Extending KKL Observer Design to Systems with Non-Unique Backward Solutions ⋆

Valentin Alleaume * **Pauline Bernard** * **Aneel Tanwani** ** **Florent Di Meglio** *

* *Centre Automatique et Systèmes, Mines Paris, Université PSL, 60 bd Saint-Michel, 75272, Paris, France, (emails : firstname.lastname@minesparis.psl.eu)*
** *LAAS–CNRS, Université de Toulouse, France, (email : aneel.tanwani@cnrs.fr)*

Abstract: Kazantzis-Kravaris-Luenberger (KKL) observers consist in finding a smooth mapping T that transforms the system dynamics into a linear filter of the output in a space of larger dimension. Indeed, an observer is then obtained by running the filter and left-inverting the transformation to recover an estimate of the state, if the mapping is injective. In this paper, we are interested in adapting this framework to systems with non-unique backward solutions, a situation which can typically occur in nonsmooth systems. In this setting, the mapping T naturally becomes set-valued which is out of the scope of the current theory and calls for more general concepts of injectivity and regularity. We prove that upper semi continuity, local boundedness and *set-valued* injectivity of this map are sufficient conditions for designing a converging KKL observer. We show that the former two are satisfied for Carathéodory ODE's and Filippov differential inclusions. We also provide examples for which set-valued injectivity is satisfied and discuss its link with *distinguishability*. Finally, we illustrate the numerical implementation of this methodology on an harmonic oscillator subject to friction.

Keywords: KKL observers; Set-valued transformations; Non-unique solutions; Nonsmooth systems.

1. INTRODUCTION

State estimation for nonlinear dynamical systems is a fundamental problem that arises in a wide range of control and signal-processing applications. Over the years, numerous techniques have been developed to reconstruct unmeasured states from available outputs, including nonlinear extensions of the Kalman filter, high-gain observers, sliding-mode observers, and set-membership approaches, see (Bernard et al., 2022) for an overview. Among these, the Kazantzis-Kravaris-Luenberger (KKL) observer has emerged as a conceptually appealing framework with a systematic design methodology. Originally proposed in Luenberger (1964) for linear systems and further developed for nonlinear systems (Andrieu and Praly, 2006), the KKL approach embeds the system into a higher-dimensional space where the dynamics take the form of a linear filter of the output. The papers (Andrieu and Praly, 2006) and (Brivadis et al., 2023) established that this transformation *exists* under regularity conditions on the system, and is *injective* under a mild *backward-distinguishability* property for almost any choice of the filter matrices of dimension $2n+1$, where n is the system dimension. Several numerical methods to compute the transformation – or more importantly its inverse – have been proposed in the literature (Ramos et al., 2020; Buisson-Fenet et al., 2023; Niazi et al., 2023; Tang, 2024).

On the other hand, hybrid and nonsmooth dynamical systems introduce significant challenges for observer design due to the presence of discontinuities, switching phenomena, and, in many cases, the lack of unique solution trajectories; see, for instance, (Alessandri and Coletta, 2001; Balluchi et al., 2013; Shim and Tanwani, 2014), which address some of these issues. Even when restricting attention to systems whose discontinuities arise solely in the vector field – so that trajectories remain continuous, as in the setting of this paper – the existing literature offers only a limited set of tools capable of handling nonsmooth continuous-time dynamics and, in particular, differential inclusions. A substantial body of work addresses special cases, such as Lur'e-type systems with multivalued feedback, using dissipativity-based LMI techniques (Osorio and Moreno, 2006), monotonicity assumptions (Doris et al., 2008; Brogliato and Heemels, 2009), or measure differential inclusions coupled with local and high-gain techniques (Tanwani et al., 2014). Observers based on sliding-mode techniques have also been considered for triangular canonical forms with bounded set-valued nonlinearities (Davila et al., 2005), though they inherently suffer from chattering and noise amplification. From an application standpoint, such structural assumptions often fail: for instance, in systems with dry friction, the trian-

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gular form breaks down when friction parameters must be estimated, and the required monotonicity conditions are violated when static friction exceeds kinetic friction. Consequently, to the best of our knowledge, the current literature lacks a general observer design methodology applicable to nonsmooth systems that may admit nonunique solutions. Recent efforts based on the KKL framework aim to bridge this gap and have been extended to hybrid settings (Tran et al., 2024); however, these approaches still critically rely on uniqueness of solutions to ensure the well-posedness of the observer dynamics.

The goal of this paper is to demonstrate the applicability of the KKL approach to a broad class of dynamical systems characterized by "not-so-regular" vector fields—that is, systems whose solutions need not be unique backwards in time. Such models are common in practice, for instance in the motion of oscillators subject to Coulomb friction or in mechanical systems with unilateral constraints. A key difficulty in extending the classical KKL methodology to this setting lies in the absence of backward uniqueness: because solution trajectories cannot, in general, be traced uniquely backward in time, the standard interpretation of the injectivity requirement for the KKL transformation becomes nontrivial and requires a more careful formulation.

