} \in [0, s] \mid m = 0, \ldots, n\right\},$$
$$(H_n^t v)_s := H_n^t(v)(s) = v\left(\theta_n^t(s)\right).$$

Using these definitions, one can now state the intermediate results which are necessary for the proof of Theorem 5.4.

Lemma 5.6. *Let $(x_t)_{t\geq 0}$ be a Markov chain with non-zero initial distribution. For each $t > 0$, there is convergence in E^z-mean:*

$$\lim_{n\to\infty} \mathsf{E}^z \left| \mathsf{E}\left[x_t \mid \sigma\left\{(H_n^t z)_{[0,t]}\right\}\right] - \mathsf{E}\left[x_t \mid \sigma\left\{z_{[0,t]}\right\}\right] \right| = 0.$$

The proof of Lemma 5.6 is a direct consequence of the arguments appearing in [Won64, Appendix 2] and is not carried out here. The vector $U\left(t, (H_n^t \widehat{z})_t\right)$ below consists of the corresponding entries $U^j\left(t, (H_n^t \widehat{z})_t\right)$ as considered in Definition 5.1.

Proposition 5.7. *Let x_t be a Markov chain with non-zero initial distribution. For each $t > 0, n \in \mathbb{N}$, there is convergence in $\mathsf{E}^\lambda \mathsf{E}^z$-mean:*

$$\lim_{\lambda\to\infty} \mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} - \mathsf{E}\left[x_t \mid \sigma\left\{H_n^t z_{[0,t]}\right\}\right] \right| = 0.$$

The proof of Proposition 5.7 appears in below, after the final statement, that we need for the proof of Theorem 5.4.

Lemma 5.8. *Let $(x_t)_{t\geq 0}$ be a Markov chain with non-zero initial distribution. For each $t > 0$ and each $\lambda > 0$ there is convergence in $\mathsf{E}^\lambda \mathsf{E}^z$−mean:*

$$\lim_{n\to\infty} \mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} - \mathsf{E}\left[x_t \mid \sigma\left\{\widehat{z}_{[0,t]}\right\}\right] \right| = 0.$$

The proof of Lemma 5.8 is provided below after the proof of Proposition 5.7.

5.3.2 Proofs

Proof of Proposition 5.7

Proof. Recall that the j-th component of the conditional expectation $\mathsf{E}[x_t \mid \sigma\{(H_n^t z)_{[0,t]}\}]$ is $\frac{U^j(t,(H_n^t z)_t)}{\|U(t,(H_n^t z)_t)\|_1}$, as showed in Section 5.2. Then it is sufficient to prove that for any $\varepsilon > 0$ there is sufficiently large $\lambda > 0$ such that

$$\mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U^j\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} - \frac{U^j\left(t, (H_n^t z)_t\right)}{\|U\left(t, (H_n^t z)_t\right)\|_1} \right| \le \varepsilon. \tag{5.3.1}$$

In what follows, we use the notation τ_k for jump times of a Poisson process and $\frac{mt}{n}$ for jump times of the 'uniform' process θ_n^t. Moreover, we recall that a Poisson process N_t is associated with $\{\tau_k\}$ as $N_t = \sup\{n \in \mathbb{N} \mid \tau_n \le t\}$ for $t \in \mathbb{R}$.

In order to make $H_n^t \hat{z}$ somehow similar to $H_n^t z$, we need the probability that at least one Poisson arrival time $\tau_k \in \{\tau_k\}_{k=0}^{N_t}$ occurs between $\frac{mt}{n} - \delta$ and $\frac{mt}{n}$ for all $m = 1, \ldots, n$ and an arbitrary $\delta \in]0, \frac{t}{n}[$. Since $\tau_k \le \frac{mt}{n}$ if $k \le N_{\frac{mt}{n}}$, this condition can be formulated as

$$\max_{m=1,\ldots,n} \min_{k=0,\ldots,N_{\frac{mt}{n}}} \left(\frac{mt}{n} - \tau_k \right) \le \delta. \tag{5.3.2}$$

The opposite event is

$$\max_{m=1,\ldots,n} \min_{k=0,\ldots,N_{\frac{mt}{n}}} \left(\frac{mt}{n} - \tau_k \right) > \delta \tag{5.3.3}$$

and its probability is the sum $\sum_{m=1}^n (1 - e^{-\lambda\delta})^{m-1} e^{-\lambda\delta}$, that is $(1 - e^{-\lambda\delta})^n$. Using this, we can decompose the expectation $\mathsf{E}^\lambda \left[\frac{U^j\left(t,(H_n^t \widehat{z})_t\right)}{\|U\left(t,(H_n^t \widehat{z})_t\right)\|_1} \right]$ into the sum

$$\begin{aligned} &(1 - e^{-\lambda\delta})^n \mathsf{E}^\lambda \left[\frac{U^j\left(t,(H_n^t \widehat{z})_t\right)}{\|U\left(t,(H_n^t \widehat{z})_t\right)\|_1} \;\middle|\; \text{eq. (5.3.3)} \right] + \\ + \;\; &\left(1 - (1 - e^{-\lambda\delta})^n\right) \mathsf{E}^\lambda \left[\frac{U^j\left(t,(H_n^t \widehat{z})_t\right)}{\|U\left(t,(H_n^t \widehat{z})_t\right)\|_1} \;\middle|\; \text{eq. (5.3.2)} \right]. \end{aligned}$$

Let us take $\lambda = 1/\sqrt{\delta}$ and define δ later. Notice that $\frac{U^j\left(t,(H_n^t \widehat{z})_t\right)}{\|U\left(t,(H_n^t \widehat{z})_t\right)\|_1} \le 1$ and $(1 - e^{-\lambda\delta})^n \to 0$ as $\lambda \to \infty$. Then it remains to prove that

$$\mathsf{E}^\lambda \mathsf{E}^z \left(\frac{U^j\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} \;\middle|\; \text{eq. (5.3.2)} \right) \to \mathsf{E}^z \left(\frac{U^j\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} \right).$$

Let us consider an arbitrary realization N_t satisfying (5.3.2). For an arbitrary fixed z_t, we obtain the process $\widehat{z}_t$. Then both processes $H_n^t \widehat{z}$ and

