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Évolution des mesures et contrôle optimal de systèmes dynamiques quasi-dissipatifs à l'aide de relaxations Moment-SOS

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To the silent emptiness of creation,
and the restless chaos of reality.

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Abstract

Optimization of nonlinear dynamical systems is challenging, and one typically exploits specific structural properties of the dynamics to analyze their evolution and design control laws. A particular class of systems that is central to this thesis is that of quasi-dissipative dynamical systems, which can be viewed as systems with a dissipative component perturbed by an additional term. Given the limitations of classical optimization methods for such dynamical systems, this thesis develops a unified framework for their analysis and optimization using measure relaxation techniques. The moment–sums-of-squares (SOS) hierarchy is the main computational motivation for this approach, as it provides globally optimal solutions with convergence guarantees.

Classical control and optimization methods become difficult to apply when system dynamics involve discontinuities, unilateral constraints, or set-valued mappings. We extend the measure-theoretic relaxation formalism to these nonsmooth settings. The first contribution establishes a rigorous formulation of measure evolution for first-order sweeping processes, proving existence, uniqueness, and a superposition representation of measure-valued solutions. A functional regularization approach and a time-discretized optimal transport scheme are developed to approximate these solutions, with convergence guarantees in Wasserstein metrics. Within this framework, we formulate a moment–SOS based semidefinite relaxation to solve the measure evolution problem.

The second contribution addresses optimal control of sweeping processes using measure relaxation techniques. We show that relaxing to a linear program in the space of measures introduces no relaxation gap in both continuous and discrete time. Using tools from optimal transport theory, we prove convergence of the discretized problem to the continuous one as the sampling interval tends to zero. A moment-SOS based semidefinite relaxation is proposed to solve the measure-valued optimal control problem.

Finally, the framework is extended to quasi-dissipative nonlinear evolution equations on Hilbert spaces, including semilinear and reaction diffusion partial differential equations. We show that the measure formulation of these infinite dimensional problems remains exact, with no relaxation gap, and we propose a convergent moment-SOS hierarchy to obtain certified numerical approximations.

Overall, the thesis combines variational analysis, optimal transport, and semidefinite programming to provide globally convergent convex formulations for a broad class of nonsmooth optimal control and PDE problems.

Keywords: measure evolution, optimal control, measure relaxation, optimal transport, sweeping process, nonsmooth dynamical systems, quasi-dissipative evolution equations, moment-SOS hierarchy, semidefinite programming

Résumé

L'optimisation des systèmes dynamiques non linéaires est difficile, et l'on exploite généralement des propriétés structurelles particulières des dynamiques pour analyser leur évolution et concevoir des lois de commande. Une classe de systèmes au cœur de cette thèse est celle des systèmes dynamiques *quasi-dissipatifs*, que l'on peut voir comme des systèmes comportant une composante dissipative perturbée par un terme additionnel. Compte tenu des limites des méthodes classiques d'optimisation pour de tels systèmes, cette thèse développe un cadre unifié pour leur analyse et leur optimisation au moyen de techniques de relaxation par mesures. La hiérarchie moments–sommes de carrés (SOS) constitue la principale motivation computationnelle de cette approche, car elle fournit des solutions globalement optimales avec des garanties de convergence.

Les méthodes classiques de commande et d'optimisation deviennent difficiles à appliquer lorsque les dynamiques comportent des discontinuités, des contraintes unilatérales ou des applications multivoques. Nous étendons le formalisme de relaxation mesure-théorique à ces contextes non lisses. La première contribution établit une formulation rigoureuse de l'évolution de mesures pour les processus de balayage du premier ordre, en démontrant l'existence, l'unicité et une représentation de superposition des solutions à valeurs mesures. Une approche de régularisation fonctionnelle et un schéma de transport optimal discrétisé en temps sont développés pour approximer ces solutions, avec des garanties de convergence pour des métriques de Wasserstein. Dans ce cadre, nous formulons une relaxation semi-définie basée sur la hiérarchie moments–SOS pour résoudre le problème d'évolution de mesures.

La deuxième contribution traite de la commande optimale des processus de balayage via des techniques de relaxation par mesures. Nous montrons que la relaxation en un programme linéaire dans l'espace des mesures n'introduit aucun écart de relaxation, en temps continu comme en temps discret. En utilisant des outils de la théorie du transport optimal, nous prouvons la convergence du problème discrétisé vers le problème continu lorsque le pas d'échantillonnage tend vers zéro. Une relaxation semi-définie fondée sur moments–SOS est proposée pour résoudre le problème de commande optimale à valeurs mesures.

Enfin, le cadre est étendu aux équations d'évolution non linéaires quasi-dissipatives sur des espaces de Hilbert, incluant des EDP semilinéaires et de type réaction–diffusion. Nous montrons que la formulation par mesures de ces problèmes de dimension infinie reste exacte, sans écart de relaxation, et nous proposons une hiérarchie moments–SOS convergente afin d'obtenir des approximations numériques certifiées.

Dans l'ensemble, cette thèse combine l'analyse variationnelle, le transport optimal et la programmation semi-définie pour fournir des formulations convexes globalement convergentes pour une large classe de problèmes de commande optimale non lisses et de problèmes d'EDP.

Keywords: évolution de mesures, commande optimale, relaxation par mesures, processus de balayage, systèmes dynamiques non lisses, équations d'évolution quasi-dissipatives, transport optimal, hiérarchie moments–SOS, programmation semi-définie

Contents

Notation

Basic sets and symbols.

$\mathbb{N}:$ Set of positive integers $\{1, 2, 3, \dots\}$.

$\mathbb{N}_0:$ Set of nonnegative integers $\{0, 1, 2, \dots\}$.

$\mathbb{Z}:$ Set of integers.

$\mathbb{Q}:$ Set of rational numbers.

$\mathbb{R}:$ Set of real numbers.

$\mathbb{R}_+:$ Nonnegative reals $[0, +\infty)$.

$\mathbb{R}^d:$ d-dimensional Euclidean space with norm $|\cdot|$.

$B_r(x):$ Open ball of radius $r > 0$ centered at x in the relevant norm.

$\mathbf{1}_A:$ Indicator function of a set A.

$\overline{A}$**,** A°**,** $\partial A:$ Closure, interior, and boundary of $A \subset \mathbb{R}^d$ (or a metric space).

Function spaces and continuity. Let X be a Banach space and $T > 0$.

$\mathcal{C}(X):$ Space of real-valued continuous functions on a topological space X.

$\mathcal{C}_b(X):$ Space of bounded continuous functions on X, with norm $\|f\|_\infty := \sup_{x\in X} |f(x)|$.

$\mathcal{C}_c(X):$ Space of continuous functions on X with compact support.

$\mathcal{C}^k(\Omega):$ Space of k-times continuously differentiable functions on an open set $\Omega \subset \mathbb{R}^d$.

$\mathcal{C}_c^1(\Omega):$ Space of C^1 functions on Ω with compact support.

$\mathcal{C}([0,T];X)$: Space of continuous maps $x(\cdot)$ from $[0,T]$ to X, with norm $\|x\|_{\mathcal{C}([0,T];X)} := \sup_{t\in[0,T]} \|x(t)\|_X$.

$\mathcal{C}^1([0,T];X)$: Space of continuously differentiable maps $x(\cdot)$ from $[0,T]$ to X.

$\mathrm{AC}([0,T];X)$: Space of absolutely continuous maps $x(\cdot)\colon [0,T] \to X$.

$\mathrm{BV}([0,T];X)$: Space of functions of bounded variation $x(\cdot)\colon [0,T] \to X$.

$L^p([0,T])$: Space of functions f such that $\int_0^T |f(t)|^p\,dt < \infty$ for $1 < p < \infty$ and $\operatorname{ess\,sup}_{t\in(0,T)} |f(t)| < \infty$ if $p = \infty$.

$W^{k,p}(\Omega):$ Space of functions $f \in L^p(\Omega)$ for Ω open in $\mathbb{R}^n$ and partial derivatives $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \le k$.

$H^k(\Omega)$: The Hilbert–Sobolev space $H^k(\Omega) := W^{k,2}(\Omega)$.

Measures and measure-valued objects. Let X be a Polish (separable complete metric) space with Borel σ-algebra $\mathscr{B}(X)$.

$\mathcal{M}(X)$: Space of finite signed Borel measures on X.

$\mathcal{M}_+(X)$: Cone of finite nonnegative Borel measures on X.

$\mathcal{P}(X)$: Space of Borel probability measures on X.

δ_x : Dirac measure at $x \in X$.

$\mathrm{supp}(\mu)$: Support of a measure $\mu \in \mathcal{M}(X)$.

$\mu(A)$: Mass of a measurable set $A \in \mathscr{B}(X)$.

$\int_X f \,\mathrm{d}\mu$: Integral of a measurable function $f\colon X \to \mathbb{R}$ w.r.t. a measure μ.

$T_\# \mu$: Push-forward of μ by a measurable map $T\colon X \to Y$: $(T_\#\mu)(B) := \mu(T^{-1}(B))$ for $B \subset Y$.

Chapter 1

Introduction

Many real-world systems are governed by dynamics with a dissipative component, possibly combined with an additional perturbation. A convenient way to capture such situations is to assume a *quasi-dissipativity* condition, which allows for controlled deviations from strict dissipativity. Importantly, this quasi-dissipative framework naturally includes nonsmooth dynamics—such as set-valued vector fields arising from unilateral constraints, impacts, frictional contact, and complementarity-type models—alongside smooth (semi)linear drifts, and in infinite-dimensional evolution equations on Hilbert spaces, including semilinear and reaction–diffusion PDEs. Classical optimization tools tailored to smooth vector fields or semilinear operators often fail to exploit this structure adequately, do not yield tractable numerical schemes, or lead to poor local optima.

Over the past two decades, measure-theoretic methods have emerged as a powerful alternative: by lifting trajectories and controls to the space of measures on state–control–time, one can reformulate the original nonlinear problems as linear optimization problems over measures. This ensemble viewpoint on dynamical systems leads to measure-valued relaxations of the underlying optimization problems, and moment–SOS hierarchies have been proposed to solve the resulting infinite-dimensional problems. These schemes have the advantage of systematically producing a sequence of lower bounds that converge to the global optimal solution.

Throughout this thesis, we repeatedly use a common paradigm: a possibly nonconvex, nonlinear optimal control or evolution problem is reformulated as a linear problem posed on a space of measures. Numerical approximations are then obtained via moment–SOS hierarchies. In Section 1.1, we illustrate this paradigm in progressively richer settings: static optimization, optimal transport, and optimal control. Section 1.2 introduces nonsmooth dynamical systems and the associated problems of measure evolution and optimal control. In Section 1.3, we introduce nonlinear evolution equations and focus on recent developments in the numerical simulation of a few relevant classes of such equations. Finally, in Section 1.4, we introduce the moment–SOS hierarchy, which is the preferred numerical scheme in this thesis and the main motivation for solving various optimization problems via measure relaxations.

1.1 Measure relaxations: from static to dynamical systems

Many engineering and scientific problems boil down to optimizing an objective function subject to some constraints. This can be expressed mathematically as:

$$J^\star \;=\; \inf_{x\in\mathcal{X}} \mathcal{L}(x) \tag{1.1}$$

where $\mathcal{X}$ is a subset of a metric space.[1] Here we have used inf to denote the possibility that the minimizer might not exist.

When $\mathcal{L}$ is a convex function and $\mathcal{X}$ is a convex set, any local minimum is global, and powerful algorithms (interior-point algorithms, first order splitting algorithms, etc) [1] exist which can provide globally optimal solutions in many important problem classes. This convex optimization paradigm has become a standard design tool across control, signal processing and machine learning [1, 2]. Yet many real-world problems violate convexity: the objective or the constraints (or both) may be nonconvex, leading to multiple local minima and, in the worst case, NP-hardness: unless P = NP, no polynomial-time algorithm can solve all instances of such problems to global optimality.

1.1.1 Static optimization

A prototypical example is the quadratically constrained quadratic program (QCQP), which reads

$$J^\star \;=\; \min_{x\in\mathbb{R}^n}\; x^\top Q_0\, x \;+\; 2\, c_0^\top x \;+\; d_0, \quad \text{s.t.} \quad x^\top Q_i\, x \;+\; 2\, c_i^\top x \;+\; d_i \;\geq\; 0, \quad i = 1,\dots,m, \tag{1.2}$$

where each $Q_i \in \mathbb{R}^{n\times n}$ is symmetric, $c_i \in \mathbb{R}^n$, and $d_i \in \mathbb{R}$. In general, this problem is NP-hard. One can *lift* the decision variable x into the rank-one matrix

$$X \;=\; \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 & x^\top \end{pmatrix} \;=\; \begin{pmatrix} 1 & x^\top \\ x & x\,x^\top \end{pmatrix} \succeq 0,$$

and define the augmented data matrices

$$\widetilde{Q}_i \;=\; \begin{pmatrix} d_i & c_i^\top \\ c_i & Q_i \end{pmatrix}, \qquad i = 0, 1, \dots, m.$$

Let $\mathbb{S}^{n+1}$ denote the space of real symmetric $(n+1)\times(n+1)$ matrices. We equip $\mathbb{S}^{n+1}$ with the Hilbert–Schmidt inner product

$$\langle A, B\rangle \;:=\; \operatorname{tr}(AB), \qquad A, B \in \mathbb{S}^{n+1},$$

and the associated Frobenius norm $\|A\|_F := \langle A, A\rangle^{1/2}$. The notation $X \succeq 0$ means that X is positive semidefinite, i.e.

$$X \succeq 0 \quad \Longleftrightarrow \quad z^\top X z \;\geq\; 0 \quad \forall\, z \in \mathbb{R}^{n+1}.$$

Each scalar quadratic form then satisfies

$$x^\top Q_i\, x + 2\, c_i^\top x + d_i \;=\; \langle \widetilde{Q}_i,\; X\rangle,$$

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[1] A metric on a set $\mathcal{X}$ is a function $\mathsf{d} : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ such that (1) $\mathsf{d}(x,x) = 0$; (2) $\mathsf{d}(x,y) = \mathsf{d}(y,x)$; (3) $\mathsf{d}(x,y) \leq \mathsf{d}(x,z) + \mathsf{d}(z,y)$. This induces the notions of open and closed sets and hence topology with respect to which we can study the convergence of sequences.

so that the original QCQP admits the equivalent reformulation

$$\begin{aligned} J^\star = \min_{X \in \mathbb{S}^{n+1}} \quad & \langle \widetilde{Q}_0,\ X \rangle \\ \text{s.t.} \quad & \langle \widetilde{Q}_i,\ X \rangle \ \geq\ 0, \quad i = 1, \dots, m, \\ & X_{00} = 1, \\ & X \ \succeq\ 0, \\ & \operatorname{rank}(X) = 1. \end{aligned}$$

Dropping the nonconvex constraint $\operatorname{rank}(X) = 1$ yields the celebrated *Shor relaxation* [3]

$$\begin{aligned} p^\star_{\text{shor}} = \min_{X \in \mathbb{S}^{n+1}} \quad & \langle \widetilde{Q}_0,\ X \rangle \\ \text{s.t.} \quad & \langle \widetilde{Q}_i,\ X \rangle \ \geq\ 0, \quad i = 1, \dots, m, \\ & X_{00} = 1, \\ & X \ \succeq\ 0, \end{aligned} \tag{1.3}$$

which is a convex semidefinite program solvable in polynomial time to any prescribed relative accuracy. Moreover, when all of the matrices Q_i in the original program are positive semidefinite the QCQP itself is convex and we obtain $p^\star_{\text{shor}} = J^\star$, i.e, the relaxation is tight. Interestingly, the tightness of the Shor relaxation has also been demonstrated in several indefinite-Q_i settings, such as hidden-convexity problems [4], trust-region subproblems [5], and certain random (and diagonal) QCQPs [6].

In order to consider polynomials of higher degrees, we assume the objective function $\mathcal{L}$ in (1.1) to be polynomial and the set $\mathcal{X}$ to be basic semialgebraic.[2] The resulting polynomial optimization problem (POP) is NP-hard in general, so one cannot expect a scalable exact algorithm for arbitrary instances. Note that the POP can be reformulated as

$$J^\star \ = \max_{c \in \mathbb{R}} \ c \quad \text{s.t.} \quad \mathcal{L}(x) - c \geq 0 \quad \forall\, x \in \mathcal{X}. \tag{1.4}$$

Here the nonnegativity condition has to be checked at all points $x \in \mathcal{X}$. For univariate polynomials, Nesterov [7] exploited the fact that nonnegative polynomials can be written as SOS. He constructed self-concordant functions for the cone of nonnegative polynomials and applied interior point method to obtain global optima. In the multivariate case, the representation theorems are more involved because not every nonnegative polynomial is a sum of squares. Indeed this was highlighted in the famous Hilbert's problem [8]. To address polynomial optimization problems, we will adopt the measure-theoretic perspective in this thesis.

To motivate this we consider a set of samples $x_k \in \mathcal{X}$, for $k = \{1, \dots, n\}$ each associated with a nonnegative weight w_k. In general, we can rewrite the problem in (1.1) as

$$p^n \ = \sum_{k=1}^{n} w_k \, \mathcal{L}(x_k) \quad \text{s.t.} \quad \sum_{k=1}^{n} w_k \ = \ 1, \quad w_k \geq 0. \tag{1.5}$$

[2] A set is called basic semialgebraic if it can be expressed as $\mathcal{X} = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0,\ j = 1, \dots, m\}$ where g_j are polynomials.

This problem can be seen as minimizing the expected value of the objective function $\mathbb{E}[\mathcal{L}(x)]$, where the expectation is taken with respect to an unknown discrete probability measure

$$\mu^n := \sum_{k=1}^{n} w_k \delta_{x_k}.$$

Here, δ_{x_k} denotes the Dirac measure at x_k, which can be thought of as a unit point mass at location x_k. Different discrete probability measures μ^n are characterized by the samples $x_k \in \mathcal{X}$ and the associated w_k. Note that if the problem admits a unique minimizer at $x_j = y$, then necessarily $w_k = 0$ for $k \neq j$ and $w_j = 1$. Rather than commit to a particular sampling, we relax the problem directly over all probability measures supported on $\mathcal{X}$:

$$p^\star \;=\; \inf_{\mu \in \mathcal{P}(\mathcal{X})} \int_{\mathcal{X}} \mathcal{L}(x) d\mu(x). \tag{1.6}$$

where $\mathcal{P}(\mathcal{X})$ denotes the set of all probability measures supported on $\mathcal{X}$. The objective functional in (1.6) is linear in the unknowns $\mu \in \mathcal{P}(\mathcal{X})$. Thus, the problem is a linear program (LP) but infinite dimensional one. For example, (1.2) can be cast as a POP with quadratic (degree-two) polynomials:

$$\begin{aligned} &\inf_{\mu \in \mathcal{P}(\mathcal{X})} \int_{\mathcal{X}} \left(x^\top Q_0 \, x \;+\; 2\, c_0^\top x \;+\; d_0 \right) d\mu(x) \\ &\quad \text{s.t. } \mathcal{X} = \{ x \in \mathbb{R}^n \mid x^\top Q_i \, x \;+\; 2\, c_i^\top x \;+\; d_i \;\leq\; 0, \quad \forall\, i = 1, \ldots, m \}. \end{aligned}$$

To see that the optimal values in (1.4) and (1.6) are equal, i.e., $p^\star = J^\star$, note first that for every $x \in \mathcal{X}$ we have $\mathcal{L}(x) \geq J^\star$. Hence, for any $\mu \in \mathcal{P}(\mathcal{X})$,

$$\int_{\mathcal{X}} \mathcal{L}(x)\, d\mu(x) \;\geq\; \int_{\mathcal{X}} J^\star \, d\mu(x) \;=\; J^\star,$$

so $p^\star \geq J^\star$. Conversely, every point $\mathsf{x} \in \mathcal{X}$ can be embedded in $\mathcal{P}(\mathcal{X})$ as a Dirac measure δ_{x}, which is a unit mass concentrated at x. For this measure,

$$\int_{\mathcal{X}} \mathcal{L}(x)\, \delta_{\mathsf{x}}(dx) = \mathcal{L}(\mathsf{x}).$$

Thus, the problem (1.1) can be written as

$$\min_{\delta_{\mathsf{x}} \in \mathcal{P}(\mathcal{X})} \int_{\mathcal{X}} \mathcal{L}(x)\, \delta_{\mathsf{x}}(dx),$$

and can be viewed as a restriction of (1.6) to the set of Dirac measures. Moreover, every feasible point of (1.1) is also feasible for (1.6), so $p^\star \leq J^\star$. Combining this with the reverse inequality obtained earlier, we conclude that $p^\star = J^\star$.

1.1.2 Transport of measures

The central role of optimal transport across probability, geometry, and analysis has been underscored by two Fields Medals awarded for advances in this area (Cédric Villani, 2010; Alessio Figalli, 2018) [9, 10]. At its core, the original Monge formulation of optimal transport is highly nonconvex; the Kantorovich relaxation turns it into a convex linear program (LP). In discrete settings, the classical assignment problem is recovered as a special case of optimal transport when masses are uniform and splitting is not allowed.

Motivating example (assignment toy model). Suppose we have m bakeries at locations $x_1, \ldots, x_m$ and n cafés at locations $y_1, \ldots, y_n$. Transporting one croissant from x_i to y_j incurs cost $c(x_i, y_j)$. In the strict assignment setting, we assume $m = n$ and each bakery serves exactly one café (and each café receives exactly one shipment). Writing the assignment as a permutation $\sigma \in \mathfrak{S}_n$[3] or, equivalently, as a permutation matrix $\Pi \in \{0,1\}^{n\times n}$ with exactly one 1 in every row and column, the discrete Monge problem is

$$\min_{\sigma\in\mathfrak{S}_n} \sum_{i=1}^{n} c\big(x_i,\, y_{\sigma(i)}\big) \quad \Longleftrightarrow \quad \min_{\Pi\in\{0,1\}^{n\times n}} \sum_{i=1}^{n}\sum_{j=1}^{n} \Pi_{ij}\, c(x_i, y_j) \ \text{ s.t. } \ \Pi\mathbf{1} = \mathbf{1},\ \Pi^\top\mathbf{1} = \mathbf{1}.$$

Here $\mathbf{1} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$ denotes the vector of all ones. This forbids splitting: a unit at x_i must go to a single y_j.

Pushforward and measure-theoretic language. We now recast the assignment problem in the language of measures.[4] [5] For a discrete measure $\mu^n = \sum_{k=1}^{n} w_k\, \delta_{x_k}$, the pushforward through a measurable map T moves each atom to its image while preserving its weight:

$$T_\#\mu^n = \sum_{k=1}^{n} w_k\, \delta_{T(x_k)}. \tag{1.7}$$

This motivates the formal definition.

> **Definition 1.1.1** (Pushforward measure)**.** Let $(\mathcal{X}, \Sigma(\mathcal{X}))$, $(\mathcal{Y}, \Sigma(\mathcal{Y}))$ be measurable spaces and $T : \mathcal{X} \to \mathcal{Y}$ measurable. The pushforward of μ by T is the measure $T_\#\mu$ on $(\mathcal{Y}, \Sigma(\mathcal{Y}))$ defined by
>
> $$T_\#\mu(A) = \mu(T^{-1}(A)) \qquad \forall\, A \in \Sigma(\mathcal{Y}). \tag{1.8}$$

To connect the discrete assignment model with the Monge formulation, define the empirical probability measures

$$\mu := \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}, \qquad \nu := \frac{1}{n}\sum_{j=1}^{n} \delta_{y_j}. \tag{1.9}$$

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[3] $\mathfrak{S}_n$ denotes the symmetric group on n elements, i.e., the set of all bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

[4] A collection $\Sigma(\mathcal{X}) \subset 2^{\mathcal{X}}$ is a σ-algebra if: (1) $\varnothing \in \Sigma(\mathcal{X})$; (2) $A \in \Sigma(\mathcal{X}) \Rightarrow A^c \in \Sigma(\mathcal{X})$; (3) if $A_n \in \Sigma(\mathcal{X})$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma(\mathcal{X})$.

[5] A measurable space is $(\mathcal{X}, \Sigma(\mathcal{X}))$. A measure $\mu : \Sigma(\mathcal{X}) \to [0, \infty]$ is a countably additive function with $\mu(\varnothing) = 0$. If $\mu(\mathcal{X}) = 1$, then μ is a probability measure, $\mu \in \mathcal{P}(\mathcal{X})$. A map $T : \mathcal{X} \to \mathcal{Y}$ is measurable if $T^{-1}(A) \in \Sigma(\mathcal{X})$ for all $A \in \Sigma(\mathcal{Y})$.

Any assignment $\sigma \in \mathfrak{S}_n$ can be encoded by a map

$$T : \{x_1, \dots, x_n\} \to \{y_1, \dots, y_n\}, \qquad T(x_i) = y_{\sigma(i)}.$$

With the empirical measures defined in (1.9), one checks that $T_{\#}\mu = \nu$ if and only if each y_j receives mass from exactly one x_i, that is, if and only if T is a bijection from $\{x_1, \dots, x_n\}$ onto $\{y_1, \dots, y_n\}$ (equivalently, T induces a permutation of the indices). In this case,

$$\frac{1}{n} \sum_{i=1}^{n} c(x_i, y_{\sigma(i)}) = \int_{\mathcal{X}} c(x, T(x))\, d\mu(x),$$

so the discrete assignment problem can be viewed as minimizing a transport cost

$$\min_{\sigma \in \mathfrak{S}_n} \sum_{i=1}^{n} c(x_i, y_{\sigma(i)}) \quad \Longleftrightarrow \quad \min_{T:\, T_{\#}\mu = \nu} \int_{\mathcal{X}} c(x, T(x))\, d\mu(x),$$

the two formulations differing only by the constant factor $1/n$ in the objective.

This suggests the following general measure-theoretic formulation. Given probability measures $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$ and a cost function $c : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$, one seeks a measurable transport map $T : \mathcal{X} \to \mathcal{Y}$ that pushes μ to ν and minimizes the total cost

$$\inf_{T:\, T_{\#}\mu = \nu} \int_{\mathcal{X}} c(x, T(x))\, d\mu(x). \tag{1.10}$$

This optimization problem is called the Monge problem. The discrete assignment model above is a special case corresponding to empirical measures supported on finitely many points. This is nonlinear and may not admit minimizers (e.g., transporting δ_0 to $\frac{1}{2}(\delta_{-1} + \delta_1)$ requires splitting the mass at 0, which no measurable map can do).

Kantorovich relaxation. In his seminal work [11], Leonid V. Kantorovich convexified the problem (1.10) by replacing transport maps with transport plans (couplings) $\theta \in \mathcal{P}(X \times Y)$ having marginals μ, ν, i.e., $\theta(A \times Y) = \mu(A) \quad \forall A \subset X,\ \theta(X \times B) = \nu(B)\ \forall B \subset Y$. In the discrete case, let $\mu = \sum_{i=1}^{m} a_i\, \delta_{x_i}$, $\nu = \sum_{j=1}^{n} b_j\, \delta_{y_j}$ with $a_i, b_j \geq 0$ and $\sum_i a_i = \sum_j b_j = 1$. A transport plan is a nonnegative matrix $\Pi \in \mathbb{R}_+^{m \times n}$ with row and column sums

$$\sum_{j=1}^{n} \Pi_{ij} = a_i, \quad \sum_{i=1}^{m} \Pi_{ij} = b_j,$$

and the relaxed problem is the LP

$$\min_{\Pi \in \mathbb{R}_+^{m \times n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \Pi_{ij}\, c(x_i, y_j) \quad \text{s.t.} \quad \Pi \mathbf{1} = a, \quad \Pi^{\top} \mathbf{1} = b.$$

When $m = n$ and $a_i = b_j = \frac{1}{n}$ for all i, j, the feasible set contains the permutation matrices (no-splitting plans) as extreme points. However, allowing general P permits splitting and yields a convex relaxation of the assignment model. In measure-theoretic terms, this reads

$$\inf_{\theta\in\mathcal{P}(\mathcal{X}\times\mathcal{Y})} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\, d\theta(x,y) \quad \text{s.t.} \quad \theta(A,\mathcal{Y}) = \mu(A), \quad \theta(\mathcal{X},B) = \nu(B) \qquad \forall A\subset\mathcal{X},\, B\subset\mathcal{Y}.$$

The case $\mathcal{X} = \mathbb{R}^n$ with cost $c(x,y) = \|x-y\|^2$ is well-studied in the literature [12, 13] and has a rich structure; it will be the focus of our discussion here. The optimal transport problem in this setting can be stated as

$$W_2(\mu,\nu) := \frac{1}{2} \min_{\theta\in\Theta(\mu,\nu)} \left(\int_{\Omega\times\Omega} |x-y|^2 \mathrm{d}\theta(x,y) \right)^{1/2} \tag{1.11}$$

where $\Theta(\mu,\nu)$ is the set of joint probability measures on $\Omega\times\Omega\subset\mathbb{R}^n\times\mathbb{R}^n$ with given marginals μ and ν, i.e., such that $\theta\in\Theta(\mu,\nu)$ satisfies $\theta(A,\Omega) = \mu(A)$ and $\theta(\Omega,A) = \nu(A)$, for every measurable $A\subset\Omega$. For example, if both the measures are Dirac measures, the W_2 distance coincides with the Euclidean distance between their atoms, i.e, $W_2(\delta_{\mathsf{x}},\delta_{\mathsf{y}}) = \|\mathsf{x}-\mathsf{y}\|$ for $\mathsf{x},\mathsf{y}\in\mathbb{R}^n$. In comparison, if we consider total variation distance (TV) between the two Dirac measures $\|\delta_{\mathsf{x}}-\delta_{\mathsf{y}}\|_{TV}$ for $x\neq y$ is always 2, so it does not encode the geometry of the underlying space.

Viewing transport as the evolution of particles, we consider a smooth flow map $\mathbb{S}(t,s): \mathbb{R}^n\to\mathbb{R}^n$ that, for all $r,s,t\in[0,T]$, satisfies $\mathbb{S}(t,s)\circ\mathbb{S}(s,r) = \mathbb{S}(t,r)$, $\mathbb{S}(t,t) = \mathbb{I}$ and is invertible, i.e., $(\mathbb{S}(t,s))^{-1} = \mathbb{S}(s,t)$. These are the standard flow properties of a (possibly time-dependent) vector field and do not hold for a general family of maps. We take a (sufficiently regular) vector field $v:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ as given, and denote by $\mathbb{S}(t,s)$ its associated flow, defined as the solution of

$$\frac{d}{dt}\mathbb{S}(t,s)x = v(t,\mathbb{S}(t,s)x), \qquad \mathbb{S}(s,s)x = x.$$

Given a particle at $x(t)\in\mathbb{R}^n$ such that $\dot{x}(t) = v(t,x(t))$, for small $h>0$ the Taylor expansion gives

$$x(t+h) = x(t) + h\,v(t,x(t)) + O(h^2) = \mathbb{S}(t+h,t)\,x(t),$$

so,

$$\mathbb{S}(t+h,t) = \mathbb{I} + h\,v(t,\cdot) + O(h^2),$$

provided v is sufficiently regular (e.g., locally Lipschitz in x and continuous in t). If the configuration at time t is described by an ensemble or cloud of particles described by a probability measure $\mu_t\in\mathcal{P}(\mathbb{R}^n)$, then we can describe the collective motion of the particles as

$$\mu_{t+h} = \mathbb{S}(t+h,t)_{\#}\mu_t = (\mathbb{I} + hv(t,\cdot))_{\#}\mu_t. \tag{1.12}$$

Next we use the change of variables formula for the pushforward measure ([14, Chapter 1]) in

(1.12) to obtain

$$\mu_{t+h}(x) = \mu_t\left((\mathbb{S}(t+h,t))^{-1}(x)\right)\left|\det\left(\frac{\partial \mathbb{S}(t+h,t)^{-1}(x)}{\partial x}\right)\right|. \tag{1.13}$$

Using the properties of map $\mathbb{S}(t+h,t)$ we obtain

$$\mu_{t+h}(x) = \mu_t(x - h\,v(t,x))\left|\det\left(\mathbb{I} - h\frac{\partial v(t,x)}{\partial x}\right)\right|. \tag{1.14}$$

The terms involving determinant can be simplified as $\left|\det(\mathbb{I} - h\,\frac{\partial v(t,x)}{\partial x})\right| = 1 - h\,\mathrm{Tr}\left(\frac{\partial v(t,x)}{\partial x}\right) = 1 - h\,(\nabla\cdot v)(t,x)$. Moreover, using the Taylor expansion of $\mu_t(x - h\,v(t,x))$ around x we obtain, $\mu_t(x - h\,v(t,x)) = \mu_t - h\,\nabla\mu_t \cdot v(t,x)$. Substituting these two expressions in (1.14) we get the first order approximation as

$$\mu_{t+h}(x) = (\mu_t(x) - h\,\nabla\mu_t(x)\cdot v(t,x))\,(1 - h(\nabla\cdot v)(t,x)) = \mu_t(x) - h\,\nabla\cdot(v\,\mu_t).$$

Rearranging the terms and letting $h \to 0$ formally yields

$$\partial_t\mu_t(x) + \nabla\cdot(v(t,x)\mu_t(x)) = 0. \tag{1.15}$$

This equation is called the *continuity* (or *Liouville*) equation. Here we implicitly identify μ_t with a density ρ_t with respect to Lebesgue measure, and we assume sufficient regularity to justify differentiation in time. By contrast, the distributional (weak) formulation of the continuity equation is written directly in terms of the curve $(\mu_t)_{t\in[0,T]}$ and test functions, and therefore does not require either the existence of a density or differentiability of $t \mapsto \mu_t$. Given a family $(\mu_t)_{t\in[0,T]}$ of measures with endpoints μ_0 and μ_T, we say it solves the continuity equation (in weak form) with velocity $v : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ if

$$\int_0^T\int_{\mathbb{R}^n}(\partial_t\phi(t,x)+\nabla_x\phi(t,x)\cdot v(t,x))\,d\mu_t(x)\,dt = \int_{\mathbb{R}^n}\phi(T,x)\,d\mu_T(x) - \int_{\mathbb{R}^n}\phi(0,x)\,d\mu_0(x) \tag{1.16}$$

for every $\phi \in C_c^1([0,T]\times\mathbb{R}^n)$.