In what follows, Section 2 provides motivating examples illustrating how nonuniqueness of backward solutions leads naturally to set-valued KKL transformations and why the classical single-valued framework is insufficient in such cases. On this basis, Section 3 establishes that the KKL observer remains asymptotically convergent when the transformation is upper semicontinuous, locally bounded, and injective in an appropriate set-valued sense. Section 4 then shows that the first two properties hold for both Carathéodory ODEs and Filippov systems, while injectivity is linked to a suitable backward-distinguishability condition. Finally, Section 5 discusses numerical implementation aspects and illustrates the proposed observer on an oscillator subject to dry friction.

Notation: We write $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{+\infty\}$, $AC(\mathbb{R};\mathbb{R}^n)$ the set of absolutely continuous function from $\mathbb{R}$ to $\mathbb{R}^n$. For a set $S \subset \mathbb{R}^n$, we denote as $d(\cdot, S)$ the distance to S.

2. PROBLEM SETTING

To formalize the problem studied in this paper, we first outline representative scenarios where a dynamical systems may exhibit nonunique backward solutions. We then recall the main ideas underlying the KKL observer construction and discuss how the lack of backward uniqueness complicates the definition and analysis of the associated state transformation.

2.1 Class of systems

Our primary objective is to address KKL-based observer design for systems where the solutions are not necessarily unique from a given initial condition. Such situations arise for dynamical systems modeled by ordinary differential equations that do not possess sufficient regularity. Without specifying a particular model class for its dynamics, we describe our dynamical system as a mapping which, to each initial conditions x_0 in a subset $\mathcal{O} \subseteq \mathbb{R}^n$ of interest associates a set of maximal solutions initialized at x_0, namely

$$\begin{aligned} \Phi : \mathcal{O} \subseteq \mathbb{R}^n &\rightrightarrows AC(\mathbb{R};\mathbb{R}^n) \\ x_0 &\mapsto \Phi(x_0) \subseteq \{\varphi : \mathbb{R} \to \mathbb{R}^n \mid \varphi(0) = x_0\}. \end{aligned} \tag{1}$$

For simplicity, we assume that for every admissible $x_0 \in \mathcal{O}$, any trajectory is defined over the entire real line. For our purposes, the set of backward trajectories is particularly relevant and we denote it by $\Phi_{\rm bw}(x_0) := \{\phi_{|(-\infty,0]},\, \phi \in \Phi(x_0)\}$. Note that neither backward nor forward uniqueness is assumed.

Two particular cases where we observe non-unique solutions are:

- ODEs with non-Lipschitz vector field: For dynamical systems of the form
$$\dot{x} = f(x);$$
with initial condition $x(0) = x_0 \in \mathbb{R}^n$, if the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is not locally Lipschitz continuous, the solutions may not be defined uniquely. Examples include the system $\dot{x} = \sqrt{|x|}$ which exhibits non unique solutions at 0.
- Secondly, non-uniqueness of solutions may arise in dynamical systems with discontinuous right-hand side. One way to express such dynamical systems is the following differential equation:
$$\dot{x} = f_\sigma(x) \tag{2}$$
with state $x \in \mathbb{R}^n$, and where σ, commonly called the *switching signal*, takes values in a finite index set. Depending on the system under consideration, σ may be time-dependent or state-dependent. Our focus in this manuscript is on systems with state-dependent switching signals. In particular, we consider a collection of closed sets $\mathcal{X}_i$, $i \in \{1,\dots,M\}$, so that, for every $x \in \mathcal{X}_i$, we have $\sigma(x) = i$. The sets $\mathcal{X}_i$ have non-empty and non-overlapping interior and they only intersect with each other at the boundary. The vector field on the right-hand side of (2) therefore becomes discontinuous on the boundary of the sets $\mathcal{X}_i$. Studying solutions of systems with state-dependent discontinuities requires us to regularize the system, and here, we work with Filippov regularization. Using the *active index set* $\mathcal{I}(x) = \{i \in \{1,\dots,M\} \mid i \in \mathcal{X}_i\}$, this regularization leads to the following differential inclusion (under certain regularity assumptions):
$$\dot{x} \in \mathcal{F}(x) = \begin{cases} f_i(x), & x \in \operatorname{int}\mathcal{X}_i \\ \operatorname{co}\{f_i(x) \mid i \in \mathcal{I}(x)\}, & \text{otherwise.} \end{cases} \tag{3}$$
As a specific example, we consider the scalar dynamical system $\dot{x} = f_i(x)$, $i \in \{1,2\}$ with $f_1(x) = -1$ for $x \geq 0$ and $f_2(x) = 1$ for $x \leq 0$. With Filippov regularization, the system is described as
$$\dot{x} \in -\operatorname{Sign}(x), \qquad x(0) = x_0 \in \mathbb{R} \tag{4}$$
where $\operatorname{Sign}(0) = [-1, 1]$. The system has unique forward complete solutions, as we note that, for each $t \geq 0$, we have (with the convention that $\operatorname{sign}(0) = 0$)
$$x(t) = \begin{cases} x_0 - \operatorname{sign}(x_0)t, & t \in [0, |x_0|], \\ 0, & t \geq |x_0|. \end{cases}$$
However, with initial condition $x_0 = 0$, we cannot uniquely define the solution backward in time. In par-

