Output Regulation in Differential Variational Inequalities using Internal Model Principle and Passivity-Based Approach

Aneel Tanwani[1,*]

[1] Department of Mathematics, Technical University of Kaiserslautern, Germany.

We consider the problem of designing state feedback control laws for output regulation in a class of dynamical systems where state trajectories are constrained to evolve within time-varying, closed, and convex sets. The first main result states sufficient conditions for existence and uniqueness of solutions in such systems. We then design a static state feedback control law using the internal model principle, which results in a well-posed closed-loop system and solves the regulation problem. As an application, we demonstrate how control input resulting from the solution of a variational inequality results in regulating the output of the system while maintaining polyhedral state constraints.

1 Introduction

Differential variational inequalities (DVIs) provide a mathematical framework to model evolution of state trajectories which, in addition to ordinary differential equations, satisfy some algebraic relations as well. Roughly speaking, DVIs comprise an ordinary differential equation to describe the motion of the state variable, and a *variational inequality* (VI) that expresses the constraints, and relations that must be satisfied by the state variable.

For the class of DVIs addressed in this paper, a set-valued mapping $\mathcal{S} : [0, \infty) \rightrightarrows \mathbb{R}^{d_s}$ is considered, and it is assumed that $\mathcal{S}(t)$ is closed, convex, and nonempty, for each $t \geq 0$. The dynamical model of the system is then described as follows:

$$\dot{x}(t) = f(t, x) + G\lambda(t) \tag{1a}$$

$$v(t) = Hx(t) + J\lambda(t), \quad v(t) \in \mathcal{S}(t), \tag{1b}$$

$$\langle v' - v(t), \lambda(t)\rangle \geq 0, \quad \forall\, v' \in \mathcal{S}(t). \tag{1c}$$

In the above equations $x(t) \in \mathbb{R}^n$ denotes the state, $\lambda(t), v(t) \in \mathbb{R}^{d_s}$, the vector field $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is absolutely continuous in first argument, and globally (and uniformly with respect to time) Lipschitz in second argument, $G \in \mathbb{R}^{n \times d_s}$, $H \in \mathbb{R}^{d_s \times n}$, $J \in \mathbb{R}^{d_s \times d_s}$ are constant matrices, and J is positive semidefinite.

In the standard references on variational inequalities, the multivalued mapping $\mathcal{S}(\cdot)$ is assumed to be stationary [5], and the case of time-varying set-valued mapping considered here is often called *quasi*-differential variational inequality; But we will refrain from this distinction in this paper, and simply use the abbreviation DVI for both cases. To make connections with the standard formulation of evolution equations with time-varying domains [4], it is seen that (1b), (1c) could be compactly written as:

$$\lambda(t) \in -\mathcal{N}_{\mathcal{S}(t)}(Hx(t) + J\lambda(t)), \tag{2}$$

where $\mathcal{N}_{\mathcal{S}}(t)(v(t))$ denotes the normal cone to the convex set $\mathcal{S}(t)$ at the point $v(t) \in \mathcal{S}(t)$, and is defined as:

$$\mathcal{N}_{\mathcal{S}(t)}(v(t)) := \{\lambda \in \mathbb{R}^{d_s} \mid \langle \lambda, v' - v(t)\rangle \geq 0, \forall\, v' \in \mathcal{S}(t)\},$$

and as convention we let $\mathcal{N}_{\mathcal{S}}(t)(v(t)) := \emptyset$, for $v(t) \notin \mathcal{S}(t)$.

One can also interpret the DVI (1) as a mathematical formalism for describing state trajectories where the motion is constrained within some prespecified set-valued map $\mathcal{S}$. To see this, let $J = 0$, then as long as $Hx(t)$ is in the interior of the set, the equation (1a) becomes $\dot{x}(t) = f(t, x(t))$ (for at least a small period of time) until $Hx(t)$ hits the boundary of the set $\mathcal{S}(t)$. At this moment, if the vector field $f(t, x(t))$ is pointed outside of the set $\mathcal{S}(t)$, then any component of this vector field in the direction normal to $\mathcal{S}(t)$ at $Hx(t)$ must be annihilated to maintain the motion of Hx within the constraint set.