Before we move on to the next part where we make connections between the W_2-metric and the continuity equation, we present an example which shows the density evolution for a 1-d nonlinear vector field.
Example (Nonlinear transport under $\dot{x} = x - x^3$). Consider the one–dimensional flow generated by the autonomous vector field

$$v(x) = x - x^3, \qquad x \in \mathbb{R}.$$

Let the initial law be the uniform probability measure on $[-1,1]$,

$$\mu_0(dx) = \rho_0(x)\,dx, \qquad \rho_0(x) = \tfrac{1}{2}\,\mathbf{1}_{[-1,1]}(x).$$

Denote by $(\mu_t)_{t\geq 0}$ the pushforward of μ_0 under the flow $\mathbb{S}(t,0)$. Using (1.13), the density ρ_t

of μ_t, supported on $[-1, 1]$, is

$$\rho_t(x) = \begin{cases} \dfrac{1}{2} \dfrac{e^{-t}}{\left(1 + x^2(e^{-2t} - 1)\right)^{3/2}}, & |x| \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad t \ge 0.$$

This family (μ_t) solves the continuity (Liouville) equation $\partial_t \mu_t + \partial_x\big((x - x^3)\mu_t\big) = 0$ with $\mu_{t=0} = \mu_0$. Figure 1.1 illustrates how the initially uniform density on $[-1, 1]$ is pushed away from the unstable equilibrium at $x = 0$ and progressively concentrates near the stable equilibrium at $x = \pm 1$.

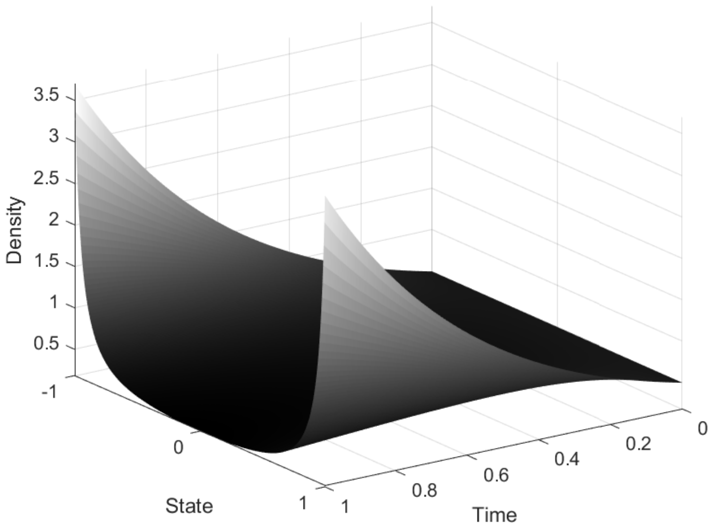


Figure 1.1: Density evolution $\rho_t(x)$ for $\dot{x} = x - x^3$, starting from $\rho_0 = \frac{1}{2}\mathbf{1}_{[-1,1]}$.

Benamou-Brenier formulation. When the transport is induced by the flow $\mathbb{S}(t, 0)$, the coupling is $\theta = (\mathrm{Id}, \mathbb{S}(1, 0))_{\#}\mu_0$ (equivalently, $\theta(\mathrm{d}x, \mathrm{d}y) = \mu_0(\mathrm{d}x)\, \delta_{\mathbb{S}(1,0)(x)}(\mathrm{d}y)$). Substituting this into (1.11) yields the Benamou–Brenier formulation

$$\begin{aligned} W_2(\mu_0, \mu_1)^2 = \min_{\mu_s, v} & \int_0^1 \int_{\mathbb{R}^n} |v(s, x)|^2 d\mu_s(x) ds \\ \text{s.t.} \quad & \partial_s \mu_s(x) + \nabla \cdot (v(s, x)\mu_s(x)) = 0. \\ & \mu_s(x)|_{s=0} = \mu_0, \quad \mu_s(x)|_{s=1} = \mu_1, \end{aligned} \tag{1.17}$$

The minimization is carried over all possible velocity fields $v(\cdot, \cdot)$ and continuous trajectories μ_s satisfying the continuity equation (1.15). This is known as the Benamou-Brenier formulation of the quadratic Wasserstein distance [15]. For a rigorous treatment of the derivation of Benamou-Brenier formulation see [16, Chapter 8].

The Benamou-Brenier formulation (1.17) admits an optimal control viewpoint, where the state variable is the evolving probability density function μ_s and the control is the velocity vector field $v(s,x)$. The continuity equation describes the dynamical equation of μ_s and the quadratic objective function is the control cost. This makes $W_2^2(\mu_0,\mu_1)$ as a value function of an optimal control problem in the space of measures.

The dynamical picture of Wasserstein distance endows the space of absolutely continuous measures with differential structure. This is at the core of interpreting various evolution equations as gradient flows in Wasserstein space. Moreover, it has led to derivation of sharp optimal inequalities [17, 18], development of unbalanced optimal transport [19] which addresses transport of arbitrary positive Radon (regular Borel) measures.

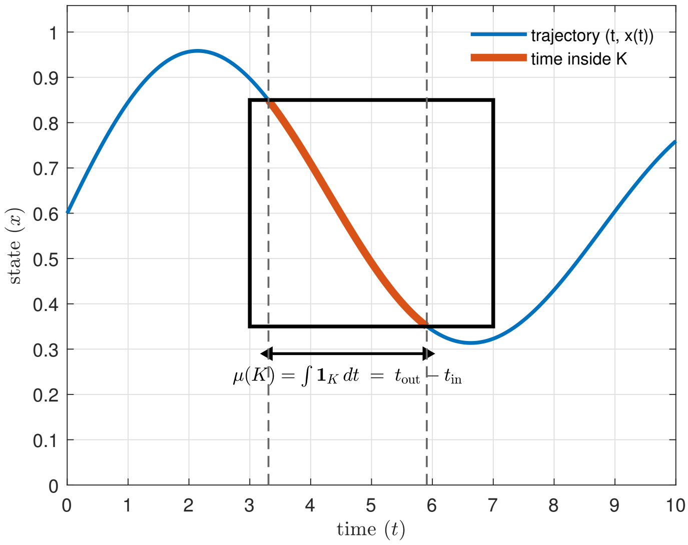


Figure 1.2: Occupation measure: time spent by the trajectory inside K (denoted by black box).

1.1.3 From dynamical formulation of OT to relaxed control

One way to reformulate the problem in (1.17) as a LP is to use *occupation measures*. Given an initial condition $x_0 \in \mathbb{R}^n$, an admissible trajectory $x(\cdot)$ and control $u(\cdot)$ for

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \qquad t \in [0,T],$$

the occupation measure μ on $[0,T] \times X \times U$ is the Borel measure defined by

$$\mu(K) \;=\; \int_0^T \mathbf{1}_K\big(t, x(t), u(t)\big)\, dt, \qquad K \subset [0,T] \times X \times U \text{ Borel.}$$

It encodes the time spent by the trajectory–control pair in subsets of $[0,T] \times X \times U$ (see Figure 1.2). Equivalently, for every bounded continuous ψ,

$$\int_{[0,T]\times X\times U} \psi(t,x,u)\, d\mu = \int_0^T \psi\big(t, x(t), u(t)\big)\, dt.$$

The quadratic Wasserstein distance admits the following linear formulation using occupation measures:

$$
\begin{aligned}
W_2(\mu_0,\mu_1)^2 = & \min_{\mu\in\mathcal{M}_+([0,1],\mathbb{R}^n,\mathbb{R}^n)} \int_0^1 \int_{\mathbb{R}^n} |v|^2 d\mu(t,x,v) \\
& \text{s.t.} \int_{[0,1]\times\mathbb{R}^n\times\mathbb{R}^n} [\partial_t\phi(t,x) + \partial_x\phi(t,x)\cdot v]\, d\mu(t,x,v) \\
& \quad = -\int \phi(0,x)d\mu(0,x) + \int \phi(1,x)d\mu(1,x), \;\; \forall\phi\in C_c^1([0,1]\times\mathbb{R}^n), \\
& d\mu(0,x) = \mu_0(x), \quad d\mu(1,x) = \mu_1(x),
\end{aligned}
\tag{1.18}
$$

Both the objective function and the dynamical constraint are linear in the unknown measure, thus the resulting program is a LP in measures. Subsequently, we see how occupation measures can be motivated as the limit of oscillating minimizing sequence of control signals.

Bolza problem. Next, we consider an optimal control problem famously known as *Bolza* problem, stated as follows:

$$
\begin{aligned}
J^\star = & \min_{x(\cdot)\in L^\infty([0,1])} \int_0^1 (x(t)^4 + (u(t)^2-1)^2)\, dt \\
& \text{s.t. } \dot{x}(t) = u(t), \quad t\in[0,1], \\
& x(0) = 0,\; x(1) = 0,
\end{aligned}
\tag{1.19}
$$

The objective is nonnegative and the minimum can be zero for $x(t) = 0$ and $u(t) = \{-1,1\}$. To achieve this we can construct a minimizing sequence of control trajectories as:

$$
u_j(t) = \sum_{i=0}^{j-1} \mathbf{1}_{\left[\frac{2i}{2j},\frac{2i+1}{2j}\right]}(t) - \mathbf{1}_{\left[\frac{2i+1}{2j},\frac{2i+2}{2j}\right]}(t). \tag{1.20}
$$

This function oscillates between 1 and -1 over the interval $[0,1]$. As seen in Figure 1.3, the trajectories converge to $x(t) = 0$ and $u(t) = \frac{\delta_{-1}+\delta_1}{2}$. The controls corresponding to $J^\star = 0$ are measure-valued and do not lie in the considered functional space. Rather we have to extend our function class to the space of measures. The issue of existence of a minimizer is mitigated by reformulating the problem in the space of measures using occupation measures. A measure relaxation of the problem (1.19) reads

$$
\begin{aligned}
J^\star = & \min_{\mu\in\mathcal{M}_+([0,1],\mathbb{R},\mathbb{R})} \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x^4 + (u^2-1)^2\right) d\mu(t,x,u) \\
& \text{s.t.} \quad \int_{[0,1]\times\mathbb{R}\times\mathbb{R}} [\partial_t\phi(t,x) + \partial_x\phi(t,x)\cdot u]\, d\mu(t,x,u) \\
& \quad = -\int \phi(t,x)d\mu_0(0,x) + \int \phi(t,x)d\mu_1(1,x) \quad \forall\phi\in C_c^1([0,1]\times\mathbb{R}), \\
& d\mu_0(0,x) = \delta_0(dt)\delta_0(dx),\; d\mu_1(1,x) = \delta_1(dt)\delta_0(dx),
\end{aligned}
\tag{1.21}
$$

where $\mathcal{M}_+([0,1]\times\mathbb{R}\times\mathbb{R})$ is the space of nonnegative Borel measures supported on $[0,1]\times\mathbb{R}\times\mathbb{R}$. The minimizing relaxed solution is attained by

$$\mu^\star(dt,dx,du) \;=\; dt\,\delta_0(dx)\left(\tfrac{1}{2}\delta_{-1}+\tfrac{1}{2}\delta_1\right)(du).$$

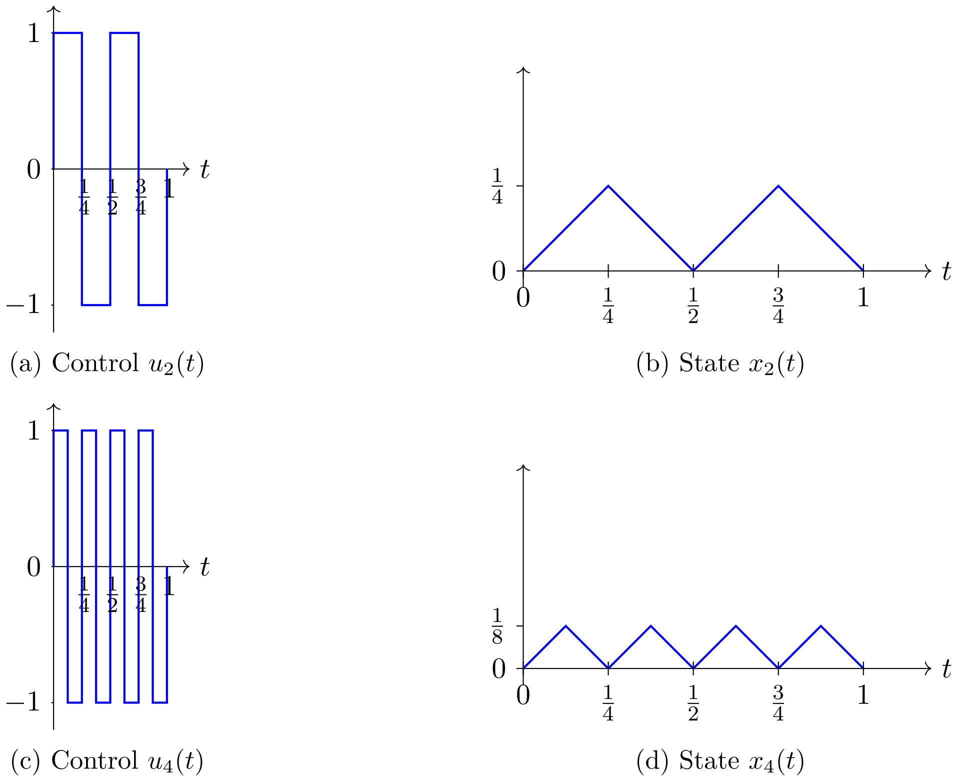


Figure 1.3: Control and state trajectories for the Bolza problem example.

For the above example (1.19), $u(t) = \frac{\delta_{-1}+\delta_1}{2}$ can be seen as capturing the oscillating behavior of the sequence (1.20). The idea of capturing the limit of oscillatory control signals by measures goes back to L.C. Young, who introduced generalized curves, commonly known as Young measure, to relax optimal control and calculus of variations problems [20, 21]. In the optimal control setting this led to relaxed controls (probability kernels over controls) and to occupation measures that encode time averages over trajectories. This relaxation scheme was applied to stochastic control problems where the controlled Markov process and controlled diffusion are recast as LP over occupation measures [22, 23]. When the system data is polynomial/semialgebraic, these relaxations further admit semidefinite relaxations and this approach was pioneered in [24]. An important theoretical issue in this scheme is: Do the original problem and its measure-based LP relaxation have the same optimal value? This issue is commonly referred to as the "relaxation gap" issue. Under standard convexity assumptions on the system data (objective and dynamics functions are convex in controls and the control set is convex), [25] showed absence of relaxation gap. Under these assumptions, the author proposed a "Superposition principle" which established that the set of occupation measures is the convex closure of the set of Young measures. The author used this principle to prove the main result of no relaxation gap. Under similar convexity assumptions on system data and for variational problem with scalar codomain and arbitrary domain dimension, [26] showed that the occupation measure relaxation is exact, i.e., no relaxation gap occurs. The authors also provide counterexample when the dimension of the domain and the codomain are greater than one, demonstrating that a relaxation gap may appear in the general case. See also [27] for further counterexamples. In [28], sufficient conditions for the absence of a relaxation gap in general variational problems are established, which essentially reduce to convexity in the state and control variables.

1.2 Quasi-dissipative nonsmooth systems

Real-world systems frequently exhibit abrupt changes arising from jumps, switches, impacts, or unilateral constraints. Such phenomena introduce corners, kinks, or flat regions into their effective "velocity" fields and occur in many familiar settings:

1. In a simple diode–resistor circuit, the device blocks current when reverse-biased and conducts with almost zero drop when forward-biased. The voltage–current characteristic thus has a sharp "knee" at zero volts, and the evolution of the circuit exhibits a nonsmooth characteristic curve.

2. When pushing a block on a table, the friction force resists the external force up to a threshold. Once this threshold is exceeded, the block starts sliding and the friction force drops to a constant kinetic friction level.

Mathematically, such phenomena can be described by a more general class of dynamical systems known as differential inclusions, which are expressed as

$$\dot{x}(t) \in F(x(t)).$$

In this thesis we focus on a specific subclass. To introduce it, let $\{S(t)\}_{t\in[0,T]}$ be a family of closed convex subsets of $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz vector field. Then the evolution of trajectories constrained to remain in $S(t)$ under the effect of the drift f is described by the following nonsmooth dynamical system:

$$\dot{x}(t) \in f(x(t)) - \mathcal{N}_{S(t)}(x(t)), \qquad x(t) \in S(t), \tag{1.22}$$

where $\mathcal{N}_{S(t)}(x)$ denotes the outward normal cone[6] to the set $S(t)$ at the point $x \in S(t)$. A broad class of nonsmooth dynamical systems can be modeled as interconnections of ordinary differential equations with monotone set-valued operators; we refer to the survey [29] for a comprehensive overview of such systems, including sweeping processes as a prominent example.

We call a trajectory $x : [0,T] \to \mathbb{R}^n$ an absolutely continuous solution of (1.22) if there exists a measurable selection $s : [0,T] \to \mathbb{R}^n$ such that

$$s(t) \in \mathcal{N}_{S(t)}(x(t)) \quad \text{for a.e. } t \in [0,T],$$

[6]The definition of the normal cone appears in Definition 2.2.3.

and

$$\dot{x}(t) = f(x(t)) - s(t) \qquad \text{for a.e. } t \in [0,T].$$

The term $-\mathcal{N}_{S(t)}(x(t))$ is a purely geometric reaction force enforcing the state constraint $x(t) \in S(t)$. Intuitively, this reaction force dissipates energy by opposing motion that would drive the trajectory outside $S(t)$.

We recall a (finite-dimensional) notion of dissipativity for set-valued vector fields that will later be extended to Hilbert spaces. A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called dissipative if

$$\langle v_1 - v_2, x_1 - x_2 \rangle \; \leq 0 \qquad \text{for all } x_1, x_2 \in \mathbb{R}^n, \; v_i \in F(x_i). \tag{1.23}$$

Equivalently, F is dissipative if $-F$ is monotone.

The drift f may inject or dissipate energy depending on the application. This leads naturally to the notion of quasi-dissipativity, where some controlled energy production is allowed. Informally, we say that F is α–quasi-dissipative if there exists $\alpha \geq 0$ such that

$$\langle v_1 - v_2, x_1 - x_2 \rangle \; \leq \; \alpha \, \|x_1 - x_2\|^2 \qquad \text{for all } x_1, x_2 \in \mathbb{R}^n, \; v_i \in F(x_i). \tag{1.24}$$

The case $\alpha = 0$ reduces to the classical notion of dissipativity. Quasi-dissipative nonsmooth systems of the form (1.22) will be a central object of study in this thesis.

1.2.1 Measure evolution

The study of how measures evolve in dynamical systems is essential for uncertainty propagation, system analysis, and optimal control. For linear systems, the first two moments, mean and covariance, admit closed-form expressions [30]. Similarly, there has been work on the propagation of ambiguity sets that yields exact expressions in the case of linear dynamics [31].

For single-valued vector fields, the problem of measure evolution can be formulated as follows: given an initial distribution $\mu_0 \in \mathcal{P}(\Omega)$ for some $\Omega \subset \mathbb{R}^n$, the evolution under a time-dependent vector field $v_t : \mathbb{R}^n \to \mathbb{R}^n$ is described by

$$\partial_t \mu_t + \operatorname{div}(\mu_t v_t) = 0, \quad \mu_{|t=0} = \mu_0. \tag{1.25}$$

This is precisely the continuity equation introduced in (1.16). For vector fields with sufficient regularity (e.g., Lipschitz continuity), measure evolution is characterized by the continuity (Liouville) equation, with solutions given by the pushforward of μ_0 under the flow; see [32, 33]. For less regular vector fields, such as those of bounded variation or belonging to suitable Sobolev spaces, the corresponding theory for measure evolution equations was developed in [34].

In contrast, the evolution of measures driven by set-valued dynamics, and in particular by nonsmooth dynamical systems such as the sweeping process (1.22), is relatively less studied. To develop an analogous theory, one needs to define an appropriate notion of measure solution associated with the underlying differential inclusion, establish well-posedness (existence, uniqueness, and stability) of the induced measure dynamics, and relate these measure trajectories to suitable continuity equations on $\mathcal{P}(\mathbb{R}^n)$. Chapter 3 addresses this gap for the specific class of quasi-dissipative finite-dimensional nonsmooth systems introduced in (1.22).

Moreover, the measure-valued dynamics thus obtained can be used to formulate optimal control problems for nonsmooth dynamical systems in the space of measures. An instance of such a problem, involving smooth vector fields, was already introduced in (1.21).

1.2.2 Optimal control

Optimal control of general nonlinear systems is a classical topic with a rich theory and a mature algorithmic toolbox. In the smooth, single-valued setting, finite-horizon optimal control problems are typically analyzed via two complementary methods: the Pontryagin Maximum Principle (yielding first-order necessary conditions) and dynamic programming/Hamilton–Jacobi–Bellman theory (yielding a value function characterization and verification machinery); see, e.g., [35, 36]. A standard formulation is the Bolza problem, in which one minimizes a terminal cost plus an integral running cost subject to a controlled differential equation and constraints. On the computational side, both indirect (shooting, adjoint-based) and direct (discretize-then-optimize) methods are widely used, often combined with convexification, continuation, and model predictive control to handle constraints and nonconvexities [37]. Similarly the optimal control of nonsmooth dynamical systems has been of interest to the research community in recent years [38, 39, 40, 41]. The optimal control problem in this case can be expressed as

$$
\begin{aligned}
J^*(x_0, t_0) = \inf_{u \in L^\infty([0,T];\mathbb{R}^m)} & \int_{t_0}^{T} \mathcal{L}(x(t), u(t))\, dt \\
\text{s.t. } & \dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + f(x(t), u(t)), \\
& x(t_0) = x_0, \quad u(t) \in U, \quad \text{for a.e. } t \in [0, T].
\end{aligned}
\tag{1.26}
$$

The discrete-time analogue of (1.26) is

$$
\begin{aligned}
J^*_\tau(x_0) = \inf_{\substack{x_k^\tau \in \mathbb{R}^n \\ u_k \in \mathbb{R}^m}} & \sum_{k=0}^{N-1} \tau\, \mathcal{L}(x_k^\tau, u_k) \\
\text{s.t. } & x_{k+1}^\tau \;=\; P_{S_{k+1}}(x_k^\tau + \tau\, f_k(x_k^\tau, u_k)), \\
& x_0^\tau = x_0, \quad u_k \in U, \quad k = 0, 1, \ldots, N-1,
\end{aligned}
\tag{1.27}
$$

where $t_k = t_0 + k\tau$, $S_k := S(t_k)$, and $f_k(x, u) := f(t_k, x, u)$ and $P_{S_{k+1}}(x)$ denotes the projection of x onto the set S_{k+1}.

Most existing approaches rely on local methods such as the Pontryagin Maximum Principle [42, 43], followed by numerical solution of the resulting optimality conditions. Similarly, there are works that use Hamilton–Jacobi–Bellman (HJB) equations to characterize optimal solutions [44]. By contrast, direct methods first discretize the problem and then apply numerical solvers [45]. The approaches typically deliver locally optimal solutions, and discretization-based schemes are prone to discretization/modeling errors [46] and require sophisticated methods to handle the nonsmooth projection/operator.

In Chapter 4, we propose a measure-based relaxation of the optimal control problems (1.26) and (1.27), providing certified global lower bounds.

1.3 Nonlinear evolution equations

We now move to infinite-dimensional dynamical systems governed by nonlinear evolution equations on Hilbert spaces. In this thesis, we focus on equations whose generator is quasi-dissipative. Such operators give rise to a rich class of evolution equations, including reaction–diffusion systems and other semilinear PDEs. In Chapter 5, we show that for this class of quasi-dissipative evolution equations, the natural measure formulation, a linear Liouville equation on probability measures, is free of relaxation gap, which in turn justifies applying an infinite-dimensional moment–SOS hierarchy.

Many physical, engineering, and biological systems can be modeled as evolution equations (or inequalities) on abstract Hilbert spaces. In an abstract form,

$$\frac{dy}{dt}(t) = f(y(t)), \qquad y(0) = y_0, \qquad \text{for a.e. } t \in [0, T], \tag{1.28}$$

where the state $y(t)$ takes values in a Hilbert space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$ and inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$, and $f : \mathcal{X} \to \mathcal{X}$ is (typically) nonlinear.

A frequently used special case is the semilinear equation

$$\frac{dy}{dt}(t) = Ay(t) + g(y(t)), \qquad y(0) = y_0, \tag{1.29}$$

where $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is (usually unbounded) and $D(A)$ denotes the operator domain, and $g : \mathcal{X} \to \mathcal{X}$ is a (nonlinear) perturbation (see Section 2.1 for related terminology and examples). Such abstract models cover, e.g., reaction–diffusion systems, damped wave equations, viscous conservation laws, and (in suitable function spaces) the incompressible Navier–Stokes equations.

Examples of nonlinear evolution equations.

- *Burgers' equation.* In one dimensional space with viscosity $\nu > 0$,

  $$\partial_t y(t, x) \;=\; \nu\, \partial_{xx} y(t, x) \;-\; y(t, x)\, \partial_x y(t, x),$$

  posed on a spatial interval with appropriate boundary conditions. For $\nu > 0$ (viscous Burgers), the Hopf–Cole transform [47, 48] yields an explicit representation in 1D; for $\nu = 0$ (inviscid Burgers),

  $$\partial_t y + y\, \partial_x y = 0,$$

  solutions generically develop shocks in finite time. Weak solutions are not unique; entropy solutions restore uniqueness and arise as vanishing-viscosity limits $\nu \to 0^+$. Numerical schemes that approximate the entropy solution include monotone/flux-limited finite volumes and other conservative, entropy-stable discretizations [49, 50].

- *Reaction–diffusion systems.* In $\mathbb{R}^d$,

  $$\partial_t y(t, x) \;=\; \mathsf{D}\, \Delta y(t, x) \;+\; \mathsf{R}\big(y(t, x)\big),$$

  where $y : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$, $\mathsf{D} \in \mathbb{R}^{m \times m}$ is a positive (semi)definite diffusion matrix, and $\mathsf{R} : \mathbb{R}^m \to \mathbb{R}^m$ is the reaction term. When $\mathsf{R}(y) = By + b$ is affine, solutions admit the variation-of-constants formula via the heat semigroup [51]. For nonlinear R, classical solution types include steady states and traveling waves; numerically, one often employs finite difference/element/volume methods, implicit or IMEX time stepping for stiffness, and structure-preserving discretizations that maintain positivity and maximum principles.

Relation to measure reformulations. Traditional numerical simulation and optimal control of PDEs rely on space–time discretization. An alternative approach, central to this thesis, is to first reformulate the problem in the space of measures and then discretize its moment representation. Following [52], measure relaxations supported on space–time yield convex formulations of optimal control for broad classes of PDEs. The question of when such linear, measure-based formulations do not introduce a relaxation gap (i.e., when the relaxation does not admit any spurious solutions) has attracted recent interest. For example, [53] proposes an LP over space–time measures together with a convergent semidefinite relaxation hierarchy.

Among these systems, in Chapter 5 we focus on a specific subclass, namely the quasi-dissipative evolution equations, characterized by

$$\langle f(x_1) - f(x_2),\, x_1 - x_2 \rangle \leq \alpha \|x_1 - x_2\|^2, \qquad \forall x_1, x_2 \in \mathcal{Y} \subset \mathcal{X}, \tag{1.30}$$

where $\alpha \geq 0$, $\mathcal{Y}$ denotes the maximally invariant set such that $y(0) \in \mathcal{Y}$ implies $y(t) \in \mathcal{Y}$ for all $t \geq 0$. Quasi-dissipative operators that are also maximal, that is, whose graphs cannot be extended without losing the quasi-dissipativity property, satisfy the Crandall–Liggett condition, which guarantees existence and uniqueness of solutions to the associated evolution equation. In Chapter 5, we study the measure relaxation of quasi-dissipative nonlinear evolution equations in the space of Borel measures supported on Hilbert space. We also show that this relaxation does not introduce any relaxation gap (i.e., it does not introduce spurious solutions).

1.4 Moment-SOS hierarchy

In the previous sections, we discussed the idea behind measure relaxation first in the context of static polynomial optimization problems, then we discussed optimal transport problem and finally arriving at optimal control problem. The measure relaxation resulted in infinite dimensional LPs in measures. Solving infinite dimensional problems is not tractable and traditional methods involve discretizing space-time coordinates to solve the resulting problem. In this thesis, we circumvent spatio-temporal gridding by working with the moments of the unknown measure and applying the moment-sums-of-squares (moment-SOS) hierarchy. The key rationale behind the technique is that the main properties of the measure like the support constraints, positivity or any other linear constraint can be expressed as linear equations on moments together with semidefiniteness of the certain matrices built from the moments.

A brief history The moment-SOS hierarchy, originally developed by Lasserre in his widely cited article [54], is the computational tool used in this thesis to solve optimization problems in the space of measures. They bring together tools from real algebraic geometry and functional analysis and formulate a plug-and-play scheme for solving optimization problems from various domains.

As we saw in Subsection 1.1, our relaxations ultimately require us to certify that certain polynomials are nonnegative. In general, deciding whether a polynomial is nonnegative on $\mathbb{R}^n$ is an NP-hard problem, meaning that no polynomial-time algorithm is known and one cannot expect a scalable exact method for arbitrary instances. By contrast, checking whether a polynomial can be written as a sum of squares (SOS) of other polynomials is more tractable, since it can be reformulated as a semidefinite program.

It is well known, however, that not every nonnegative polynomial is a SOS. A classical counterexample is the Motzkin polynomial

$$p_M(x, y) := x^4y^2 + x^2y^4 - 3x^2y^2 + 1,$$

which is nonnegative on $\mathbb{R}^2$ but cannot be written as a SOS.

The question of positivity of polynomials on a subset of $\mathbb{R}^n$ has been addressed through numerous works in real algebraic geometry, leading to various Positivstellensätze (Psatz) [55]. A Psatz expresses the problem of checking nonnegativity on a semialgebraic set into the existence of algebraic certificates involving sums of squares of polynomials. Using these results from real algebraic geometry and fundamental functional analysis arguments in [54] solved the polynomial optimization problem.

Later, this hierarchy was applied to solve a variety of problems. In [56], the authors extended it to optimal control problems with polynomial data. They used measure relaxation techniques to formulate the optimal control problem in terms of occupation measures, and then constructed a moment-SOS hierarchy whose relaxations provide guaranteed lower bounds on the optimal value. The same framework was used for problems involving volume approximation of semialgebraic sets [57]. In [58], the authors further extended these techniques to study the region of attraction of dynamical systems. For an extensive introduction to the moment-SOS hierarchy and its numerous applications, see [24].

1.5 Outline and contributions

The thesis is organized as follows:

- Chapter 2 introduces the technical preliminaries which are used throughout the thesis. Specifically, we introduce basics of dissipative operators, nonsmooth dynamical systems, continuity equations, the issue of relaxation gap, and the moment-SOS hierarchy.

- In Chapter 3, we study the evolution of measures in nonsmooth dynamical systems using three complementary approaches. (1) In Section 3.3, invoking the superposition principle, we formulate the continuity equation for a measure that admits a disintegration into a probability measure supported on the set of admissible vector fields, and a time-indexed family of probability measures describing the distribution of system trajectories. (2) In Section 3.4, we regularize the maximal monotone map that defines the nonsmooth vector field, and study the corresponding regularized measure trajectories.

We show that the desired measure trajectory arises as the limit of these regularized trajectories as the regularization parameter vanishes. In addition, we derive quantitative bounds in the Wasserstein-1 metric between the solutions of the regularized field and the limiting measure associated with the nonsmooth field. For measures with bounded support, we further obtain analogous bounds in the Wasserstein-2 metric. (3) In Section 3.5, we propose a time-stepping scheme for the discretized evolution of measures and prove that the absolutely continuous measure-valued trajectories of the continuity equation are recovered as the time-step tends to zero. Section 3.6 provides numerical validations of each formalism: it computes approximate moments using the moment-SOS hierarchy (first formalism) illustrates the closed-form Wasserstein bounds on an academic example (second formalism), and demonstrates the time-stepping scheme showing measure concentration effects (third formalism). The results presented in this chapter are taken from the article [59].

- Next in Chapter 4, we address the optimal control problem for the class of systems of interest presented in Chapter 3. In Section 4.2, we discuss the relaxation of the continuous-time optimal control problem, first using Young measures and then using occupation measures. We show that this does not introduce any relaxation gap, that is, the optimal value remains the same for the reformulated LP. In Section 4.3, we formulate the relaxation of the discrete-time problem in the space of measures, ensuring that the relaxation does not introduce any gap between the optimal values of the problems. In Section 4.4, we analyze the convergence of the relaxed discrete-time optimal control problem to its continuous-time counterpart. Finally, in Section 4.5, we introduce relaxations based on semidefinite programming for the relaxed discrete-time problem and illustrate the proposed approach with an academic example. The content of the chapter is based on the articles [60] and [61].

- In Chapter 5, we study the linear reformulation of the nonlinear evolution equation in the space of measures. We first provide the details of this reformulation, along with its connections to semigroup theory, in Sections 5.2–5.4. In Section 5.5, we prove that there is no relaxation gap between a quasi-dissipative nonlinear evolution equation in a Hilbert space and its linear Liouville equation reformulation on probability measures. In other words, strong and generalized solutions of the resulting continuity equations are unique in the class of measure-valued solutions. As a major consequence, optimization over these non-linear partial differential equations can be carried out with the infinite-dimensional moment-SOS hierarchy with global convergence guarantees and we provide the details about the moment-SOS hierarchy in Section 5.6. The article [62] is the main source of this chapter.

- In Chapter 6, we first provide concluding remarks summarizing the contribution of the thesis. Then we provide three research directions which we see as natural extensions to the problems addressed in this thesis.

Chapter 2

Technical Preliminaries

In this chapter, we collect the technical preliminaries used throughout the thesis. Section 2.1 recalls the notion of dissipativity for linear operators and its connection with C_0-semigroups. Section 2.2 introduces quasi-dissipativity for finite-dimensional dynamical systems, together with tools from nonsmooth dynamics that will be further developed in Chapters 3 and 4. The quasi-dissipativity concepts introduced there will be extended in Chapter 5 to infinite-dimensional evolution equations in Hilbert spaces. Section 2.3 discusses basic notions related to the continuity equation. In Section 2.4.1, under suitable convexity assumptions, we prove a no-relaxation-gap result and present an example exhibiting a relaxation gap. Finally, Section 2.5 summarizes the essentials of the moment–SOS hierarchy used in the remainder of the thesis. A discussion of weak convergence and basic functional-analytic notions is deferred to Appendix A.