ticular, we note that for any $t_1 \in (-\infty, 0]$, $t \mapsto \phi^\theta(t)$ for $\theta \in \{1, -1, 0\}$ defined on $\mathbb{R}_{\leq 0}$ as

$$\phi^\theta_{t_1}(t) = \begin{cases} 0 & t \geq t_1, \\ -\theta(t - t_1), & t < t_1, \end{cases} \tag{5}$$

are all solutions. It follows that $\Phi_{\mathrm{bw}}(0) := \{\phi^\theta_{t_1}, t_1 \in (-\infty, 0], \theta \in \{1, -1, 0\}\}$ is set-valued.

For the observer design problem, we associate an observation equation with this dynamical system which provides partial measurement of the state:

$$y = h(x) \tag{6}$$

where $h : \mathbb{R}^n \to \mathbb{R}^{n_y}$ is assumed to be continuously differentiable. For a given initial condition x_0, we associate to each possible solution $\phi \in \Phi(x_0)$, its corresponding output trajectory defined as $h \circ \phi$.

2.2 KKL observer design

For a "regular" dynamical system described by an ordinary differential equations (ODE)

$$\dot{x} = f(x) \tag{7}$$

with locally Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^n$ and output (6), an observer can be designed by looking for a state transformation $\mathbb{R}^n \ni x \mapsto z = T(x) \in \mathbb{R}^{n_z}$, with $n_z \geq n$, such that $z(t) = T(x(t))$ follows dynamics of the form

$$\dot{z} = Az + By \tag{8}$$

along the system solutions of interest, with matrices $A \in \mathbb{R}^{n_z \times n_z}$ Hurwitz and $B \in \mathbb{R}^{n_z \times n_y}$ to be chosen. By choosing A to be Hurwitz, any solution $t \mapsto z(t)$ to (8) converges to $t \mapsto T(x(t))$, and if the transformation T is injective, an estimate $\widehat{x}(t)$ of $x(t)$ may be asymptotically recovered via a left-inverse of T, denoted $T^\star$, leading to an observer of the form

$$\dot{z} = Az + By, \qquad \widehat{x} = T^\star(z). \tag{9}$$

In fact, under appropriate backward-completeness assumptions, it can be shown that a possible choice of the mapping T is defined as

$$T(x) = \int_{-\infty}^{0} \exp(-As) B(h(X(s; x))) ds \tag{10}$$

where $X(s; x)$ denotes the (unique) solution of the system at time s starting with initial condition x, that is, $X(0; x) = x$. This mapping T can be shown to be well-posed and injective under mild regularity assumptions on the system data, as well as *backward distinguishability*, namely the fact that initial conditions can be distinguished from the knowledge of their backward outputs. The goal of this paper is to demonstrate the possible extension of this methodology to nonsmooth systems with possibly nonunique backward solutions.

3. KKL DESIGN WITH NONUNIQUE SOLUTIONS

In this section, we present the main theoretical developments of the paper. We first construct the set-valued transformation T that generalizes the classical KKL mapping to systems with possibly nonunique trajectories. We then derive sufficient conditions –expressed in terms of regularity and structural properties of T – that ensure the asymptotic reconstruction of the system state.

3.1 Set-valued transformation

A natural generalization of (10) is to consider a *set-valued* map T defined on a subset $\mathcal{O}$ as

$$\begin{aligned} &T : \mathcal{O} \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^{n_z} \\ &T(x) = \left\{ \int_{-\infty}^{0} e^{-As} Bh(\phi(s)) ds, \phi \in \Phi_{\mathrm{bw}}(x) \right\} \end{aligned} \tag{11}$$

provided the solutions are backward complete and the corresponding output does not grow faster than a given exponential as defined in assumption 1. Observe that this set-valued extension is not necessary for systems that only exhibit forward non-uniqueness

Assumption 1. There exist $\mathcal{O} \subseteq \mathbb{R}^n$, $\rho > 0$ and $\psi \in L_1(\mathbb{R}_{\leq 0})$ such that for any $x \in \mathcal{O}$, any backward solution $\phi \in \Phi_{\mathrm{bw}}(x)$, we have $\forall s \in \mathbb{R}_{\leq 0}, |e^{\rho s} h(\phi(s))| \leq \psi(s)$.

Remark 1. (Backward saturation). If the solutions to be estimated with the observer remain in forward time in a bounded set $\mathcal{X}$, it is possible to modify the system dynamics and solutions outside of $\mathcal{X}$ without changing the estimation problem. Thus, one can always satisfy Assumption 1 by saturating the system solutions and dynamics outside of $\mathcal{X}$. But this should be done with care to preserve the required regularity and observability properties, see (Andrieu and Praly, 2006; Brivadis et al., 2023) for a lengthier discussion.