The variational inequalities expressed in (1c) find utility across many applications [3] and may be used to express optimality conditions, and certain physical systems [1]. In particular, electrical circuits with nonsmooth devices (such as diodes and transistors) can be modeled as follows:

$$\dot{x}(t) = Ax + Bu(t) + G\lambda(t) \tag{3a}$$

$$0 \leq \lambda(t) \perp Hx(t) + J\lambda(t) + Du(t) \geq 0 \tag{3b}$$

* Corresponding author: E-mail tanwani@mathematik.uni-kl.de, Phone +49 (0)631 205 2735, Fax +49 (0)631 205 4986.

where the notation $0 \leq a \perp b \geq 0$ is a short-hand for saying that each component of the vectors a and b is nonnegative, and the inner product $\langle a, b \rangle = 0$. By using the notation $\mathbb{R}_+^{d_s}$ for the positive orthant of $\mathbb{R}^{d_s}$, system (3) can be rewritten in the form of (1) by letting

$$\mathcal{S}(t) := \{z \in \mathbb{R}^{d_s} \mid z + Du(t) \in \mathbb{R}_+^{d_s}\}.$$

From control perspective, DVIs form an important class of nonsmooth dynamical systems and lately several control-theoretic problems have been studied for such systems [6–8]. The problem of output regulation relates to designing control laws for asymptotically tracking a reference trajectory or rejecting disturbances. Studying this problem in the context of DVIs is useful because it provides a useful framework for

- generating and tracking certain nonsmooth signals, and
- maintaining state constraints while achieving a certain control objective.

The problem of output regulation for a simpler class of DVIs was introduced in our work [7, 8] and in this paper, we present a summary of results for a more generalized system class based on the recent research work. While VIs can also be written using the notion of (set-valued) normal cone operators so that the differential inclusion resulting from (1) has maximal monotone operators, the introduction of matrices G, H, J and the vector field $f(\cdot, \cdot)$ may not preserve monotonicity which makes it difficult to study the solution of such systems. From control point of view, the discontinuities in the description of such systems are state-dependent which introduces several complexities in designing feedback control laws.

To address these issues, we first derive conditions to study the existence and uniqueness of solutions in such systems. It is shown that under certain conditions on the system data, the given system could be equivalently written as a differential inclusion where the right-hand side is the sum of a multivalued maximal monotone operator and a Lipschitz-continuous function. Such inclusions are then shown to possess unique solutions. For the output regulation problem, we let $f(t, x) := Ax + Bu(t) + Fx_r(t)$, where x_r is an exogenous signal generated by an exosystem, and our objective is to design feedback control u such that the resulting closed-loop system is well-posed and the state x asymptotically tracks x_r. The derivation of control laws is based on the use of internal model principle, and for the sake of simplicity, only the design of a static feedback control law is presented when full state measurements are available. Under the condition that the closed-loop system can be rendered passive, we show that the overall system is indeed well-posed and the desired error variable indeed converges to the origin. As applications, we demonstrate how control input resulting from the solution of a variational inequality results in regulating the output of the system while maintaining polyhedral state constraints.

2 Problem Formulation

As stated in the introduction, we basically consider two problems related to system class (1).

1. **Well-posedness of DVI** (1)**:** First, we are interested in knowing under what conditions on the system dynamics, a unique solution exists in the following sense:

   Definition 2.1 For each initial condition $x(0)$ satisfying $Hx(0) \in \mathcal{S}(0) + \operatorname{range} J$, there exists a locally absolutely continuous function $x : [0, \infty) \to \mathbb{R}^n$, such that $x(\cdot)$ satisfies (1a) for (Lebesgue-almost) every $t \geq 0$, and $Hx(t) \in \mathcal{S}(t) + \operatorname{range} J$.

2. **Output Regulation:** For this problem, we restrict ourselves to the case of linear vector fields, and the system class in particular is defined as follows:

   $$\dot{x}(t) = Ax(t) + Bu(t) + Fx_r(t) + G\lambda(t); \qquad \lambda(t) \in -\mathcal{N}_{\mathcal{S}(t)}(Hx(t) + J\lambda(t)) \tag{4}$$

   where $x_r : [0, \infty) \to \mathbb{R}^{d_r}$ is the reference signal that is generated from the following equations:

   $$\dot{x}_r(t) = A_r x_r(t) + G_r \lambda_r(t); \qquad \lambda_r(t) \in -\mathcal{N}_{\mathcal{S}(t)}(H_r x_r(t) + J_r \lambda_r(t)). \tag{5}$$

   The output regulation variable $w(\cdot)$ is defined as:

   $$w(t) = Cx(t) - C_r x_r(t). \tag{6}$$

   It will be assumed throughout the paper that system (5) admits a solution (not necessarily unique) in the sense of Definition 2.1. We say that the output regulation is achieved if there exists a control input $u(\cdot)$ such that the following properties are satisfied:

   - **Well-posedness:** For each initial condition $x(0)$ satisfying $Hx(0) \in \mathcal{S}(0) + \operatorname{range} J$, there exists a unique solution in the sense of Definition 2.1.
   - **Regulation:** It holds that $\lim_{t \to \infty} w(t) = 0$.
   - **Closed-loop stability:** The plant and controller dynamics have a globally asymptotically stable equilibrium at the origin when $x_r \equiv 0$.