2.1 Basics of dissipative operators

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space with norm $\| \cdot \|_{\mathcal{H}}$. We use the Riesz isomorphism to identify $\mathcal{H}$ with its dual. If $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, a linear operator is a map

$$A : D(A) \subset \mathcal{X} \to \mathcal{Y},$$

defined on a linear subspace $D(A) \subset \mathcal{X}$. A subset $D \subset \mathcal{X}$ is *dense in* $\mathcal{X}$ if its closure in $\mathcal{X}$ equals $\mathcal{X}$, i.e., $\overline{D}^{\mathcal{X}} = \mathcal{X}$. Accordingly, a (possibly unbounded) linear operator A is called *densely defined* if its domain $D(A)$ is dense in $\mathcal{X}$, that is,

$$\overline{D(A)}^{\mathcal{X}} = \mathcal{X}.$$

Bounded operators and operator norm. We say A is *bounded* if $D(A) = \mathcal{X}$ and there exists $C > 0$ such that

$$\|Ax\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}} \qquad \forall\, x \in \mathcal{X}.$$

In this case A is continuous and its *operator norm* is

$$\|A\|_{\mathrm{op}} := \sup_{\|x\|_{\mathcal{X}} \leq 1} \|Ax\|_{\mathcal{Y}} < \infty.$$

We denote the space of all bounded linear operators by

$$\mathrm{B}(\mathcal{X}, \mathcal{Y}) := \{A : \mathcal{X} \to \mathcal{Y} \text{ linear and bounded}\}, \qquad \mathrm{B}(\mathcal{X}) := \mathrm{B}(\mathcal{X}, \mathcal{X}).$$

Adjoint (densely defined case). Assume now that $A : D(A) \subset \mathcal{X} \to \mathcal{Y}$ is *densely defined.*

Its (Hilbert space) *adjoint* $A^* : D(A^*) \subset \mathcal{Y} \to \mathcal{X}$ is the operator with domain

$$D(A^*) := \Big\{ z \in \mathcal{Y} : \exists\, w \in \mathcal{X} \text{ such that } \langle Ax, z\rangle_{\mathcal{Y}} = \langle x, w\rangle_{\mathcal{X}} \ \ \forall\, x \in D(A) \Big\},$$

and $A^*z := w$ for such z. Then

$$\langle Ax, z\rangle_{\mathcal{Y}} = \langle x, A^*z\rangle_{\mathcal{X}} \qquad \forall\, x \in D(A),\ \forall\, z \in D(A^*).$$

Graph, closedness, and closability. The graph of A is

$$\operatorname{graph}(A) := \{(x, Ax) : x \in D(A)\} \subset \mathcal{X} \times \mathcal{Y}.$$

We say that A is *closed* if $\operatorname{graph}(A)$ is closed in $\mathcal{X} \times \mathcal{Y}$ (with the product norm). We say that A is *closable* if the closure $\overline{\operatorname{graph}(A)}$ is the graph of some operator, denoted $\overline{A}$, called the *closure* of A. Equivalently, A is closable iff: if $x_n \in D(A)$, $x_n \to 0$ in $\mathcal{X}$ and $Ax_n \to y$ in $\mathcal{Y}$, then $y = 0$.

Kernel and range. For later use we set $\ker A := \{x \in D(A) : Ax = 0\}$ and $\operatorname{Range}(A) := A(D(A)) \subset \mathcal{Y}$.

We will be interested in dissipative operators in this thesis and next we provide a formal definition of the same.

Definition 2.1.1 (Dissipativity)**.** A densely defined $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is *dissipative* if

$$\langle Ax, x\rangle_{\mathcal{H}} \le 0 \qquad \forall\, x \in D(A).$$

It is *maximally dissipative* (or *m-dissipative*) if it is dissipative and, for all $\lambda > 0$,

$$\operatorname{Range}(\mathrm{I}_{\mathcal{H}} - \lambda A) = \mathcal{H}.$$

Equivalently, $(\mathrm{I}_{\mathcal{H}} - \lambda A)^{-1}$ exists on $\mathcal{H}$ and is a contraction for all $\lambda > 0$.

Example: Dissipativity of the Laplacian on $L^2(\mathbb{R}^n)$. Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and consider the Laplacian $A = \Delta$ with domain $D(A) = H^2(\mathbb{R}^n)$. Recall that $H^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, so A is densely defined. For $y \in H^2(\mathbb{R}^n)$, an integration by parts (justified by standard density/approximation arguments) yields

$$\langle \Delta y, y\rangle_{L^2} = \int_{\mathbb{R}^n} (\Delta y(x))\, \overline{y(x)}\, dx = -\int_{\mathbb{R}^n} |\nabla y(x)|^2\, dx = -\|\nabla y\|_{L^2}^2 \ \le 0,$$

so Δ is dissipative on $L^2(\mathbb{R}^n)$.

We now verify the range condition for m-dissipativity. Given $f \in L^2(\mathbb{R}^n)$, consider the resolvent equation

$$(\mathrm{I}_{\mathcal{H}} - \Delta)u = f.$$

Taking Fourier transforms and using $\widehat{\Delta u}(\xi) = -|\xi|^2 \widehat{u}(\xi)$, we obtain

$$(1 + |\xi|^2)\, \widehat{u}(\xi) = \widehat{f}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

so

$$\widehat{u}(\xi) = \frac{1}{1+|\xi|^2}\,\widehat{f}(\xi).$$

The multiplier $(1+|\xi|^2)^{-1}$ is bounded and strictly positive, hence $\widehat{u} \in L^2(\mathbb{R}^n)$ whenever $\widehat{f} \in L^2(\mathbb{R}^n)$, and by Plancherel's theorem this defines a unique $u \in L^2(\mathbb{R}^n)$. Moreover,

$$(1+|\xi|^2)\,\widehat{u}(\xi) \in L^2(\mathbb{R}^n),$$

so $u \in H^2(\mathbb{R}^n) = D(A)$ and indeed satisfies $(\mathrm{I}_{\mathcal{H}} - \Delta)u = f$. Thus

$$\mathrm{Range}(\mathrm{I}_{\mathcal{H}} - \Delta) = L^2(\mathbb{R}^n).$$

We have shown that Δ is dissipative and that $\mathrm{Range}(\mathrm{I}_{\mathcal{H}} - \Delta)$ is all of $\mathcal{H}$, so Δ is m-dissipative.

The next tool we would like to introduce is the *resolvent*, which encodes both spectral and dynamical information about A in a form that is often easier to estimate than A itself.

Definition 2.1.2 (Resolvent set and resolvent)**.** The *resolvent set* of A, denoted by $\rho(A)$, is the set of complex numbers $\lambda \in \mathbb{C}$ for which $(\lambda \mathrm{I}_{\mathcal{H}} - A) : D(A) \to \mathcal{H}$ is bijective and has a bounded inverse. For $\lambda \in \rho(A)$, the resolvent of A is

$$R(\lambda, A) := (\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1}. \tag{2.1}$$

Dissipative operators can be equivalently characterized using the properties of resolvent. On Hilbert spaces, dissipativity admits the norm inequality

$$\|(\lambda \mathrm{I}_{\mathcal{H}} - A)x\| \;\geq\; \lambda\|x\| \qquad \forall\, x \in D(A),\ \forall\, \lambda > 0, \tag{2.2}$$

and whenever $\lambda \in \rho(A)$ this is equivalent to the resolvent estimate

$$\|\lambda\, R(\lambda, A)\| \;\leq\; 1. \tag{2.3}$$

Moreover, A is m-dissipative iff $(0,\infty) \subset \rho(A)$ and (2.3) holds for all $\lambda > 0$.

Example: Resolvent estimate for the Laplacian Δ on $L^2(\mathbb{R}^n)$ Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and Δ be the Laplacian with domain $H^2(\mathbb{R}^n)$. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$, consider the resolvent operator

$$R(\lambda, \Delta) := (\lambda - \Delta)^{-1},$$

defined (for such λ) by solving

$$(\lambda - \Delta)u = f, \qquad f \in L^2(\mathbb{R}^n).$$

Expressing this in Fourier space, we obtain

$$\widehat{u}(\xi) = m(\xi)\,\widehat{f}(\xi),$$

where the multiplier is

$$m(\xi) = \frac{1}{\lambda + |\xi|^2}.$$

We now estimate the modulus of $m(\xi)$. Since $\operatorname{Re}(\lambda + |\xi|^2) = \alpha + |\xi|^2$ and

$$|m(\xi)| = \frac{1}{|\lambda + |\xi|^2|} \leq \frac{1}{\operatorname{Re}(\lambda + |\xi|^2)} = \frac{1}{\alpha + |\xi|^2} \leq \frac{1}{\alpha},$$

we obtain

$$\|\widehat{u}\|_{L^2}^2 = \int_{\mathbb{R}^n} |m(\xi)|^2 \, |\widehat{f}(\xi)|^2 \, d\xi \leq \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi.$$

By Plancherel's theorem, this yields

$$\|u\|_{L^2} \leq \frac{1}{\alpha} \, \|f\|_{L^2},$$

that is,

$$\|R(\lambda, \Delta)\| \leq \frac{1}{\alpha} = \frac{1}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0.$$

Equivalently,

$$\|\lambda R(\lambda, \Delta)\| \leq 1 \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > 0,$$

which is precisely the resolvent contractivity estimate characterizing generators of contractive C_0-semigroups in the Lumer–Phillips theorem.

The following cornerstone result connects dissipativity to well-posed linear dynamics via contractive C_0-semigroups. Very roughly, a (strongly continuous) semigroup is a family of linear operators describing the time evolution of the system, with composition corresponding to the addition of times; precise definitions are collected in Appendix A.

Theorem 2.1.1 (Lumer–Phillips). *Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be densely defined and closed. The following are equivalent:*

1. *A generates a contractive C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ on $\mathcal{H}$ (i.e., $\|\mathbb{S}(t)\| \leq 1$ for all $t \geq 0$).*
2. *A is maximally dissipative.*

In this case, for $y_0 \in D(A)$ the Cauchy problem $\dot{y}(t) = Ay(t)$, $y(0) = y_0$, has a unique strong solution $y \in C^0([0,\infty); D(A)) \cap C^1((0,\infty); \mathcal{H})$ given by $y(t) = \mathbb{S}(t)y_0$.

By Theorem 2.1.1, in the case $A = \Delta$, A generates the heat semigroup $e^{t\Delta}$ with $\|e^{t\Delta}\| \leq 1$. For more details on semigroup theory and dissipative operators in the context of control theory, we refer to [63].

Remark 2.1.2. If $y_0 \notin D(A)$, the trajectory $t \mapsto \mathbb{S}(t)y_0$ is in general not differentiable at $t = 0$ and solves $\dot{y} = Ay$ only in the mild sense (via the variation-of-constants formula / semigroup theory).

We will revisit strong, mild, and weak solutions for nonlinear dissipative dynamics in Chapter 5. Nonlinear evolution equations will require a modified notion of dissipativity compared to Definition 2.1.1 for linear operators; this will be presented in Section 5.2.

2.2 Nonsmooth dynamical systems

The class of nonsmooth dynamical system we will address in this thesis can be generally written as

$$\dot{x}(t) \in -F(x(t)) \tag{2.4}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map (also called a set-valued operator). An important case is when F is a maximally monotone operator.

Definition 2.2.1 (Dissipative and maximally dissipative set-valued maps)**.** Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. We say that F is dissipative if

$$\langle x_1 - x_2,\ y_1 - y_2 \rangle \leq 0 \qquad \forall (x_i, y_i) \in \operatorname{graph}(F).$$

We say that F is maximally dissipative if it is dissipative and there is no dissipative $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that

$$\operatorname{graph}(F) \subsetneq \operatorname{graph}(G).$$

Equivalently, F is dissipative (resp. maximally dissipative) if and only if $-F$ is monotone (resp. maximally monotone).

Remark 2.2.1. This notion of dissipativity can be extended to the Hilbert space and is formally presented in Section 5.2.

The next theorem collects the basic well-posedness and contraction properties of (2.4).

Theorem 2.2.2. *Let $T > 0$ and let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be such that $-F$ is maximal dissipative. For any $x_0 \in \operatorname{dom} F$, there exists a unique solution $x \in AC([0,T];\mathbb{R}^n)$ to (2.4) with $x(0) = x_0$ and $x(t) \in \operatorname{dom} F$ for all $t \in [0,T]$. Moreover,*

$$\|\dot{x}\|_{L^\infty(0,T;\mathbb{R}^n)} \leq \|\operatorname{proj}(0, F(x_0))\|.$$

If x_1, x_2 are two solutions with initial data $x_1(0), x_2(0) \in \operatorname{dom} F$, then

$$|x_2(t) - x_1(t)| \leq |x_2(0) - x_1(0)| \qquad \forall t \in [0,T].$$

In the above theorem $\operatorname{proj}(0, F(x_0))$ can be understood as the minimal norm element of $F(x_0)$. The proofs of existence generally follow two routes and the following are the ideas behind the two routes:

- Treat the set-valued dynamics in (2.4) as a limit of Moreau–Yosida regularized dynamics when the regularization parameter tends to zero. The Moreau–Yosida regularization of a maximal monotone operator is defined as

$$Y_\lambda := \frac{1}{\lambda}\,(I - J(\lambda, F)), \tag{2.5}$$

where $J(\lambda, F) = (I + \lambda F)^{-1}$ is the resolvent of F for $\lambda > 0$ [1] When F is maximally monotone, $J(\lambda, F)$ is single-valued and defined on all of $\mathbb{R}^n$, and the map Y_λ is single-valued and λ^{-1}-Lipschitz. For a given initial condition, Picard's theorem establishes existence and uniqueness of the solution to the Lipschitz ODE

$$\dot{x}_\lambda(t) + Y_\lambda(x_\lambda(t)) = 0, \qquad x_\lambda(0) = x_0.$$

Moreover, the sequence of trajectories $x_\lambda(\cdot)$ converges uniformly to $x(\cdot)$ (solution to (2.4)) on compact time intervals, i.e.,

$$\lim_{\lambda \to 0} \|x_\lambda(\cdot) - x(\cdot)\|_{L^\infty([0,T])} = 0.$$

- Alternatively, one can interpret the dynamics as the limit of an implicit Euler (proximal/resolvent) discretization of (2.4) as the time step tends to zero. For a given time step $h > 0$, the iterates are defined by the backward Euler rule

$$\frac{x_{k+1} - x_k}{h} + F(x_{k+1}) \ni 0, \qquad \text{equivalently} \qquad x_{k+1} = J(h, F)(x_k),$$

  where $J(h, F) := (I + hF)^{-1}$ is well defined and nonexpansive for maximal monotone F. Passing to the limit $h \to 0$ along the piecewise constant (or piecewise linear) interpolants yields the solution of (2.4), and the nonexpansiveness ensures the contractivity estimate in Theorem 2.2.2.

For more details and proofs for general class of nonsmooth dynamical systems, we refer to [29].

The specific class of dynamical systems considered here is the Moreau sweeping process, as introduced in (1.22). In order to study it, we first recall some basic notions of cones, normal cones, and tangent cones.

Definition 2.2.2 (Cone)**.** A nonempty subset $\mathcal{K}$ of $\mathbb{R}^n$ is called a *cone* if

$$x \in \mathcal{K} \implies \lambda x \in \mathcal{K} \quad \forall\, \lambda \geq 0.$$

Definition 2.2.3. (Polar cone and normal cone) The polar cone $\mathcal{K}^o$ of a nonempty subset $\mathcal{K} \subset \mathbb{R}^n$ is defined as

$$\mathcal{K}^o := \{p \in \mathbb{R}^n \mid \langle p, x\rangle \leq 0 \quad \forall x \in \mathcal{K}\}\,.$$

[1] $J(\lambda, F)$ is the resolvent in the sense of monotone operator theory, not to be confused with the spectral resolvent of a linear operator A, namely $R(\mu, A)$ defined in (2.1); when $F = A$ is linear, the two are related by $J(\lambda, A) = (I + \lambda A)^{-1} = -\frac{1}{\lambda}\,R(-\frac{1}{\lambda}, A)$.

The normal cone to set $S \subset \mathbb{R}^n$ at a point $x \in S$ is defined as

$$\mathcal{N}_S(x) := \{p \in \mathbb{R}^n \mid \langle p, y - x \rangle \leq 0 \quad \forall y \in S\}$$

Consider the sweeping process

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)), \qquad x(0) \in S(0),$$

where $S(t)$ is assumed to be convex and compact for every $t \in [0, T]$. Under suitable regularity assumptions on the map $t \mapsto S(t)$, the existence and uniqueness of solutions to this Cauchy problem are well understood; see, for instance, [64, 29]. Conceptually, these existence results can be viewed as particular cases of Theorem 2.2.2, but because of the time-varying sets $S(t)$ the proof is more involved and is carefully addressed in [64].

A natural extension is to include a gradient drift of the form $\nabla f(x)$ for some convex function $f : \mathbb{R}^n \to \mathbb{R}$. In this case the (single-valued) operator $x \mapsto \nabla f(x)$ is monotone, and more generally the subdifferential ∂f is maximally monotone. Existence and uniqueness for such perturbed sweeping processes have been studied using the above tools; see, e.g., [65].

In this thesis we will study measure evolution and optimal control problems for first-order Moreau sweeping processes with Lipschitz continuous perturbations f:

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + f(x(t)). \tag{2.6}$$

For each fixed t, the right-hand side of (2.6) defines a (possibly set-valued) vector field, and we will see that these perturbed sweeping processes can be viewed as quasi-dissipative vector fields, in the following sense.

Definition 2.2.4 (Quasi-dissipative vector field)**.** A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called α–quasi-dissipative for some $\alpha \geq 0$ if

$$\langle v_1 - v_2, x_1 - x_2 \rangle \;\leq\; \alpha \, \|x_1 - x_2\|^2 \qquad \text{for all } x_1, x_2 \in \mathbb{R}^n,\ v_i \in F(x_i). \tag{2.7}$$

Equivalently, F is α–quasi-dissipative if the shifted operator

$$x \;\mapsto\; F(x) + \alpha x$$

is dissipative in the sense of (1.23). The case $\alpha = 0$ reduces to classical dissipativity.

We now relate this notion to the nonsmooth system (1.22). Assume that:

- the normal cone mapping $x \mapsto \mathcal{N}_{S(t)}(x)$ is maximally monotone for each fixed t (which holds for closed convex sets), and
- the drift f satisfies a one-sided Lipschitz condition

  $$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \;\leq\; \alpha \, \|x_1 - x_2\|^2 \qquad \forall x_1, x_2 \in \mathbb{R}^n$$

  for some $\alpha \geq 0$.

Define the set-valued mapping

$$F_t(x) := f(x) - \mathcal{N}_{S(t)}(x).$$

Let $x_1, x_2 \in S(t)$ and choose $v_i = f(x_i) - \xi_i$ with $\xi_i \in \mathcal{N}_{S(t)}(x_i)$. Then

$$\begin{aligned}\langle v_1 - v_2, x_1 - x_2\rangle &= \langle f(x_1) - f(x_2), x_1 - x_2\rangle - \langle \xi_1 - \xi_2, x_1 - x_2\rangle \\ &\leq \alpha\|x_1 - x_2\|^2 - \langle \xi_1 - \xi_2, x_1 - x_2\rangle.\end{aligned}$$

By monotonicity of the normal cone mapping, we have

$$\langle \xi_1 - \xi_2, x_1 - x_2\rangle \; \geq 0,$$

so that

$$\langle v_1 - v_2, x_1 - x_2\rangle \; \leq \; \alpha\|x_1 - x_2\|^2.$$

Hence F_t is α–quasi-dissipative in the sense of Definition 2.2.4 for each fixed t.

In other words, the sweeping process (1.22) provides a canonical example of a nonsmooth quasi-dissipative system: the reaction term $-\mathcal{N}_{S(t)}$ is (maximally) dissipative, and the drift f is a perturbation that is controlled in a one-sided Lipschitz sense. This finite-dimensional picture will later be extended in Chapter 5 to quasi-dissipative operators on Hilbert spaces.

Example (1-d sweeping process). Let $x : [0, T] \to \mathbb{R}$ satisfy

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + c, \tag{2.8}$$

where $c \in \mathbb{R}$ is a constant and

$$S(t) = (-\infty, a(t)] \quad \text{with } |a(t_2) - a(t_1)| \leq L_s|t_2 - t_1| \quad \forall\, t_1, t_2 \in [0, T].$$

Then $S(t)$ is convex and is Lipschitz in time with respect to the Hausdorff metric:

$$d_H(S(t_2), S(t_1)) = |a(t_2) - a(t_1)| \leq L_s|t_2 - t_1|.$$

The normal cone for $S(t)$ is

$$\mathcal{N}_{S(t)}(x) = \begin{cases}\{0\}, & x < a(t), \\ [0, \infty), & x = a(t),\end{cases}$$

so $-\mathcal{N}_{S(t)}(x) = \{0\}$ in the interior and $-\mathcal{N}_{S(t)}(a(t)) = (-\infty, 0]$ at the boundary. Hence:

- If $x(t) < a(t)$, (2.8) reduces to $\dot{x}(t) = c$.
- If $x(t) = a(t)$, then $\dot{x}(t) \in (-\infty, 0] + c$, i.e., $\dot{x}(t) \leq c$.

In particular, on the boundary the normal cone term can compensate the constant drift c so that the trajectory remains in the moving set $S(t)$. Under the standing assumptions, and by [29], the solution trajectories of (2.8) are absolutely continuous.

In Chapter 3, we will further explore the results relevant for problems related to the study of measure evolution through these dynamical systems. Furthermore in Chapter 4, we study the optimal control problem related to these dynamical systems using measure relaxations and measure-valued formulations tailored to these quasi-dissipative nonsmooth systems.

2.3 Continuity equation

This section recalls the continuity (Liouville) equation already introduced in (1.16), summarizes the geodesic structure of the quadratic Wasserstein space, and records a simple well-posedness result under spatial Lipschitz regularity of the vector field.

2.3.1 Weak formulation

Let $T > 0$ and let $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a (time-dependent) Borel-measurable vector field. For a measure curve $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}(\mathbb{R}^n)$, the *continuity equation*

$$\partial_t \mu_t + \nabla\cdot(v(t,\cdot)\,\mu_t) = 0, \qquad \mu_{|t=0} = \mu_0 \tag{2.9}$$

is understood in the weak sense described below.

We use the narrow topology on $\mathcal{P}(\mathbb{R}^n)$, i.e., the weakest topology making $\mu \mapsto \int_{\mathbb{R}^n} \phi\, d\mu$ continuous for every $\phi \in C_b(\mathbb{R}^n)$. A curve $t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^n)$ is said to be *narrowly continuous* if $t \mapsto \int \phi\, d\mu_t$ is continuous on $[0,T]$ for all $\phi \in C_b(\mathbb{R}^n)$. With this notation, a (narrowly continuous) curve $(\mu_t)_{t\in[0,T]}$ is a *weak solution* of (2.9) with initial datum μ_0 if for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ the map $t \mapsto \int \varphi\, d\mu_t$ is absolutely continuous and satisfies

$$\frac{d}{dt}\int \varphi\, d\mu_t = \int \nabla\varphi(x)\cdot v(t,x)\, d\mu_t(x) \quad \text{for a.e. } t \in [0,T], \qquad \mu_{|t=0} = \mu_0.$$

Equivalently, in space–time distributional form: for every $\psi \in C_c^1([0,T]\times\mathbb{R}^n)$,

$$\int_0^T\!\!\int_{\mathbb{R}^n} \big(\partial_t\psi(t,x) + \nabla_x\psi(t,x)\cdot v(t,x)\big)\, d\mu_t(x)\, dt \;+\; \int_{\mathbb{R}^n} \psi(0,x)\, d\mu_0(x) \;=\; \int_{\mathbb{R}^n} \psi(T,x)\, d\mu_T(x).$$

Assumption 2.3.1. There exist constants $L, G \geq 0$ such that for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^n$,

$$|v(t,x) - v(t,y)| \leq L|x-y|, \qquad |v(t,x)| \leq G\,(1+|x|).$$

Under Assumption 2.3.1, the Carathéodory ODE $\dot{x}(t) = v(t,x(t))$ generates a unique flow $\mathbb{S}(t,s) : \mathbb{R}^n \to \mathbb{R}^n$ with the group property $\mathbb{S}(t,s)\circ\mathbb{S}(s,r) = \mathbb{S}(t,r)$ and $\mathbb{S}(t,t) = \mathrm{id}$.

2.3.2 Absolutely continuous curves and Wasserstein geodesics

Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of Borel probability measures on $\mathbb{R}^n$. Define

$$\mathcal{P}_2(\mathbb{R}^n) := \Big\{\mu \in \mathcal{P}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |x|^2\, d\mu(x) < \infty\Big\}.$$

For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$, let the set of *transport plans* (couplings) be

$$\Pi(\mu,\nu) := \Big\{\theta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) : (\pi_1)_\#\theta = \mu,\ (\pi_2)_\#\theta = \nu\Big\},$$

where $\pi_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are the coordinate projections and $(\pi_i)_\#\pi$ denotes the push-forward of π by π_i. The 2-Wasserstein distance between μ and ν is

$$W_2(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\, d\pi(x,y)\right)^{1/2}.$$

For a curve $(\mu_s)_{s\in[0,1]}$ in the metric space $(\mathcal{P}_2(\mathbb{R}^n), W_2)$, its *metric derivative* at s is

$$|\mu'|(s) := \lim_{h\to 0} \frac{W_2(\mu_{s+h},\mu_s)}{|h|} \quad \text{(when the limit exists)}.$$

A curve is *absolutely continuous* in $(\mathcal{P}_2(\mathbb{R}^n), W_2)$ if there exists $g \in L^1(0,1)$ with $W_2(\mu_s,\mu_t) \le \int_s^t g(r)\,dr$ for all $0 \le s \le t \le 1$; in this case $|\mu'|(s)$ exists for a.e. $s \in [0,1]$, so that

$$W_2(\mu_s,\mu_t) \le \int_s^t |\mu'|(r)\,dr, \qquad \text{Length}(\mu_\cdot) = \int_0^1 |\mu'|(r)\,dr.$$

In particular, for absolutely continuous curves the following characterization holds: if $\partial_s\mu_s + \nabla\cdot(v_s\mu_s) = 0$ with $v_s \in L^2(\mu_s)$, then

$$\|v_s\|_{L^2(\mu_s)} \ge |\mu'|(s) \quad \text{for a.e. } s,$$

and equality holds for a suitable (minimal) choice of v_s, yielding $|\mu'|(s) = \|v_s\|_{L^2(\mu_s)}$ a.e. Recalling the Benamou–Brenier dynamic formulation from Chapter 1,

$$W_2^2(\mu_0,\mu_1) = \inf\Big\{\int_0^1\int_{\mathbb{R}^n} |v_s(x)|^2\, d\mu_s(x)\, ds :\ \partial_s\mu_s + \nabla\cdot(v_s\mu_s) = 0,\ \mu_{|s=0} = \mu_0,\ \mu_{|s=1} = \mu_1\Big\}, \tag{2.10}$$

see (1.16). Minimizers (μ_s, v_s) are constant-speed geodesics; in particular, for any absolutely continuous $(\mu_s)_{s\in[0,1]}$ in W_2 one can select a Borel field $v_s \in L^2(\mu_s)$ such that

$$\partial_s\mu_s + \nabla\cdot(v_s\mu_s) = 0 \quad \text{and} \quad |\mu'|(s) = \|v_s\|_{L^2(\mu_s)} \quad \text{for a.e. } s \in [0,1]. \tag{2.11}$$

Conversely, any pair (μ_s, v_s) satisfying (2.11) with $v_s \in L^2(\mu_s)$ yields an absolutely continuous curve, and geodesics are precisely those for which $s \mapsto \|v_s\|_{L^2(\mu_s)}$ is constant and attains the minimum in (2.10). This identification between metric derivatives and kinetic energy is the backbone of our later use of geodesic interpolations (see Section 3.5 for the discrete-time construction and McCann interpolation). For a rigorous treatment we refer to [16, Chapter 8].

2.3.3 Well-posedness by method of characteristics

We recall a classical well-posedness result for the continuity equation under Lipschitz vector fields; we sketch the proof and refer to [16, Ch. 8] for details.

Theorem 2.3.1. *Let Assumption 2.3.1 hold and let* $\mu_0 \in \mathcal{P}_2(\mathbb{R}^n)$*. Then the curve*

$$\mu_t := \mathbb{S}(t,0)_{\#}\mu_0, \qquad a.e.\ t \in [0,T],$$

is the unique narrowly continuous solution of (2.9)*. Moreover* $t \mapsto \mu_t$ *is absolutely continuous in* $(\mathcal{P}_2(\mathbb{R}^n), W_2)$ *and, for every* $\varphi \in C_c^\infty(\mathbb{R}^n)$*,*

$$\frac{d}{dt}\int \varphi\, d\mu_t = \int \nabla\varphi \cdot v(t,\cdot)\, d\mu_t \quad \text{for a.e. } t \in [0,T].$$

Proof sketch. Existence. Let $\mu_t := \mathbb{S}(t,0)_{\#}\mu_0$. For $\varphi \in C_c^\infty(\mathbb{R}^n)$ and a.e. t,

$$\frac{d}{dt}\varphi(\mathbb{S}(t,0)x) = \nabla\varphi(\mathbb{S}(t,0)x) \cdot v(t, \mathbb{S}(t,0)x)\,.$$

Integrating against μ_0 and changing variables yields the weak formulation. The linear growth bound propagates second moments, so $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^n)$ and $t \mapsto \mu_t$ is absolutely continuous in $(\mathcal{P}_2, W_2)$.

Uniqueness. Let (ν_t) be any weak solution with $\nu_0 = \mu_0$. For $\psi \in C_c^\infty(\mathbb{R}^n)$, set $\varphi_t(x) := \psi(\mathbb{S}(0,t)x)$. Then, by the chain rule and the flow identity $\partial_t\mathbb{S}(0,t)x = -v(t, \mathbb{S}(0,t)x)$,

$$\partial_t\varphi_t(x) + \nabla\varphi_t(x) \cdot v(t,x) = 0.$$

Hence

$$\frac{d}{dt}\int \varphi_t\, d\nu_t = \int (\partial_t\varphi_t + \nabla\varphi_t \cdot v)\, d\nu_t = 0,$$

so $\int \psi\, d\nu_t = \int \psi\, d(\mathbb{S}(t,0)_{\#}\mu_0)$ for all ψ, i.e. $\nu_t = \mu_t$. □

Remark 2.3.2. Low-regularity settings (Sobolev/BV vector fields, one-sided Lipschitz, differential inclusions) require DiPerna–Lions/Ambrosio or superposition techniques and will appear later in the thesis. For the present bridge to measure relaxations, the Lipschitz case suffices as a baseline.

2.4 Measure relaxation

In Section 1.1.3, we introduced Young measures and occupation measures in the context of a Bolza optimal control problem. In this section, we first present the general ideas behind measure relaxation, and then use these ideas to relax a convex optimal control problem via occupation measures. The guiding idea is to enlarge both the feasible space of solutions and the class of functionals acting on them in such a way that the objective becomes continuous and the relaxed feasible set enjoys suitable compactness properties, thereby preventing non-attainment phenomena.

Idea and intuition behind Young measures. Assume the original optimization problem suffers from non-attainment, for instance numerical experiments suggest oscillatory minimizing sequences, or existing existence theorems do not apply. We then study properties of oscillating minimizing sequences $\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{X}$ with

$$\mathcal{L}(x_j) \longrightarrow J^\star := \inf_{x\in\mathcal{X}} \mathcal{L}(x).$$

The Young–measure relaxation enlarges the feasible set and extends the functional so that the infimum is attained in the relaxed space.

Abstract construction: Fix a class of test functionals $\mathscr{L}$ with $\mathcal{L} \in \mathscr{L}$. We consider admissible sequences in $\mathcal{X}$ and restrict attention to those sequences $\{x_j\}$ for which the limit $\lim_{j\to\infty} \mathcal{F}(x_j)$ exists for every $\mathcal{F} \in \mathscr{L}$. We then declare two such sequences $\{x_j\}$ and $\{y_i\}$ equivalent if

$$\{x_j\} \;\equiv\; \{y_i\} \quad \Longleftrightarrow \quad \lim_{j\to\infty} \mathcal{F}(x_j) = \lim_{i\to\infty} \mathcal{F}(y_i) \quad \text{for every } \mathcal{F} \in \mathscr{L}.$$

Let $\mathcal{X}_r$ be the set of equivalence classes $\tilde{x} = [\{x_j\}]$ of such sequences. In particular, $\mathcal{X}$ embeds canonically into $\mathcal{X}_r$ by identifying $x \in \mathcal{X}$ with the constant sequence $(x, x, \dots)$, so that we may regard $\mathcal{X} \subset \mathcal{X}_r$.

For each $\mathcal{F} \in \mathscr{L}$ define its relaxed extension $\widehat{\mathcal{F}} : \mathcal{X}_r \to \mathbb{R}$ by

$$\widehat{\mathcal{F}}(\tilde{x}) \;:=\; \lim_{j\to\infty} \mathcal{F}(x_j) \qquad \text{for any representative } \tilde{x} = [\{x_j\}],$$

which is well defined by the equivalence relation. We equip $\mathcal{X}_r$ with the coarsest topology for which all maps $\widehat{\mathcal{F}}$, $\mathcal{F} \in \mathscr{L}$, are continuous. Equivalently, a sequence $(\tilde{x}_j) \subset \mathcal{X}_r$ converges to $\tilde{x} \in \mathcal{X}_r$ if and only if

$$\lim_{j\to\infty} \widehat{\mathcal{F}}(\tilde{x}_j) = \widehat{\mathcal{F}}(\tilde{x}) \quad \forall \mathcal{F} \in \mathscr{L}.$$

In particular, we write $\mathcal{L}_r := \widehat{\mathcal{L}}$.