Although T no longer is a single-valued transformation that maps the system dynamics into the observer dynamics (8), we can still show that solutions to (8) converge to the image by T of the system solutions in the following sense.

Theorem 1. Suppose that Assumption 1 holds. Consider a pair $(A, B) \in \mathbb{R}^{n_z \times n_z} \times \mathbb{R}^{n_z \times n_y}$ such that $A + \rho I$ is Hurwitz for some $\rho > 0$, and the map T defined in (11). Then, for any system solution $t \mapsto x(t)$ initialized in $\mathcal{O}$, there exists a C^1 selection $t \mapsto z(t) \in T(x(t))$ solution to (8) (fed with the corresponding output $t \mapsto y(t) = h(x(t))$), such that any other solution $t \mapsto \hat{z}(t)$ to (8) verifies

$$\lim_{t \to \infty} \|\hat{z}(t) - z(t)\| = 0,$$

that is,

$$\lim_{t \to \infty} d(\hat{z}(t), T(x(t)) = \lim_{t \to \infty} \inf_{\tilde{z} \in T(x(t))} \|\hat{z}(t) - \tilde{z}\| = 0.$$

Proof. Pick a system solution $t \mapsto x(t)$ initialized in $\mathcal{O}$ and consider its output $t \mapsto y(t) = h(x(t))$. Under Assumption 1, the map T is well-defined and one can define $z : \mathbb{R} \to \mathbb{R}^m$ as

$$z(t) = \int_{-\infty}^{0} e^{-As} By(t + s) ds.$$

By definition, we have $z(t) \in T(x(t))$ for all t. Besides, for any t, and any $h \neq 0$,

$$\begin{aligned} z(t + h) &= \int_{-\infty}^{0} e^{-As} By(t + h + s) ds \\ &= \int_{-\infty}^{h} e^{-A(s-h)} By(t + s) ds \\ &= e^{Ah} \int_{-\infty}^{h} e^{-As} By(t + s) ds \end{aligned}$$

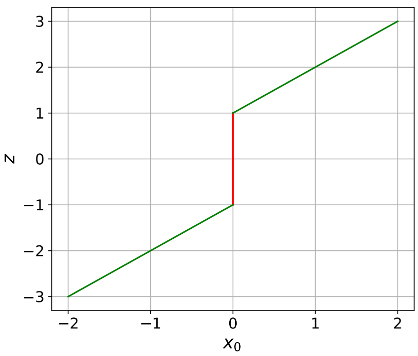


Fig. 1. Graph of T defined in (12): set-valuedness does not hinder injectivity.

so that

$$\frac{1}{h}(z(t+h)-z(t)) = \frac{e^{Ah}-I}{h}z(t)+\frac{e^{Ah}}{h}\int_0^h e^{-As}By(t+s)ds$$

which indeed tends to $Az(t)+By(t)$ as h tends to 0. It follows that z is C^1 and solution to (8). Since A is Hurwitz, any other solution to (8) converges to z, hence the result.

It follows that running (8) from any initial condition provides an estimate $\hat{z}$ that asymptotically approaches the image of the system solution by T. Despite its set-valuedness, T may very well be *injective* (in a sense to be defined), admit a left inverse, and thus allow for the asymptotic reconstruction of the system solution.

Example 1. Consider the dynamics (4) with output $h(x) = x$. Pick a one-dimensional pair $(A, B) = (\lambda, 1)$ with $\lambda < 0$. From the expression of the backward solutions (5), the map T defined in (11) writes as

$$T(x) = \int_{-\infty}^{0} e^{-\lambda s}\left(x - \mathrm{sign}(x)s\right) ds = -\frac{x}{\lambda} + \frac{\mathrm{sign}(x)}{\lambda^2}$$

for $x \neq 0$, and

$$T(0) = \left\{\frac{\theta}{\lambda^2}e^{-\lambda t_1}, \quad \theta \in \{-1, 1, 0\}, \quad t_1 \in (-\infty, 0]\right\}$$

leading to

$$T(x) = \begin{cases} -\dfrac{x}{\lambda} + \dfrac{\mathrm{sign}(x)}{\lambda^2} & \text{if } x \neq 0 \\ \left[-\dfrac{1}{\lambda^2}, \dfrac{1}{\lambda^2}\right] & \text{if } x = 0. \end{cases} \tag{12}$$

The graph of T is given in Figure 1. We can see that this set-valued map is *injective* on $\mathbb{R}$ in the sense that any point z in the image $\mathbb{R}$ of T determines a unique x such that $z \in T(x)$.

3.2 Sufficient conditions for observer convergence

We have seen in Theorem 1 that running (8) from any initial condition provides an estimate $\hat{z}$ that asymptotically approaches the image of the system solution by T. We next provide sufficient injectivity and regularity conditions, see (Aubin and Frankowska, 2009) on the map T to guarantee the existence of a left-inverse providing a converging estimate in the x-coordinates.