$H_n^t z$ are pure jump and all jumps occur at times $\frac{mt}{n}$ for $m = 1, \ldots, n$ only. Hence the two processes $U^j(\cdot, H_n^t \widehat{z})$ and $U^j(\cdot, H_n^t z)$ differ in their exponent terms only. For $U^j\left(t, (H_n^t z)_t\right)$ we have jump times only at $\left\{\frac{mt}{n}\right\}$ and the exponent term straightforwardly contains the following sum

$$-\sum_{m=1}^{n}\left(z_{\frac{mt}{n}} - z_{\frac{(m-1)t}{n}} - \int_{\frac{(m-1)t}{n}}^{\frac{mt}{n}} h(\bar{x}_s)\mathrm{d}s\right)^2 \tag{5.3.4}$$

For $U^j\left(\cdot, H_n^t \widehat{z}\right)$, employing the condition (5.3.2), we obtain that for each $m = 1, \ldots, n$, there is a certain $\delta_m \in (0, \delta)$ such that $H\left(\widehat{z}_{\frac{mt}{n}}\right) = \widehat{z}_{\frac{mt}{n}} = z_{\frac{mt}{n} - \delta_m}$. So the exponent term of $U^j\left(t, (H_n^t \widehat{z})_t\right)$ contains a similar sum

$$-\sum_{m=1}^{n}\left(z_{\frac{mt}{n}-\delta_m} - z_{\frac{(m-1)t}{n}-\delta_{m-1}} - \int_{\frac{(m-1)t}{n}}^{\frac{mt}{n}} h(\bar{x}_s)\mathrm{d}s\right)^2 \tag{5.3.5}$$

Thus a realization of Poisson process N_t defines only the sequence of δ_m, which are uniformly bounded by δ. It immediately yields the convergence of each component:

$$\mathsf{E}^\lambda \mathsf{E}^z \left\{\left|U^j(t, H_n^t(z)) - U^j(t, H_n^t(\widehat{z}))\right|\right\} \to 0 \qquad \text{as } \delta \to 0, \text{ i.e. } \lambda \to \infty.$$

To get the convergence of normalized values, notice that both fractions are less than one and the denominators are bounded away from zero in the following sense. Denote $\omega = \min_i p_i(0)$; by linearity of the Kolmogorov equation, none of $p_{i,j}(t)$ equals zero for $t > 0$. Hence, for each $t > 0$ there is a pair (i, j) such that $p_i(0)p_{i,j}(t) \geq \omega/d$, and

$$\|U\left(t, H_n^t(\widehat{z})\right)\|_1 \geq U^j\left(t, H_n^t(\widehat{z})\right) \geq \omega/d\ \mathsf{E}\left[\exp\frac{\text{eq. (5.3.5)}}{2Rt/n}\ \Big|\ x_0 = a_i, x_t = a_j\right].$$

Definition (5.1.2) of the process z implies the following chain of estimations:

$$\begin{aligned}
&\left(\int_{-M}^{M}\frac{\exp\left\{-\frac{1}{2}\left(\frac{x}{R(t/n+\delta)^2}\right)\right\}}{\sqrt{2\pi R(t/n+\delta)}}\mathrm{d}x\right)^n\\
&\leq \mathsf{P}[\max_m |\eta_{\frac{mt}{n}-\delta_m} - \eta_{\frac{(m-1)t}{n}-\delta_{m-1}}| \leq M]\\
&\leq \mathsf{P}[|\text{eq. (5.3.5)}| \leq n(M + 2\max_l a_l)^2(t/n+\delta)^2]\\
&\leq \mathsf{P}\left[\|U\left(t, (H_n^t\widehat{z})_t\right)\|_1 \geq \kappa_M\right] \to 1 \qquad \text{as } M \to \infty
\end{aligned}$$

where $\kappa_M = \omega/d\exp\left\{\frac{n(M+2\max_l a_l)^2(t/n+\delta)^2}{2Rt/n}\right\}$ and tends to 0 as $M \to \infty$. For $\|U\left(t, (H_n^t\widehat{z})_t\right)\|_1$ we have the same estimation, but with zeros instead of δ_m and δ. These inequalities defines a suitable $\delta = \delta(\varepsilon)$ so that (5.3.1) holds. $\square$

Proof of Lemma 5.8

Proof. In Section 5.2, we showed that $\frac{U^j\left(t,\widehat{z}_t\right)}{\|U\left(t,\widehat{z}_t\right)\|_1}$ is the j-th component of

$$\mathsf{E}\left[x_t \mid \sigma\left\{\widehat{z}_{[0,t]}\right\}\right].$$

Then it is sufficient to prove that for any ε, there exists $D = D(\varepsilon) > 0$ such that for any $n > t/D$

$$\mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U^j\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} - \frac{U^j\left(t, \widehat{z}_t\right)}{\|U^j\left(t, \widehat{z}_t\right)\|_1} \right| \leq \varepsilon. \tag{5.3.6}$$

If $t/D =: M_D \in \mathbb{N}$, we can consider M_D disjoint intervals $\left]\frac{tM}{M_D}, \frac{t(M+1)}{M_D}\right]$ of length D (for $M = 1, \ldots, M_D$). Now we are interested in the case where at most one jump of N_t occurs at each interval $\left]\frac{tM}{M_D}, \frac{t(M+1)}{M_D}\right]$, i.e.

$$\max_{M=1,\ldots,M_D} N_{\frac{tM}{M_D}} - N_{\frac{t(M-1)}{M_D}} \leq 1. \tag{5.3.7}$$

The probability of the event (5.3.7) is $e^{-\lambda t}(1+\lambda D)^{\frac{t}{D}}$, and so it tends to 1 as $D \to 0$. Therefore, for any $D > 0$, we decompose and estimate the expectations of (5.3.6) as

$$2\left(1 - e^{-\lambda t}(1+\lambda D)^{\frac{t}{D}}\right) + e^{-\lambda t}(1+\lambda D)^{\frac{t}{D}} \sum_i p_i(0) p_{ij}(t)$$
$$\times \mathsf{E}^\lambda \mathsf{E}^z \left[\left| \frac{U^j\left(t, (H_n^t \widehat{z})_t\right)}{\|U\left(t, (H_n^t \widehat{z})_t\right)\|_1} - \frac{U^j\left(t, \widehat{z}_t\right)}{\|U^j\left(t, \widehat{z}_t\right)\|_1} \right| \middle| \text{eq. (5.3.7)}\right],$$