Relaxed problem: With these ingredients, the relaxed problem reads

$$\text{find } \; \tilde{x}^\star \in \mathcal{X}_r \quad \text{such that} \quad \mathcal{L}_r(\tilde{x}^\star) = r^\star := \inf_{\tilde{x}\in\mathcal{X}_r} \mathcal{L}_r(\tilde{x}). \tag{2.12}$$

Because $\mathcal{X} \subset \mathcal{X}_r$ and $\mathcal{L}_r|_{\mathcal{X}} = \mathcal{L}$, we immediately have $r^\star \leq J^\star$. Conversely, by construction of $\mathcal{X}_r$, for any $\tilde{x} = [\{x_j\}]$,

$$\mathcal{L}_r(\tilde{x}) = \lim_{j\to\infty} \mathcal{L}(x_j) \geq J^\star,$$

hence $r^\star \geq J^\star$. Therefore

$$r^\star = J^\star,$$

and any $\tilde{x}$ realized by a minimizing sequence $\{x_j\}$ with $\mathcal{L}(x_j) \to J^\star$ is a minimizer of the relaxed problem (2.12). In this sense, all sequences generating the same $\tilde{x}$ are equivalent and are minimizing sequences.

In summary, the abstract construction above provides an enlarged space $\mathcal{X}_r$ that contains $\mathcal{X}$ and captures generalized (oscillatory) limits of admissible sequences, together with a relaxed functional $\mathcal{L}_r$ that coincides with $\mathcal{L}$ on $\mathcal{X}$ and is continuous with respect to the induced topology. In particular, the relaxed problem on $\mathcal{X}_r$ has the same optimal value $r^\star = J^\star$ as the original problem, while being well posed with actual minimizers in the relaxed space.

In Chapter 4 we instantiate this framework for optimal control of set-valued dynamical systems. There, $\mathcal{X}_r$ will be represented via Young measures (capturing fast oscillations), and

$\mathcal{L}_r$ becomes the Young–measure (or occupation–measure) relaxation of the original cost.

2.4.1 Measure relaxation of a convex optimal control problem

In Section 1.1.3, we reviewed measure relaxations of optimal control problems and discussed recent advances on relaxation gaps, which are a central theoretical issue in this framework. In this section, we present a no–relaxation–gap result for an optimal control problem in which the vector field is convex in the control variable and the control set is convex, following [28]. For the sake of completeness, and to illustrate the main ideas behind such arguments, we provide here a self-contained version of the proof from [28] specialized to the optimal control setting.

2.4.2 Proof of no relaxation gap

Consider the optimal control problem

$$\begin{aligned} J(x_0) = \min_{u(\cdot)} & \int_0^T \mathcal{L}(x(t), u(t))\, dt \\ \text{s.t. } & \dot{x}(t) = f(t, x(t), u(t)), \\ & x(\cdot) \in W^{1,\infty}([0,T]; \mathbb{R}^n), \\ & u(\cdot) \in L^\infty([0,T]; \mathbb{R}^m), \\ & x(t) \in X \subset \mathbb{R}^n, \;\; u(t) \in U \subset \mathbb{R}^m \quad \forall t \in [0,T], \end{aligned} \tag{2.13}$$

where we assume that both X and U are compact and convex.

Assumption 2.4.1. The function

$$\bar{\mathcal{L}}(t, x, z) := \min \{\mathcal{L}(x, u) \,|\, u \in U, \; z = f(t, x, u)\}$$

is locally bounded and convex in (x, z), and the set of minimizers is nonempty for all (t, x, z) in its domain.

The relaxation of problem (2.13) using occupation measures, as introduced in Section 1.1.3, leads to the following linear problem:

$$\begin{aligned} J_r(x_0) = \min_{\mu} & \int_{[0,T]\times X\times U} \mathcal{L}(x, u)\, d\mu(t, x, u) \\ \text{s.t. } & \int_{[0,T]\times X\times U} \Big[\partial_t \phi(t, x) + \partial_x \phi(t, x) \cdot f(t, x, u)\Big]\, d\mu(t, x, u) \\ & \qquad = \int_X \phi(T, x)\, d\mu_T(x) - \phi(0, x_0), \quad \forall \phi \in C^1([0,T] \times X), \\ & \mu \in \mathcal{M}_+([0,T] \times X \times U), \end{aligned} \tag{2.14}$$

where $\mathcal{M}_+(A)$ denotes the cone of nonnegative Borel measures on A and μ_T is the terminal-state measure.

Theorem 2.4.1. *Let $x_0 \in X$ be fixed and let us assume that the Assumption 2.4.1 is satisfied for the problem* (2.13)*. Then the optimal value $J(x_0)$ of* (2.13) *is equal to the optimal value of the occupation measure relaxed optimal control problem* (2.14)*, i.e., $J(x_0) = J_r(x_0)$ for all $x_0 \in X$.*

Proof. We divide the proof in two parts: (1) Proof of $(J(x_0) \geq J_r(x_0))$ and (2) Proof of $(J_r(x_0) \geq J(x_0))$.

(1) (Proof of $(J(x_0) \geq J_r(x_0))$): For given initial data $x_0 \in X$, let $t \mapsto (x(t), u(t))$ be an admissible trajectory for (2.13). Then the measures

$$d\mu(t,x,u) = dt\, \delta_{x(t)}(dx)\, \delta_{u(t)}(du), \quad \mu_T(dx) = \delta_{x(T)}(dx)$$

are admissible for (2.14) and

$$J_r(x_0) \leq \int_{[0,T]\times X\times U} \mathcal{L}(x,u) d\mu(t,x,u) = \int_0^T \mathcal{L}(x(t),u(t))dt.$$

Minimizing over admissible trajectories in (2.13) yields $J_r(x_0) \leq J(x_0)$.

(2) (Proof of $(J(x_0) \leq J_r(x_0))$): To prove this we have to show that an admissible pair of measures (μ, μ_T) for (2.14) can generate an admissible trajectory $(t, x(t), u(t))$ for (2.13).

First observe that with a particular test function $\phi(t,x) = \psi(t)$ in (2.14) that does not depend on x, it holds

$$\int_X \psi(T) d\mu_T(x) - \psi(0) = \int_{[0,T]\times X\times U} \dot{\psi}(t) d\mu(t,x,u) \tag{2.15}$$

and this implies that the t marginal of μ is the Lebesgue measure on $[0,T]$, i.e.

$$d\mu(t,x,u) = dt\, d\mu_t(x,u) \tag{2.16}$$

for some $\mu_t \in \mathcal{M}_+(X \times U)$. With the choice $\psi(t) = t - T$ in (2.15), we obtain $\int d\mu = T$ and hence $\int d\mu_t = 1$, i.e., μ_t is a probability measure.

Define

$$x(t) := \int_{X\times U} x\, d\mu_t(x,u), \qquad z(t) := \int_{X\times U} f(t,x,u)\, d\mu_t(x,u). \tag{2.17}$$

Fix $w \in \mathbb{R}^n$ and take the scalar test function $\phi_w(t,x) := \langle w, x\rangle\, \psi(t)$ with $\psi \in C_c^1((0,T))$ in (2.14). Then

$$\int_{[0,T]\times X\times U} \Big(\langle w, x\rangle\, \dot{\psi}(t) + \langle w, f(t,x,u)\rangle\, \psi(t) \Big)\, d\mu(t,x,u) = 0,$$

and, after disintegrating μ as in (2.16) and using (2.17),

$$\int_0^T \langle w, x(t)\rangle\, \dot{\psi}(t)\, dt = -\int_0^T \langle w, z(t)\rangle\, \psi(t)\, dt.$$

Since this holds for all w, we conclude

$$\int_0^T x(t)\,\dot{\psi}(t)\,dt = -\int_0^T z(t)\,\psi(t)\,dt,$$

i.e., the weak derivative of $x(t)$ is $z(t)$.

Moreover, from (2.17) and the fact that μ_t is a probability measure,

$$\|z(t)\| = \Big\|\int f(t,x,u)\,d\mu_t(x,u)\Big\| \le \int \|f(t,x,u)\|\,d\mu_t(x,u) \le M_f,$$

where

$$M_f := \sup_{(t,x,u)\in[0,T]\times X\times U} \|f(t,x,u)\| < \infty,$$

which follows from f being polynomial and $X \times U$ compact. By convexity of X, it follows that $x(t) \in X$. Therefore $\dot{x}(\cdot) = z(\cdot) \in L^\infty([0,T];\mathbb{R}^n)$ and hence $x(\cdot) \in W^{1,\infty}([0,T];\mathbb{R}^n)$.

For each $t \in [0,T]$, consider the set-valued map

$$U_t := \arg\min_{u\in U} \mathcal{L}(x(t),u) \quad \text{s.t.} \quad z(t) = f(t,x(t),u). \tag{2.18}$$

For each fixed $t \in [0,T]$, the constraint set

$$A_t := \{u \in U : f(t,x(t),u) = z(t)\}$$

is nonempty and closed (hence compact), since $u \mapsto f(t,x(t),u)$ is continuous and $z(t) \in f(t,x(t),U)$ by construction. The continuous cost $u \mapsto \mathcal{L}(x(t),u)$ attains its minimum on A_t, so U_t is nonempty and compact. Moreover, the graph

$$\mathrm{graph}(U_\cdot) = \big\{(t,u) \in [0,T] \times U : f(t,x(t),u) = z(t),\ \mathcal{L}(x(t),u) = \bar{\mathcal{L}}(t,x(t),z(t))\big\}$$

is Borel-measurable, since it is defined by equalities of measurable maps. Hence the multifunction $t \mapsto U_t$ is measurable with nonempty closed values [66, Def. 8.1.1, Thm. 8.1.4]. By the measurable selection theorem [66, Thm. 8.1.3], there exists a measurable selection $t \mapsto u(t) \in U_t$, that is, $u(\cdot) \in L^\infty([0,T];U)$.

Thus we obtain a trajectory $t \mapsto (x(t),u(t))$ which is admissible for problem (2.13). Thus we get

$$\begin{aligned} J(x_0) &\le \int_0^T \mathcal{L}(x(t),u(t))\mathrm{d}t = \int_0^T \bar{\mathcal{L}}(t,x(t),z(t))\mathrm{d}t \\ &\le \int_{[0,T]\times X\times U} \bar{\mathcal{L}}(t,x,f(t,x,u))\mathrm{d}\mu_t(x,u)\mathrm{d}t \le \int_{[0,T]\times X\times U} \mathcal{L}(x,u)\mathrm{d}\mu(t,x,u) \end{aligned} \tag{2.19}$$

where the first equality follows from the definition of $u(t)$, the second inequality uses Assumption 2.4.1 and the last inequality uses Jensen's inequality. For the last inequality, we observe that if we define the set $\mathscr{U}_z := \{u \in U \mid z = f(t,x,u)\}$ then $U = \cup_z \mathscr{U}_z$ and for each (x,z), $\bar{\mathcal{L}}(t,x,z) \le \mathcal{L}(x,u)$. Minimizing over admissible measures in (2.14) yields $J(x_0) \le J_r(x_0)$. □

Theorem 2.4.1 illustrates a central theme of this thesis: under suitable structural assumptions, measure relaxations of control problems preserve the optimal value, i.e., they introduce no relaxation gap. In Chapters 3 and 4, we exploit the quasi-dissipative structure of sweeping processes to obtain analogous no-gap results for measure evolution and optimal control. In Chapter 5, we prove an infinite-dimensional counterpart for quasi-dissipative evolution equations on Hilbert spaces.

2.4.3 Example with relaxation gap

Next we provide a well-known example [58, Appendix 3] where the nonconvexity in the state and the feasible control sets produces a relaxation gap.

Fix $R > 0$ and $L > R$. Let

$$D_L := \left\{(x,y) \in \mathbb{R}^2 : \ (x+L)^2 + y^2 \leq R^2\right\}, \quad D_R := \left\{(x,y) \in \mathbb{R}^2 : \ (x-L)^2 + y^2 \leq R^2\right\},$$

and let the two disks be connected only by the line segment

$$L_{LR} := \{(x,0) : \ -L \leq x \leq L\}.$$

Define the (nonconvex) admissible state set

$$X \ := \ D_L \cup L_{LR} \cup D_R.$$

Consider the control system

$$\dot{x}(t) = u_x(t), \qquad \dot{y}(t) = u_y(t), \qquad (u_x(t), u_y(t)) \in U := [-1,1] \times \{-1,1\},$$

such that $u \in L^\infty([0,T];\mathbb{R}^2)$. We define the initial condition

$$(x(0), y(0)) = \left(-L + \tfrac{R}{2},\, 0\right) \in D_L,$$

and target point $X_T := (L + \frac{R}{2}, 0)$.

Consider, for instance, the terminal-cost problem

$$\begin{aligned} &\min_{u(\cdot)} \|(x(T), y(T)) - X_T\|, \\ &\qquad \text{s.t.} \quad \dot{x}(t) = u_x(t), \qquad \dot{y}(t) = u_y(t), \\ &\qquad\qquad (u_x(t), u_y(t)) \in U := [-1,1] \times \{-1,1\}, \qquad \text{a.e. } t \in [0,T] \\ &\qquad\qquad (x(t), y(t)) \in X, \qquad \text{a.e. } t \in [0,T]. \end{aligned}$$

Claim 1 (no feasible transfer with classical controls). No admissible control can steer the state from D_L to D_R while remaining in X. Indeed, any trajectory leaving D_L and entering D_R must traverse L_{LR}, along which one must have $y(t) \equiv 0$ on a time interval of positive length. However, since $\dot{y}(t) = u_y(t) \in \{-1,1\}$ almost everywhere, $y(\cdot)$ cannot remain identically zero on any time interval. Thus, the transfer is impossible, and the optimal value is strictly positive (in particular, it is $+\infty$ for a minimum-time formulation).

Claim 2 (feasible transfer for the relaxed/measure problem). In the Young measure relaxation, one replaces the control $u(\cdot)$ by a measurable family of probability measures ν_t on U, and the dynamics are governed by the averaged velocity

$$(\dot{x}(t), \dot{y}(t)) \;=\; \int_U (u_x, u_y)\, d\nu_t(u_x, u_y) \;\in\; [-1,1] \times \mathrm{co}\{-1,1\} \;=\; [-1,1] \times [-1,1].$$

Choose, for t corresponding to motion along L_{LR},

$$\nu_t \;=\; \tfrac{1}{2}\,\delta_{(1,1)} \;+\; \tfrac{1}{2}\,\delta_{(1,-1)},$$

which yields the averaged control $(\dot{x}, \dot{y}) = (1, 0)$. Then $y(t) \equiv 0$ and $x(t)$ increases at unit speed: the state moves along L_{LR} from the left disk to the right disk while remaining within X. Hence, the relaxed problem admits a feasible solution that reaches the target X_T (achieving zero terminal distance), whereas the original problem does not. The Young measure relaxation thus *convexifies* the vertical control component, allowing averaged velocities (e.g., $\dot{y} = 0$) that are unattainable by any classical control. Consequently, the relaxed optimal value is strictly smaller than the classical one so this leads to a *relaxation gap*. This example shows that, in the absence of suitable structural conditions, measure relaxations may strictly lower the optimal value. The quasi-dissipativity assumptions imposed in later chapters can be viewed as conditions that rule out such pathological behavior.

2.5 Moment–SOS hierarchy

In Chapter 1, we introduced several optimization problems from the viewpoint of measure relaxations. The relaxation procedure yields a linear program posed on an infinite-dimensional space of measures. To compute approximations, one seeks finite-dimensional truncations, either by discretizing in space–time or by working in the space of moments. In this thesis, we adopt the moment-SOS hierarchy, which casts the resulting finite-dimensional relaxations as semidefinite programs (SDPs). This section presents the key objects and constructions that arise when solving these relaxed problems in the space of measures. The central object in the moment–SOS hierarchy is the moment sequence $\mathbf{m} = \{m_\alpha\}_{\alpha \in \mathbb{N}^n}$. This sequence is associated to a measure μ via

$$m_\alpha = \int \mathbf{x}^\alpha \, d\mu(\mathbf{x}).$$[2]

However, not every moment sequence arises from a measure, and an important question is: *When does a moment sequence z correspond to a measure?* This is called the K-moment problem and is formally stated as follows:

Definition 2.5.1. (K-moment problem) Given a real sequence $\mathbf{m} = \{m_\alpha\}$ and a nonempty set K, the K-moment problem addresses the existence of a measure $\mu \in \mathcal{M}(K)_+$ satisfying

$$\int_K \mathbf{x}^\alpha d\mu(x) = m_\alpha$$

where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

[2] Multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

To address this problem, we introduce the Riesz functional on the space of polynomials.

Definition 2.5.2 (Riesz functional)**.** Given a moment sequence $\mathbf{m} = (m_\alpha)_{\alpha\in\mathbb{N}^n} \in \mathbb{R}^{\mathbb{N}^n}$, the *Riesz functional* $\ell_{\mathbf{m}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ is defined by

$$\ell_{\mathbf{m}}(p) \;=\; \sum_{\alpha\in\mathbb{N}^n} m_\alpha\, p_\alpha,$$

for every polynomial $p(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^n} p_\alpha\, \mathbf{x}^\alpha \in \mathbb{R}[\mathbf{x}]$.

Example. Let $p(x) = 1 - 4x + 5x^2$ and $\mathbf{m} = (1, -1, 2)$, i.e., $m_0 = 1$, $m_1 = -1$, $m_2 = 2$. Then

$$\ell_{\mathbf{m}}(p) \;=\; 1\cdot 1 \;+\; (-1)\cdot(-4) \;+\; 2\cdot 5 \;=\; 1 + 4 + 10 \;=\; 15.$$

Moreover, if the moment sequence $\mathbf{m}$ is generated by a measure $\mu \in \mathcal{P}(\mathbb{R}^n)$, then

$$\ell_{\mathbf{m}}(p) \;=\; \int_{\mathbb{R}^n} p(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}),$$

i.e., in this case the Riesz functional coincides with integration with respect to μ.

The Riesz–Haviland theorem provides a criterion for the existence of a representing measure.

Theorem 2.5.1 (Riesz–Haviland)**.** *Let $K \subset \mathbb{R}^n$ be closed. A moment sequence $\mathbf{m}$ admits a nonnegative Radon measure μ supported on K (i.e., $\mu \in \mathcal{M}_+(K)$ and $m_\alpha = \int_K \mathbf{x}^\alpha\, \mathrm{d}\mu$ for all α) if and only if*

$$\ell_{\mathbf{m}}(p) \;\geq\; 0$$

for every polynomial $p \in \mathbb{R}[\mathbf{x}]$ such that $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in K$.

In practice, verifying global nonnegativity of a polynomial on K is generally intractable. A common workaround is to replace the nonnegativity constraint by membership in the cone of sums-of-squares (SOS) polynomials, which are manifestly nonnegative. Since not every nonnegative polynomial is SOS, we employ a Positivstellensatz to certify nonnegativity through SOS conditions; see [67] and the expositions [68, 55].

Theorem 2.5.2 (Dual Putinar-type certificate)**.** *Let K be a basic semialgebraic set of the form*

$$K \;=\; \{\mathbf{x} \in \mathbb{R}^n \;|\; g_k(\mathbf{x}) \geq 0,\ k = 0, \ldots, \ell,\ \textit{with } g_0(\mathbf{x}) = R^2 - \|\mathbf{x}\|^2\}.$$

Then there exists a (representing) measure $\mu \in \mathcal{M}_+(K)$ for the moment sequence $\mathbf{m}$ if and only if, for all $p \in \mathbb{R}[\mathbf{x}]$,

$$\ell_{\mathbf{m}}(p^2) \;\geq\; 0, \qquad \ell_{\mathbf{m}}(g_k\, p^2) \;\geq\; 0 \quad \textit{for } k = 0, \ldots, \ell. \tag{2.20}$$

The presence of the ball constraint $g_0(\mathbf{x}) = R^2 - \|\mathbf{x}\|^2$ enforces the *Archimedean* condition.

If K is a compact basic semialgebraic set, then $K \subset \{\mathbf{x} : \|\mathbf{x}\| \leq R\}$ for R large enough, so g_0 is a redundant but convenient constraint to ensure Archimedean condition.

The inequalities (2.20) are stated in terms of the Riesz functional $\ell_{\mathbf{m}}$ acting on squares and on weighted squares. A tractable numerical method is obtained by truncating the moment sequence to a finite-dimensional vector and rewriting these positivity conditions as semidefinite constraints on certain matrices built from the moments. We refer to the finite-dimensional relaxation of the K-moment problem in Definition 2.5.1 as the *truncated K-moment problem.*

For $d \in \mathbb{N}$, let $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : \ |\alpha| \leq d\}$. We work with the truncated monomial basis $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ indexed by $\alpha \in \mathbb{N}^n_d$. Concretely, once a polynomial p is identified with its vector of coefficients, the quantity $\ell_{\mathbf{m}}(p^2)$ becomes a quadratic form in these coefficients whose Gram matrix is precisely the *moment matrix* $M_d(\mathbf{m})$; likewise, $\ell_{\mathbf{m}}(q\,p^2)$ is represented by the corresponding *localizing matrix.* This leads to the following definitions.

Definition 2.5.3. Given a truncated sequence $\mathbf{m} = (m_\alpha)_{|\alpha| \leq 2d} \in \mathbb{R}^{\mathbb{N}^n_{2d}}$ we define

- *Moment matrix* $M_d(\mathbf{m}) \in \mathbb{R}^{\mathbb{N}^n_d} \times \mathbb{R}^{\mathbb{N}^n_d}$ by

$$(M_d(\mathbf{m}))_{\alpha,\beta} := m_{\alpha+\beta}$$

- For a polynomial $q(x) = \sum_\gamma q_\gamma x^\gamma$, the *localizing matrix* $M_{d-t}(q\mathbf{m})$ with $t := \lceil \deg q/2 \rceil$

$$(M_{d-t}(q\mathbf{m}))_{\alpha,\beta} := \sum_\gamma q_\gamma m_{\alpha+\beta+\gamma}.$$

The following theorem provides the necessary condition, in terms of semidefiniteness of the moment and the localizing matrix, for a sequence to solve the truncated K-moment problem.

Theorem 2.5.3. *Let $\mathbf{m} = (m_\alpha)_{|\alpha| \leq 2d}$ be a truncated sequence. If $\mathbf{m}$ admits a representing measure $\mu \in \mathcal{M}_+(K)$ with $m_\alpha = \int \mathbf{x}^\alpha d\mu(x)$ for all $|\alpha| \leq 2d$, then the following conditions hold:*

$$M_d(\mathbf{m}) \succeq 0, \quad M_{d-d_k}(g_k\mathbf{m}) \succeq 0, \qquad \forall k = 0, \dots, l,$$

where $d_k := \lceil \deg(g_k)/2 \rceil$

Note the converse statement is not true in general at a fixed order d; however it becomes true when the *flatness condition* holds. Next we illustrate the above mentioned concepts through an QCQP example introduced in (1.2).

2.5.1 Example

To illustrate the basic idea behind the moment–SOS hierarchy, we consider the moment relaxation of the QCQP introduced in (1.2):

$$\inf_{x\in\mathcal{X}} x^\top Q_0\, x \;+\; 2\,c_0^\top x \;+\; d_0 \tag{2.21}$$

$$\text{s.t. } g_i(x) := x^\top Q_i\, x \;+\; 2\,c_i^\top x \;+\; d_i \;\geq\; 0, \quad i=1,\dots,l, \tag{2.22}$$

where

$$\mathcal{X} := \{x\in\mathbb{R}^n : g_i(x)\geq 0,\; i=1,\dots,l\}.$$

The measure relaxation replaces the decision variable x by a probability measure $\mu\in\mathcal{P}(\mathcal{X})$:

$$\inf_{\mu\in\mathcal{P}(\mathcal{X})} \int_{\mathcal{X}} \big(x^\top Q_0\, x \;+\; 2\,c_0^\top x \;+\; d_0\big)\, d\mu(x) \tag{2.23}$$

$$\text{s.t. } \int_{\mathcal{X}} g_i(x)\, d\mu(x) \;\geq\; 0, \quad i=1,\dots,l. \tag{2.24}$$

We now truncate the moment sequence of μ. Let

$$m_1(x) := \begin{pmatrix}1\\ x\end{pmatrix} \in \mathbb{R}^{n+1},$$

and define the degree-1 moment matrix

$$M := M_1(\mu) := \int_{\mathcal{X}} m_1(x)\, m_1(x)^\top\, d\mu(x) = \begin{pmatrix} 1 & \int x^\top d\mu(x) \\ \int x\, d\mu(x) & \int x\, x^\top\, d\mu(x)\end{pmatrix} \in \mathbb{S}^{n+1}.$$

By construction, $M \succeq 0$ because it is an integral of rank-one PSD matrices. The cost and constraints can be written linearly in M. Indeed, define the lifted matrices

$$\widetilde{Q}_i := \begin{pmatrix} d_i & c_i^\top \\ c_i & Q_i\end{pmatrix}, \qquad i=0,1,\dots,l.$$

Then

$$\int_{\mathcal{X}} g_i(x)\, d\mu(x) = \int_{\mathcal{X}} \begin{pmatrix}1\\ x\end{pmatrix}^\top \widetilde{Q}_i \begin{pmatrix}1\\ x\end{pmatrix} d\mu(x) = \langle \widetilde{Q}_i, M\rangle, \qquad i=0,\dots,l.$$

Thus the degree-1 moment relaxation of (1.2) reads

$$\begin{aligned} J^\star = \min_{M\in\mathrm{Sym}_{n+1}} \quad & \langle \widetilde{Q}_0,\; M\rangle \\ \text{s.t.} \quad & \langle \widetilde{Q}_i,\; M\rangle \;\geq\; 0, \quad i=1,\dots,l, \\ & M \;\succeq\; 0. \end{aligned}$$

This is exactly the Shor relaxation (1.3). Note that since $\deg g_i = 2$, at order $d=1$ the localizing matrix constraint $M_{d-1}(g_i\,\mathbf{m}) \succeq 0$ reduces to the scalar inequality $M_0(g_i\,\mathbf{m}) = \int_{\mathcal{X}} g_i(x)\, d\mu(x) = \langle \widetilde{Q}_i, M\rangle \geq 0$.

In Chapters 3 and 4, we apply the finite-dimensional moment–SOS hierarchy to approximate measure relaxations of optimization problems related to quasi-dissipative nonsmooth systems, leading to semidefinite programs that compute approximate moments of occupation measures and terminal distributions. In Chapter 5, we extend these ideas to an infinite-dimensional moment framework for quasi-dissipative evolution equations on Hilbert spaces. In all cases, the absence of a relaxation gap ensures that the hierarchy converges to the original nonconvex optimal control or PDE problem.

Chapter 3

Evolution of measures

This chapter develops mathematical formalisms and provides numerical methods for studying the evolution of measures in nonsmooth dynamical systems using the continuity equation. The nonsmooth dynamical system is described by an evolution variational inequality and we derive the continuity equation associated with this system class using three different formalisms. The first formalism consists of using the superposition principle to describe the continuity equation for a measure that disintegrates into a probability measure supported on the set of vector fields and another measure representing the distribution of system trajectories at each time instant. The second formalism is based on the regularization of the nonsmooth vector field and describing the measure as the limit of a sequence of measures associated with the regularization parameter. In doing so, we obtain quantitative bounds on the Wasserstein metric between measure solutions of the regularized vector field and the limiting measure associated with the nonsmooth vector field. The third formalism uses a time-stepping algorithm to model a time-discretized evolution of the measures and show that the absolutely continuous trajectories associated with the continuity equation are recovered in the limit as the sampling time goes to zero. We also validate each formalism with numerical examples. For the first formalism, we use polynomial optimization techniques and the moment-SOS hierarchy to obtain approximate moments of the measures. For the second formalism, we illustrate the bounds on the Wasserstein metric for an academic example for which the closed-form expression of the Wasserstein metric can be calculated. For the third formalism, we illustrate the time-stepping based algorithm for measure evolution on an example that shows the effect of the concentration of measures. This chapter builds on the measure-relaxation viewpoint introduced in Chapter 1 and on the technical preliminaries of Chapter 2—especially quasi-dissipativity for sweeping processes and the continuity (Liouville) equation (Sections 2.2 and 2.3). These tools provide the structural and PDE backbone for the three formalisms developed below.

3.1 Introduction

The study of evolution of measures in finite dimensional systems has found relevance in the design of optimal control problems, understanding the system behavior under uncertainties, and several other applications. The primary step in this direction is to understand how the probabilistic initial conditions evolve in time under the action of a vector field. Such questions have been fairly well studied for single-valued dynamical systems with sufficient regularity (such as Lipschitz continuity) of the vector field. However, when we relax the regularity assumptions on the vector field, the question of evolution of measures brings forth some interesting questions which are of relevance for the applications as well. We are thus motivated to study the evolution of probability measures for a class of dynamical systems described by differential inclusions and in particular where the differential inclusion models the trajectories constrained to a pre-specified set. We present different mathematical formalisms to study measure evolution for such dynamical systems and provide corresponding numerical algorithms for simulations.

For an autonomous dynamical system described by an ordinary differential equation (ODE) with Lipschitz continuous vector field, the time evolution of the measure describing the initial condition is governed by a linear partial differential equation (PDE), commonly called the continuity equation or the Liouville equation [14, Section 5.4]. The solution to this PDE, that is the probability measure describing the distribution at a given time, is the push-forward or image of the initial probability measure through the flow map at that time. Lipschitz continuity of the vector field ensures that the flow map of the ODE is invertible, which in turn ensures that the push-forward measure is the unique solution to the continuity equation. The Cauchy problem for continuity equation with Sobolev fields was studied by [34]. Continuity equations corresponding to one-sided Lipschitz vector fields have been studied in [32, 33]. In [69] and [16], the authors consider fields of bounded variation and discuss the potential nonuniqueness of solutions to the continuity equation by introducing the notion of superposition principle. For the differential inclusions with convex set-valued mappings, the reference [70] provides a generalized superposition principle. For our purposes, the solutions based on the superposition principle are useful for numerical purposes. We propose a vector field selection from a time-varying differential inclusion from which we derive a continuity equation suitable for numerical algorithms. We use the converse statement of the superposition principle to characterize all possible solutions to the proposed continuity equation.

In this chapter, we are particularly concerned with a class of dynamical systems where the nonsmoothness arises due to the modeling of constraints on state trajectories. Such systems are described by the inclusion

$$\dot{x}(t) \in f(x(t)) - \mathcal{N}_{S(t)}(x(t)) \tag{3.1}$$

where $\mathcal{N}_S(x) \in \mathbb{R}^n$ denotes the outward normal cone to the set S at the point $x \in \mathbb{R}^n$. Since the normal cone takes a zero value in the interior of S, it is clear that the right-hand side of (3.1) is potentially discontinuous at the boundary of the set S. One can also think of (3.1) as an evolution variational inequality, described as

$$\langle \dot{x}(t) - f(x(t)), y - x(t) \rangle \geq 0,$$

for all $y \in S$, $x(t) \in S$, $t \in [0, T]$, where the brackets denote the inner product between vectors. Such dynamical systems have been a matter of extensive study in past decades due to their relevance in engineering and physical systems. The survey article [29], and a research monograph [71], provide an overview of different research oriented directions in the literature pertaining to system (3.1) and its connections to different classes of nonsmooth mathematical models. Analysis of such systems requires tools from variational analysis, nonsmooth analysis, set-valued analysis [66, 72, 73]. For a fixed initial condition, $x(0) \in S$, the question of existence and uniqueness of solution to system (3.1) has already been well-established in the literature, and the origins of such works can be found in [74], see [75] for a recent exposition.

However, if we consider the initial conditions described by a probability measure, then the evolution of this measure under the dynamics of (3.1) has received much less attention in the literature. One can study such problems by considering stochastic versions of (3.1) by adding a diffusion term on the right-hand side. Such systems first came up in the study of variational inequalities arising in stochastic control [76], and in the literature, we can find results on existence and uniqueness of solutions in appropriate function space. In [77], this is done by considering Yosida approximations of the maximal monotone operator, whereas [78] provides a proof based on time-discretization of system (3.1). These approaches have been generalized for prox-regular set S in [79], and the case where the drift term contains Young measures [80, 81]. In [82], the authors provide a constructive approximation of measures associated with system (3.1) with $f \equiv 0$, which are based on a generalization of time-stepping algorithm and involves projecting the density function onto the constraint set with respect to the Wasserstein metric.

The main contribution of this chapter is to provide different formalisms for describing the evolution of measures for the class of systems considered in (3.1). In particular, our contribution lies in studying three different techniques for describing the propagation of probabilistic initial conditions for system (3.1) and we provide numerical methods for each of these techniques.

The first approach is based on using the previously mentioned superposition principle. Here, we consider a continuity equation where the velocity vector field is obtained by a selection of the set-valued mapping in system dynamics, which results in a (possibly non-unique) solution to the measure evolution. We develop a converse result which actually shows that all possible solutions can be associated with a selection of the vector field. The tools used in the process are similar to the ones appearing in [70], but we develop a specific representation of the continuity equation in terms of a measure which can be computed numerically using the moment-SOS (polynomial sums of squares) hierarchy [24] and semi-definite programming based techniques.

The second approach builds on our recent work in [65] where we approximate the dynamics of system (3.1) by ODEs with Lipschitz continuous right-hand side. The solution of the continuity equation associated with each ODE provides a sequence of measures which allows us to approximate the solution of the measure evolution problem. We show that the limiting measure can be represented by the pushforward of the unique flow map of system (3.1) and we develop quantitive bounds on the Wasserstein distance between the limiting measure and its approximations obtained from the regularization method.

Finally, another approach we adopt for studying the evolution of measures subject to constrained dynamics is based on computing an approximation of the transport maps for system (3.1) via time discretization. Time discretization based techniques are well known for constructing solutions to evolution PDEs using the gradient flow structure of the Wasserstein space [83]. Time discretization schemes have been recently used in [84], which exploits the gradient flow structure for the system class (3.4) in the Euclidean space to construct solutions for constrained optimization problems. For sweeping processes without the perturbation term, this approach was adopted in [82] and it generalizes the classical time-stepping algorithm proposed in [74] to the setting of measures. We use these techniques to construct the solutions of the continuity equation associated with system (3.1). In particular, one computes the distribution at discrete time instants by interpolating the distribution through the perturbation term, and then projecting it onto the constraint set with respect to the Wasserstein metric. This scheme is built on the dynamic viewpoint of optimal transport problems where the absolutely continuous curves in Wasserstein space satisfy the continuity equation. Under certain conditions, we show that the sequence of time-discretized measures converges to a solution described by the push-forward of the initial distribution under the transport map.