Definition 1. (Def 5.4.9 in Aubin and Frankowska (2009)). A set-valued map T is said to be injective on $\mathcal{X}$ if

$$\forall x_a, x_b \in \mathcal{X}^2, x_a \neq x_b \implies T(x_a) \cap T(x_b) = \emptyset. \tag{13}$$

Lemma 1. Let $\mathcal{X} \subset \mathbb{R}^n$ be compact and $T : \mathcal{X} \rightrightarrows \mathbb{R}^{n_z}$ an injective, upper semicontinuous and locally bounded map. Then, there exists $T^\star : \mathbb{R}^{n_z} \to \mathbb{R}^n$ uniformly continuous such that

$$T^\star(z) = x \quad \forall x \in \mathcal{X}, \quad \forall z \in T(x). \tag{14}$$

Proof. First, notice that the injectivity property on $\mathcal{X}$ allows us to define a single-valued map $T^\star : T(\mathcal{X}) \to \mathbb{R}^n$ as

$$T^\star(z) = \text{“unique } x \in \mathcal{X} \text{ such that } z \in T(x)\text{”}.$$

We then prove that there exists a concave $\mathcal{K}$-map ρ such that

$$|x_a - x_b| \leq \rho\left(\Delta_T(x_a, x_b)\right), \qquad \forall (x_a, x_b) \in \mathcal{X} \times \mathcal{X}, \tag{15}$$

where

$$\Delta_T(x_a, x_b) := \min_{\substack{z_a \in T(x_a) \\ z_b \in T(x_b)}} |z_a - z_b|.$$

For that, define for $s \in \mathbb{R}_{\geq 0}$,

$$\rho_0(s) := \sup_{\substack{(x_a, x_b) \in \mathcal{X} \times \mathcal{X} \\ \Delta_T(x_a, x_b) \leq s}} |x_a - x_b| \tag{16}$$

and observe that ρ_0 is non-decreasing and by injectivity, $\rho_0(0) = 0$. Then, let us show that ρ_0 is continuous at 0. For that, define the set-valued map $F : \mathcal{X} \times \mathcal{X} \rightrightarrows \mathbb{R}^{n_z}$ the following way : $F(x_a, x_b) = \{z_a - z_b, z_a \in T(x_a), z_b \in T(x_b)\}$. F is upper semicontinuous thanks to upper semicontinuity of T. Let $g : v, z \in \mathrm{Graph}(F) \mapsto -|z| \in \mathbb{R}$. Then, the map $\Delta_T : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ writes as

$$\Delta_T(v) = -\sup_{z \in F(v)} g(v, z) \tag{17}$$

and is thus upper semicontinuous (as a real-valued function) using (Aubin and Frankowska, 2009, Theorem 1.4.16) (F and g being upper semicontinuous and F having compact values by upper semicontinuity and local boundedness). We then proceed as in (Bernard, 2019, Lemma A.9) to show that ρ_0 defined in (16) is upper semicontinuous at 0, and thus continuous since it is only defined on nonnegative values. Indeed, let us fix a sequence $(s_k) \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ converging to 0. We can extract from this sequence $s_{\psi(k)}$ such that $s_{\psi(k)}$ is decreasing. By monotony and positivity of ρ_0 we know that $\lim_{k\to\infty} \rho_0(s_{\psi(k)})$ exists in $\mathbb{R}$. By definition of the sup in ρ_0, we can find sequences $(x_a^k) \in \mathcal{X}^{\mathbb{N}}$, $(x_b^k) \in \mathcal{X}^{\mathbb{N}}$ such that

$$\Delta_T(x_a^k, x_b^k) \leq s_{\psi(k)} \tag{18}$$

$$\rho_0(s_{\psi(k)}) - \frac{1}{k} \leq |x_a^k - x_b^k| \leq \rho_0(s_{\psi(k)}). \tag{19}$$

With $\mathcal{X}$ being compact, we can extract a converging subsequence from $(x_a^k), (x_b^k)$ and denote their respective limits by x_a^*, x_b^*. From (18), we get $\lim_{k\to\infty} \Delta_T(x_a^k, x_b^k) = 0$. Upper semicontinuity and positivity of Δ_T then give $\lim_{k\to\infty} \Delta_T(x_a^k, x_b^k) = \Delta_T(x_a^*, x_b^*) = 0$. Local boundedness and upper semicontinuity of T ensure that $T(x_a^*), T(x_b^*)$ are compact. By injectivity of T, we get that $\Delta_T(x_a^*, x_b^*) = 0 \implies x_a^* = x_b^*$ and using (19) it follows $\lim_{k\to\infty} \rho_0(s_{\psi(k)}) =$

0. We conclude that ρ_0 is continuous at 0. Proceeding as in the proof of (Bernard, 2019, Lemma A.6), we know there exists a regularization of ρ_0 into a concave class $\mathcal{K}$-map ρ so that (15) holds. Then, from (15), and by definition of $T^\star$, we get that

$$|T^\star(z_a) - T^\star(z_b)| \leq \rho(|z_a - z_b|), \quad \forall (z_a, z_b) \in T(\mathcal{X}) \times T(\mathcal{X}),$$

namely $T^\star$ is uniformly continuous on $T(\mathcal{X})$. Finally, using McShane extension theorem (McShane, 1934, Theorem 2) to each component of $T^\star$, we obtain a uniformly continuous extension of $T^\star$ on $\mathbb{R}^{n_z}$, such that

$$|T^\star(z_a) - T^\star(z_b)| \leq c\rho(|z_a - z_b|), \quad \forall (z_a, z_b) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_z}, \tag{20}$$

for some $c > 0$.