and the first addend is less than $\varepsilon/4$ for every $D \in]0, D^*[$ for a certain D^*. In the same way, we can separate two cases: if total number of jumps of $\widehat{z}$ on $[0,t]$ is more than $K = K(\lambda, \varepsilon)$ or if it is less or equal. The expectations of (5.3.6) conditioned on the first case are less than $\varepsilon/4$ if K is such that $e^{-\lambda t}\sum_{i=K}^\infty \frac{(\lambda t)^i}{i!} < \varepsilon/8$. Indeed, it means that $\varepsilon/4$ is greater than the value $\left| \frac{U^j\left(t,(H_n^t \widehat{z})_t\right)}{\|U\left(t,(H_n^t \widehat{z})_t\right)\|_1} - \frac{U^j\left(t,\widehat{z}_t\right)}{\|U^j\left(t,\widehat{z}_t\right)\|_1} \right|$ times the probability that $\widehat{z}$ has more than K jumps. Finally, let us get rid of the case of Poisson processes that contain relatively small inter-arrival times, i.e. let us find $L > 0$ such that $\varepsilon/4$ bounds from below the expectations of (5.3.6) conditioned on Poisson process with inter-arrival time less than L. For a Poisson process, the probability of containing all inter-arrival times greater than L is not less than $e^{-\lambda LK}$ (since K is the maximum number of jumps), i.e. it suffices to take $L = L(\lambda, \varepsilon)$ such that $1 - e^{-\lambda LK} \leq \varepsilon/8$.

Let us denote by $\mathcal{A}$ the event that correspond to the inequality (5.3.7), the condition that the number of jumps of $\widehat{z}$ is not more than K and the condition that the inter-arrival times are greater than L. Consider a realization of N_t from $\mathcal{A}$. Denote by τ_k the corresponding jump times of $H_n^t\widehat{z}$. Then jumps of $\widehat{z}$ occur at $\tau_k - \delta_k$. So that $H_n^t\widehat{z}(\frac{tM}{M_D}) = \widehat{z}(\frac{tM}{M_D})$ for $M = 1, \dots, M_D$ and so the exponent term in $U^j(t, H_n^t\widehat{z})$ becomes

$$\exp \sum_k \frac{-(\widehat{z}(\tau_k) - \widehat{z}(\tau_{k-1}) - \int_{\tau_{k-1}}^{\tau_k} h(\bar{x}_s)\mathrm{d}s)^2}{2R(\tau_k - \tau_{k-1})}$$

while the exponent term in $U^j(t, H_n^t\widehat{z})$ contains same values at times $\tau_k - \delta_k$. Thus it is easy to see that we have convergence

$$\mathsf{E}^\lambda \mathsf{E}^z \left[\left| U^j\left(t, (H_n^t\widehat{z})_t\right) - U^j\left(t, \widehat{z}_t\right) \right| \,\middle|\, \mathcal{A} \right] \to 0$$

for all components j. To get (5.3.6) it remains to estimate the denominators. Denote $\omega = \min_i p_i(0)$; by linearity of the Kolmogorov equation, none of $p_{i,j}(t)$ equals zero for $t > 0$. Hence, for each $t > 0$ there is a pair (i, j) such that $p_i(0)p_{i,j}(t) \geq \omega/d$, and so

$$\begin{aligned}
&\|U(t, H_n^t(\widehat{z}))\|_1 \\
&\geq U^j\left(t, H_n^t(\widehat{z})\right) \\
&\geq \omega/d \; \mathsf{E}\left[\exp \sum_k \frac{-(\widehat{z}(\tau_k) - \widehat{z}(\tau_{k-1}) - \int_{\tau_{k-1}}^{\tau_k} h(\bar{x}_s)\mathrm{d}s)^2}{2R(\tau_k - \tau_{k-1})} \,\middle|\, \begin{matrix} x_0 = a_i, \\ x_t = a_j \end{matrix} \right].
\end{aligned}$$

Here the sum contains not more than K elements, $\tau_k - \tau_{k-1} > L - D$, x_s are from a finite set S, and the only remaining randomness in η is bounded as

$$\left(\int_{-M}^{M} \frac{\exp\left\{-\frac{1}{2}(\frac{x}{Rt^2})\right\}}{\sqrt{2\pi Rt}} \mathrm{d}x \right)^K \leq (\mathsf{P}[|\eta_t - \eta_0| \leq M])^K \leq \mathsf{P}[\max_k |\eta_{\tau_k} - \eta_{\tau_{k-1}}| \leq M].$$

Thus, we have $\mathsf{P}[\|U(t, H_n^t(\widehat{z}))\|_1 \geq \kappa] \to 0$, as $\kappa \to 0$. The same holds for $\|U(t, \widehat{z})\|_1$. This allows us to choose a suitable $D \in]0, D^*[$ and conclude the proof. $\square$

Proof of Theorem 5.4

The proof is essentially based on the schematic shown in Figure 5.1.

Proof. By Proposition 5.7, for any ε_1 there is large enough $\lambda_1 = \lambda(\varepsilon_1)$ such that for any $\lambda \geq \lambda_1$,

$$\mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U\big(t,(H_n^t \widehat{z})_t\big)}{\|U\big(t,(H_n^t \widehat{z})_t\big)\|_1} - \mathsf{E}\left[x_t \mid \sigma\left\{(H_n^t z)_{[0,t]}\right\}\right] \right| \leq \varepsilon_1.$$

By Lemmas 5.6 and 5.8, for any ε_2, we can find $n_2 = n(\varepsilon_2)$, such that for every $n \geq n_2$, the following two inequalities hold:

$$\begin{aligned} \mathsf{E}^\lambda \mathsf{E}^z \left| \mathsf{E}\left[x_t \mid \sigma\left\{(H_n^t z)_{[0,t]}\right\}\right] - \mathsf{E}\left[x_t \mid \sigma\left\{z_{[0,t]}\right\}\right] \right| &\leq \varepsilon_2, \\ \mathsf{E}^\lambda \mathsf{E}^z \left| \frac{U\big(t,(H_n^t \widehat{z})_t\big)}{\|U\big(t,(H_n^t \widehat{z})_t\big)\|_1} - \mathsf{E}\left[x_t \mid \sigma\left\{\widehat{z}_{[0,t]}\right\}\right] \right| &\leq \varepsilon_2. \end{aligned}$$

In the first of these two inequalities, the expectation E^λ does not change anything as inner parts are independent of N_t.