Finally, we address the computational aspects for each of the three formalisms with the help of academic examples. The proposed continuity equation could be seen as an infinite dimensional linear problem (LP) in the space of measures. We use the moment-SOS hierarchy to approximate the moments of the measures and we provide an illustration of this method. For the second formalism based on functional regularization, we explicitly, for a one dimensional case, compute the Wasserstein distance between the measure solutions to the nonsmooth system and to approximation obtained by regularization method. For the last formalism based on time-stepping algorithm for measures, we consider an example of a two-dimensional system based on time-space discretization and then evolving the measures using the given algorithm.

3.2 Preliminaries and overview

3.2.1 Measure evolution

Consider the dynamical system described by an ordinary differential equation (ODE):

$$\dot{x} = f(t, x). \tag{3.2}$$

If the vector field $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $f(\cdot, x)$ is Lebesgue measurable for each $x \in \mathbb{R}^n$ and $f(t, \cdot)$ is Lipschitz continuous for each $t \in \mathbb{R} \geq 0$, then there exists a unique absolutely continuous function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ that solves (3.2). Consequently, we consider the *flow map* $X_t : \mathbb{R}^n \to \mathbb{R}^n$ parameterized by $t \in \mathbb{R}_{\geq 0}$ having the property that $x(t) = X_t(x_0)$ for each $x_0 \in \mathbb{R}^n$. It is also of interest to study the evolution of probability measures for system (3.2) when the initial condition is described by a probability distribution on $\mathbb{R}^n$, that is, $x(0) \sim \mu_0$, where $\mu_0 \in \mathcal{P}(\mathbb{R}^n)$, the set of probability measures on $\mathbb{R}^n$. In words, the law of the random variable $x(0)$ is the probability measure μ_0. The resulting measure $\mu_t \in \mathcal{P}(\mathbb{R}^n)$, for $t \in \mathbb{R}_{\geq 0}$, is defined by the *continuity equation*, also called the Liouville equation, a linear partial differential equation (PDE) which models the transport of a distribution along the flow of trajectories of the underlying system and preserves the mass of the distribution. For the cases where the vector field $f(t, x)$ is Lipschitz in x for each t, the continuity equation reads

$$\partial_t \mu_t + \nabla \cdot (f(t, \cdot)\mu_t) = 0 \tag{3.3}$$

where $\nabla\cdot$ is the divergence operator. The equation is to be understood in the weak sense, i.e. $\int [\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot f(t, x)] \mathrm{d}\mu(t, x) = 0$ where $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}^n)$ and ∇_x is the gradient operator. Furthermore, a measure μ_t solving (3.3) can be represented as the push-forward of μ_0 under the mapping X_t, denoted $\mu_t = X_{t\#}\mu_0$. Here, and throughout this chapter, for a function $g : \mathbb{R}^n \to \mathbb{R}^m$ and a measure μ_0 supported on a set in $\mathbb{R}^n$ the push-forward of μ_0 under the mapping g is denoted by $g_\#\mu_0$ and it is defined as $g_\#\mu_0(A) := \mu_0(\{x \in R^n : g(x) \in A\})$ for every measurable set $A \subset \mathbb{R}^m$.

In this work, we particularly consider the class of following differential inclusions:

$$\dot{x}(t) \in f(t,x) - \mathcal{N}_{S(t)}(x), \quad x(0) \sim \mu_0 \tag{3.4}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, $S : [0,T] \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, and $\mathcal{N}_{S(t)}(x)$ denotes the outward normal cone to the convex set $S(t)$ at $x \in S(t)$. We impose the following assumptions on system class (3.4) so that the system is well-posed.

Assumption 3.2.1. There exists $L_f > 0$ such that

$$|f(t,x_1)| \leq L_f(1 + |x_1|),$$
$$|f(t,x_1) - f(t,x_2)| \leq L_f|x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^n$.

Assumption 3.2.2. The mapping $S : [0,T] \rightrightarrows \mathbb{R}^n$ is closed and convex-valued for each $t \in [0,T]$, and $S(\cdot)$ varies in a Lipschitz continuous manner with time, i.e., there exists a constant L_s such that

$$d_H(S(t), S(s)) \leq L_s|t - s|$$

where $d_H(A,B) := \max\left\{\sup_{x\in B} \operatorname{dist}(x,A),\ \sup_{x\in A} \operatorname{dist}(x,B)\right\}$ is the Hausdorff distance between the sets A and B.

Under these two assumptions, several references in the literature prove the existence and uniqueness of solutions to (3.4) with $x(0) \in S(0) \subset \mathbb{R}^n$, see for example [29] for an overview. In this chapter, we are interested in studying the evolution of measures for system class (3.4). The PDE considered in (3.3) cannot be readily obtained in that case and we study three different principles to describe the evolution of measures for our system (3.4). In the remainder of this section, we provide an overview of these techniques from the existing literature. In the later sections, we develop each of these techniques for system class (3.4).

3.2.2 Superposition principle

In the first instance, we look at (3.4) as a differential inclusion with a set-valued right-hand side in the dynamics. In this regard, we see that the evolution of measures is described using the *superposition principle* for the differential inclusions of the form

$$\dot{x}(t) \in F(t, x(t)) \tag{3.5}$$

where $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. Let us explain briefly and informally what is the superposition principle. A *selection* of F is a mapping $(t,x) \mapsto \overline{f}(t,x) \in F(t,x)$. Associated with a selection is an absolutely continuous solution $\gamma \in AC([0,T];\mathbb{R}^n)$ with $\gamma(0) = x(0)$ such that $\dot{\gamma}(t) = \overline{f}(t,\gamma(t))$ for Lebesgue a.e. $t \in \mathbb{R}_{\geq 0}$. Let us consider the set of all admissible curves

$$\Gamma_T := \{\gamma \in AC([0,T];\mathbb{R}^n) : \dot{\gamma} = \overline{f}(t,\gamma), \overline{f} \text{ a selection of } F\}.$$

The evaluation map is defined as a Borel measurable map $e_t : \mathbb{R}^n \times \Gamma_T \to \mathbb{R}^n$ such that

$$e_t(x, \gamma) := \gamma(t) \ \forall t \in [0, T] \text{ and } \gamma(0) = x, \gamma \in \Gamma_T. \tag{3.6}$$

Let θ be a probability measure such that $\theta \in \mathcal{P}(\mathbb{R}^n \times \Gamma_T)$. Under some mild integrability condition [16, Theorem 8.2.1], the measure solutions μ_t to a continuity equation associated with (3.5) (under some selection of vector field from $F(t, x)$) can be represented as

$$\mu_t = e_{t\#}\theta \tag{3.7}$$

which for any continuous function

$$\phi : \mathbb{R}^n \to \mathbb{R} \text{ satisfies } \int \phi(x) \mathrm{d}\mu_t(x) = \int \phi(e_t(x, \gamma)) d\theta(x, \gamma).$$

The solutions μ_t can be understood as a superposition over solution trajectories $\gamma \in \Gamma_T$, where the superposition is captured by the measure θ. The solutions to differential inclusion (3.5) are possibly nonunique and hence μ_t in (3.7) is also not necessarily unique for a given initial measure.

3.2.3 Functional regularization

The basic idea of the regularization is to consider a sequence of ODEs with a parameter λ:

$$\dot{x}^\lambda(t) = g_t^\lambda(x(t))$$

so that the solutions $x^\lambda(t)$ approach the solution $x(t)$ that solves (3.4), under the constraint $x^\lambda(0) = x(0)$. Here, for each $\lambda > 0$ and for each $t \in [0, T]$, $g_t^\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued Lipschitz continuous function, whose construction is provided in Section 3.4. One can derive the classical continuity equation (3.3) to these ODEs and obtain a parameterized sequence of measures μ_t^λ as follows:

$$\partial_t \mu_t^\lambda + \nabla \cdot (g_t^\lambda(\cdot)\mu_t) = 0.$$

An obvious candidate for describing the measure solving (3.4) is to take the limit of $\{\mu_t^\lambda\}$ as $\lambda \to 0$. In Section 3.4, we study the limit of this sequence using the Wasserstein metric to quantify the distance between μ_t^λ and the limiting measure.

We measure convergence of probability measures using the Wasserstein distance. We refer to Section 2.3 for the necessary background on optimal transport (including the definition of W_p) and its connection with continuity equations.

3.2.4 Time discretization and optimal transport

Time discretization based techniques are well known for constructing solutions to evolution PDEs by using the gradient flow structure of the Wasserstein space. It is based on partitioning a time interval into finitely many nodes (discrete times) and describing the measure at those times as a function of the initial distribution through appropriate mappings using the system data. The interpolation between the two measures (described at two consecutive times) is based on the principles of optimal transport and provides an approximation to the measure evolution problem for system (3.4).

We refer to Section 1.1.2 for the Monge and Kantorovich formulations of optimal transport and for the definition of the Wasserstein distance. Recall that $W_2(\cdot,\cdot)$ endows the set of probability measures with finite second moment with a metric structure; this metric space is the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n)$. One interesting property which will be of interest is that any absolutely continuous curve in Wasserstein space $\mathcal{W}_2$ is a solution to a continuity equation [83]. In [85] the author proved that if there exists a pair of measures $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^n)$ with μ_0 absolutely continuous with respect to the Lebesgue measure, then there exists a (uniquely determined) constant speed geodesic between these measures and such constant speed geodesics satisfy a continuity equation. It is possible to construct an approximation of an absolutely continuous curve by defining measures at discrete time instants and using an interpolation via constant speed geodesics between successive time instants. In [82], the authors use time-discretization to approximate the measure solution of continuity equation associated with (3.4) without the drift term $f(\cdot)$. The method is based on recursively defining measures at different time instants using an optimal transport map which transports the measures from one time instant to the next. Considering suitable interpolation schemes, one constructs the trajectory and shows that it converges to the solution of the continuity equation. We will use a time-stepping scheme for (3.4) to construct measures at different time instants starting from an initial distribution. Using appropriate interpolation, we will prove that the interpolated curves converge to the absolutely continuous curves which will be the measure valued solutions to the continuity equation associated to (3.4).

3.3 Superposition principle

In this section, we consider a general system class described by a differential inclusion. Starting from a vector field selection of this differential inclusion, we propose a continuity equation driven by this selection and we characterize all possible solutions to this equation.

3.3.1 Differential inclusion

Consider a dynamical system governed by the differential inclusion:

$$\dot{x} \in F(t, x) \tag{3.8}$$

where $F : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. For an initial condition $x_0 \in \mathbb{R}^n$, we denote the solution to (3.8) at time $t \in [0, T]$ by $X_t(x_0)$, where X_t represents the flow map for system (3.8). For solutions to be well-defined, F satisfies the following:

Assumption 3.3.1. The set $F(t, x)$ is convex for every $t \in [0, T]$ and every $x \in \mathbb{R}^n$.

Assumption 3.3.2. If there exists a solution to (3.8) corresponding to a selection $\overline{f}$ of F, then it holds that,

$$|\overline{f}(t, x)| \leq \beta(t)(1 + |x|) \tag{3.9}$$

where, $\beta(\cdot) \in L^1([0,T];\mathbb{R}_+)$.

In what follows, we consider a selection $\overline{f}_\eta(t,x)$ of $F(t,x)$ defined using a probability measure $\eta(\cdot|t,x) \in \mathcal{P}(F(t,x))$ as follows:[1]

$$\overline{f}_\eta(t,x) := \int_{F(t,x)} v \mathrm{d}\eta(v|t,x), \quad \eta(\cdot|t,x) \in \mathcal{P}(F(t,x)). \tag{3.10}$$

Due to the convexity of $F(t,x)$, it follows that $\overline{f}_\eta(t,x) \in F(t,x)$. We do not make any further assumptions on the regularity of $\overline{f}_\eta(t,x)$ and thus an ODE system $\dot{x} = \overline{f}_\eta(t,x)$ may admit multiple solution trajectories from a given initial condition. We also let Γ_T^η denote the set of trajectories associated with the selection $\overline{f}_\eta$, that is,

$$\Gamma_T^\eta := \{\gamma \in AC([0,T];\mathbb{R}^n) : \dot{\gamma} = \overline{f}_\eta(t,\gamma)\}. \tag{3.11}$$

Remark 3.3.1. In the case where the set $F(t,x)$ is finitely generated (that is, for each (t,x), it is represented by a linear combination of finitely many vector fields $f_i(x)$ for $i \in \{1,...,n\}$), the vector field (3.10) reduces to a *Fillipov differential inclusion*. For example, given a piecewise smooth system $\dot{x}(t) = f_i(x(t))$ for $x \in \mathcal{R}_i$, where $\mathcal{R}_i$ are disjoint regions covering $\mathbb{R}^n$, the Fillipov differential inclusion for such system would result in $\dot{x}(t) = \mathrm{conv}(f_i(x))$, where conv denotes the convex combination of the vector fields. Any absolutely continuous solution $x(t)$ would satisfy $\dot{x}(t) = \sum_{i\in I} w_i(x(t)) f_i(x(t))$ where $I(x)$ denotes the active set at $x \in \mathbb{R}^n$, and the weights are such that $\sum w_i = 1$ and $w_i \geq 0$. In [87], the authors propose a LP to compute the weights and thus solve the differential equation using an *active-set* method. In this case of piecewise smooth vector fields, the measure η will be discretely supported on the set $\{f_i(t,x)\}_{i\in I(x)}$, and the vector field in (3.10) yields

$$\overline{f}_\eta(x) = \sum_{v\in\{f_i(x)\}, i\in I(x)} v w_i(x) = \sum_{i\in I(x)} f_i(x) w_i(x).$$

Thus, $w = (w_1, \ldots, w_m)$ with $\sum_i w_i = 1$ can be seen as the discrete version of the measure η.

3.3.2 Vector field selection

The vector field selection in (3.10) is used to define the continuity equation for the measure evolution problem. Before doing so, we make some connections with related literature to provide an interpretation of $\overline{f}_\eta(t,x)$ as the weighted average of the vector fields associated with solution trajectories of (3.8).

$$e_t^{-1}(x) := \{(y,\gamma) \text{ s.t. } \gamma \in \Gamma_T, \gamma(0) = y, \gamma(t) = x\}. \tag{3.12}$$

[1] If Assumptions 3.3.1 and 3.3.2 hold, there exists a measurable selection $\overline{f}_\eta(t,x) \in F(t,x)\ \forall (t,x)$ [86]. Measurability of the proposed vector field can be checked by first replacing the integrand with the indicator function I_S, then $\overline{f}_\eta(t,x) = \eta(S|t,x)$ which is a measurable function for every fixed $S \subset F(t,x)$. One can use standard measure theoretic arguments to approximate the integral using simple functions.

A disintegration $\theta_{t,x}(y,\gamma)$ of $\theta(y,\gamma)$ w.r.t. e_t is such that, for $\psi \in \mathcal{C}(\mathbb{R}^n \times \Gamma_T; \mathbb{R})$:

$$\int_{\mathbb{R}^n \times \Gamma_T} \psi(y,\gamma)\mathrm{d}\theta(y,\gamma) = \int_{\mathbb{R}^n} \int_{e_t^{-1}(x)} \psi(y,\gamma)\mathrm{d}\theta_{t,x}(y,\gamma)\mathrm{d}\mu_t(x). \tag{3.13}$$

Then a mean-vector field is introduced as follows:

$$\widetilde{f}(t,x) := \int_{e_t^{-1}(x)} \dot{\gamma}(t)\mathrm{d}\theta_{t,x}(y,\gamma). \tag{3.14}$$

The velocity vector (3.14) can be understood as a weighted mean of all the velocity vectors $\dot{\gamma}(t)$ over the curves γ passing through point x at time t. Note that the convexity of $F(t,x)$ ensures that the mean-velocity (3.14) belongs to the set $F(t,x)$. We show that the vector field defined in (3.14) is equivalent to the vector field defined in (3.10) for some appropriate choice of $\eta \in \mathcal{P}(\mathbb{R}^n)$. To establish this, we introduce a velocity evaluation operator $\mathbf{d}_t : \mathbb{R}^n \times \Gamma_T \to \mathbb{R}^n$, which is a Borel measurable map defined by

$$\mathbf{d}_t(y,\gamma) := \dot{\gamma}(t) \text{ with } \gamma(0) = y. \tag{3.15}$$

Using this mapping, we define:

$$\eta(\cdot \mid t,x) := \mathbf{d}_{t\#}\theta_{t,x}(\cdot). \tag{3.16}$$

Proposition 3.3.2. *Let $\mathbf{d}_t$ be the velocity evaluation operator in* (3.15)*. Then, tor each $t \in [0,T]$ and $x \in \mathbb{R}^n$, it holds that $\mathbf{d}_t^{-1}(F(t,x)) = e_t^{-1}(x)$. Moreover, for the measures η defined in* (3.16)*, the associated vector field in* (3.10) *is equal to* (3.14)*.*

Proof. The proof of the first claim follows from the definition, i.e.,

$$\mathbf{d}_t^{-1}(F(t,x)) = \{(\gamma(0),\gamma); \gamma(t) = x\} \tag{3.17}$$

$$e_t^{-1}(x) = \{(\gamma(0),\gamma); \gamma(t) = x\}. \tag{3.18}$$

So the two sets are the same. Next, we prove the equivalence of the two vector fields (3.14) and (3.10). Using the definition of $\overline{f}_\eta$ in (3.10) and the equality in (3.16), we get

$$\overline{f}_\eta(t,x) = \int\limits_{F(t,x)} \dot{\gamma}(t)\ \mathrm{d}(\mathbf{d}_{t\#}\theta_{t,x})(y,\gamma).$$

Under the change of variables in the above equation,

$$\overline{f}_\eta(t,x) = \int\limits_{\mathbf{d}_t^{-1}(F(t,x))} \dot{\gamma}(t)\ \mathrm{d}\theta_{t,x}(y,\gamma).$$

Now using $\mathbf{d}_t^{-1}(F(t,x)) = e_t^{-1}(x)$, we get

$$\overline{f}_\eta(t,x) = \int_{e_t^{-1}(x)} \dot{\gamma}(t)\, \mathrm{d}\theta_{t,x}(y,\gamma) = \tilde{f}(t,x)$$

for each $t \in [0,T]$ and $x \in \mathbb{R}^n$. □

Thus, the set of trajectories $\Gamma_T^\eta \subset \Gamma_T$ for (3.10) and (3.14) are the same under the constraint prescribed in (3.16).

3.3.3 Continuity equation and its measure solution

We now state the main results of this section concerning the formulation of the continuity equation. In Proposition 3.3.3, we show that, for every $\eta(\cdot|t,x) \in \mathcal{P}(F(t,x))$ and every θ concentrated on $\mathbb{R}^n \times \Gamma_T^\eta$, the image measure $\mu_t = e_{t\#}\theta$ satisfies the continuity equation driven by $\overline{f}_\eta(t,x)$ in (3.10). Starting from this equation, in Theorem 3.3.4 we discuss the converse statement and characterize all the measure solutions to the derived continuity equation. This characterization of the solutions is especially important as later we propose a numerical method for the simulation of measure evolution through nonsmooth dynamical systems as the solution of this continuity equation.

Proposition 3.3.3. *Consider system* (3.8) *under Assumption 3.3.1 and Assumption 3.3.2. For each $t \in [0,T]$ and $x \in \mathbb{R}^n$, let $\eta(\cdot|t,x)$ be a probability measure supported on $F(t,x)$, and let $\overline{f}_\eta$ and Γ_T^η be defined as in* (3.10) *and* (3.11)*, respectively. Then, for every $\theta \in \mathcal{P}(\mathbb{R}^n \times \Gamma_T^\eta)$, the measure $\mu_t := e_{t\#}\theta$ satisfies the following continuity equation driven by $\overline{f}_\eta(t,x)$, i.e.*

$$\int_{\mathbb{R}^n} \varphi(T,x)\mathrm{d}\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0,x)\mathrm{d}\mu_0(x) = \int_{[0,T]\times\mathbb{R}^n} \Big[\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot \overline{f}_\eta(t,x)\Big]\, \mathrm{d}\mu_t(x)\mathrm{d}t \quad (3.19)$$

for every compactly supported $\varphi \in \mathcal{C}^1([0,T]\times\mathbb{R}^n;\mathbb{R})$.

Proof. To derive (3.19), we start by proving that the mapping $t \mapsto \int \phi(x)\mathrm{d}\mu_t(x)$ is absolutely continuous,[2] for compactly supported $\phi \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$. We then use the property of almost everywhere differentiability of absolutely continuous functions to differentiate $\int \phi(x)\mathrm{d}\mu_t(x)$ w.r.t. time.

Absolute continuity of $\int \phi \mathrm{d}\mu_t$: Consider the pairwise disjoint intervals $(\underline{t}_i, \overline{t}_i) \subset [0,T]$, such that $\sum_{i=1}^N(\overline{t}_i - \underline{t}_i) < \delta$, for a given $\delta > 0$. Choose $\phi \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$, then for any $\gamma \in \Gamma_T^\eta$ we

[2] We consider test functions $\varphi(t,x) = \rho(t)\phi(x)$ which are dense in $\mathcal{C}^1(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$, and then the differentiability of $\int \varphi(t,x)\mathrm{d}\mu_t(x)$ depends on the absolute continuity of $\int \phi(x)\mathrm{d}\mu_t(x)$ since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \rho(t)\phi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}^n} \partial_t(\rho(t))\phi(x)\mathrm{d}\mu_t(x) + \int_{\mathbb{R}^n} \rho(t)\frac{\mathrm{d}}{\mathrm{d}t}\phi(x)\mathrm{d}\mu_t(x).$$

So we need to prove that $t \mapsto \int \phi(x)\mathrm{d}\mu_t(x)$ is absolutely continuous.

have

$$\sum_{i=1}^{N} \phi(\gamma(\bar{t}_i)) - \phi(\gamma(\underline{t}_i)) = \sum_{i=1}^{N} \int_{(\underline{t}_i, \bar{t}_i)} \Big(\nabla_x \phi(\gamma(t)) \Big) \cdot \overline{f}_\eta(\gamma(t)) \mathrm{d}t. \tag{3.20}$$

Integrating (3.20) with $\theta \in \mathcal{P}(\mathbb{R}, \Gamma_T^\eta)$ leads to

$$\sum_{i=1}^{N} \int_{\mathbb{R}^n \times \Gamma_T^\eta} \Big[\phi(\gamma(\bar{t}_i)) - \phi(\gamma(\underline{t}_i)) \Big] \mathrm{d}\theta(x, \gamma)$$
$$= \sum_{i=1}^{N} \int_{(\underline{t}_i, \bar{t}_i)} \int_{\mathbb{R}^n \times \Gamma_T^\eta} \Big(\nabla_x \phi(\gamma(t)) \Big) \cdot \overline{f}_\eta(\gamma(t)) \mathrm{d}\theta(x, \gamma) \mathrm{d}t. \tag{3.21}$$

Now using $\mu_t = e_{t\#}\theta$ on the left side of the above equation, taking the absolute values on both sides and then using Hölder's inequality we get

$$\sum_{i=1}^{N} \Big| \int_{\mathbb{R}^n} \phi(x) \mathrm{d}\mu_{\bar{t}_i}(x) - \int_{\mathbb{R}^n} \phi(x) \mathrm{d}\mu_{\underline{t}_i}(x) \Big|$$
$$\leq ||\nabla_x \phi||_\infty \sum_{i=1}^{N} \int_{(\underline{t}_i, \bar{t}_i)} \int_{\mathbb{R}^n \times \Gamma_T^\eta} |\overline{f}_\eta(\gamma(t))| \mathrm{d}\theta(x, \gamma) \mathrm{d}t. \tag{3.22}$$

Using the growth bounds on the vector field (Assumption 3.3.2), we can derive the estimate $\int_{\mathbb{R}^n \times \Gamma_T^\eta} |\overline{f}_\eta(\gamma(t))| \mathrm{d}\theta(x, \gamma) \leq K\beta(t)$ for some $K > 0$; refer to Appendix 3.7 for details. Substituting this inequality in (3.22), we get

$$\sum_{i=1}^{N} \Big| \int \phi(x) \mathrm{d}\mu_{\bar{t}_i}(x) - \int \phi(x) \mathrm{d}\mu_{\underline{t}_i}(x) \Big| \leq K \|\nabla_x \phi\|_\infty \sum_{i=1}^{N} \int_{(\underline{t}_i, \bar{t}_i)} \beta(t) \mathrm{d}t.$$

Since β is integrable and $\sum_{i=1}^{N} (\bar{t}_i - \underline{t}_i) < \delta$ for an arbitrary $\delta > 0$, the right-hand side can be made arbitrarily small. This proves the absolute continuity of $t \mapsto \int \phi(x) d\mu_t(x)$.

Next, we differentiate $\int \varphi(t, x) d\mu_t(x)$ for any $\varphi(t, x) \in \mathcal{C}_c^1([0, T] \times \mathbb{R}^m)$ and we obtain the following (refer to Appendix 3.7 for details),

$$\int_{\mathbb{R}^n} \varphi(T, x) \mathrm{d}\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0, x) \mathrm{d}\mu_0(x)$$
$$= \int_{[0,T] \times \mathbb{R}^n} \Big(\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \overline{f}_\eta(t, x) \Big) \mathrm{d}\mu_t(x) \mathrm{d}t$$

which shows the desired relation. □

We now rewrite equation (3.19) in a form which we will use in the optimization formulation proposed later in Section 3.6. Substituting the expression for $\overline{f}_\eta(t, x)$ from (3.10) in (3.19), we get

$$\int_{\mathbb{R}^n} \varphi(T,x)\mathrm{d}\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0,x)\mathrm{d}\mu_0(x) = \int_{[0,T]\times\mathbb{R}^n} \Big(\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot \int_{F(t,x)} v\ \mathrm{d}\eta(v|t,x)\Big)\mathrm{d}\mu_t(x)\mathrm{d}t. \tag{3.23}$$

Rearranging the terms in the above equations and defining $\mathrm{d}\hat{\mu}(t,x,v) = \mathrm{d}\eta(v|t,x)\mathrm{d}\mu_t(x)dt$, we get

$$\int_{\mathbb{R}^n} \varphi(T,x)\mathrm{d}\mu_T - \int_{\mathbb{R}^n} \varphi(0,x)\mathrm{d}\mu_0 = \int\limits_{[0,T]\times\mathbb{R}^n}\int\limits_{F(t,x)} \Big[\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot v\Big]\ \mathrm{d}\hat{\mu}(t,x,v). \tag{3.24}$$

Equation (3.24) will be the starting point for the next result as we will characterize all possible solutions $\hat{\mu}$ to it. The solutions $\hat{\mu}$ will determine the vector field (3.10) (by defining η) and the solutions to the continuity equation driven by this vector field will lead to a measure concentrated on the trajectories of the derived vector field.

Theorem 3.3.4. *Consider system* (3.8) *under Assumption 3.3.1 and Assumption 3.3.2. Any measure $\hat{\mu}$ that solves the continuity equation* (3.24) *is of the form*

$$\mathrm{d}\hat{\mu}(t,x,v) = d\eta(v|t,x)d\mu_t(x)\mathrm{d}t \tag{3.25}$$

where $\eta(\cdot|t,x) \in \mathcal{P}(F(t,x))$ and μ_t solves (3.19).

Proof. In Euclidean space $\mathbb{R}^n$, we can use the disintegration theorem [88, Corollary 10.4.13] to write $\mathrm{d}\hat{\mu}(t,x,v) = \mathrm{d}\eta(v|t,x)d\overline{\mu}(t,x)$ where $\eta(\cdot|t,x) \in \mathcal{P}(F(t,x))$. Using this we can rewrite

$$\int_{\mathbb{R}^n} \varphi(T,x)\mathrm{d}\mu_T - \int_{\mathbb{R}^n} \varphi(0,x)\mathrm{d}\mu_0 = \int\limits_{[0,T]\times\mathbb{R}^n}\int\limits_{F(t,x)} \Big[\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot \zeta\Big]\ \mathrm{d}\eta(\zeta|t,x)\mathrm{d}\overline{\mu}(t,x). \tag{3.26}$$

Rearranging the terms results in

$$\int_{\mathbb{R}^n} \varphi(T,x)\mathrm{d}\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0,x)\mathrm{d}\mu_0(x) = \int_{[0,T]\times\mathbb{R}^n} \Big(\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot \overline{f}_\eta(t,x)\Big)\mathrm{d}\overline{\mu}(t,x) \tag{3.27}$$

where $\overline{f}_\eta$ is defined as

$$\overline{f}_\eta(t,x) = \int_{F(t,x)} \zeta \mathrm{d}\eta(\zeta|t,x) \text{ for } \eta(\cdot|t,x) \in \mathcal{P}(F(t,x)) \tag{3.28}$$

and spans the set $F(t,x)$ as we have assumed $F(t,x)$ to be convex for every $t \in [0,T]$. *Decomposition of $\overline{\mu}$:* Next we show that the marginal of $\overline{\mu}(t,x)$ w.r.t. time is a Lebesgue measure. This can be shown by taking $\varphi(t,x) = t^k$ for some $k \geq 0$ in (3.27), then we get

$$\mu_T(\mathbb{R}^n)T^k - \int_{\mathbb{R}^n} t^k \mathrm{d}\mu_0 = \int_{[0,T]\times\mathbb{R}^n} kt^{k-1}\mathrm{d}\overline{\mu}(t,x) \tag{3.29}$$

where taking $k = 0$ gives $\mu_T(\mathbb{R}^n) = \mu_0(\mathbb{R}^n)$ and for $k \geq 1$ results in

$$\frac{\mu_T(\mathbb{R}^n)T^k}{k} = \int_{[0,T]\times\mathbb{R}^n} t^{k-1}\mathrm{d}\overline{\mu}(t,x).$$

So, up to scaling we can write $\overline{\mu}(\mathrm{d}t,\mathrm{d}x) = \mu_t(\mathrm{d}x)\mathrm{d}t$. Substituting this we arrive at the following continuity equation

$$\int_{\mathbb{R}^n} \phi(T,x)\mathrm{d}\mu_T - \int_{\mathbb{R}^n} \phi(0,x)\mathrm{d}\mu_0 = \int\limits_{[0,T]\times\mathbb{R}^n} \Big[\partial_t\phi(t,x) + \nabla_x\phi(t,x)\cdot\overline{f}_\eta(t,x)\Big]\mathrm{d}\mu_t(x)\mathrm{d}t. \tag{3.30}$$

Now using the results in [69], the only solutions to the continuity equation have a representation in terms of measure $\theta \in \mathcal{P}(\mathbb{R}^n \times \Gamma)$ (as defined in Section 3.2.2) as

$$\mu_t = e_{t\#}\theta$$

where θ are concentrated on the solution trajectories to system $\dot{x}(t) = \overline{f}_\eta(t, x(t))$. □

3.3.4 Measure Evolution for Constrained Systems

Using the equations defined in (3.24), we arrive at the continuity equation for the system defined in (3.4), i.e., the equation

$$\begin{aligned}\int_{S(T)} \phi(T,x)d\mu_T(x) - \int_{S(0)} \phi(0,x)d\mu_0(x)& \\ = \int\limits_{[0,T]\times S(t)} \int\limits_{f(t,x)-\mathcal{N}_{S(t)}(x)} \Big[\partial_t\phi(t,x) + \nabla_x\phi(t,x)\cdot\zeta\Big]\, d\hat{\mu}(t,x,\zeta)&\end{aligned} \tag{3.31}$$

holds for every compactly supported $\phi \in \mathcal{C}^1([0,T]\times\mathbb{R}^n;\mathbb{R})$.

Using Theorem 3.3.4, it follows that the measure solutions to the continuity equation have a representation (3.25). In this decomposition, $\eta(\cdot|t,x)$ represents the selection from the set $f(t,x) - \mathcal{N}_{S(t)}(x)$ so that the resulting trajectories evolve within the set $S(t)$. Corresponding to such selections, the solutions to ODE (3.4) are unique with $x(0) \in S(0)$, see for example [29, Section 5]. Consequently, we have that $\theta = \delta_{X_t(x_0)}$ and due to Proposition 3.3.3, the measure $\mu_t = e_{t\#}\theta$ corresponds to

$$\mu_t := X_{t\#}\mu_0 \tag{3.32}$$

where X_t denotes the flow map associated with system (3.4).

3.4 Functional regularization

As noted earlier, the right-hand side of system (3.4) is possibly discontinuous, and this introduces complexity in writing the transport equation for measures. The second approach that we propose relies on working with Lipschitz continuous approximations of the right-hand side of (3.4) to generate a sequence of approximate solutions $\{x^\lambda\}_{\lambda>0}$ parameterized by $\lambda > 0$. In particular, we work with the so-called *Moreau-Yosida (MY) regularization* (refer to Section 2.2 for introduced to MY regularization), which for system (3.4) takes the following form:

$$\dot{x}^\lambda(t) = g_t^\lambda(x^\lambda(t)) := f(t, x^\lambda(t)) - \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S(t))) \tag{3.33}$$

where we take $x^\lambda(0) \in S(0)$, and $\text{proj}(x^\lambda(t), S(t))$ refers to the projection of the vector $x^\lambda(t)$ onto the set $S(t)$ with respect to the Euclidean distance. It is well known that the solution curve μ_t^λ of the continuity equation with the Lipschitz regular vector field (3.33) satisfies the following pushforward relationship

$$\mu_t^\lambda = X_t^\lambda{}_\# \mu_0 \tag{3.34}$$

where $X_t^\lambda(x_0) := x^\lambda(t, x_0)$ is the flow map associated with (3.33). In [65], the authors show that the solution $x^\lambda(t)$ solving (3.33) converges uniformly to the solution $x(t)$ of (3.4) when $x^\lambda(0) = x(0) \in S(0)$, and that the measures μ_t^λ converge in weak star topology to the measure solutions $\mu_t = X_{t\#}\mu_0$, with X_t being the flow map of (3.4). In this section, we provide quantitative bounds on the Wasserstein distance between measures μ_t and μ_t^λ.