From the combination of Theorem 1 and Lemma 1, we obtain the following convergence result for observer (9).

Theorem 2. Let $\mathcal{X} \subseteq \mathcal{O} \subseteq \mathbb{R}^n$ be compact. Suppose that Assumption 1 holds and that T defined in (11) is injective, upper semicontinuous and locally bounded on $\mathcal{X}$. Consider $T^\star$ given by Theorem 1. Then, for any system solution $t \mapsto x(t)$ defined on $\mathbb{R}$ and remaining in $\mathcal{X}$ for all forward times,any solution $\hat{z}$ to the observer (8) fed with the corresponding system output $t \mapsto y(t)$ verifies

$$\lim_{t \to \infty} \|T^\star(\hat{z}(t)) - x(t)\| = 0.$$

Proof. Direct consequence of Theorem 1 combined with (14) and (20).

Note that those conditions are satisfied in Example 1. We next study another example.

Example 2. Consider the differential equation $\dot{x} = \sqrt{|x|}$ with output $h(x) = x$. For negative initial conditions x_0, solutions in backward-time are unique, given by

$$x(t) = -\frac{1}{4}(-2\sqrt{|x_0|} + t)^2$$

while for nonnegative initial conditions x_0, solutions in backward-time are nonunique given by

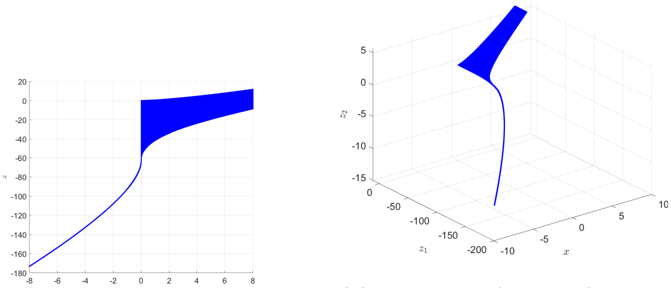
$$x(t) = \begin{cases} \frac{1}{4}(2\sqrt{|x_0|} + t)^2, & -2\sqrt{|x_0|} \leq t \leq 0 \\ 0, & -t_1 \leq t + 2\sqrt{|x_0|} \leq 0 \\ -\frac{1}{4}(t + t_1 + 2\sqrt{|x_0|})^2, & t \leq -t_1 - 2\sqrt{|x_0|} \end{cases}$$

for any $t_1 \in [0, +\infty)$ parametrizing the time period where the solution stays at 0. Then, it can be checked that, for $(A, B) = (\lambda, 1)$ with $\lambda < 0$, the map T defined in (11) writes as

$$T(x) = \begin{cases} \dfrac{1}{2\lambda^3} - \dfrac{x}{\lambda} - \dfrac{\sqrt{|x|}}{\lambda^2} & \text{if } x < 0 \\ \left\{ \dfrac{\gamma_\lambda(x, e^{t_1})}{2\lambda^3} - \dfrac{x}{\lambda} - \dfrac{\sqrt{|x|}}{\lambda^2}, \, t_1 \in \overline{\mathbb{R}}_{\geq 0} \right\} & \text{if } x \geq 0 \end{cases}$$

where $\gamma_\lambda(u_1, u_2) := e^{2\lambda\sqrt{|u_1|}}(1 + u_2^\lambda) - 1$ with the convention that $e^{-\infty} = 0$, leading to

$$T(x) = \begin{cases} \kappa_\lambda(x) + \dfrac{\gamma_\lambda(0,1)}{2\lambda^3} & \text{if } x < 0 \\ \left[\kappa_\lambda(x) + \dfrac{\gamma_\lambda(x,0)}{2\lambda^3}, \kappa_\lambda(x) + \dfrac{\gamma_\lambda(x,1)}{2\lambda^3} \right] & \text{if } x \geq 0, \end{cases} \tag{21}$$



(a) $A = -0.2$, $B = 1$

(b) $A = \text{diag}(-0.2, -1)$, $B = (1, 1)$

Fig. 2. Graph of T for $\dot{x} = \sqrt{|x|}$ with different dimensions for the pair (A, B).

where $\kappa_\lambda(x) := -\frac{x}{\lambda} - \frac{\sqrt{|x|}}{\lambda^2}$. The graph of this set-valued map T is shown in Figure 2 for two choices of the pair (A, B). T is indeed upper semicontinuous and locally bounded. However, one sees that for (A, B) of dimension 1, T is not injective since values of z are in the image of several distinct points x. On the other hand, it seems that picking (A, B) of dimension 2 allows to spread the set-valuedness in a way that makes T injective.