So for a given $\varepsilon > 0$, taking $\varepsilon_1 = \varepsilon_2 = \varepsilon/3$ and the corresponding λ_1 and n_2, we immediately obtain that there exists $\lambda(\varepsilon)$ such that for any $\lambda \geq \lambda(\varepsilon)$ the required convergence property holds:

$$\mathsf{E}^\lambda \mathsf{E}^z \left| \mathsf{E}\left[x_t \mid \sigma\{\widehat{z}_{[0,t]}\}\right] - \mathsf{E}[x_t \mid \sigma\{z_{[0,t]}\}] \right| \leq \varepsilon.$$

□

5.3.3 Summary

In this chapter, we discussed the design of optimal filters for a continuous-time Markov chain with a finite state space, coupled with an observation process subjected to additive Gaussian noise, where measurements are available at randomly sampled time instants. We initiate by articulating the optimal filter within this context, followed by deriving a recursive formula encapsulating it as a continuous-discrete filter. Our principal contribution lies in benchmarking the efficiency of our proposed filter against its continuous-time analog the classical Wonham filter, which is based on a continuous observation process.

We demonstrate that by configuring the sampling scheme around a Poisson counter and incrementally increasing the mean sampling rate, the expected value of the posterior conditional distribution of our continuous-discrete filter progressively aligns with the posterior distribution obtained from a continuous Wonham filter.

Chapter 6

Conclusions

6.1 Contribution

This work seeks to contribute to the field of filtering for continuous-time systems, particularly focusing on challenges arising from limitations in output observations due to continuous white noise and time-based sampling. It specifically addresses the added complexity introduced by output observations arriving randomly at discrete instances. Through this lens, we examine strategies for deriving bounds on the covariance of state estimation errors for select classes of dynamical systems, employing our proposed filtering algorithms.

Our findings cover the derivation of optimal estimators for both linear dynamics and state processes governed by continuous Markov chains. In the linear scenario, we investigated the estimator's stability and its sensitivity to changes in Poisson intensity, which reflects the observation frequency. For the Markov chain model, we demonstrated the L_1-convergence of our estimator to the optimal estimator (Wonham filter) as the Poisson process intensity increases.

Furthermore, for nonlinear systems featuring Lipschitz nonlinear dynamics, we proposed formulas for a sub-optimal estimator, taking cues from the optimal estimator in the linear case. This necessitated measuring the sub-optimal filter's efficiency relative to its optimal counterpart, especially under the regime of random discrete observations where tuning the estimator could enhance its performance or reduce computational demands. Our research elucidates sufficient conditions that guarantee the boundedness of the expected error covariance matrix.

Aiming to investigate the nonlinear problem, we opted for the ensemble filter method, acknowledging its significance in large-scale models requiring noisy data estimation. However, at this stage the obtained results on ensemble filters are restricted to the linear case. Our contributions include the first derivation of a continuous-discrete McKean–Vlasov type diffusion process with additive Gaussian noise in the observation model. This process is validated as an exact filter, where the conditional expectation is same as the optimal filter. Subsequently, this process is approximated by particle systems interconnected through empirical covariance. Notably, while these ensemble filters draw parallels with established methods in a continuous framework, they exhibit unique dependencies on system parameters.

To summarize, this project contributes to the field of filtering for continuous-time systems by providing new approaches for addressing the challenges associated with random sampling of output data. Additionally, it enhances the existing methodologies for the analysis of both linear and nonlinear systems.

6.2 Further directions

6.2.1 Ensemble filters

Three immediate generalizations are important for the presented study on ensemble filters.

Analysis of a 'mixed' ensemble filter

We showed that a mean-field type process is an exact filter if it satisfies the (sufficient) conditions listed in Proposition 4.2. In addition to the vanilla and transport-inspired ensemble filters considered in Chapter 4, one can consider another possible option. The conditions are met if $\Omega_t = 0$, $\Delta_t = 0$, $\Gamma_t = G$ and

$$\Xi_{\tau_k}: \quad Q_{\tau_k}\Xi_{\tau_k}^\top + \Xi_{\tau_k}Q_{\tau_k}\Xi_{\tau_k}^\top + \Xi_{\tau_k}Q_{\tau_k} = -L_{\tau_k}CQ_{\tau_k}.$$

Hence, the exact filter becomes

$$\mathrm{d}s_t := As_t\mathrm{d}t + G\mathrm{d}w_t + \Big(L_t y_t - L_t C\widehat{s}_t + \Xi_t(s_t - \widehat{s}_t)\Big)\mathrm{d}N_t,$$

where $Q_t = \mathsf{E}[(s_t - \widehat{s}_t)(s_t - \widehat{s}_t)^\top]$ and $\widehat{s} = \mathsf{E}[s_t \mid \mathcal{Y}_t]$ for the σ-algebra $\mathcal{Y}_t$ generated by observation process y up to time t.

Its structure combines the 'continuous' part of the vanilla filter and the 'jump' part of the transport-inspired filter. It allows us to avoid the heavy computation of inverted covariance matrices in contrast to the case of the transport-inspired filter. Using independent noises instead, similar to the vanilla filter, seems reasonable for the continuous part. Moreover, the non-random choice of the jump part also seems feasible, and the price consists of computation of the matrix Ξ_{τ_k}. Since the times of computations are discrete and isolated, the performance should improve. Nonetheless, the corresponding ensemble filter requires investigation.

From scalar to multidimensional case

The principles of filtering theory using Poisson-distributed observations for single-variable (or scalar) cases have provided key insights into the operations and characteristics of optimal and ensemble filters. These insights cover various aspects, including the conditions for steady states and stability. Although there has been considerable advancement in understanding and applying these principles in scalar cases, the complexities of the real world often require an approach that considers higher dimensional systems.

In real-world scenarios, systems often involve multiple interacting components or variables. This complexity means that extending our research to consider multiple variables at once is not just beneficial but essential for accurately modeling such systems. Moving from single-variable to multi-variable (or multi-dimensional) cases brings forth specific challenges and opportunities. Particularly, our understanding of steady states and stability in the single-variable context needs to be carefully adjusted to fit the complexities of multi-variable systems. This adjustment involves expanding the mathematical definitions of stability to cater for multi-dimensional situations and exploring how steady states can exist and be unique in such contexts.

Moving forward into multi-dimensional studies requires us to modify and improve existing research methods. This may include upgrading computational methods to manage more complex calculations and developing new theoretical models that can accurately represent multi-dimensional dynamics. It is especially important to focus on how ensemble filters can be scaled and how optimal filters perform when dealing with multiple dimensions.