Remark 3.4.1. As a parallel to the regularization technique presented, we find an approach based on mollification in [69] to study the evolution of measures for nonsmooth dynamical systems. Such mollification is carried out by using a convolution kernel $\psi : \mathbb{R}^n \to [0, \infty)$ with the properties that $\psi(x)$ is bounded, measurable with $\psi(x) = \psi(-x)$, $\int \psi(x)dx = 1$. Let $\psi_\epsilon := \frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon})$, and the corresponding convolution with a measure μ as $(\psi_\epsilon * \mu)(x) := \int \psi_\epsilon(x-y)d\mu(y)$. For $\mu^\epsilon := \mu * \psi_\epsilon$, it can be shown that $W_2(\mu^\epsilon, \mu) \le \epsilon \int |\psi(x)|^2 \mathrm{d}x$. In [69], a similar mollification technique was used and the narrow convergence[3] of the measures was proven by working with a smooth vector field $g_\epsilon := \frac{(g\mu)*\psi_\epsilon}{\mu^\epsilon}$ with the corresponding continuity equation $\partial_t \mu_t^\epsilon + \nabla \cdot (g_t^\epsilon \mu_t^\epsilon) = 0$.

Theorem 3.4.2. *Let $\mu_t^\lambda \in \mathcal{P}(\mathbb{R}^n)$ be defined as in* (3.34)*, and let $\mu_t = X_{t\#}\mu_0$, with $\mu_0 \in \mathcal{P}(S(0))$ and X_t being the flow map of* (3.4)*. Then, the W_1 distance between μ_t and μ_t^λ satisfies the following bound:*

$$W_1(\mu_t, \mu_t^\lambda) \le C_1 \sqrt{\frac{L_f \lambda (e^{L_f t} - 1)}{2}} \int_{S(0)} |x_0| d\mu_0(x_0) \tag{3.35}$$

[3]Note: Family of measures μ_n converges narrowly to measure μ if $\lim_{n\to\infty} |\int f d\mu_n - \int f d\mu| \to 0$ for a bounded $f \in \mathcal{C}(\Omega; \mathbb{R})$, where Ω is any Polish space. Note that the definition is different from weak* convergence where the convergence is defined w.r.t. compactly supported continuous functions [16]. When the underlying space Ω is compact both the notions of convergence coincide.

where $C_1 = L_f(1+\kappa) + L_S$ *and* $\kappa := (e^{2L_fT} - 1)\frac{2L_f+L_s}{2L_f}$.

Remark 3.4.3. Under the assumption that $\text{supp}(\mu_0)$ is compact, $\text{supp}(\mu_t)$ and $\text{supp}(\mu_t^\lambda)$ will be compact as these are the push-forwards of Lipschitz continuous operators. This results in $W_2(\mu_t^\lambda, \mu_t) \le C_2 W_1(\mu_t^\lambda, \mu_t)$ for some $C_2 > 0$ and hence a qualitatively similar bound holds for W_2 metric as the one indicated in (3.35).

We will use the following lemma in the proof of Theorem 3.4.2.

Lemma 3.4.4. *Let* $x^\lambda(t)$ *be the solution to* (3.33) *and* $x(t)$ *be the solution to* (3.4) *with* $x^\lambda(0) = x(0) \in S(0)$. *Then, for each* $t \in [0,T]$, *it holds that*

$$|x^\lambda(t) - x(t)| \le (L_f(1+\kappa+|x_0|) + L_s)\sqrt{\frac{\lambda(e^{2L_ft}-1)}{2L_f}}$$

where $\kappa = (e^{2L_fT} - 1)\frac{2L_f+L_s}{2L_f}$.

Proof. Let $x^\lambda(\cdot)$ and $x^\nu(\cdot)$ be solutions to (3.33) corresponding to λ and ν as regularization parameters. Then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) &= \langle x^\lambda(t) - x^\nu(t)\ ,\ \dot{x}^\lambda(t) - \dot{x}^\nu(t)\rangle \\ &= \Big\langle x^\lambda(t) - x^\nu(t), f(t, x^\lambda(t)) - \frac{1}{\lambda}\Big(x^\lambda(t) - \text{proj}(x^\lambda(t), S)\Big) - f(t, x^\nu(t)) \\ &\qquad - \frac{1}{\nu}(x^\nu(t) - \text{proj}(x^\nu(t), S))\Big\rangle.\end{aligned}$$

Using the Lipschitz property of $f(t,x)$ and the Cauchy-Schwarz inequality for the terms involving $\langle x^\lambda(t) - x^\nu(t), f(t, x^\lambda(t)) - f(t, x^\nu(t))\rangle$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) \le L_f|x^\lambda(t) - x^\nu(t)|^2 - \Big\langle x^\lambda(t) - x^\nu(t)\ , \\ - \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S)) + \frac{1}{\nu}(x^\nu(t) - \text{proj}(x^\nu(t), S))\Big\rangle.\end{aligned} \tag{3.36}$$

We rewrite $x^\lambda(t) - x^\nu(t) = x^\lambda(t) - \text{proj}(x^\lambda(t), S) + (\text{proj}(x^\lambda(t), S) - \text{proj}(x^\nu(t), S)) - (x^\nu(t) - \text{proj}(x^\nu(t), S))$ and substitute it in (3.36) to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) &\le L_f|x^\lambda(t) - x^\nu(t)|^2 \\ &+ \Big\langle x^\lambda(t) \pm \text{proj}(x^\lambda(t), S) - x^\nu(t) \pm \text{proj}(x^\nu(t), S), \\ &- \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S)) + \frac{1}{\nu}\left(x^\nu(t) - \text{proj}(x^\nu(t), S)\right)\Big\rangle.\end{aligned} \tag{3.37}$$

For notational convenience, we denote $Y_\lambda(x^\lambda(t)) = \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S))$, $Y_\nu(x^\nu(t)) =$

$\frac{1}{\nu}(x^\nu(t) - \mathrm{proj}(x^\nu(t), S))$ and substitute these in (3.37) to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) \le L_f|x^\lambda(t) - x^\nu(t)|^2 + \\ \Big\langle \lambda Y_\lambda(x^\lambda(t)) + (\mathrm{proj}(x^\lambda(t), S) - \mathrm{proj}(x^\nu(t), S)) \\ - \nu Y_\nu(x^\nu(t)), -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t))\Big\rangle. \quad (3.38)\end{aligned}$$

It is known that $Y_\lambda(x^\lambda(t))$ and $Y_\nu(x^\nu(t))$ satisfy the monotonicity property, i.e.

$$\Big\langle -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t)), (\mathrm{proj}(x^\lambda(t), S) - \mathrm{proj}(x^\nu(t), S))\Big\rangle \le 0 \quad (3.39)$$

since $Y_\lambda(x^\lambda(t))$ and $Y_\nu(x^\nu(t))$ are Moreau-Yosida regularizations of the $\mathcal{N}_S(x)$ operator. We substitute (3.39) in (3.38) and get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) \le L_f|x^\lambda(t) - x^\nu(t)|^2 \\ + \langle \lambda Y_\lambda(x^\lambda(t)) - \nu Y_\nu(x^\nu(t)), -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t))\rangle. \quad (3.40)\end{aligned}$$

Using the Cauchy-Schwartz inequality for the second term on the right-hand side of (3.40), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) \le L_f|x^\lambda(t) - x^\nu(t)|^2 - \lambda|Y_\lambda(x^\lambda(t))|^2 - \\ \nu|Y_\nu(x^\nu(t))|^2 + (\lambda+\nu)|Y_\lambda(x^\lambda(t))| \cdot |Y_\nu(x^\nu(t))|. \quad (3.41)\end{aligned}$$

Next we use Young's inequality for the term $|Y_\lambda(x^\lambda(t))| \cdot |Y_\nu(x^\nu(t))|$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) &\le L_f|x^\lambda(t) - x^\nu(t)|^2 - \lambda|Y_\lambda(x^\lambda(t))|^2 - \nu|Y_\nu(x^\nu(t))|^2 \\ &\quad + \frac{(\lambda+\nu)}{2}(|Y_\lambda(x^\lambda(t))|^2 + |Y_\nu(x^\nu(t))|^2) \\ &\le L_f|x^\lambda - x^\nu(t)|^2 + \frac{\nu}{2}|Y_\lambda(x^\lambda(t))|^2 + \frac{\lambda}{2}|Y_\nu(x^\nu(t))|^2. \quad (3.42)\end{aligned}$$

In [65], uniform bounds on $|x^\lambda(t)|$ were obtained, which leads to $|Y_\lambda(x^\lambda(t))| \le L_f(1+\kappa+e^{2L_fT}|x^\lambda(0)|) + L_s$, where $\kappa = (e^{2L_fT} - 1)\frac{2L_f+L_s}{2L_f}$ and a similar bound on $Y_\nu(x^\nu(t))$ (see Appendix 3.7). We use this bound in (3.42) and obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|x^\lambda(t) - x^\nu(t)|^2) \le L_f|x^\lambda(t) - x^\nu(t)|^2 \\ + \frac{(\nu+\lambda)}{2}|L_f(1+\kappa+e^{2L_fT}|x^\lambda(0)|) + L_s|^2 \quad (3.43)\end{aligned}$$

where we have used the fact that $x^\lambda(0) = x^\nu(0) = x(0)$. Now applying Gronwall's lemma we get

$$|x^\lambda(t) - x^\nu(t)|^2 \leq |L_f(1 + \kappa + e^{2L_f T}|x^\lambda(0)|) + L_s|^2(\nu + \lambda)\frac{e^{2L_f t} - 1}{2L_f}$$

where the term involving $|x^\lambda(0) - x^\nu(0)|$ in the right-hand side is zero. Next we use the fact that $\lim_{\nu \to 0} x^\nu(t) = x(t)$ [65] holds pointwise and $x^\nu(0) = x(0) = x_0$ to obtain the desired bound. □

Proof of Theorem 3.4.2. To get a bound on the distance between μ_t and μ_t^λ, we use the dual characterization of the W_1 distance [83]:

$$W_1(\mu_t^\lambda, \mu_t) = \sup_{\substack{\phi \in \mathcal{C}(\Omega;\mathbb{R}),\\ ||\phi||_{\text{Lip}} \leq 1}} \int \phi \, \mathrm{d}(\mu_t^\lambda - \mu_t) \tag{3.44}$$

where $\|\phi\|_{\text{Lip}}$ denotes the Lipschitz modulus of ϕ and Ω is a measurable set containing all $S(t)$ for $t \in [0, T]$. We use the representation formula for μ_t^λ and μ_t to obtain,

$$\sup_{\substack{\phi \in C(\Omega;\mathbb{R})\\ ||\phi||_{\text{Lip}} \leq 1}} \int \phi \, \mathrm{d}(\mu_t^\lambda - \mu_t) = \sup_{\substack{\phi \in C(\Omega;\mathbb{R})\\ ||\phi||_{\text{Lip}} \leq 1}} \int \phi(x^\lambda(t, x_0))\mathrm{d}\mu_0(x_0) - \int \phi(x^\lambda(t, x_0)|_{\lambda=0})\mathrm{d}\mu_0(x_0)$$

where $x^\lambda(t, x_0)|_{\lambda=0} := \lim_{\lambda \to 0} x^\lambda(t, x_0)$. Using the first order Taylor expansion of $\phi(x^\lambda(t, x_0))$ w.r.t. λ, for each $\phi \in \mathcal{C}(\Omega; \mathbb{R})$ s.t. $||\phi||_{\text{Lip}} \leq 1$, we get,

$$\begin{aligned}
&\int \phi(x^\lambda(t, x_0))\mathrm{d}\mu_0 - \int \phi(x^\lambda(t, x_0)|_{\lambda=0})\mathrm{d}\mu_0 \\
&\leq \int \Big[\phi(x^\lambda(t, x_0)|_{\lambda=0}) + \nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0} \cdot (x^\lambda(t, x_0) - \\
&\qquad x(t, x_0))\Big]\mathrm{d}\mu_0 - \int \phi(x(t, x_0))\mathrm{d}\mu_0 \\
&= \int \nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0}(x^\lambda(t, x_0) - x(t, x_0))\mathrm{d}\mu_0 \\
&\leq \int |\nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0}| \; |(x^\lambda(t, x_0) - x(t, x_0))|\mathrm{d}\mu_0.
\end{aligned} \tag{3.45}$$

Using the fact that ϕ is of Lipschitz constant 1, the above equation reduces to

$$W_1(\mu_t^\lambda, \mu_t) \leq \int_{S(0)} |(x^\lambda(t, x_0) - x(t, x_0))| d\mu_0(x_0). \tag{3.46}$$

Using Lemma 3.4.4 in (3.46), we get the inequality in (3.35). □

3.5 Time Discretization and Optimal Transport

In this section, we provide a construction of solutions to the continuity equation for (3.4) using a time discretization scheme. Throughout, we work in the Wasserstein space $(\mathcal{P}_2(\mathbb{R}^n), W_2)$ of probability measures with finite second moment. We refer to Section 2.3 for the general definitions of the Wasserstein distance, the continuity equation, and absolutely continuous curves of measures, and to [16] for further background. Here we only recall that solutions will be represented as absolutely continuous curves $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^n)$ solving the continuity equation in the sense of distributions.

3.5.1 Construction of curves in Wasserstein space

Next we propose a construction of curves in $\mathcal{W}_2(\mathbb{R}^n)$ through an interpolation between measures defined at discrete time instants using a discretization of the nonsmooth dynamical system (3.4). Consider a partition $\{0 = t_0, t_1, \dots, t_N = T\}$ of the time interval $[0, T]$ such that $t_{k+1} - t_k = \tau$. For a fixed value of τ, we define the measures $\{\mu_k^\tau\}_{k=0}^N$ recursively.

Let $S_k := S(t_k)$, which is a closed convex set under Assumption 3.2.2. We denote by P_{S_k} the Euclidean projection onto S_k, and we consider the map $G^k : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G^k(x) := P_{S_{k+1}}(x + \tau f_k(x)), \qquad f_k(x) := f(t_k, x).$$

The successor of μ_k^τ is now defined as its push-forward under the map G^k:

$$\mu_{k+1}^\tau := (G^k)_\# \mu_k^\tau = \Big[P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})\Big]_\# \mu_k^\tau. \tag{3.47}$$

Similarly, for each $x \in S(t_k)$, the velocity vector at time t_{k+1} is defined as

$$v_{k+1}^\tau(x) := \frac{G^k(x) - x}{\tau}.$$

Next we consider the following two different interpolation curves, which will serve different purposes.

(1) Geodesic (McCann) interpolation. Fix $k \in \{0, \dots, N-1\}$. Let

$$\theta_k^\tau \in \Pi_{\mathrm{opt}}(\mu_k^\tau, \mu_{k+1}^\tau)$$

be an optimal transport plan between μ_k^τ and μ_{k+1}^τ for the quadratic cost. For $t \in (t_k, t_{k+1}]$, define the rescaled time

$$s = s(t) := \frac{t - t_k}{\tau} \in (0, 1] \quad \text{and} \quad \mathcal{I}_s(x, y) := (1 - s)x + sy.$$

The McCann (constant-speed) interpolated curve is then defined by

$$\mu_t^\tau := (\mathcal{I}_s)_\# \theta_k^\tau, \qquad t \in (t_k, t_{k+1}]. \tag{3.48}$$

In particular, μ_t^τ is a constant-speed W_2-geodesic connecting μ_k^τ to μ_{k+1}^τ, and

$$|(\mu^\tau)'|(t) = \frac{W_2(\mu_k^\tau, \mu_{k+1}^\tau)}{\tau} \qquad \text{for a.e. } t \in (t_k, t_{k+1}]. \tag{3.49}$$

To define the interpolated velocity field without assuming the existence of an optimal transport map, we disintegrate θ_k^τ with respect to the map $\mathcal{I}_s$ (equivalently, with respect to

μ_t^τ): there exists a Borel family of probability measures $\{\theta_{t,z}^\tau\}_{z\in\mathbb{R}^n}$ such that

$$\theta_k^\tau = \int_{\mathbb{R}^n} \theta_{t,z}^\tau \, d\mu_t^\tau(z), \qquad \text{and each } \theta_{t,z}^\tau \text{ is supported on } \mathcal{I}_s^{-1}(\{z\}).$$

Then we define the (minimal) geodesic velocity by

$$v_t^\tau(z) := \frac{1}{\tau}\int_{\mathbb{R}^n\times\mathbb{R}^n} (y-x)\, d\theta_{t,z}^\tau(x,y), \qquad \text{for } \mu_t^\tau\text{-a.e. } z, \quad t\in(t_k,t_{k+1}]. \tag{3.50}$$

The corresponding momentum measure is

$$E_t^\tau := v_t^\tau \, \mu_t^\tau, \tag{3.51}$$

and the pair (μ_t^τ, E_t^τ) satisfies the continuity equation $\partial_t\mu_t^\tau + \nabla\cdot E_t^\tau = 0$ in the distributional sense on $(t_k,t_{k+1})\times\mathbb{R}^n$. Moreover, one can choose v_t^τ so that

$$\|v_t^\tau\|_{L^2(\mu_t^\tau)} = \frac{W_2(\mu_k^\tau,\mu_{k+1}^\tau)}{\tau} \quad \text{for a.e. } t\in(t_k,t_{k+1}],$$

cf. [83, Chapter 5].

(2) Piecewise constant interpolation. Define

$$\hat{\mu}_t^\tau = \mu_{k+1}^\tau, \quad \text{for } t\in(t_k,t_{k+1}], \tag{3.52}$$

$$\hat{v}_t^\tau = v_{k+1}^\tau, \quad \text{for } t\in(t_k,t_{k+1}]. \tag{3.53}$$

We also define the corresponding momentum vector as

$$\hat{E}_t^\tau := \hat{v}_t^\tau \, \hat{\mu}_t^\tau. \tag{3.54}$$

We will use the piecewise constant interpolation to show that the limit velocity belongs to (3.4).

3.5.2 Convergence result

Next we illustrate the convergence of the constructed curves μ_t^τ, $\hat{\mu}_t^\tau$ to μ_t which is solution to the continuity equation associated with (3.4).

Theorem 3.5.1. *Consider system* (3.4) *under Assumption 3.2.1 and Assumption 3.2.2, with* $\mu_0\in\mathcal{P}(S(0))$*. For* $\tau>0$*, and* $t\in[0,T]$*, let* μ_t^τ *and* v_t^τ *be defined as in* (3.48) *and* (3.50)*, respectively. Then, as* $\tau\to 0$*, we get the following two convergence results:*

- *The measures* μ_t^τ *in* (3.48) *and* $\hat{\mu}_t^\tau$ *in* (3.52) *converge uniformly in the* W_2 *metric to* $\mu_t = X_{t\#}\mu_0$*, where* X_t *is the flow map associated with* (3.4)*;*
- *Momentum vectors* $E_t^\tau = v_t^\tau\mu_t^\tau$ *(defined in* (3.51)*) and* $\hat{E}_t^\tau = \hat{v}_t^\tau\hat{\mu}_t^\tau$ *(defined in* (3.54)*) converge to* $E_t := v_t\mu_t$ *in the weak star sense. Moreover, the velocity* v_t *is such that* $v_t\in f(t,x)-\mathcal{N}_{S(t)}(x)$*.*

Proof. We split the proof into four parts: (1) Proof of convergence of μ_t^τ to μ_t and $\hat{\mu}_t^\tau\to\mu_t$; (2)

Proof of convergence of E_t^τ and $\hat{E}_t^\tau$ to E_t; (3) Absolute continuity of E_t with respect to μ_t such that $E_t = v_t\mu_t$; and (4) Convergence of v_t^τ to v_t with the property that $v_t \in f(t,x) - \mathcal{N}_{S(t)}(x)$. The first of these four items follows from the following result:

Lemma 3.5.2. *Let μ_k^τ and μ_{k+1}^τ be defined as in* (3.47) *and* (3.48)*. Then, it holds that*

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \tau(L_f C_{\max} + L_s) \tag{3.55}$$

where the constants L_f, L_s are defined in Assumption 3.2.1 and $C_{\max}$ is a constant that captures the uniform bound on $|x_k|$ at time instant t_k, independently of $k \in \mathbb{N}$.

Proof of Lemma 3.5.2. The mapping $G^k(x)$ defines a feasible transport map between μ_k^τ and μ_{k+1}^τ. As the Wasserstein distance is defined to be the infimum over all feasible transport maps, we have

$$W_2^2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \int_{S_k} |P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})(x) - x|^2 d\mu_k^\tau(x). \tag{3.56}$$

Next we use the triangle inequality to obtain

$$W_2^2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \int_{S_k} |P_{S_{k+1}}(\tau f_k(x) + x) - P_{S_{k+1}}(x)|^2 d\mu_k^\tau(x) + |P_{S_{k+1}}(x) - x|^2 \mathrm{d}\mu_k^\tau(x). \tag{3.57}$$

Projection operators on convex sets satisfy the nonexpansive property and we use this fact for the first term. For the second term, we use the definition of the Hausdorff distance to obtain

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \Big(\int_{S_k} |\tau f_k(x)|^2 d\mu_k^\tau(x)\Big)^{1/2} + \tau \mathrm{d}_H(S_k, S_{k+1}). \tag{3.58}$$

Using Assumption 3.2.1 for the drift term $f_k(x)$, it holds

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \Big(\int_{S_k} \tau^2 |(L_f(1 + |x|))^2| \mathrm{d}\mu_k^\tau(x)\Big)^{1/2} + \tau d_H(S_k, S_{k+1}). \tag{3.59}$$

Next, we establish a bound on the first term on the right-hand side of (3.59). We know that $\mu_k^\tau = G_\#^k \circ G_\#^{k-1} \circ \cdots \circ G^1{}_\# \mu_0 = (G^1 \circ G^2 \circ \cdots \circ G^k)_\# \mu_0$. Let $G^{k \dots 1} := G^k \circ G^{k-1} \circ \cdots \circ G^1$, then it follows that

$$\int_{S_k} (1 + |x|)^2 d\mu_k^\tau(x) = \int_{S_0} (1 + |G^{k\dots 1}(x_0)|)^2 \mathrm{d}\mu_0^\tau(x_0). \tag{3.60}$$

Letting $x_k := G^{k\dots 1} x_0$, it has been shown in [29, Section 5] that

$$|x_k| \leq \widetilde{C}_1 |x_0| + \widetilde{C}_2 \tag{3.61}$$

for some constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ depending on the system data. Now using this uniform bound on $|x_k|$ in the first term on the right-hand side of (3.59), we get

$$\Big(\int_{S_k}\tau^2 L_f^2(1+|x|)^2\mathrm{d}\mu_k^\tau(x)\Big)^{1/2} = \tau L_f\Big(\int_{S_0}(1+|x_k|)^2\mathrm{d}\mu_0^\tau(x_0)\Big)^{1/2}$$

$$\le \tau L_f\Big(\int_{S_0}(1+\widetilde{C}_1|x_0|+\widetilde{C}_2)^2\mathrm{d}\mu_0^\tau(x_0)\Big)^{1/2} \tag{3.62}$$

$$=: \tau L_f C_{\max} \tag{3.63}$$

where the equality between (3.62) and (3.63) holds because $\mu_0^\tau \in \mathcal{P}(S_0)$. Thus

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \le \tau L_f C_{\max} + \tau d_H(S_k, S_{k+1}) = \tau(L_f C_{\max} + L_s)$$

where we used Assumption 3.2.2 in the last equality. □

(1) Proof of convergence of μ_t^τ to μ_t: The convergence is based on computing the bounds on $W_2(\mu_t^\tau, \mu_s^\tau)$ for $s, t \in [0, T]$, where we recall that μ_t^τ are the interpolated measures. This is done by using the characterization of absolutely continuous curves (**??**), i.e.

$$W_2(\mu_t^\tau, \mu_s^\tau) \le \int_s^t |(\mu^\tau)'(r)|\mathrm{d}r \tag{3.64}$$

and using Hölder's inequality leads to

$$\int_s^t |(\mu^\tau)'|(r)dr \le (t-s)^{1/2}\Big(\int_s^t |(\mu^\tau)'(r)|^2\mathrm{d}r\Big)^{1/2} \le (t-s)^{1/2}\Big(\sum_k \tau\Big(\frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau}\Big)^2\Big)^{\frac{1}{2}}. \tag{3.65}$$

Using the bounds on $W_2(\mu_{k+1}^\tau, \mu_k^\tau)$ from (3.55), we get

$$\sum_k \tau\Big(\frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau}\Big)^2 \le \sum_k (L_s + L_f C_{\max})^2\tau = (L_s + L_f C_{\max})^2 T.$$

Substituting (3.5.2) in (3.65) we obtain

$$W_2(\mu_t^\tau, \mu_s^\tau) \le (t-s)^{1/2}(L_s + L_f C_{\max})T^{1/2}. \tag{3.66}$$

Thus the curves are uniformly $\frac{1}{2}$Hölder continuous. Moreover, the curves μ_t^τ lie in the W_2 space for each $t \in [0, T]$ which is compact. Thus, we can apply the Ascoli-Arzelà theorem (for the Hölder continuous functions) i.e., there exists a subsequence τ_j for which $\mu_t^{\tau_j} \to \mu_t$ uniformly in W_2 space and the limit curve μ_t is absolutely continuous.

Similar to (3.66) one can derive bounds for $\hat{\mu}_t^\tau$ and conclude that $\hat{\mu}_t^{\tau_j} \to \hat{\mu}_t$. Moreover the limit curves are the same as

$$W_2(\hat{\mu}_t^\tau, \mu_t^\tau) \le (\tau)^{1/2}(L_f C_{\max} + L_s)T^{1/2}.$$

This holds because the curves $\hat{\mu}_t^\tau$ coincide with μ_t^τ at $k\tau$ and they are constant on the interval $(k\tau, (k+1)\tau]$. Thus both curves converge to the same limit curve μ_t.

(2) Proof of convergence of E_t^τ: In order to study the convergence properties of the velocity vector, we need to investigate the convergence properties of a family of momentum vectors

$E_t^\tau = v_t^\tau \mu_t^\tau$ which is a vector measure[4] $E_t^\tau \in \mathcal{M}^n(\Omega)$, where Ω is a measurable set which contains all $S(t)$, for all $t \in [0,T]$. We define $m^\tau \in \mathcal{M}^n([0,T]\times\Omega)$ as $m^\tau := v_t^\tau \mu_t^\tau \mathrm{d}t$.

Lemma 3.5.3. *The norm of m^τ satisfies the following bound:*

$$|m^\tau|([0,T]\times\Omega) \le T^{\frac{3}{2}}(L_f C_{\max} + L_s). \tag{3.67}$$

Proof. By definition

$$|m^\tau|([0,T]\times\Omega) = \int_{[0,T]} \mathrm{d}t \int_\Omega |v_t^\tau| d\mu_t^\tau.$$

Using the Cauchy-Schwarz inequality and then (3.49) we get

$$|m^\tau|([0,T]\times\Omega) \le T^{1/2} \int_{[0,T]} ||v_t^\tau||_{L^2(\mu_t^\tau)} dt \le T^{1/2} \sum_k \frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau}.$$

Using Lemma 3.5.2, we further obtain

$$|m^\tau|([0,T]\times\Omega) \le T^{1/2} \sum_k (L_f C_{\max} + L_s) = T^{\frac{3}{2}}(L_f C_{\max} + L_s)$$

which is the desired bound. □

So, m^τ is uniformly bounded and thus compact under weak convergence in the space of vector valued measures on $[0,T]\times\Omega$. We conclude that up to a subsequence $m^\tau \rightharpoonup m$ and thus $E_t^\tau \rightharpoonup E_t$. For $\hat{m}^\tau := \hat{v}_t^\tau \hat{\mu}_t^\tau$, a similar bound holds, i.e. $|\hat{m}^\tau| \le T^{\frac{3}{2}}(L_f C_{\max} + L_s)$. Using the same arguments one concludes $\hat{m}^\tau \rightharpoonup \hat{m}$ and thus $\hat{E}_t^\tau \rightharpoonup \hat{E}_t$. Moreover, using [83, Lemma 8.9] we conclude that $\hat{E}_t = E_t$.

Next we discuss about the properties of the limit object E_t and show that E_t is absolutely continuous with respect to μ_t, such that $E_t = v_t \mu_t$, for each $t \in [0,T]$.

(3) Absolute continuity of E_t: At this point, we recall the properties of then Benamou-Brenier functional $\mathcal{B}(\mu, E)$ defined as follows. For $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $E \in \mathcal{M}^n(\mathbb{R}^n)$, let

$$\mathcal{B}(\mu,E) := \sup_{\substack{a\in\mathcal{C}(\mathbb{R};\mathbb{R}), b\in\mathcal{C}(\mathbb{R}^n;\mathbb{R}^n) \\ a,b \text{ are bounded} \\ a+\frac{1}{2}|b|^2 \ge 0 \text{ pointwise}}} \int_{\mathbb{R}^n} a(x)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^n} b(x)\,\mathrm{d}E(x)$$

which has the following properties:

- $\mathcal{B}(\cdot,\cdot)$ is convex, lower semicontinuous, and non-negative;
- $\mathcal{B}(\mu, E) = \frac{1}{2}\int |v|^2 \mathrm{d}\mu$ only if $E = v\mu$ is absolutely continuous with respect to μ and $\mathcal{B}(\mu,E)$ is ∞ otherwise.

Note that $\mathcal{B}(\mu_t^\tau \mathrm{d}t, E_t^\tau \mathrm{d}t) = \int_{[0,T]} \int_\Omega |v_t^\tau|^2 \mathrm{d}\mu_t^\tau \mathrm{d}t$. Now using the uniform bound on $|m^\tau|$ (thus

[4]The space of vector measures $\mathcal{M}^n(\Omega)$ is a normed space dual to $\mathcal{C}(\Omega;\mathbb{R}^n)$. Under this duality the notion of weak star convergence is defined which further implies that bounded sets in $\mathcal{M}^n(\Omega)$ are weak star compact.

on $\int_{[0,T]}\int_\Omega |v_t^\tau|^2 \mathrm{d}\mu_t^\tau \mathrm{d}t)$ in (3.67) and the lower semi-continuity of $\mathcal{B}(\cdot,\cdot)$, we get

$$\mathcal{B}(\mu_t \mathrm{d}t, \mathrm{d}m) \le \liminf_{\tau\to 0} \mathcal{B}(\mu_t^\tau \mathrm{d}t, \mathrm{d}m^\tau) < \infty$$

where $\mathrm{d}m$ represents the limit of $\mathrm{d}m^\tau$, as $\tau \to 0$. We can now invoke the second property mentioned above which implies that $\mathrm{d}m$ is absolutely continuous with respect to $\mathrm{d}\mu_t \mathrm{d}t$. Thus, there exists v_t such that $\mathrm{d}m = v_t \mu_t \mathrm{d}t$. Similarly, E_t is absolutely continuous with respect to μ_t and $E_t = v_t \mu_t$.

(4) *Proof that* $v_t(x) \in f(t,x) - \mathcal{N}_{S(t)}(x)$*:* Next we show that the velocity v_t belongs to the admissible set of velocities. By construction of the velocities v_{k+1}^τ, for every $x \in S_k$, and every $y \in S_{k+1}$, we have

$$\langle y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})(x)), \frac{(P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})(x) - x)}{\tau} \rangle \ge 0$$

which results in

$$\langle y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})(x)), v_{k+1}^\tau(x) - f_k(x) \rangle \ge 0,$$

i.e., $v_{k+1}^\tau(x) - f_k(x)$ is in the normal cone to set S_{k+1}. The above condition should hold in the integral form for any smooth positive function $h(t,x)$ i.e.

$$\int_{[0,T]}\int_\Omega \Big\langle h(t_k,x)(y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \mathrm{id})x), \\ (v_{k+1}^\tau(x) - f_k(x)) \mathrm{d}\mu_k^\tau(x) \Big\rangle \ge 0, \quad \forall y \in S_{k+1}. \quad (3.68)$$

Next we can extend (3.68) to piecewise constant interpolated curves (3.52) and corresponding velocities (3.53). We make a passage from discrete time $k\tau$ to $t \in (k\tau, (k+1)\tau]$ by identifying $v_{k+1}^\tau \mathrm{d}\mu_k^\tau = \hat{v}_t^\tau \mathrm{d}\hat{\mu}_t^\tau = \mathrm{d}\hat{E}_t^\tau$ and similarly, $\mathrm{d}\mu_k^\tau = \mathrm{d}\hat{\mu}_t^\tau$. Next, we use the convergence results established in the first two steps of the proof, so that, in the limit as $\tau \to 0$, we obtain

$$\int_{[0,T]}\int_\Omega \Big\langle h(t,x)(y-x), \mathrm{d}E_t(x) \Big\rangle - \\ \int_{[0,T]}\int_\Omega \Big\langle h(t,x)(y-x), f(t,x) \mathrm{d}\mu_t(x) \Big\rangle \ge 0, \quad \forall y \in S(t).$$

As we have already established that $E_t = v_t \mu_t$ and since $h(t,x)$ is an arbitrary positive function, we get

$$\langle y - x, v(t,x) - f(t,x) \rangle \ge 0, \quad \forall\, y \in S(t).$$

Thus, $v(t,x) - f(t,x) \in -\mathcal{N}_{S(t)}(x)$.

Uniqueness of solutions: The uniqueness of solutions follows from the same argument as presented in Section 3.3.4 based on the superposition principle. Furthermore, it also states that the solution has a representation formula as $\mu_t = X_{t\#}\mu_0$. □

3.6 Numerical results

3.6.1 Moment-SOS hierarchy

In Section 3.3, we derived the continuity equation (3.24) associated with dynamical system (3.4) using the superposition principle. The problem of computing the evolution of a probability measure through a dynamical system can be interpreted as a feasibility problem where the feasible set is an affine section (modeled by the continuity equation) of the cone of non-negative measures (supported on the trajectories). For notational convenience, we rewrite the continuity equation in (3.24) as

$$\partial_t \mu + \nabla \cdot (\overline{f}_\eta \mu) + \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 \tag{3.69}$$

where $\overline{f}_\eta$ is defined in (3.10) and the equation should be understood in the weak sense, i.e. when integrated against sufficiently smooth test functions as in (3.31). Given an initial distribution μ_0 with $\text{supp}(\mu_0) \subset X$, we formulate the problem of evolution of measures in (3.4) as the following infinite dimensional optimization problem:

$$\begin{aligned}
&\text{Find } \mu, \mu_{\mathrm{T}} \text{ such that}\\
&\partial_t \mu + \nabla \cdot (\overline{f}_\eta \mu) + \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0\\
&\mu \geq 0, \mu_0 \geq 0, \mu_T \geq 0\\
&\text{supp}(\mu) \subset [0,T] \times B, \quad \text{supp}(\mu_T) \subset X_T,
\end{aligned} \tag{3.70}$$

where $B = \{(x,v) | x \in \Omega \subset \mathbb{R}^n, v \in F(x)\}$. This is an infinite dimensional LP. In this section, we present numerical results obtained using the moment-SOS hierarchy (refer to Section2.5 for basic notions of moment-SOS hierarchy) to solve this LP in terms of approximate moments of measures μ and μ_T. The moments are generated by integration on a dense set of functions ϕ in $\mathcal{C}^1([0,T], \mathbb{R}^n)$ as in (3.24). For notational convenience we use the monomial basis.