4. REGULARITY AND INJECTIVITY OF T

As shown in the previous section, Theorem 2 provides sufficient conditions on the mapping T for the KKL observer to be asymptotically convergent. In this section, we are interested in translating them into appropriate conditions on *the system*. We first show that continuity of solutions implies local boundedness and upper semicontinuity. Thereafter, we discuss the link between backward distinguishability and injectivity.

4.1 Local boundedness and upper semicontinuity of T

Firstly, for the regularity of T, we require the following assumption on the system dynamics which describes the continuity of solutions with respect to initial conditions over compact intervals.

Assumption 2. For every converging sequence $x_k \to x$, and every compact interval $I_K \subset (-\infty, 0]$, and the corresponding sequence of solutions $\phi_k : I_K \to \mathbb{R}^n$ with $\phi_k(0) = x_k$, it holds that ϕ_k converges uniformly to a solution $\phi : I_K \to \mathbb{R}^n$, with $\phi(0) = x$.

Proposition 1. Suppose that Assumption 1 and Assumption 2 hold, then T defined in (11) is locally bounded and upper semicontinuous.

Before providing the proof of Proposition 1, we note that the property listed in Assumption 2 is often called the *sequential compactness* of trajectories. Let us provide some system classes where the convergence property stated in Assumption 2 naturally holds.

- *Carathéodory ODEs:* Consider the case of ordinary differential equations of the form $\dot{x} = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Carathéodory conditions: i) $f(t, \cdot)$ is continuous for almost all $t \in \mathbb{R}$; ii) the function $f(\cdot, x)$ is measurable for each x, and iii) For every compact set $K \subset \mathbb{R}^n$, and every interval $I \subset \mathbb{R}$, there exists an integrable function $\psi_K : I \to \mathbb{R}$ such that $|f(t,x)| \le \psi_K(t)$. Then, it follows from (Filippov, 1988, Theorem 5, Page 9) that Assumption 2 holds for differential equations where the right-hand side satisfies Carathéodory conditions. In particular, among the examples given earlier, the trajectories of the system $\dot{x} = \sqrt{|x|}$ satisfies Assumption 2.

- *Filippov Differential Inclusions:* Let us consider the case where the state trajectories are defined by the differential inclusion of the form

$$\dot{x} \in \mathcal{F}(t,x), \quad x \in \mathcal{O} \subset \mathbb{R}^n$$

where $\mathcal{F} : \mathbb{R}\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is nonempty, convex- and closed-valued, and upper semicontinuous in x. In that case, Assumption 2 holds due to (Filippov, 1988, Theorem 3, Page 79). As an example, we see that $\dot{x} \in -\mathrm{Sign}(x)$ satisfies the aforementioned conditions, and thus, Assumption 2 holds in that case.

Proof. [Proof of Proposition 1] To prove local boundedness, take $x \in \mathcal{O}$ and a neighborhood $\mathcal{U} \subset \mathcal{O}$ of x. For any $\bar{x} \in \mathcal{U}$, and $\phi \in \Phi_{\mathrm{bw}}(\bar{x})$, with ρ and ψ defined in Assumption 1, we have

$$\left\| \int_{-\infty}^{0} e^{-As} Bh(\phi(s))ds \right\| \le \int_{-\infty}^{0} \left\| e^{-(A+\rho I)s} B \right\| \psi(s)ds < \infty.$$

To show upper semicontinuity, let us pick a sequence of initial conditions x_k that converges to x. Using the backward completeness stated in Assumption 1 , $T(x_k)$ is nonempty and we choose a sequence $z_k \in T(x_k)$ such that $z_k \to z$, for some $z \in \mathbb{R}^m$. For each z_k, there is $\phi_k \in \Phi_{\mathrm{bw}}(x_k)$ such that, $z_k = \int_{-\infty}^{0} e^{-As} Bh(\phi_k(s))ds$. From Assumption 2, it follows that there exists $\phi \in \Phi_{\mathrm{bw}}(x)$ such that $\phi_k(s) \to \phi(s)$ for each $s \in (-\infty, 0]$. The same assumption allows us to dominate the integrand $e^{-As}Bh(\phi_k(\cdot))$ by an integrable function, and invoke dominated convergence theorem. Thus, we get

$$\lim_{k\to\infty} z_k \to z = \int_{-\infty}^{0} e^{-As} Bh(\phi(s))ds \in T(x)$$

which proves the desired claim.

4.2 Discussion regarding injectivity

In the classical KKL setting, *backward distinguishability* of the system is essential to prove injectivity of the mapping T. The formulation of this property can be adapted to our setting where backward trajectories need not be unique.