From linear to nonlinear case

Unlike in linear systems, where convergence typically refers to moments converging to true values, convergence in nonlinear systems often needs to be understood in the weak sense. This means that the convergence pertains to the distributions themselves, necessitating a nuanced approach. Specifically, researchers can utilize the generators corresponding to the nonlinear filters to demonstrate this form of convergence, providing a mathematical framework for understanding how predictions from filters approximate the real system behavior over time. However, the exactness property seems feasible using approaches similar to those employed in the linear case. Exactness, in this context, refers to the filter's ability to precisely estimate the state of a system based on observations. However, the methodologies to demonstrate this property in nonlinear systems adapt to focus on the distributions that constitute solution to the filtering problem, not just their moments.

Another vital concept in the study of nonlinear filters is the propagation of chaos property. This property describes how, in a large system, the randomness or 'chaos' in the state of individual elements becomes independent as the system evolves. Proving this property for nonlinear filters is crucial because it can help validate the filter's performance in complex, dynamically changing environments. A precise and rigorous proof of the propagation of chaos in the context of nonlinear filtering remains an area for further investigation, while it seems promising and plausible.

The exploration of nonlinear ensemble filters presents a rich field of study, diverging from linear case through its focus on distributions and requiring innovative approaches for proving convergence and exactness. The challenges in proving convergence in the weak sense and the propagation of chaos property highlight the need for continued research and methodological development in this area.

6.2.2 Generalizing the sampling process

The current research framework is based on the assumption that observations are sampled through a homogeneous Poisson process. This foundational assumption prompts a compelling avenue for future exploration, particularly in terms of how the nature of the sampling process influences the outcomes of the study. Two major directions are proposed for further study along this line:

1. *Examination of Interarrival Time Assumptions:* The initial aspect of this exploration involves a critical assessment of the assumptions regarding the expected interarrival times within the sampling process. A key question here is the sufficiency of bounding these interarrival times for the purposes of the study. Does the mere limitation of these intervals guarantee robustness in the findings, or is there a requisite for the sampling process to adhere to the strong Markov property to ensure that the future and past are conditionally independent given the present? This inquiry will delve into the implications of these assumptions and identify the necessary conditions for the validity of the sampling process.

2. *Investigation into Non-homogeneous Poisson Processes:* Another promising research direction is the investigation into non-homogeneous Poisson processes as an alternative to the homogeneous sampling assumption. This exploration would involve treating the intensity function of the Poisson process as a variable parameter, thus allowing for an examination of how variations in intensity affect the outcome of our study. Within this context, there exists a potential for imposing restrictions or penalties on the changes in intensity, adding another layer of complexity to the analysis. Such penalties could pertain to practical limitations in the sampling technique or reflect theoretical constraints intended to maintain certain conditions within the study.

By venturing into these areas, the research aims to deepen the understanding of how different sampling assumptions and modifications impact the reliability and applicability of the findings. Each direction not only presents a methodological challenge but also offers the potential to uncover new insights into the dynamics of the processes under study.

6.2.3 Markov chain setup

Studying filtering problem for Markov chains helps to understand the dynamics of systems exhibiting nonlinear behavior. The key advantage is the finiteness of the distributions that is feasible for numerical experiments and is theoretical analysis. Such investigations provide insights into the properties and performance of filters within nonlinear frameworks.

A significant focus within this realm is the study of the optimal estimator and the impact of incoming observations on various aspects of system estimation. Each observation carries the potential to significantly ameliorate the estimator's performance by refining its predictions. In contrast to the continuous-time case, one can investigate how different properties are affected by observations and which of them reflect the advantage of obtaining new information provided by observations. The conjectures are the following.

1. *Covariance Reduction:* The arrival of observations may lead to a reduction in the covariance associated with the estimator. This reduction signifies an increase in confidence regarding the state's prediction, as the estimator's uncertainty decreases with additional information.

2. *Entropy Reduction:* Observations may also lead to a decrease in entropy, which is a measure of the uncertainty or randomness of the system's state. A lower entropy value post-observation indicates a more informed and less uncertain estimate of the system's state.

3. *Distance to True Distribution:* Another critical measure is the distance between the estimated distribution and the true distribution of the system's state. Minimizing such a distance, one thereby enhances the accuracy of the estimation process. Appropriate notion of a distance also remains to discuss.

Preliminary analytical studies did not lead to a meaningful solution so far, pointing out the need for more detailed investigation.

In addition, the approximation of nonlinear processes using Markov chains has become a fundamental approach in efficiently modeling complex behaviors that linear models cannot capture. This method's efficiency in various domains suggests a promising avenue for extending Markov chain-based estimators into the realm of nonlinear processes. However, the intricacy of nonlinear dynamics demands a thoughtful selection of methods and mathematical frameworks for this expansion.

Appendix A

Auxiliary Results

Following results from the literature have been used in the analysis carried out in Chapter 3.

Lemma A.1 ([Won68, Lemma 4.1]). *Consider the equation,* $GG^\top + KK^\top = \overline{G}\overline{G}^\top$*, and let* F *be an arbitrary matrix of suitable dimension.*

1. *If* (A, G) *is controllable, then* $(A + KF, \overline{G})$ *is controllable.*

2. *If* (A, G) *is stabilizable, then* $(A + KF, \overline{G})$ *is stabilizable.*

Lemma A.2 ([Won68, Lemma 5.1]). *Let* (A, G) *be controllable,* $\mathcal{P}_0 \geq 0$ *and let* $\mathcal{P}$ *satisfy*

$$\dot{\mathcal{P}}_t = A\mathcal{P}_t + \mathcal{P}_t A^\top + \Pi(\mathcal{P}_t) + GG^\top, \quad \mathcal{P}(0) = \mathcal{P}_0.$$

If there exists a constant matrix $\mathcal{Q} \geq 0$ *such that* $A\mathcal{Q} + \mathcal{Q}A^\top + \Pi(\mathcal{Q}) + GG^\top = 0$*, then* $\mathcal{P}_t \to \mathcal{Q}$ *as* $t \to +\infty$*.*