Numerical example. Consider the nonsmooth system

$$\dot{\mathbf{x}}(t) \in (1,0) - \mathcal{N}_S(\mathbf{x}(t)) \tag{3.71}$$

where $S = \{\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. For this problem, we decompose the measure solution μ of (3.31) into two measures: μ_S supported in the disc and $\mu_{\partial S}$ supported on the boundary of the disc. In particular, we let $X_S \coloneqq \text{supp}(\mu_S) = \{0 \leq t \leq 1\} \times \{(\mathbf{x},\mathbf{v}) : x_1^2 + x_2^2 \leq 1, v_1 = 1, v_2 = 0\}$, and $X_{\partial S} \coloneqq \text{supp}(\mu_{\partial S}) = \{0 \leq t \leq 1\} \times \{(\mathbf{x},\mathbf{v}) : x_1^2 + x_2^2 = 1, x_1 v_2 = x_2 v_1\}$. The continuity equation (3.31) can be expressed as

$$\begin{aligned}
&\int_{X_T} \phi(T,\mathbf{x}) d\mu_T(\mathbf{x}) - \int_{X_0} \phi(0,\mathbf{x}) d\mu_0(\mathbf{x})\\
&= \int\limits_{X_S} \Big[\partial_t \phi(t,\mathbf{x}) + \nabla_{\mathbf{x}} \phi(t,\mathbf{x}) \cdot \mathbf{v}\Big]\, d\mu_S(t,\mathbf{x},\mathbf{v}) + \int\limits_{X_{\partial S}} \Big[\partial_t \phi(t,\mathbf{x}) + \nabla_{\mathbf{x}} \phi(t,\mathbf{x}) \cdot \mathbf{v}\Big]\, d\mu_{\partial S}(t,\mathbf{x},\mathbf{v}).
\end{aligned} \tag{3.72}$$

To approximate the moments of the measures satisfying the above equation we use the monomial basis. Let $R[\mathbf{x}]$ be the ring of multivariate polynomials and $R_k[\mathbf{x}] \subset R[\mathbf{x}]$ be the vector space of polynomials of degree not exceeding k. Then the monomial basis of $R_k[\mathbf{x}]$ can be expressed as $\phi(t,\mathbf{x}) := t^a\mathbf{x}^b := t^a x_1^{b_1} x_2^{b_2}..x_n^{b_n}$ where $a + b_1 + b_2 + .. + b_n \leq k$.

Proposition 3.6.1. *Let* $m^0_{a,b} := \int_0^T \int_{\mathbb{R}^n} t^a\mathbf{x}^b d\mu_0(t,\mathbf{x})$ *be the moments of* μ_0*. Then using the monomial basis in* (3.72) *we get*

$$m^T_{a,b} - m^0_{a,b} = am^S_{a-1,b} + am^{\partial S}_{a-1,b} + \sum_{i=1}^n \int_{X_{\partial S}} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_{\partial S}(t,\mathbf{x},\mathbf{v}) + \sum_{i=1}^n \int_{X_S} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_S(t,\mathbf{x},\mathbf{v}) \quad (3.73)$$

where $m^S_{a,b} := \int_{X_S} t^a\mathbf{x}^b d\mu(t,\mathbf{x})$, $m^{\partial S}_{a,b} := \int_{X_{\partial S}} t^a\mathbf{x}^b d\mu(t,\mathbf{x})$, $m^T_{a,b} := T^a \int_{X_T} \mathbf{x}^b d\mu_T(\mathbf{x})$ *and* $e_i = (0,..,1,..,0)$ *is the vector with one at the ith entry.*

Proof. The proof is a straightforward extension of [65, Proposition 4.3] □

The moment-SOS hierarchy allows us to evaluate approximate moments $m_{a,b} := m^S_{a,b} + m^{\partial S}_{a,b}$ and $m^T_{a,b}$ related to occupation measures μ and μ_T. For details about the moment-SOS hierarchy we refer the readers to [65, Section 4.3] or [24]. The initial distribution is $\mu_0 = \delta_{(0,0.5)}$, i.e. the Dirac measure at coordinates $(0, 0.5)$ whose moments $m^0_{a,b}$ are readily available. The simulation results are displayed in Figure 4.3, where we plot the first order moments of terminal measures μ_T, for different terminal times T and computed for a given relaxation order using `GloptiPoly` and `SeDuMi`[5]. We can observe the effect of the boundary on the measure even before it hits the boundary. We attribute this effect to the numerical inaccuracy due to finite order truncation. The approximate moments obtained, sometimes called pseudo-moments, may not represent the true moments but the accuracy increases as the relaxation order increases.

3.6.2 Bound on Wasserstein Distance

In Section 3.4, we used the Moreau-Yosida regularization to approximate the solution $(\mu_t)_{t\geq 0} \in \mathcal{P}(\mathbb{R}^n)$ of the continuity equation associated with the nonsmooth dynamical system (3.4) with measures $(\mu^\lambda_t)_{t\geq 0} \in \mathcal{P}(\mathbb{R}^n)$ which are solutions to the continuity equation with regularized vector field (3.33). We showed that the measures μ_t, μ^λ_t, which have representations of the form (3.32) and (3.34) respectively, satisfy a bound on their Wasserstein distance of the form $W_1(\mu^\lambda_t, \mu_t) < C_W\sqrt{\lambda}$ for each time $t \in [0,T]$ (the explicit expression of the constant C_W is in (3.35)). Now let us validate this bound for the following example:

$$\dot{x}(t) \in -1 - \mathcal{N}_{\mathbb{R}^+}(x(t)). \quad (3.74)$$

[5]GloptiPoly is a MATLAB toolbox for modeling and solving generalized moment problems via the moment–SOS hierarchy, and SeDuMi is a solver for conic optimization problems (in particular semidefinite programs) used here to solve the resulting relaxations.

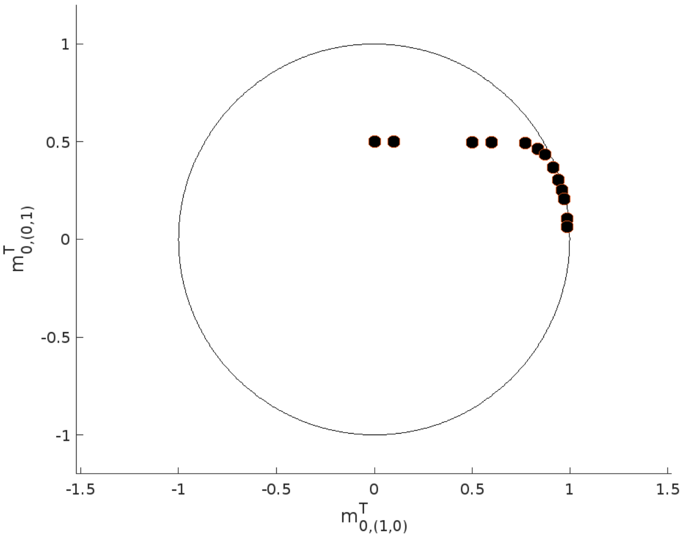


Figure 3.1: Moment-SOS hierarchy approximations of the first degree moments of the terminal measure μ_T solving the continuity equation (3.69) for the nonsmooth system (3.71), with different terminal times ranging from $T = 0$ (Dirac mass at $(0, 0.5)$) to $T = 3$ (Dirac mass approximately at $(1, 0)$).

Its Moreau-Yosida regularization is:

$$\dot{x}^\lambda(t) = -1 - \frac{1}{\lambda}(x(t) - \max(x(t), 0)). \tag{3.75}$$

We consider an initial value problem for (3.74) and (3.75) with $x(0) = 0.5$. The solution to the initial value problem for (3.74) is

$$x(t) = \begin{cases} 0.5 - t & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } t \geq \frac{1}{2} \end{cases} \tag{3.76}$$

and the solution for (3.75) is

$$x^\lambda(t) = \begin{cases} 0.5 - t & \text{for } 0 < t < \frac{1}{2} \\ \lambda(\exp(-(t - 0.5)/\lambda) - 1) & \text{for } t \geq \frac{1}{2}. \end{cases} \tag{3.77}$$

Proposition 3.6.2. *Given $\mu_t^\lambda, \mu_t \in \mathcal{P}(\mathbb{R})$ and $\mu_0 = \mu_0^\lambda = \delta_{x=a}$ for some $a > 0$, then*

$$W_1(\mu_t^\lambda, \mu_t) = |x^\lambda(t) - x(t)|. \tag{3.78}$$

Proof. It is known that in one dimension $W_1(\mu, \nu) = \int_{\mathbb{R}} |F_\mu(s) - F_\nu(s)| \mathrm{d}s$ [83] where $F_\mu(\cdot)$ and $F_\nu(\cdot)$ are the cumulative distribution functions of measures μ and ν respectively. Using this formula, we get

$$W_1(\mu_t^\lambda, \mu_t) = \int_{\mathbb{R}} |F_{\mu_t^\lambda}(s) - F_{\mu_t}(s)| \mathrm{d}s. \tag{3.79}$$

Given $\mu_0 = \mu_0^\lambda = \delta_{x=a}$, the measures μ_t^λ and μ_t can be computed as $\mu_t^\lambda = \delta_{x^\lambda(t)}$ and $\mu_t = \delta_{x(t)}$ and thus $F_{\mu_t^\lambda}(s) = \Theta_{x^\lambda(t)}(s)$ and $F_{\mu_t}(s) = \Theta_{x(t)}(s)$. Here, $\Theta : \mathbb{R} \to \mathbb{R}$ is such that $\Theta_{x(t)}(s) = 1$ for $x(t) \leq s$ and $\Theta_{x(t)}(s) = 0$ for $x(t) \geq s$. Substituting $F_{\mu_t^\lambda}$ and F_{μ_t} in (3.79), we get

$$W_1(\mu_t^\lambda, \mu_t) = \int_{\mathbb{R}} |\Theta_{x^\lambda(t)}(s) - \Theta_{x(t)}(s)| \mathrm{d}s. \tag{3.80}$$

This quantity can be seen as the area under $|\Theta_{x^\lambda(t)}(s) - \Theta_{x(t)}(s)|$ at time t and performing the integration (3.80) we get the desired result. □

Given (3.76), (3.77) and using Proposition 3.6.2 we obtain for $a = 0.5$:

$$W_1(\mu_t^\lambda, \mu_t) = \begin{cases} 0 & \text{for } 0 < t < \frac{1}{2} \\ |\lambda(e^{-(t-0.5)/\lambda} - 1)| & \text{for } t \geq \frac{1}{2}. \end{cases} \tag{3.81}$$

Next we obtain the bound in (3.35) for system (3.74) with $L_f = 1$, $L_s = 0$ and $\mu_0 = \delta_{x=0.5}$. Substituting these values we get

$$W_1(\mu_t^\lambda, \mu_t) \leq \frac{3}{2} e^2 \sqrt{\frac{\lambda}{2}(e^t - 1)}. \tag{3.82}$$

In Figure 3.2, we show the plots of the analytical distance (3.81) and its upper bound (3.82) for different values of λ at four different time instants.

In general, we can also look at the difference between the moments associated with the measures μ_t^λ and μ_t.

Proposition 3.6.3. *Consider the moment sequences $m_k(t)$ resp. $m_k^\lambda(t)$ for μ_t resp. μ_t^λ. Then, for a fixed $k \in \mathbb{N}$ and $t \in [0, T]$,*

$$|m_k(t) - m_k^\lambda(t)| \leq C_k W_1(\mu_t, \mu_t^\lambda)$$

where $C_k := k \max_{z \in \Omega} z^{k-1}$.

Proof. By the definition of the moments

$$|m_k(t) - m_k^\lambda(t)| \leq \Big| \int_\Omega x^k \mathrm{d}\mu_t(x) - \int_\Omega y^k \mathrm{d}\mu_t^\lambda(y) \Big|. \tag{3.83}$$

Let us consider an optimal transport plan $\gamma \in \mathcal{P}(\Omega \times \Omega)$ for compact $\Omega \subset \mathbb{R}^n$ such that $\pi_{x\#}\gamma = \mu_t$ and $\pi_{y\#}\gamma = \mu_t^\lambda$. We use this transport plan in (3.83) to obtain

$$\Big| \int_\Omega x^k \mathrm{d}\mu_t(x) - \int_\Omega y^k \mathrm{d}\mu_t^\lambda(y) \Big| \leq \Big| \int_\Omega (x^k - y^k) \mathrm{d}\gamma(x, y) \Big| \leq \int_\Omega |x^k - y^k| \mathrm{d}\gamma(x, y). \tag{3.84}$$

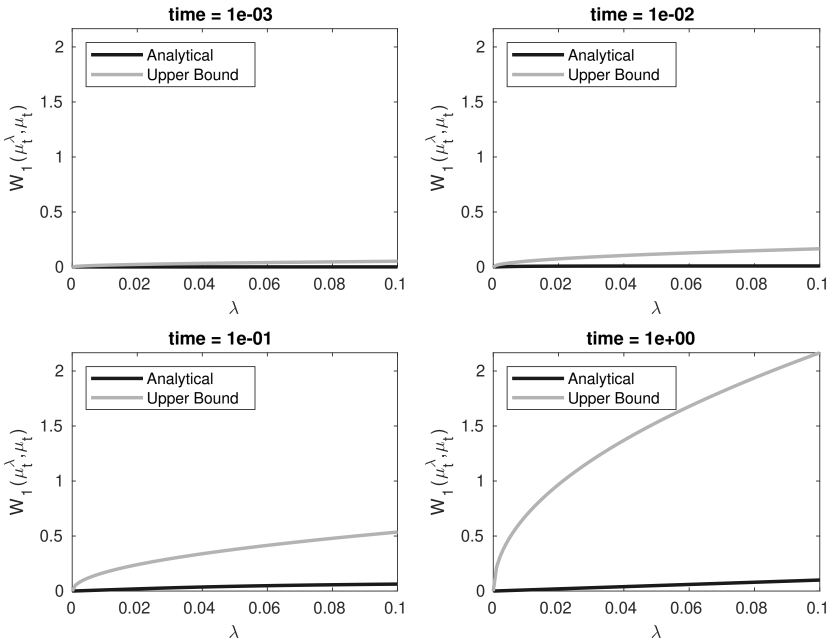


Figure 3.2: W_1 distance (3.81) (in black) and its upper bound (3.82) (in gray) between the image measures μ_t resp. μ_t^λ through the flow (3.32) resp. the regularized flow (3.34), at various time instants.

Next, using the mean value theorem, we have $x^k - y^k = (x-y)kz^{k-1}$ for some $z \in [x, y]$. Using this formula in (3.84), we obtain

$$|m_k(t) - m_k^\lambda(t)| \leq C_k \int_\Omega |x-y| \mathrm{d}\gamma(x,y) = C_k W_1(\mu_t, \mu_t^\lambda) \sim O(\sqrt{\lambda})$$

where $C_k := k \max_{z \in \Omega} z^{k-1}$. □

So we expect that the distance of the moment sequence corresponding to the measures of the form (37) and (39) is $O(\sqrt{\lambda})$. Using the moment-SOS hierarchy for such an approximation scheme as presented in [65], we have a quantitative bound on the numerical error introduced by the functional regularization.

3.6.3 Time-discretized measure evolution

In Section 3.5, we used geodesic interpolation and piecewise constant interpolation for the time discretized curves and showed convergence to the solution to the continuity equation (3.24). Let us apply the time-stepping algorithm to implement a time discretized evolution of the measure solutions to the continuity equation associated with the nonsmooth dynamical system (3.4).

Let τ be the time step between two discretized measures and let μ_k^τ be the measure at time k. Then from Section 3.5:

$$\mu_0^\tau = \mu_0; \quad \mu_{k+1}^\tau = P_{S_{k+1}\#}(\tau f_k(\cdot) + id)_\# \mu_k. \tag{3.85}$$

We model the measure as mass distributed on a space discretized grid. Then the time-stepping scheme consists of the following two steps:

- Step 1: compute the pushforward measure

$$\tilde{\mu}_k := (\tau f_k(\cdot) + id)_{\#}\mu_k; \tag{3.86}$$

- Step 2: project each cell lying outside S_{k+1} back onto S_{k+1}. This operation can be formulated as

$$\mu_{k+1} = P_{S_{k+1}\#}\tilde{\mu}_k \tag{3.87}$$

  and it can be understood as a projection of measures on the set of measures with support S_{k+1}.

In Figure 3.3 we illustrate the scheme for the following example

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t))$$

where $S(t)$ is a 4×4 grid square, with each cell assigned with mass of $1/16$ at time $t = 0$.

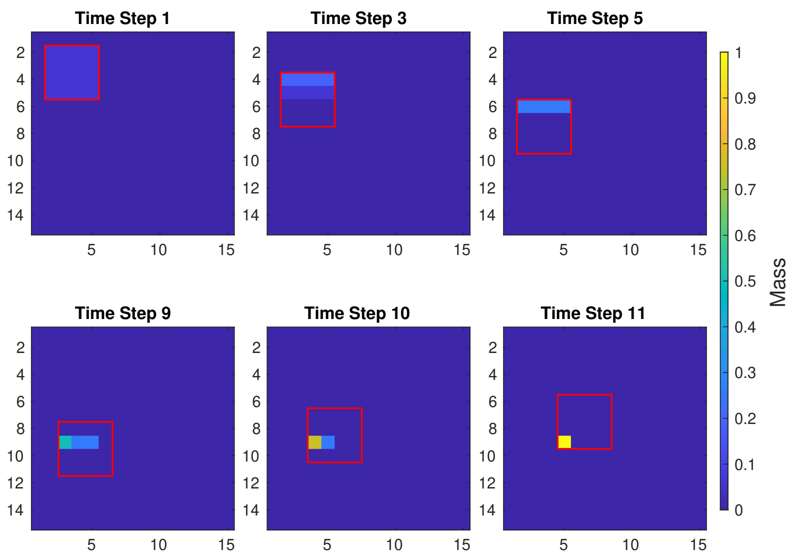


Figure 3.3: Snapshots of time evolution of a uniform probability distribution on the moving square. The mass is initially distributed on a 4×4 square grid with $1/16$ mass on each cell. For the first 7 time steps the square moves downward and the mass can be seen to concentrate on the top edge of the square. Afterwards the square moves diagonally and the mass concentrates on the left edge until it completely concentrates at the bottom left corner at time step 12.

Remark 3.6.4. If we consider the numerical techniques for each formalism, we observe that each formalism handles the selection of velocity in a different way. This will affect the numerical accuracy and computational efficiency. For the first formalism, we observe that $\mu_S, \mu_{\partial S}$ in (81) are dependent of the velocity vector. Thus, while specifying the support of the measure, the normal cone has to be defined explicitly. Though we can directly write the continuity equation (without using approximation) with this approach, the measure depends on the velocity vector. The increase in number of variables (to be computed) reduces the computational efficiency.

When we approximate the solutions of the continuity equation corresponding to the regularized dynamics (38), the projection of the velocity vector onto the set must be defined. There is no need to select the velocity vector as the vector field is smoothened. However, numerical errors are introduced due to the approximation scheme used.

Similarly, the numerical approximation of the discrete-time problem is based on (95). One needs to define explicitly the projection map onto the set at each time instant. The measures at each time instant do not depend on an extra velocity vector, thus avoiding increasing the number of variables encountered with the first formalism.

3.7 Conclusion

We addressed the problem of evolution of measures in a nonsmooth dynamical system modeled by evolution variation inequalities using three different formalisms. For the time-discretization formalism presented in Section 5, one can also draw some similarities with the literature on constrained sampling for stochastic differential equations, see for example [89]. In these algorithms, a sequence of random variables is simulated in a recursive manner by projecting one-step of Euler-Maruyama interpolation onto a closed convex set, and under some regularity assumptions, it can be shown that the probability law associated with the limit of these random variables converges to the invariant distribution with respect to total variation. In our case, we restrict our analysis over finite time intervals and compare the time-discretized probability measure with the actual solution without assuming the existence of an invariant distribution.

Appendix

Bounds used in proof of Proposition 3.3.3

In this appendix, we provide estimates for the bounds $\int |\dot\gamma(t)| d\theta(x,\gamma)$. To do so, we first get bounds on $|\gamma(t)|$ using the linear growth in Assumption 3.3.2,

$$\begin{aligned} |\gamma(t)-\gamma(0)| &\leq \int |\dot\gamma(s)|\mathrm{d}s \leq \int \beta(s)(1+|\gamma(s)|)\mathrm{d}s \\ &\leq \int_0^t \beta(s)(1+|\gamma(0)|)\mathrm{d}s + \int_0^t \beta(s)(|\gamma(s)-\gamma(0)|)\mathrm{d}s. \end{aligned}$$

Applying Gronwall inequality by assuming first term is non-decreasing,

$$|\gamma(t)-\gamma(0)| \leq \Big[\int_0^t \beta(s)(1+|\gamma(0)|)\mathrm{d}s\Big]\, e^{\left(\int_0^t \beta(\sigma)\mathrm{d}\sigma\right)}. \tag{3.88}$$

Let $\overline{\beta}_t := \int_0^t \beta(\sigma)\mathrm{d}\sigma$, then

$$|\gamma(t)-\gamma(0)| \leq \overline{\beta}_t \exp(\overline{\beta}_t)(1+|\gamma(0)|) \leq \overline{\beta}_T \exp(\overline{\beta}_T)(1+|\gamma(0)|).$$

We use this last inequality to get the desired bound as follows:

$$\begin{aligned}\int_{\mathbb{R}^n\times\Gamma_T^\eta} |\dot{\gamma}(t)|\mathrm{d}\theta(x,\gamma) &\leq \int_{\mathbb{R}^n\times\Gamma_T^\eta} \beta(t)(1+|\gamma(t)|)\mathrm{d}\theta(x,\gamma)\\ &\leq \int_{\mathbb{R}^n\times\Gamma_T^\eta} \beta(t)(1+|\gamma(0)|)(\overline{\beta}_T\exp(\overline{\beta}_T)+1)\mathrm{d}\theta(x,\gamma) \leq K\beta(t),\end{aligned}$$

where $K := \int_{\mathbb{R}^n\times\Gamma_T^\eta}(1+|\gamma(0)|)(\overline{\beta}_T\exp(\overline{\beta}_T)+1)\mathrm{d}\theta(x,\gamma)$.

Differentiating first moment with time dependence

In this appendix, we derive the Liouville equation (3.19). For any $\varphi \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ we differentiate $\int \varphi(t,x)\mathrm{d}\mu_t(x)$ and get,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\varphi(t,x)\mathrm{d}\mu_t(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\varphi(t,x)\mathrm{d}\mathbf{e}_{t\#}\theta(x,\gamma)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n\times\Gamma_T^\eta}\varphi(t,\gamma(t))\mathrm{d}\theta(x,\gamma)\\ &= \int_{\mathbb{R}^n\times\Gamma_T^\eta}\Big(\partial_t\varphi(t,\gamma(t))+\nabla_x\varphi(t,x)\dot{\gamma}(t)\Big)\mathrm{d}\theta(x,\gamma)\\ &= \int_{\mathbb{R}^n}\Big(\partial_t\varphi(t,x)+\nabla_x\varphi(t,x)\cdot f(t,x)\Big)\mathrm{d}\mu_t(x).\end{aligned}$$

Integrating w.r.t. t on both sides gives the desired result.

A property for Moreau-Yosida regularization for the mapping induced by normal cone

In [65], the bound on $|Y_\lambda|$ was established as follows,

$$|Y_\lambda(x^\lambda(t))| \leq \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}(L_f + L_f(|x^\lambda(t)|)+L_s)\mathrm{d}s. \tag{3.89}$$

It was demonstrated that x^λ satisfy uniform bound $|x^\lambda(t)| \leq e^{2L_fT}|x^\lambda(0)| + \kappa$ where, $\kappa = (e^{2L_fT}-1)\frac{2L_f+L_s}{2L_f}$ and T is the time interval for the existence of trajectory. Upon substituting this uniform bound on $|x^\lambda|$ in (3.89) we get,

$$|Y_\lambda(x^\lambda(t))| \leq L_f(1+\kappa+e^{2L_fT}|x^\lambda(0)|)+L_s. \tag{3.90}$$

Chapter 4

Optimal control

In this chapter, we address the optimal control problem for a class of dynamical systems with constrained state trajectories. These systems are modeled by a differential inclusion with a drift term and a normal cone mapping associated with the constraint set. The optimal control problem is considered in continuous-time and discrete-time, where the latter provides a computational advantage over the former. In both cases, the nonlinear problem is reformulated as an infinite-dimensional linear program (LP) over occupation measures. We show that this does not introduce any relaxation gap, that is, the optimal value remains the same for the reformulated LP. Using appropriate tools from functional analysis and optimal transport, we also show the convergence of the optimal value of the discrete problem to the optimal value of the continuous problem. We propose finite-dimensional convex optimization algorithms based on the moment-SOS hierarchy to provide numerical approximations of the proposed infinite-dimensional LPs.

4.1 Introduction

We now build on Chapter 3's measure-evolution formalisms for sweeping processes, and we rely throughout on Chapter 2's preliminaries on quasi-dissipativity and continuity equations (Sections 2.2 and 2.3), as well as the discussion of relaxation gaps and moment–SOS tools (Sections 2.4 and 2.5). With this foundation, we formulate and analyze measure relaxations of the optimal control problem, and we develop their tractable moment–SOS relaxations.

Optimal control of nonlinear systems has remained a problem of interest for the control community over several decades. Developments on the theoretical side as well as numerical methods for efficiently solving such problems continue to attract attention of researchers from different community. In this chapter, we consider the optimal control problem for a class of nonsmooth systems, and propose numerically tractable algorithms for our approach. In particular, the systems that we consider are described by a differential inclusion where the right-hand side comprises a drift term and a normal cone mapping associated to a time-varying constraint set; see [29] for an overview of such models and their relevance in various applications that include robotics, mechanics, electrical circuits, market equilibrium, and power systems with switching.

Let us recall that the optimal control of nonlinear systems with well-posedness properties is a well-studied problem [90, 35]. In contrast, optimal control for nonsmooth dynamical systems with set-valued right-hand side is more complicated and has been an area of active research over the last two decades because of its theoretical and practical relevance [91].

Among the existing work on optimal control problems for the class of nonsmooth systems (which are closely related to the ones studied here), we note that the authors in [42] consider an evolution variational inequality with a closed convex constraint set. By deriving the first-order

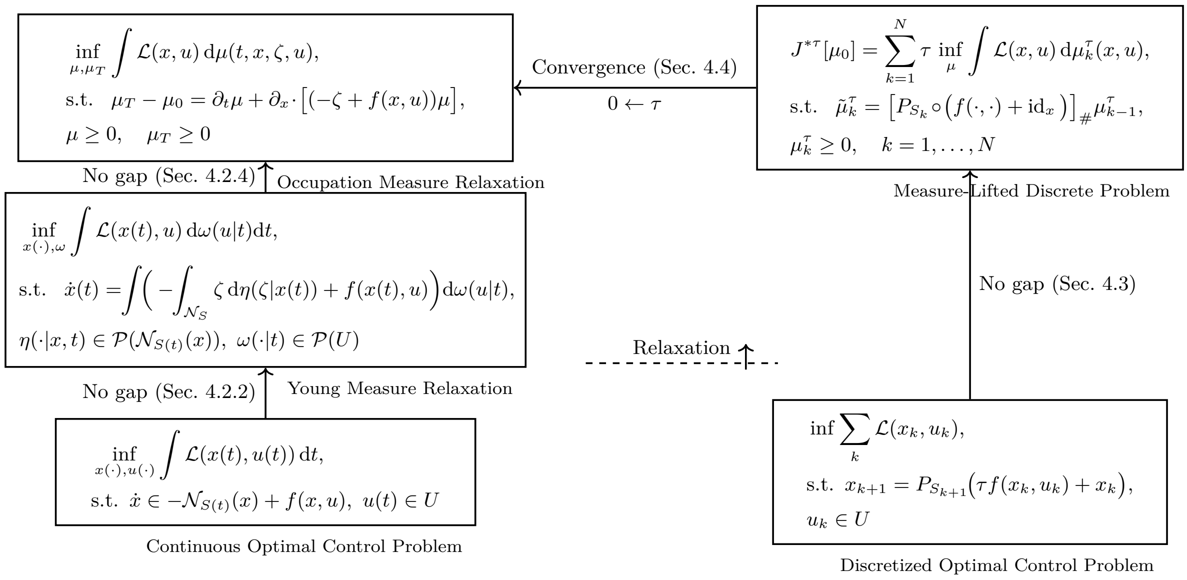


Figure 4.1: Outline of the approach developed in this chapter for the optimal control of nonsmooth dynamical systems. The continuous-time problem is relaxed via Young and occupation measures (no relaxation gap). The discrete-time formulation is lifted to the cone of measures and convergence as $\tau \to 0$ is established.

optimality conditions for the regularized problem, and taking the limit as the regularization parameter tends to zero, the corresponding optimality conditions for the original nonsmooth system are obtained. Pontryagin Maximum Principle based optimality conditions for discrete approximations of such systems are derived in [38] and sophisticated tools from variational analysis are used to derive the limit equation. In [39], the authors use exact penalization-based techniques to extend the optimality conditions to more general problems with weaker assumptions on the system data. In a more recent article, [92] uses distance to the constraint set as the penalization which allows the authors to obtain uniform, w.r.t. control estimates and they show strong convergence of the approximating trajectories in $W^{1,2}$ metric.

Direct approach based on discretizing the problem and solving the resulting static optimization have several fundamental issues. In [46], the authors pointed out that the error in gradients of the simulation results are independent of the step sizes, which means that optimization problems based on such discretization schemes often fail. The authors propose an optimal control strategy that approximates the discontinuous system with a smooth right-hand side and use the Euler integrator, ensuring that the gradients approach the true values under sufficiently small step sizes relative to the smoothing parameter. Finite-element-based discretization along with time-freezing methods have been recently proposed in software packages[45]. Further in [40], the authors extend the finite element with switch detection method to address optimal control problems with nonsmooth differential equations by transforming them into dynamic complementarity systems. For a class of linear complementarity systems (closely related to the formalism adopted in our present work), but where the right-hand side of the differential equation is a Lipschitz continuous map, the paper [43] presents first-order optimality conditions and some numerical results.

On the other hand, global methods for nonlinear control based on occupation measures have gained popularity in the last decade because of their powerful modeling capabilities and the availability of efficient algorithms and semidefinite programming (SDP) solvers[24]. These methods reformulate a finite-dimensional nonlinear optimal control problem into a primal/dual pair of infinite-dimensional LPs. The primal problem is expressed in the cone of nonnegative Borel measures and the dual problem is expressed in the cone of nonnegative continuous functions[25, 56]. The infinite-dimensional LPs admit a sequence of finite dimensional relaxations using the moment-sums-of-square(SOS) hierarchy[24]. Each relaxation is formulated as an SDP and can be solved using off-the-shelf SDP solvers. Under mild assumptions, the sequence yields a monotonically non-decreasing sequence of lower bounds converging to the true optima. Occupation measures were also studied in the context of control of stochastic systems [93, 94]. One more advantage of occupation measure-based methods is the ability to easily handle uncertainty in the initial distribution.

In [95], the authors solve an optimal transport problem for a Lagrangian-based cost obtained from a dynamical system with nonholonomic constraints. The problem of uniqueness and the existence of solutions for Lagrangian cost derived from linear quadratic regulator were studied in [96]. Similar ideas have been pursued in [97]. In these works, the problem of mass transport for Lagrangian systems is seen as a decoupled problem where the lower level computes the optimal control for the finite-dimensional particle system and then this cost is used to find an optimal map to transport the initial measures to the final configuration.

In the present work, we address the problem of optimal control of nonsmooth dynamical systems. We use occupation measures to relax both the discrete-time problem and the continuous-time problem into an infinite-dimensional LP defined in the cone of nonnegative Borel measures. Under some mild assumptions, we show that this does not produce any relaxation gap. This means that the value of the LP is equal to the value of the original optimal control problem. Further, we use tools from optimal transport theory to show the convergence of the discrete-time problem to the continuous-time problem when the time step size goes to zero. The overall layout is summarized in Figure 4.1. Apart from the independent theoretical interest, the convergence of the discrete-time problem to the continuous-time problem is also of practical importance, as solving the discrete-time problem provides an approximation to the continuous-time problem. We also propose numerical techniques which provide finite dimensional approximation of the relaxed discrete-time problem.

4.1.1 Problem setting

We consider a controlled nonsmooth dynamical system modeled as an evolution variational inequality:

$$\dot{x}(t) \in f(x(t), u(t)) - \mathcal{N}_{S(t)}(x(t)), \quad x(0) = x_0. \tag{4.1}$$

Here $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given map (the vector field or drift), and $\mathcal{N}_{S(t)}(x(t))$ denotes the outward normal cone to the compact convex set $S(t)$ at $x(t) \in \mathbb{R}^n$, for each $t \in [0, T]$. Admissible controls satisfy $u(t) \in U$ for a.e. $t \in [0, T]$, where $U \subset \mathbb{R}^m$ is convex and compact. We make the following assumptions, which are the same as those in Chapter 3 (see Assumptions 3.2.1 and 3.2.2), to ensure existence and uniqueness of solutions.