3 Existence of Solutions

The main result concerning the existence and uniqueness of solutions to system class (1) now follows.

Theorem 3.1 *There exists a unique solution to* (1)*, in the sense of Definition 2.1, if the following conditions hold:*

(C1) *The matrix J is positive semidefinite and there exists a symmetric positive definite matrix P such that*

$$\ker(J + J^\top) \subseteq \ker(PG - H^\top).$$

(C2) *For each $t \geq 0$,* $\operatorname{range} H \cap (\mathcal{S}(t) + \operatorname{range} J) \neq \emptyset$.

(C3) *The mapping $\mathcal{S} : [0, \infty) \rightrightarrows \mathbb{R}^{d_s}$ is closed and convex valued with nonempty relative interior for each $t \geq 0$, and varies in an absolutely continuous manner with time, that is, $\exists$ an absolutely continuous function $\nu(\cdot) : [0, \infty) \to \mathbb{R}_+$, such that,*

$$|d(v, \mathcal{S}(t_1)) - d(v, \mathcal{S}(t_2))| \leq |\nu(t_1) - \nu(t_2)|, \ \forall\, t_1, t_2 \geq 0.$$

(C4) *The function $f(t, \cdot)$ is globally Lipschitz (uniformly in time), that is, there exists a constant $\mu > 0$ such that $\forall\, t \geq 0$:*

$$|f(t, x_1) - f(t, x_2)| \leq \mu |x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}^n,$$

and $f(\cdot, x)$ is absolutely continuous for each $x \in \mathbb{R}^n$.

4 Output Regulation with Full State Feedback

The most classical reference on regulation of output in linear time-invariant systems is [2]. In this section, it is assumed that all the states of the plant (1) and (5) are available for feedback and a control law with static state feedback is sought which achieves stability and regulation. In the formulation of our results, the following terminology is used: A quadruple of matrices (A, B, C, D) is called *strictly passive* if there exist a scalar $\gamma > 0$ and a symmetric positive definite matrix P such that

$$\begin{bmatrix} A^\top P + PA + \gamma P & PB - C^\top \\ B^\top P - C & -(D + D^\top) \end{bmatrix} \leq 0. \tag{7}$$

Theorem 4.1 *Consider systems* (4)*,* (5) *under assumptions* **(C2)** *and* **(C3)***. Suppose that a matrix K renders the triplet $(A + BK, G, H, J)$ strictly passive, and that there exist matrices $\Pi \in \mathbb{R}^{n \times d_r}$ and $M \in \mathbb{R}^{d_u \times d_r}$ such that*

$$\Pi A_r = A\Pi + BM + F \quad \text{and} \quad C\Pi - C_r = 0 \tag{8a}$$

$$\Pi G_r = G, \quad H\Pi = H_r \quad \text{and} \quad J_r = J. \tag{8b}$$

Then the output regulation problem is solvable with the following static feedback control law:

$$u(t) = Kx(t) + (M - K\Pi)x_r(t). \tag{9}$$

When it is assumed that the full states $x(\cdot)$ and $x_r(\cdot)$ are not available for feedback, and only the regulation error $w(\cdot)$ is available to the controller, then one can design a dynamic compensator which only takes $w(\cdot)$ as the input. Using the *certainty equivalence principle*, the compensator first estimates the state variables $x(\cdot)$ and $x_r(\cdot)$ and then defines the control law as a function of these estimates. For details, see [8].

5 Case Study: Viability Control and Regulation

As an application, we consider the problem of finding a control input which maintains predefined constraints on the state trajectories of a dynamical system while achieving output regulation. Stated more precisely, suppose that we are given a plant

$$\dot{x}(t) = Ax(t) + Bu(t) + Fx_r(t) \tag{10}$$

and we would like to find a control $u(\cdot)$ which not only tracks a reference trajectory generated by the exosystem of form (5), but also results in the state satisfying the constraint that $Hx(t) \in \mathcal{S}(t)$, for all $t \geq 0$, where $\mathcal{S}(\cdot)$ is some predefined closed and convex set-valued map. This could be achieved by decomposing u as $u := u_{\text{reg}} + u_\lambda$, where we choose $u_\lambda(t)$ as the solution of the following variational inequality:

$$u_\lambda(t)^\top (v' - Hx(t)) \geq 0, \quad \forall\, v' \in \mathcal{S}(t). \tag{11}$$

This choice of control input transforms the plant equation (10) as follows:

$$\dot{x}(t) = Ax(t) + Bu_{\text{reg}}(t) + Bu_\lambda(t) + Fx_r(t). \tag{12}$$

For any $u_\lambda(\cdot)$ that satisfies (11), it now holds for the trajectories of the closed-loop system (12) that $v(t) := Hx(t) \in \mathcal{S}(t)$. When $\mathcal{S}(\cdot)$ is a time-varying polytope, (11) is formulated as a linear complementarity problem which could be solved very efficiently using standard softwares [7].