Appendix B

Additional Calculations

B.1 Proof of Proposition 3.10

Proof. To prove the existence of solution to equation (3.3.4), we first consider the differential equation (3.3.1) with a time-varying $K(\cdot)$. The resulting time-varying differential equation, subject to the initial condition $\mathcal{P}(0) = \mathcal{P}_0$, has the solution

$$\mathcal{P}_t = \varphi(t, \mathcal{P}, M_{\mathcal{P}}, K) := \Phi(t,0)\mathcal{P}_0\Phi^\top(t,0) + \int_0^t \Phi(t,s)\Big(GG^\top + \Pi(\mathcal{P}_s) + \lambda K_s M_{\mathcal{P}_s} K_s^\top\Big)\Phi^\top(t,s)\mathrm{d}s.$$

For any given $K(\cdot)$ and a positive definite matrix $M_{\mathcal{P}} \in \mathbb{R}^{p\times p}$ (affine in $\mathcal{P}$), we can find a solution of the Volterra equation $\mathcal{P}_t = \varphi(t, \mathcal{P}, M_{\mathcal{P}}, K)$ by successive approximation. Let us construct a sequence of these solutions P_i for $i \geq 1$ such that $M_0 = I_{p\times p}$, $K_0 = C^\top$, $\mathcal{P}_{i+1}(t) = \varphi(t, \mathcal{P}_{i+1}, M_i, K_i)$, $M_{i+1} = (1+c)C\mathcal{P}_{i+1}C^\top + (2c+c^2)\operatorname{tr} P_{i+1} I_{p\times p} + V$, and $K_{i+1} = M_{i+1}^{-1}C^\top\mathcal{P}_{i+1}$. Notice that all $\mathcal{P}_i$ are positive semidefinite, all M_i are positive definite and so invertible, all matrices $K_i, M_i, \mathcal{P}_i$, $i \in \mathbb{N}$, are measurable and bounded with respect to t. Consider $Q_i(t) = \mathcal{P}_i(t) - \mathcal{P}_{i+1}(t)$ with $Q_i(0) = 0$. From (3.3.1) and (3.4.4), we get

$$\begin{aligned}
\dot{Q}_i &= \Psi(\mathcal{P}_i, M_i, K_i) - \Psi(\mathcal{P}_{i+1}, M_{i+1}, K_{i+1}) \\
&\geq \Psi(\mathcal{P}_i, M_i, K_i) - \Psi(\mathcal{P}_{i+1}, M_{i+1}, K_i) \\
&= (A - \lambda K_i C)Q_i + Q_i(A - \lambda K_i C)^\top + \Pi(Q_i) \\
&\qquad + \lambda K_i(M_i - M_{i+1})K_i^\top \\
&= (A - \lambda K_i C)Q_i + Q_i(A - \lambda K_i C)^\top + \Pi(Q_i) \\
&\quad + \lambda(1+c)K_i C Q_i C^\top K_i^\top + \lambda \operatorname{tr}(Q_i)(2c+c^2)K_i K_i^\top
\end{aligned}$$

$$= (A - \lambda K_i C)Q_i + Q_i(A - \lambda K_i C)^\top + \Pi(Q_i) \\ + \lambda K_i\Big((1+c)CQ_iC^\top + \mathrm{tr}\,(Q_i)(2c+c^2)I_p\Big)K_i^\top$$

so there exists a positive semidefinite matrix R such that

$$\dot{Q}_i = (A - \lambda K_i C)Q_i + Q_i(A - \lambda K_i C)^\top + \Pi(Q_i) \\ + \lambda K_i\Big((1+c)CQ_iC^\top + \mathrm{tr}\,(Q_i)(2c+c^2)I_p\Big)K_i^\top + R.$$

Hence, $Q_i(t)$ can be rewritten as a solution of the equation

$$Q_i(t) = \int_0^t \Phi(t,s)(\Pi(Q_i) + \lambda K_i M_{Q_i} K_i^\top + R)\Phi^\top(t,s)\mathrm{d}s,$$

implying that $Q_i(t) \geq 0$ for each $i \geq 1$ and $t \in [0,T]$, as it can be calculated by successive approximation with $Q_i(0) = 0$. Consequently, the sequence $\{\mathcal{P}_i(t)\}_{i\in\mathbb{N}}$ is non-increasing for each $t \in [0,T]$. It also means that on the finite segment $[0,T]$, $\mathcal{P}_i(t)$ is uniformly bounded by the number, $\sup\{|\mathcal{P}_1(t)| \mid t \in [0,T]\}$, for every $i \geq 1$ and $t \in [0,T]$; consequently $M_i(t)$ and $K_i(t)$ are uniformly bounded as well. Moreover, we let $\mathcal{P}_t = \lim_{i\to\infty} \mathcal{P}_i(t)$, $M_t = \lim_{i\to\infty} M_i(t) = (1+c)C\mathcal{P}_tC^\top + (2c+c^2)\,\mathrm{tr}(\mathcal{P}_t)I_{n\times n} + V$, $K_t = \lim_{i\to\infty} K_i(t) = \mathcal{P}C^\top M^{-1}$. So, the Volterra equation has the form: $\mathcal{P}_t = \varphi(t, \mathcal{P}, M, K)$ for M_t and K_t defined above, meaning that the chosen solution $\mathcal{P}$ is well-defined and unique. □

B.2 Proof of Lemma 3.13

Let $\mathcal{K}_{\mathcal{P}_t} = \mathcal{P}_t C^\top M_{\mathcal{P}_t}^{-1}$ and let $\Phi(t,s)$ denote the fundamental matrix associated with $A - \lambda\mathcal{K}_{\mathcal{P}_t}C$. Then

$$\mathcal{P}_t = \int_0^t \Phi(t,s)(\Pi(\mathcal{P}_s) + GG^\top + \lambda\mathcal{K}_{\mathcal{P}_s}M_{\mathcal{P}_s}\mathcal{K}_{\mathcal{P}_s}^\top)\Phi^\top(t,s)\mathrm{d}s. \tag{B.2.1}$$