Definition 2. (Backward Distinguishable). The dynamical system is backward distinguishable on $\mathcal{X}$ if for all $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$ such that $x_a \neq x_b$, for all $\phi_a \in \Phi_{\mathrm{bw}}(x_a)$ and all $\phi_b \in \Phi_{\mathrm{bw}}(x_b)$, there exists $t \le 0$ such that

$$h \circ \phi_a(t) \neq h \circ \phi_b(t).$$

Observe that if there exists $x_a \neq x_b$, $\phi_a \in \Phi_{\mathrm{bw}}(x_a)$, $\phi_b \in \Phi_{\mathrm{bw}}(x_b)$ such that $h \circ \phi_a = h \circ \phi_b$, then, from the definition of T in (11), we get $T(x_a) \cap T(x_b) \neq \emptyset$ and T is not injective on $\mathcal{X}$. In other words, backward-distinguishability is necessary for injectivity. Whether it is sufficient remains an open question for future research.

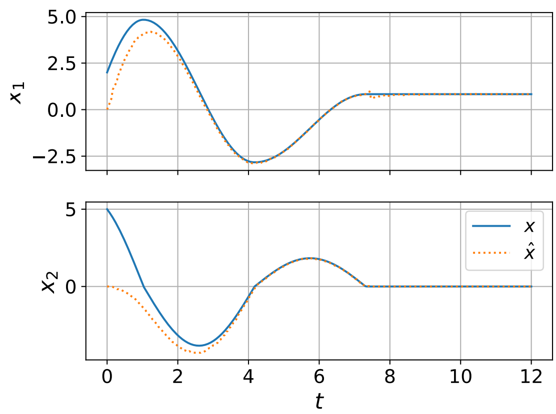


Fig. 3. Asymptotic convergence of the KKL observer for system (22).

5. NUMERICAL OBSERVER IMPLEMENTATION

In (Buisson-Fenet et al., 2023, section 4.1), a systematic *supervised* numerical methodology is exposed to implement the KKL observer (9) in the context of smooth systems. It consists in learning a numerical approximation for the mapping T, and more importantly its inverse $T^\star$. It only requires the ability to compute solutions of the system in backward and forward time. The backward integration step being optional, one can ignore it in order to adapt this method in our context where backward solutions may be nonunique. A numerical solver in forward time is still needed as shown in the following example.

Example 3. Consider an oscillator subject to friction

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 - \mu \,\mathrm{sign}(x_2) \end{cases}, \quad y = x_1. \tag{22}$$

One can check that this system is forward unique and nonsmooth, we consequently decide to simulate it using an implicit scheme. On the other hand, it is backward unique everywhere but on $[-\frac{\mu}{\omega^2}, \frac{\mu}{\omega^2}] \times 0$. Direct derivation of $T, T^\star$ is more tedious than in the previous examples, which motivates the use of the numerical method. We choose $\mu = \omega^2 = 1$, $A = \mathrm{diag}(-5, -1, -2)$, $B = [1, 1, 1]^\top$. For a uniform array of 10^6 initial conditions (x_0, z_0) obtained from sampling the square $[-10, 10]^2$, we integrate the (x, z) cascade dynamics (22)-(8) and drop the pair of points associated to the transient of z, i.e. $(x(t), z(t))$ such that $t \le 5s$. We store the remaining pairs of points (x_i, z_i) as a lookup table. Because the transient of z was dropped, we know from Theorem 2 that $d(z_i, T(x_i))$ is close to zero. Moreover, system (22) being a piecewise linear Filippov system, it verifies Assumptions 1 and 2. One can check that it is backward distinguishable, which leads us to conjecture that T is injective. Then, by Lemma 1 we get the existence of a left inverse $T^\star$ and by Theorem 2, we have $x_i \approx T^\star(z_i)$. Therefore, the previously constructed dataset gives an approximation of $T^\star$ and can be used as look-up table (or to fit a numerical model of $T^\star$, e.g. through neural netwoks). One can then implement the KKL observer by running (8) with an arbitrary initial condition z_0 and applying the learned $T^\star$. The performance of the corresponding observer (9)(with the dataset used as look-up table) can be seen in Figure 3.

Remark 2. (PDE-based methods). In the classical KKL setting with smooth systems, the transformation T is known to satisfy a PDE which can be leveraged in an *unsupervised* learning approach through an autoencoder with z as a latent space. The knowledge of this PDE is also used in Physics Informed Neural Network (Niazi et al., 2023). In this paper, considering non-Lipschitz systems and a set-valued map T, it remains unclear whether this PDE is still relevant in a certain sense, and consequently adapting the related numerical technique requires more work.

6. CONCLUSION

In this paper, we show that the KKL methodology for designing observers extends naturally to nonsmooth systems. Indeed, we provide a set of theoretical conditions under which one can design an asymptotic observer and show that those conditions are verified on some illustrative examples. Such extension does not introduce additional numerical difficulties: the resulting observer can be implemented with the same computational structure as in the smooth case, enabling us to demonstrate its effectiveness on a physically relevant system exhibiting nonunique backward trajectories. This work motivates further research aimed at characterizing classes of such systems for which injectivity of the KKL transformation can be guaranteed through suitable backward-distinguishability conditions.

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