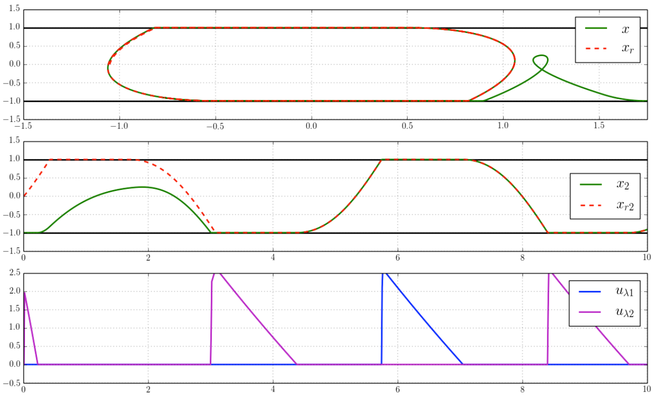

Fig. 1: The top plot shows the phase portrait of the trajectories of the plant and the exosystem. The middle plot confirms that x_2 converges to x_{r2} while staying within the set $\mathcal{S}$. The bottom plot shows the values of discontinuous component of the control input which only become nonzero when x_2 is on the boundary of the set $\mathcal{S}$.

Example 5.1 As an illustration of foregoing discussion and Theorem 4.1, let us consider an example. The plant to be controlled is a second order linear system described by the equations:

$$\dot{x}_1 = -0.1x_1 + x_2\,; \qquad \dot{x}_2 = u.$$

The exosystem is defined by the following linear complementarity system:

$$\dot{x}_r := \begin{pmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \end{pmatrix} = \begin{bmatrix} -0.1 & 1 \\ -2 & 1 \end{bmatrix} x_r + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \lambda_r\,; \qquad 0 \le \lambda_r \perp \begin{pmatrix} -x_{r2} \\ x_{r2} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ge 0. \tag{13}$$

Consider the set $\mathcal{S} := \{z \in \mathbb{R} \,:\, z + 1 \ge 0\}$ and the matrix $H := \left[\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right]$, then the relation (13) is equivalently expressed as $\lambda_r \in -\mathcal{N}_{\mathcal{S}\times\mathcal{S}}(Hx_r)$, see also [7, §5]. We are interested in designing a control input u, such that $\lim_{t\to\infty} |x_2(t) - x_{r2}(t)| = 0$, and $\forall\, t \ge 0$, $|x_2(t)| \le 1$, or equivalently $Hx(t) \in \mathcal{S} \times \mathcal{S}$. Verbally speaking, the exosystem has been chosen so that the plot of x_{r2} (versus time) resembles a sine wave clipped at the value 1, see Fig. 1. The control objective is to guarantee $|x_2(t)| \le 1$ and that x_2 converges asymptotically to x_{r2}. Decomposing the input as $u := u_{\text{reg}} + u_\lambda$ results in the closed-loop system of the form (12). In the notation of Theorem 4.1, we let $\Pi = I_{2\times 2}$, $P = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $K = [-2 \;\; -2]$, and $M = [-2 \;\; 1]$, so that $u_{\text{reg}}(t) := -2x_1(t) - 2x_2(t) + 3x_{r2}(t)$ follows from (9). The discontinuous component of the input[1] $u_\lambda := -u_{\lambda 1} + u_{\lambda 2}$ is obtained as a solution of the following complementarity problem:

$$0 \le \begin{pmatrix} u_{\lambda 1} \\ u_{\lambda 2} \end{pmatrix} \perp \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ge 0. \tag{14}$$

The results of the simulation are shown in Fig. 1.

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[1] We are implicitly using the fact that the two constraints imposed in this problem, $x_2 \le 1$ and $x_2 \ge -1$, are not active simultaneously. Thus, the complementarity formulation (14) ensures that $u_\lambda = -u_{\lambda 1}$ if $x_2 = 1$, and $u_\lambda = u_{\lambda 2}$ if $x_2 = -1$, otherwise $u_\lambda = 0$.