Let us fix an arbitrary $\tau \geq 0$ and define $\widetilde{\mathcal{K}}_{\mathcal{P}_t} = \mathcal{K}_{\mathcal{P}_{t+\tau}}$. Further, let $\widetilde{\Phi}(t,s)$ be the fundamental matrix associated with $A - \lambda\widetilde{\mathcal{K}}_{\mathcal{P}_t}C$; then $\widetilde{\Phi}(t,s) = \Phi(t+\tau, s+\tau)$. Let $\widetilde{\mathcal{P}}_t$ be the solution of

$$\dot{\widetilde{\mathcal{P}}}_t = \Psi(\widetilde{\mathcal{P}}_t, M_{\widetilde{\mathcal{P}}_t}, \widetilde{\mathcal{K}}_{\mathcal{P}_t}), \quad \widetilde{\mathcal{P}}_0 = 0,$$

then $\mathcal{P}_t \leq \widetilde{\mathcal{P}}_t$, for each $t \geq 0$, by minimum property. Moreover, we have the following representations of $\widetilde{\mathcal{P}}_{t-\tau}$:

$$\widetilde{\mathcal{P}}_{t-\tau} = \int_0^{t-\tau} \widetilde{\Phi}(t-\tau,s)(\Pi(\widetilde{\mathcal{P}}_s) + GG^\top + \lambda\widetilde{\mathcal{K}}_{\mathcal{P}_s}M_{\widetilde{\mathcal{P}}_s}\widetilde{\mathcal{K}}_{\mathcal{P}_s}^\top)\widetilde{\Phi}^\top(t-\tau,s)\mathrm{d}s$$

$$
\begin{aligned}
&= \int_0^{t-\tau} \Phi(t,s+\tau)(\Pi(\widetilde{\mathcal{P}}_s) + GG^\top + \lambda \mathcal{K}_{\mathcal{P}_{s+\tau}} M_{\widetilde{\mathcal{P}}_s} \mathcal{K}^\top_{\mathcal{P}_{s+\tau}})\Phi^\top(t,s+\tau)\mathrm{d}s \\
&= \int_\tau^{t} \Phi(t,s)(\Pi(\widetilde{\mathcal{P}}_{s-\tau}) + GG^\top + \lambda \mathcal{K}_{\mathcal{P}_{s}} M_{\widetilde{\mathcal{P}}_{s-\tau}} \mathcal{K}^\top_{\mathcal{P}_{s}})\Phi^\top(t,s)\mathrm{d}s \\
&\leq \int_0^{t} \Phi(t,s)(\Pi(\widetilde{\mathcal{P}}_{s-\tau}) + GG^\top + \lambda \mathcal{K}_{\mathcal{P}_{s}} M_{\widetilde{\mathcal{P}}_{s-\tau}} \mathcal{K}^\top_{\mathcal{P}_{s}})\Phi^\top(t,s)\mathrm{d}s.
\end{aligned}
$$

The last inequality, combined with (B.2.1), yields

$$
\mathcal{P}_t - \widetilde{\mathcal{P}}_{t-\tau} \geq \int_0^t \Phi(t,s)(\Pi(\mathcal{P}_s - \widetilde{\mathcal{P}}_{s-\tau}) + \lambda \mathcal{K}_{\mathcal{P}_s} M_{(\mathcal{P}_s - \widetilde{\mathcal{P}}_{s-\tau})} K^\top_{\mathcal{P}_s})\Phi^\top(t,s)\mathrm{d}s
$$

and hence we get $\mathcal{P}_s - \widetilde{\mathcal{P}}_{s-\tau} \geq 0$, for each $s \geq \tau$. By letting $s = t+\tau$, we obtain the desired monotonicity, that is, $\mathcal{P}_t \leq \widetilde{\mathcal{P}}_t \leq \mathcal{P}_{t+\tau}$, and this completes the proof.

B.3 Properties of a scalar nonlinear ODE

Lemma B.1. *Let $\mathcal{Q}_0 > 0$ and let $\mathcal{Q}\colon \mathbb{R}_{geq0} \to \mathbb{R}$ satisfy the following differential equation*

$$
\dot{\mathcal{Q}}_t = 2A\mathcal{Q}_t + F^2 - \lambda \frac{\mathcal{Q}_t^2 C^2}{\mathcal{Q}_t C^2 + V} \tag{B.3.1}
$$

where $V > 0$, $F \neq 0$, $\lambda > 0$ and $\lambda > 2A$. The solution $\mathcal{Q}$ has one positive steady state $\mathcal{Q}^*$ that is

$$
\mathcal{Q}^* = \frac{2AV + F^2C^2}{2\lambda - 4A} + \sqrt{\left(\frac{2AV + F^2C^2}{2\lambda - 4A}\right)^2 + \frac{F^2}{\lambda - 2A}}. \tag{B.3.2}
$$

Moreover, $\dot{\mathcal{Q}}|_{\mathcal{Q}=F^2/(\lambda-2A)} > 0$ and

$$
\mathcal{Q}_t \geq \min\{\mathcal{Q}_0, F^2/(\lambda - 2A)\} \qquad \textit{for all } \ t \geq 0.
$$

Proof. Notice that $C^2\mathcal{Q}_t + V > 0$ for $\mathcal{Q}_t \geq 0$. The equation (B.3.1) can be written as follows

$$
\dot{\mathcal{Q}}_t = \frac{(2A-\lambda)C^2\mathcal{Q}_t^2 + (2AV + F^2C^2)\mathcal{Q}_t + F^2}{C^2\mathcal{Q}_t + V}, \tag{B.3.3}
$$

The parabola

$$
(2A-\lambda)C^2\mathcal{Q}^2 + (2AV + F^2C^2)\mathcal{Q} + F^2
$$

given by the right-hand side of (B.3.3) is concave because $(2A - \lambda) < 0$ and is positive at $\mathcal{Q} = 0$. Hence there is only one non-negative steady state $\mathcal{Q}^*$, that is computed directly as in (B.3.2). It also means that for any $\mathcal{Q}_0 > 0$ and for all $t \geq 0$

$$\mathcal{Q}_t \geq \min\{\mathcal{Q}_0,\ \mathcal{Q}^*_\lambda\},$$
$$\mathcal{Q}_t \leq \max\{\mathcal{Q}_0,\ \mathcal{Q}^*_\lambda\}.$$

In addition, plugging $\mathcal{Q} = F^2/(\lambda - 2A)$ into (B.3.1) one can obtain the following expression

$$\mathcal{Q} = \lambda F^2 V(V(\lambda - 2A) + F^2G^2),$$

that is positive due to the assumptions on V, F and λ. Thus we obtain the last statement of the lemma. □

Lemma B.2. *Let $\mathcal{Q}^{\mathrm{m}}_0 > 0$ and let for $t > 0$ the scalar derivative $\dot{\mathcal{Q}}^{\mathrm{m}}$ satisfy*

$$\dot{\mathcal{Q}}^{\mathrm{m}}_t = 2A\mathcal{Q}^{\mathrm{m}}_t + F^2 - \lambda\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2}{\mathcal{Q}^{\mathrm{m}}_tC^2 + V} + \frac{\lambda}{\mathrm{m}}\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2V}{(\mathcal{Q}^{\mathrm{m}}_tC^2 + V)^2}. \tag{B.3.4}$$

where $V, \lambda, m > 0$ and moreover $F \neq 0$ and $\lambda \geq 2A + F^2/\mathcal{Q}^0$. The solution $\mathcal{Q}^{\mathrm{m}}$ has a positive steady state and

$$\mathcal{Q}^{\mathrm{m}}_t \geq \min\{\mathcal{Q}_0, F^2/(\lambda - 2A)\} \qquad t \geq 0.$$

Proof. Since the dependence $\dot{\mathcal{Q}}^{\mathrm{m}}$ of $\mathcal{Q}^{\mathrm{m}}$ is continuous, the existence of a positive steady state follows from the facts that $\dot{\mathcal{Q}}^{\mathrm{m}}|_{\mathcal{Q}^{\mathrm{m}}=0} > 0$ and

$$\lim_{Z\to\infty} \dot{\mathcal{Q}}^{\mathrm{m}}|_{\mathcal{Q}^{\mathrm{m}}=Z} = -\infty.$$

The assumption $\lambda \geq 2A + F^2/\mathcal{Q}^{\mathrm{m}}_0$ follows both $\lambda > 2A$ and $\mathcal{Q}^{\mathrm{m}}_0 > F^2/(\lambda - 2A)$. The term $\frac{\lambda}{\mathrm{m}}\frac{(\mathcal{Q}^{\mathrm{m}}_t)^2C^2V}{(\mathcal{Q}^{\mathrm{m}}_tC^2+V)^2}$ is always positive. It allows get the comparison

$$\dot{\mathcal{Q}}^{\mathrm{m}}|_{\mathcal{Q}^{\mathrm{m}}=Z} \geq \dot{\mathcal{Q}}|_{\mathcal{Q}=Z}$$

for the process $\mathcal{Q}$ defined by (B.3.1). Hence Lemma B.1 implies that

$$\dot{\mathcal{Q}}^{\mathrm{m}}|_{\mathcal{Q}^{\mathrm{m}}=F^2/(\lambda-2A)} > 0.$$

Combining this with condition $\mathcal{Q}^{\mathrm{m}}_0 > F^2/(\lambda-2A)$ one has that $\mathcal{Q}^{\mathrm{m}}_t \geq F^2/(\lambda-2A)$ for all $t \geq 0$.

□

Appendix C

McKean–Vlasov Itô–Skorohod equations

This section is about stochastic equation of the McKean–Vlasov type, which depends on both a stochastic process and its law. The considered case is not general since it is applied for linear filtering equations with constant diffusion.

Below $\Pi(Z)$ denotes the space of probability laws on Z. Following the results of [Gra92] and applying our notation (from Section 2.2), we obtain the following.

Definition C.1 ([Gra92, Def. 2.1])**.** *Let $\sigma, b, g \colon \mathbb{R}^d \times \Pi(\mathbb{R}^d) \to \mathbb{R}$. An $\mathbb{R}^n$-valued process $(X_t)_{t\geq 0}$ is a solution to the McKean–Vlasov stochastic differential equation starting at X_0 if*

$$\mathrm{d}X_t = \sigma\left(X_t, \bar{\mathbb{P}}_t\right)\mathrm{d}\omega_t + b\left(X_t, \bar{\mathbb{P}}_t\right)\mathrm{d}t + g\left(X_{t-}, \bar{\mathbb{P}}_t\right)\mathrm{d}N_t \tag{C.0.1}$$

where $\bar{\mathbb{P}}_t = \bar{\mathbb{P}}_0 \circ X_t^{-1}$ is the law of X_t.

Despite the fact that it is unclear how to solve such equations, one may study its approximation given by interacting particles.

Existence and uniqueness conditions are provided by [Gra92, Th. 2.1] both for the particle system and the nonlinear McKean–Vlasov equation; the assumptions are the following.

The m-particle system is

$$\mathrm{d}X_t^i = \sigma\left(X_t^i, S_t^{\mathrm{m}}\right)\mathrm{d}\omega_t^i + b\left(X_t^i, S_t^{\mathrm{m}}\right)\mathrm{d}t + g\left(X_{t-}^i, S_t^{\mathrm{m}}\right)\mathrm{d}N_t^i \tag{C.0.2}$$

where the i-th particle starts at X_0^i, Wiener process $(\omega^i)_{t\geq 0}$ and Poisson process $(N_t^i)_{t\geq 0}$ are independent; the empirical measure is

$$S_t^{\mathrm{m}} = \frac{1}{\mathrm{m}}\sum_{i=1}^{\mathrm{m}} \delta_{X_t^i}.$$

We wish to know what happens when the number of particles m goes to infinity. It can be studied in terms of chaocity.

Definition C.2 ([Gra92, eq. (4.4)]). *If E is a Polish space, $\mathbb{Q}$ is a probability measure on E, $\mathbb{Q}^{\mathrm{m}}$ are symmetrical probability measures on E^{m}, we say that $(\mathbb{Q}^{\mathrm{m}})_{\mathrm{m}\in\mathbb{N}}$ is $\mathbb{Q}$-chaotic if for any $n \in \mathbb{N}, f_1, \dots, f_n$ continuous bounded on E,*

$$\lim_{\mathrm{m}\to\infty} \left\langle \mathbb{Q}^{\mathrm{m}}, f_1 \otimes \cdots \otimes f_n \otimes 1^{\otimes \mathrm{m}-\mathrm{n}} \right\rangle = \prod_{i=1}^{n} \langle \mathbb{Q}, f_i \rangle$$

This means a fixed finite number of coordinates behave in the limit as if they were independent, of law $\mathbb{Q}$. It is equivalent to the fact that the sequence of the (random) empirical measures of the m coordinates converges in law, under the $\mathbb{Q}^{\mathrm{m}}$, to (the deterministic) $\mathbb{Q}$; see [Szn91].

Theorem C.3 ([Gra92, Th. 4.1]). *Assume the sequence of the initial laws for* (C.0.2) *is $\mathbb{Q}$-chaotic. Then the sequence $\left(\bar{\mathbb{Q}}^K\right)_{K\geqslant 1}$ solving* (C.0.2) *is $\bar{\mathbb{P}}_t$-chaotic, where $\bar{\mathbb{P}}_t$ is the McKean measure solving* (C.0.1) *starting at $\mathbb{Q}$.*

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