Assumption 4.1.1. There exist a compact set Ω and constant $L_f > 0$ such that, $S(t) \subset \Omega$ for each $t \in [0,T]$, and for every $x, x_1, x_2 \in \Omega$, we have

$$\max_{u \in U} |f(x,u)| \leq L_f(1+|x|)$$
$$|f(x_1,u) - f(x_2,u)| \leq L_f|x_1 - x_2| \quad \forall u \in U.$$

Assumption 4.1.2. The set-valued mapping $S : [0,T] \rightrightarrows \mathbb{R}^n$ satisfies the following:

- For every $t \in [0,T]$, $S(t)$ is nonempty, closed and convex.
- $S(\cdot)$ is Lipschitz continuous w.r.t. the Hausdorff distance: there exists $L_s \geq 0$ such that $d_H(S(t), S(s)) \leq L_s|t-s|$ for $t, s \in [0,T]$ where $d_H(A,B) = \max\{\sup_x \operatorname{dist}(x,A), \sup_x \operatorname{dist}(x,B)\}$ is the Hausdorff distance between sets $A, B \subset \mathbb{R}^n$.

An optimal control problem for such a system, over a finite interval $[0,T]$, can be formulated as

$$\begin{aligned} J^*(x_0) = &\inf_{u \in L^\infty([0,T];\mathbb{R}^m)} \int_0^T \mathcal{L}(x(t),u(t))\mathrm{d}t \\ &\text{s.t. } \dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + f(x(t),u(t)), \\ &\qquad x(0) = x_0, \quad u(t) \in U, \quad \forall t \in [0,T] \end{aligned} \tag{4.2}$$

where $\mathcal{L}(x,u)$ is a continuous cost function in x, u. Further assumptions on the cost function and the drift term, that are necessary for our purposes, are formulated as follows:

Assumption 4.1.3. For each $x \in \mathbb{R}^n$, the image set $f(x,U)$ is convex. The cost $\mathcal{L}(x,u)$ is convex in u for each x, and satisfies the growth bound:

$$\mathcal{L}(x,u) \leq l(x)(1+|u|^2) \quad \forall x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m \tag{4.3}$$

where l is locally bounded and Borel-measurable.

In order to study the discrete-time version of (4.2), consider a partition of $[0,T] = \{0 = t_0, t_1, ...t_i, ...t_N = T\}$ with $t_k - t_{k-1} = \tau$ being the time step between two samples. We use a time-stepping-based algorithm to define the evolution of states [29]. Let x_k^τ resp. x_{k+1}^τ be the state at time $k\tau$ resp. $(k+1)\tau$, and let u_k be the control at time $k\tau$. They are related by

$$x_{k+1}^\tau = P_{S_k}(\tau f_k(x_k^\tau, u_k) + x_k^\tau) \tag{4.4}$$

where P_{S_k} is the (unique) projection onto the compact convex set $S_k := S(k\tau)$.

The discrete-time optimal control problem can then be written as

$$
\begin{aligned}
J_\tau^*(x_0) = \inf_{\substack{x_k^\tau \in \mathbb{R}^n \\ u_k \in \mathbb{R}^m}} & \sum_{k=0}^{N} \tau \mathcal{L}(x_k^\tau, u_k) \\
\text{s.t.} \quad & x_{k+1}^\tau = P_{S_k}(\tau f_k(x_k^\tau, u_k) + x_k) \\
& x(0) = x_0, \quad u_k \in U, \quad \forall k = 0, 1, \ldots, N.
\end{aligned}
\tag{4.5}
$$

4.1.2 Contribution

The primary problem studied in this chapter concerns the continuous-time problem (4.2) and discrete-time problem (4.5) for the nonsmooth system (4.1). In particular, we provide an alternative formulation for these problems and study the relation between them as the sampling time tends to zero. The main contributions can be summarized as follows:

- We study the optimal control problem in both continuous-time (4.2) and discrete-time (4.5) by formulating or relaxing them to linear optimization problems in occupation measures.
- We prove that this relaxation introduces no gap, meaning the value of the LP on measures matches the value of the original optimal control problem.
- We analyze the convergence of the relaxed discrete-time problem to the continuous-time problem using interpolation schemes based on optimal transport.
- We propose numerical techniques for solving approximately the infinite-dimensional LPs using the moment-SOS hierarchy, a family of finite-dimensional semidefinite programs of increasing size.

4.1.3 Outline

The remainder of the chapter is organized as follows: In Section 4.2, we discuss the relaxation of the continuous-time optimal control problem, first using Young measures and then using occupation measures. We also examine key aspects of this relaxation process, such as the absence of a relaxation gap. In Section 4.3, we formulate the relaxation of the discrete-time problem in the space of measures, ensuring that the relaxation does not introduce any gap between the optimal values of the problems. In Section 4.4, we analyze the convergence of the relaxed discrete-time optimal control problem to its continuous-time counterpart. Finally, in Section 4.5, we introduce relaxations based on semidefinite programming for the relaxed discrete-time problem and illustrate the proposed approach with an academic example.

4.2 Relaxed continuous-time optimal control

In this section, our aim is to describe the relaxation of the continuous time optimal control problem (4.2) in the space of nonnegative Borel measures. As a first step in this direction, in Theorem 4.2.2, we study weak convergence of the sequence of inputs (u_j) using Young measures, and show that the corresponding sequence of state trajectories (x_j) is strongly convergent. We then introduce the relaxation of problem (4.2) using Young measures and demonstrate in Theorem 4.2.3 that this does not alter the optimal value, i.e. there is no relaxation gap. Then, we introduce occupation measures, which allow us to further relax the optimal control problem so that the resulting problem becomes linear with respect to the measures. Finally, in Theorem 4.2.6, we prove that this last reformulation does not introduce any relaxation gap.

4.2.1 Minimizing sequences and Young measures

Let us consider a minimizing sequence $(x_j(\cdot), u_j(\cdot))_{j\in\mathbb{N}}$ of control and state trajectories. By that, we mean that each state trajectory $x_j(\cdot)$ is a solution to (4.1) for the given control $u_j(\cdot)$, and the value of the problem (4.2) decreases when j increases. Next we consider the control sequence of this minimizing sequence to be weakly convergent and we study the convergence properties of the sequence of associated state trajectories.

Note however that the nonlinearity of the dynamics in (4.1) does not preserve weak convergence. For a simple illustrating example, consider a sequence of functions $(g_j)_{j\in\mathbb{N}}$ defined as follows:

$$g_j(t) = \sum_{i=0}^{j-1} \mathbf{1}_{\left[\frac{2i}{2j},\frac{2i+1}{2j}\right]}(t) - \mathbf{1}_{\left[\frac{2i+1}{2j},\frac{2i+2}{2j}\right]}(t) \tag{4.6}$$

where $\mathbf{1}_I(t)$ is equal to one if $t \in I$ and zero otherwise. Given $p \in [1, \infty)$, the sequence $(g_j)_{j\in\mathbb{N}}$ converges weakly to zero in $L^p([0, 1])$, i.e. $\lim_{j\to\infty} \int \phi(t)g_j(t)\mathrm{d}t = 0$ for all $\phi \in L^q([0, 1])$, where $q \in [1, \infty)$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. But if we consider a nonlinearity $f(u) = |u|$, then $f(g_j(t)) = 1$ for all $j \in \mathbb{N}$. So, we observe that $\lim_{j\to\infty} f(g_j) \neq f(\lim_{j\to\infty} g_j) = f(0) = 0$. Rather, the oscillating behavior of the weakly converging sequence can be captured by Young measures [98].

Definition 4.2.1 (Young measure)**.** Let $I = [0, T]$ and $X = \mathbb{R}^n$. A *Young measure* on I with values in X is a mapping $t \mapsto \nu_t \in \mathcal{P}(X)$ such that for every $\phi \in \mathcal{C}_b(X)$ the function

$$t \longmapsto \int_X \phi(x)\, d\nu_t(x)$$

is Borel measurable on I.

In the previous example, the sequence generates a family of homogeneous (i.e. independent of t) Young measures $\omega_t = \frac{\delta_{-1}+\delta_1}{2}$ which capture correctly the weak-$\star$ convergence

$$\begin{aligned}\lim_{j\to\infty} \int_0^T \phi(t)f(g_j(t))\mathrm{d}t &= \int_0^T \int_{\mathbb{R}} \phi(t)f(\mathsf{u})\mathrm{d}\omega_t(\mathsf{u})\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \phi(t)|\mathsf{u}|\frac{\delta_{-1}+\delta_1}{2}\mathrm{d}t = \int_0^T \phi(t)\mathrm{d}t\end{aligned}$$

where $\phi \in L^1([0, T])$. Thus, we obtain the correct weak-$\star$ limit of the compositions, $f(g_j(\cdot)) \overset{*}{\rightharpoonup} 1$. The next result formalizes this mode of convergence: Young measures encode the limits of compositions $\psi(t, x_j(t), u_j(t))$ when (x_j, u_j) converge weakly and ψ is a Carathéodory integrand. A function $\psi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is called *Carathéodory* if it is measurable in t for each fixed (x,u), and continuous in (x,u) for almost every $t \in [0,T]$.

Proposition 4.2.1 ([98, Theorem 6.2]). *Let $(x_j, u_j)_{j\in\mathbb{N}}$ be a sequence of measurable and equibounded functions, and let $\psi : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. For every weakly convergent sequence $(\psi(t, x_j(t), u_j(t)))_{j\in\mathbb{N}}$ in $L^1([0,T])$, there exists a subsequence (without relabelling), and a family of probability measures $(\nu_t)_{t\in[0,T]}$, such that*

$$\lim_{j\to\infty} \int_0^T \psi(t, x_j(t), u_j(t))\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n\times\mathbb{R}^m} \psi(t, \mathsf{x}, \mathsf{u})\mathrm{d}\nu_t(\mathsf{x}, \mathsf{u})\mathrm{d}t.$$

The above result can be understood as a statement about the convergence properties of sequences of compositions of nonlinear functions, with Young measures encoding the limiting behaviour. More precisely, it realizes the abstract construction of Young measures introduced in Chapter 2: the map

$$(x(\cdot), u(\cdot)) \mapsto \left(t \mapsto \delta_{(x(t),u(t))}\right)$$

embeds trajectories into the dual space $L^\infty([0,T]; \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m))$, and weak-$\star$ compactness yields (up to subsequences) a limit Young measure $t \mapsto \nu_t$; see Chapter 2, Section 2.4. In this context:

- The sequence $(x_j, u_j)_{j\in\mathbb{N}}$ can be seen as generating a sequence of Dirac measures $(\delta_{(x_j,u_j)})_{j\in\mathbb{N}}$ in the space $L^\infty([0,T]; \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m))$. [1]
- The weak-$\star$ convergence of $\delta_{(x_j,u_j)}$ in this measure space yields a Young measure $(\omega_t)_{t\in[0,T]}$ which captures the limiting behavior of (x_j, u_j) when tested against functions in $L^1([0,T]; \mathcal{C}_0(\mathbb{R}^n \times \mathbb{R}^m))$.

Young measures also help to establish strong convergence of state trajectories for certain class of dynamical systems even under weak convergence of control sequences. For example, consider the following dynamical system:

$$\dot{x}(t) = |u(t)| \tag{4.7}$$

with the sequence of inputs $(u_j)_{j\in\mathbb{N}} = (g_j)_{j\in\mathbb{N}}$ where g_j is defined in (4.6) and $x(0) = 0$. The sequence converges weakly to zero in $L^p([0,T])$ for each $p \in [1,\infty)$, and in weak-$\star$ sense for $p = \infty$. Thus, we might expect that, in the limit, $\dot{x}(t) = |u(t)| = 0$ and hence $x(t) = 0$ for $t \in [0,T]$. But the sequence of solutions to (4.7), corresponding to u_j, is given by $x_j(t) = t$ which uniformly converges to $x(t) = t$. This sort of convergence can be explained by the use of Young measures, where we describe the limiting control by a measure and system (4.7)

[1] The spaces $L^1([0,T]; \mathcal{C}_0(\mathbb{R}^n))$ and $L^\infty([0,T]; \mathcal{M}(\mathbb{R}^n))$ form a dual pair, with their duality relation given by

$$\langle \nu, \phi \rangle = \int_0^T \int_{\mathbb{R}^n} \phi(t,x)\, \nu_t(\mathrm{d}x)\, \mathrm{d}t,$$

for $\nu \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^n))$ and $\phi \in L^1([0,T]; \mathcal{C}_0(\mathbb{R}^n))$.

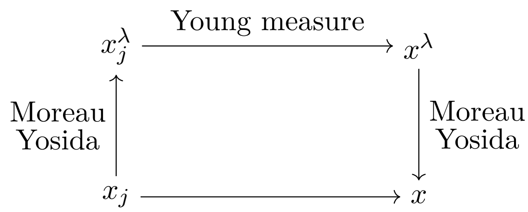


Figure 4.2: Schematic diagram of the proof of Theorem 4.2.2.

with such inputs is interpreted as follows:

$$\dot{x}(t) = \int |\mathsf{u}| \mathrm{d}\omega_t(\mathsf{u}) = \int |\mathsf{u}| \frac{\delta_{-1} + \delta_1}{2} = 1. \tag{4.8}$$

The last equation clearly implies that $x(t) = t$. Note that system (4.8) is a relaxation of system (4.7) in the sense that system (4.8) corresponds to the particular choice of Young measure $\mathrm{d}\omega_t(\mathsf{u}) = \delta_{u(t)}$ in system (4.8). We therefore expect system (4.8) to have a larger set of solutions than system (4.7).

We next show that, given a weakly convergent control sequence $(u_j(\cdot))_{j\in\mathbb{N}}$, the corresponding sequence of state trajectories $(x_j(\cdot))_{j\in\mathbb{N}}$ is strongly convergent when interpreting limiting controls in the sense of Young measure. Moreover, each pair $(x_j(\cdot), u_j(\cdot))$ satisfies (4.1) and the limit trajectory $x(\cdot)$ satisfies

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + \int_U f(x(t), \mathsf{u})\, \mathrm{d}\omega(\mathsf{u}|t). \tag{4.9}$$

Theorem 4.2.2. *Let $(u_j)_{j\in\mathbb{N}}$ be a weakly converging sequence, in $L^\infty([0,T];\mathbb{R}^m)$, which generates the limiting Young measure $\mathrm{d}\omega_t$. Then, up to a subsequence, the sequence $(x_j)_{j\in\mathbb{N}}$ of associated solution trajectories* (4.1) *converges uniformly to x satisfying* (4.9).

Proof. Let us consider the weakly converging sequence $(u_j)_{j\in\mathbb{N}}$ with the Young measure ω_t as the limit (in the sense of Proposition 4.2.1) and associate with it a sequence of absolutely continuous functions $(x_j)_{j\in\mathbb{N}}$ where each $x_j : [0,T] \to \mathbb{R}^n$ is obtained as a solution to (4.1) by applying the control input u_j, $j \in \mathbb{N}$. We aim to prove that the sequence $(x_j)_{j\in\mathbb{N}}$ is uniformly convergent to x, where x is described by (4.9). The proof consists of three steps: (1) approximation by regularization, (2) convergence to Young measure, (3) uniform convergence of the relaxed dynamics.

1. *Approximation by regularization*: We consider the Moreau-Yosida regularization of the dynamics in (4.1)

$$\dot{x}_j^\lambda(t) = f(x_j^\lambda(t), u_j(t)) - \frac{1}{\lambda}(x_j^\lambda(t) - P_{S(t)}(x_j^\lambda(t))). \tag{4.10}$$

For λ small enough, $x_j^\lambda(t) \in \Omega$[2] and we borrow the following inequalities from [65, Theorem 3.1]:

$$|x_j^\lambda(t)| \leq \exp(2L_f T)|x_j^\lambda(0)| + (\exp(2L_f T) - 1)\frac{2L_f + L_s}{2L_f}, \tag{4.11}$$

$$|\dot{x}_j^\lambda(t)| \leq 2L_f + L_f|x_j^\lambda(t)| + L_f \max_{0\leq s\leq t} |x_j^\lambda(s)| + L_s \tag{4.12}$$

which establish the bounds on $x_j^\lambda(t)$ and $\dot{x}_j^\lambda(t)$ are uniform in j and λ. So, considering a sequence of $x_j^\lambda(\cdot)$ for $\lambda \to 0$, we can use the Arzelà-Ascoli theorem to conclude that the sequence converges uniformly to $x_j(\cdot)$ satisfying (4.1) with $u = u_j(t)$.

2. *Convergence to Young measure*: Now we address the convergence of sequence $x_j^\lambda(\cdot) \to x^\lambda(\cdot)$ as $j \to \infty$. Using the bounds in (4.11), we conclude that the sequences $x_j^\lambda(\cdot), \dot{x}_j^\lambda(\cdot)$ are uniformly bounded in j. So, using the Arzelà-Ascoli theorem, we conclude that x_j^λ converges uniformly (up to a subsequence) to some x^λ. To characterize the limit $x^\lambda(t)$, we use Proposition 4.2.1 to infer that the sequence $\{x_j^\lambda(\cdot), u_j(\cdot)\}$ generates a Young measure $\nu^\lambda(\cdot,\cdot|t) \in \mathcal{P}(\mathbb{R}^n \times U)$ such that

$$\dot{x}^\lambda(t) = \int_{\mathbb{R}^n\times U} \Big[-\frac{1}{\lambda}(\mathsf{x} - P_{S(t)}(\mathsf{x})) + f(\mathsf{x},\mathsf{u})\Big] \mathrm{d}\nu^\lambda(\mathsf{x},\mathsf{u}|t). \tag{4.13}$$

In order to show that the trajectories $x_j^\lambda(t)$ converge pointwise to $x^\lambda(t)$ at each time t, we consider

$$\begin{aligned}|x^\lambda(t) - x_j^\lambda(t)| = \Big| x_0 + \int_0^t \int_{\mathbb{R}^n\times U} \Big[f(\mathsf{x},\mathsf{u}) - \frac{1}{\lambda}(\mathsf{x} - P_{S(t)}(\mathsf{x}))\Big] \mathrm{d}\nu^\lambda(\mathsf{x},\mathsf{u}|s)\mathrm{d}s \\ - x_0 - \Big[\int_0^t f(x_j^\lambda(t), u_j(t)) - \frac{1}{\lambda}(x_j^\lambda(t) - P_{S(t)}(x_j^\lambda(t)))\Big]\Big|\end{aligned}$$

where we have used the fact that $x_j^\lambda(0) = x^\lambda(0) = x_0$. The pointwise convergence $\dot{x}_j^\lambda(t)$ to $\dot{x}^\lambda(t)$, implied by Proposition 4.2.1, and it being integrable allow us to use the dominated convergence theorem to conclude that $|x_j^\lambda(t) - x^\lambda(t)| \to 0$ pointwise in t as $j \to \infty$. Moreover, since we have pointwise convergence of the whole sequence and uniform convergence up to a subsequence, we can conclude that each subsequence converges uniformly to the same limit $x^\lambda(t)$. The Young measure ν_t^λ can thus be written as $\nu^\lambda(\mathrm{d}x, \mathrm{d}u|t) = \delta_{x^\lambda(t)}(\mathrm{d}x)\omega(\mathrm{d}u|t)$ [98, Theorem 6.2]. Consequently, the dynamics in (4.13) reduces to

$$\dot{x}^\lambda(t) = -\frac{1}{\lambda}(x^\lambda(t) - P_{S(t)}(x^\lambda(t))) + \int_U f(x^\lambda(t),\mathsf{u})\mathrm{d}\omega(\mathsf{u}|t). \tag{4.14}$$

3. *Uniform convergence of the relaxed dynamics*: Next, we consider the convergence of the sequence $x^\lambda(t) \to x(t)$ as $\lambda \to 0$. We note that the term $\int_U f(x^\lambda(t),\mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)$ in (4.14) is Lipschitz and it satisfies the linear growth condition. Thus, the dynamics in (4.14) satisfies bounds similar to (4.11) which ensure uniform convergence of $x^\lambda(t) \to x(t)$ using the Arzelà-Ascoli theorem. Using arguments similar to those in [65, Lemma 3.4],

[2] In Assumption 4.1.1, the set Ω can be chosen such that $x_j^\lambda(t) \in \Omega$ holds for small enough λ and all $t \in [0, T]$.

we conclude that the limit trajectory $x(\cdot)$ satisfies (4.9). Thus we have the following inequality

$$\|x_j(\cdot) - x(\cdot)\| \leq \left\|x_j(\cdot) - x_j^\lambda(\cdot)\right\| + \left\|x_j^\lambda(\cdot) - x^\lambda(\cdot)\right\| + \left\|x^\lambda(\cdot) - x(\cdot)\right\|$$

and we have already established uniform convergence for all the three terms on the right-hand side of the inequality. By applying the $\epsilon/3$ argument, we establish that $x_j(\cdot)$ converges uniformly to $x(\cdot)$, where $x(\cdot)$ is the solution of system (4.9). □

4.2.2 Relaxation with Young measures

In the previous section, the formalism of Young measures was introduced to describe the selection of optimal control inputs. The dynamical system (4.1) also contains a multi-valued mapping for which we need to consider a selection rule. Following [59, Section 3], we make a selection from $\mathcal{N}_{S(t)}(x)$ by choosing a probability measure $\eta(\cdot|t, x(t)) \in \mathcal{P}(\mathcal{N}_{S(t)}(x(t)))$. Then, (4.1) can be expressed as

$$\dot{x}(t) = -\int_{\mathcal{N}_{S(t)}(x(t))} \zeta \mathrm{d}\eta(\zeta|t, x(t)) + f(x(t), u(t)) =: g(x(t), u(t)) \quad \text{s.t. } x(0) = x_0. \tag{4.15}$$

Let $(x_j(\cdot), u_j(\cdot))_{j\in\mathbb{N}}$ be a minimizing sequence of state and controls of the problem defined in (4.2) such that $\lim_{j\to\infty} \int \mathcal{L}(x_j(t), u_j(t))\mathrm{d}t \to J^*(x_0)$. Under Assumption 4.1.3, the function $\mathcal{L}(\mathsf{x}, \mathsf{u})$ is upper bounded by $|\mathsf{u}|^2$ and the uniform boundedness of $u_j(\cdot)$ implies that $(\mathcal{L}(x_j(\cdot), u_j(\cdot)))_{j\in\mathbb{N}}$ is uniformly integrable. Thus using Proposition 4.2.1, we get

$$\lim_{j\to\infty} \int_0^T \mathcal{L}(x_j(t), u_j(t))\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n\times U} \mathcal{L}(\mathsf{x}, \mathsf{u})\mathrm{d}\nu_t(\mathsf{x}, \mathsf{u})\mathrm{d}t.$$

Following the results stated in Theorem 4.2.2 and [98, Theorem 6.2], ν_t can be written as $\nu_t = \delta_{x(t)}\omega_t$. Thus,

$$\lim_{j\to\infty} \int_0^T \mathcal{L}(x_j(t), u_j(t))\mathrm{d}t = \int_0^T \int_U \mathcal{L}(x(t), \mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}t.$$

Using this relaxation of the cost function and Theorem 4.2.2, the optimal control problem defined in (4.2) can be relaxed as follows

$$J_r^*(x_0) := \inf_{x(\cdot),\ \omega(\cdot|\cdot)} \int_0^T \int_U \mathcal{L}(x(t), \mathsf{u})\ \mathrm{d}\omega(\mathsf{u}|t)\ \mathrm{d}t \tag{4.16}$$

subject to,

$$\dot{x}(t) = \int_U f(x(t), \mathsf{u})\ \mathrm{d}\omega(\mathsf{u}|t) - \int_{\mathcal{N}_{S(t)}(x(t))} \zeta\ \mathrm{d}\eta(\zeta|t, x(t)), \tag{4.17}$$
$$x(0) = x_0, \quad \eta(\cdot|t, x(t)) \in \mathcal{P}(\mathcal{N}_{S(t)}(x(t))), \quad \omega(\cdot|t) \in \mathcal{P}(U),\ \forall t \in [0, T].$$

For this relaxed optimal control problem, our next result shows that this relaxation does not introduce any gap in the optimal value.

Theorem 4.2.3. *The optimal value $J^*(x_0)$ of optimal control problem* (4.2) *and the optimal value $J_r^*(x_0)$ of Young's measure relaxed optimal control problem* (4.16) *are equal.*

Proof. (Proof of $J_r^*(x_0) \leq J^*(x_0)$): We observe that if $(x(\cdot), u(\cdot))$ is feasible for (4.2), then it is feasible for (4.17). So if we consider an optimal solution $(x^*(\cdot), u^*(\cdot))$ of (4.2) then $(x^*(\cdot), \delta_{u^*(\cdot)})$ is feasible for (4.17), so

$$J^*(x_0) = \int_0^T \mathcal{L}(x^*(t), u^*(t))\mathrm{d}t = \int_0^T \mathcal{L}(x^*(t), \mathsf{u})\delta_{u^*(\cdot)}\mathrm{d}t \geq J_r^*(x_0)$$

where for the inequality we have used the fact that $(x^*(\cdot), \delta_{u^*(\cdot)})$ is a feasible solution to (4.16).

(Proof of $J^*(x_0) \leq J_r^*(x_0)$): Since we know that $\mathcal{L}(\cdot, \mathsf{u})$ is a convex function in u (based on Assumption 4.1.3), we can use Jensen's inequality to obtain

$$J_r^*(x_0) = \int_0^T \int_U \mathcal{L}(x(t), \mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}t \geq \int_0^T \mathcal{L}(x(t), u(t))\mathrm{d}t.$$

Now we invoke Assumption 4.1.3 which ensures that $g(x, U)$ is convex, so $\int g(x(t), \mathsf{u})d\omega(\mathsf{u}|t) \in g(x(t), U)$. There exists a measurable selection $u(t) \in U$ such that $\dot{x}(t) = g(x(t), u(t)) = \int g(x(t), \mathsf{u})d\omega(\mathsf{u}|t)$ [66, Theorem 8.1.3]. The pair $(x(\cdot), u(\cdot))$ thus obtained is feasible for (4.2). So, we get

$$J_r^*(x_0) = \int_0^T \int_U \mathcal{L}(x(t), \mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}t \geq \int_0^T \mathcal{L}(x(t), u(t))\mathrm{d}t \geq J^*(x_0). \qquad \square$$

The problem (4.16)–(4.17) above is still nonlinear in $x(\cdot)$, and in the next subsection, we use occupation measures which will lead us to a LP in the space of nonnegative Borel measures. To establish this, we make a passage from the Young measure to *occupation measures* by embedding the Young measure in the space of linear functionals.

4.2.3 Occupation measures

Each triplet $(x(t), \eta(\cdot|t, x(t)), \omega(\cdot|t))$ for $t \in [0, T]$ which satisfy (4.17) can be associated with a pair of measures $(\hat{\mu}, \xi) \in (\mathcal{M}(\mathfrak{B} \times U), \mathcal{M}(S(T)))$ where $\mathcal{M}(\mathfrak{B} \times U)$ denotes the space of Borel measures on $\mathfrak{B} \times U$ for

$$\mathfrak{B} := \{(t, x, \zeta) \mid t \in [0, T],\ x \in \Omega \subset \mathbb{R}^n,\ \zeta \in \mathcal{N}_{S(t)}(x)\}, \tag{4.18}$$

and $\Omega \supset \bigcup_{t \in [0,T]} S(t)$. Let the pair $(\hat{\mu}, \xi)$ satisfy the following relation

$$\langle \hat{\mu}, h \rangle + \langle \xi, g \rangle = \int_0^T \int_U h(t, x(t), \mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}t + g(T, x(T)) \tag{4.19}$$

for all $h \in \mathcal{C}([0,T]\times\Omega\times U)$ and $g \in \mathcal{C}(S(T))$. Let us define

$$\mathfrak{R} := \left\{ \begin{array}{l} (\hat{\mu},\xi) \in \mathcal{M}_+(\mathfrak{B}\times U)\times\mathcal{M}_+(S(T)) \Big| (4.19) \text{ holds for} \\ \text{some } \omega(\cdot|t)\in\mathcal{P}(U) \text{ and } x(\cdot), \\ \text{with } x(\cdot) \text{ satisfying (4.17) for the same } \omega(\cdot|t) \\ \text{for some } \eta(\cdot|t,x(t))\in\mathcal{P}(\mathcal{N}_S(x(t)))\ \forall t\in[0,T]. \end{array} \right\} \tag{4.20}$$

The set $\mathfrak{R}$ provides an embedding of the triplet $(x(t),\eta(\cdot|t,x(t)),\omega(\cdot|t))$ for $t\in[0,T]$ into the space of measures $\mathcal{M}(\mathfrak{B}\times U)\times\mathcal{M}(S(T))$. A canonical embedding is given by

$$\mathrm{d}\hat{\mu}(t,x,\zeta,u) = \mathrm{d}t\,\delta_{x(t)}(\mathrm{d}x)\,\nu(\mathrm{d}\zeta,\mathrm{d}u \mid t,x(t)), \quad \xi := \delta_{x(T)},$$

with $\nu(\cdot \mid t,x(t)) \in \mathcal{P}(\mathcal{N}_{S(t)}(x(t))\times U)$, whose conditional marginals satisfy $\pi_{\zeta\#}\nu = \eta$ and $\pi_{u\#}\nu = \omega$. Specifically, using (4.19), we derive the following bound on the norm of $\hat{\mu}$:

$$\|\hat{\mu}\| = \sup_{\|h\|_\infty\le 1}\Big|\int_0^T\int_U h(t,x(t),\mathsf{u})\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}t\Big| \le \int_0^T \|\omega\|_{TV}\,\mathrm{d}t \le T$$

where $\|\omega\|_{TV}$ is the total variation norm of $\omega(\cdot|t)\in\mathcal{P}(U)$ (see [99] for the definition of total variation norm). Similarly, ξ satisfies the following bound:

$$\|\xi\| = \sup_{\|g\|_\infty\le 1}|\langle\xi,g\rangle| = \sup_{\|g\|_\infty\le 1}|g(T,x(T))| \le 1$$

and thus any pair $(\hat{\mu},\xi)\in\mathfrak{R}$ satisfies:

$$\|\hat{\mu}\|\le T; \quad \|\xi\|\le 1; \quad \hat{\mu},\xi\ge 0. \tag{4.21}$$

Because the pair $(\hat{\mu},\xi)$ satisfy (4.9) and (4.19), and both relations depend nonlinearly on the state trajectory $x(\cdot)$, convex combinations of admissible triplets need not be admissible; hence the set $\mathfrak{R}$ in (4.20) is not convex.

We observe that for every triplet $(x(t),\eta(\cdot|t,x(t)),\omega(\cdot|t))$ satisfying Young measure relaxed dynamics (4.17), the associated measure $\mathrm{d}\mu(t,\mathsf{x},\zeta,\mathsf{u}) = \mathrm{d}t\delta_{x(t)}\mathrm{d}\omega(\mathsf{u}|t)\mathrm{d}\eta(\zeta|x(t))$ satisfies the following equation for every $\phi\in C^1([0,T]\times\Omega)$:

$$\int_{[0,T]\times S(t)}\int_{\mathcal{N}_{S(t)}(x)}\int_U \Big[\partial_t\phi(t,\mathsf{x})\mathrm{d}\mu(t,\mathsf{x},\zeta,\mathsf{u}) + \partial_x\phi(t,\mathsf{x})\cdot(-\zeta+f(\mathsf{x},\mathsf{u}))\mathrm{d}\mu(t,\mathsf{x},\zeta,\mathsf{u})\Big] = \int_{S(T)}\phi(T,\mathsf{x})\mathrm{d}\mu_T(\mathsf{x}) - \phi(0,x_0). \tag{4.22}$$

where $\mu\in\mathcal{M}(\mathfrak{B}\times U)$, $\mu_T\in\mathcal{M}(S(T))$. Equation (4.22) is called *continuity equation* or *Liouville equation* and the equation obtained is linear in (μ,μ_T).

Next, we characterize all possible solutions to (4.22). Fix a constant $C_\eta>0$ such that every admissible triplet $(x(t),\eta(\cdot \mid t,x(t)),\omega(\cdot \mid t))$ that satisfies the dynamics (4.17) also satisfies $\int_0^T\int_{\mathcal{N}_{S(t)}(x(t))\times U}|\zeta|\,\mathrm{d}\eta(\zeta,u|t,x(t))\,\mathrm{d}t\le C_\eta$ (the constant can be derived from the fact that $\dot{x}(\cdot)$ has uniform norm bound and the fact that we have $\int_U f(x(t),u)\mathrm{d}\omega(u|t)$ is bounded uniformly from Assumption 4.1.1). Consider the set of pairs of measures (μ, μ_T) such that

$$\mathfrak{O} := \Big\{(\mu, \mu_T) \in \mathcal{M}_+(\mathfrak{B} \times U) \times \mathcal{M}_+(S(T)) \mid \\ \mu, \mu_T \geq 0,\ \|\mu\| \leq T,\ \|\mu_T\| \leq 1, \\ (4.22) \text{ holds},\ \textstyle\int_{\mathfrak{B}\times U} |\zeta| \, \mathrm{d}\mu \leq C_\eta \Big\}. \quad (4.23)$$

Since the measure $\mathrm{d}\mu(t, x, \zeta, u) = \mathrm{d}t\delta_{x(t)}\mathrm{d}\nu(\zeta, u|t, x(t))$ with $\pi_{\zeta\#}\nu = \eta$ and $\pi_{u\#}\nu = \omega$, satisfies the continuity equation (4.22) and the uniform first order moment bound on η, this establishes the inclusion $\mathfrak{R} \subset \mathfrak{O}$. Subsequently, we show that the set $\mathfrak{O}$ is convex and weak-$\star$ compact. The convexity of set $\mathfrak{O}$ follows from the linearity of the governing equation in (μ, μ_T).

Lemma 4.2.4. *The set $\mathfrak{O}$ defined in (4.23) is weak-$\star$ sequentially compact.*

The proof can be found in Appendix 4.7.1.

4.2.4 Relaxation with occupation measures

In the following subsection, we present an occupation measure-based reformulation of the problem (4.2). We note that any occupation measure satisfying (4.22) can be disintegrated as $\mathrm{d}\mu(t, x, \zeta, u) = \mathrm{d}t\, \mathrm{d}\mu(x|t)\, \mathrm{d}\nu(\zeta, u|t, x)$ with $\nu \in \mathcal{P}(\mathcal{N}_{S(t)}(x) \times U)$ has conditional marginals $\pi_{\zeta\#}\nu = \eta$, $\pi_{u\#}\nu = \omega$. Using this occupation measure framework, the problem in (4.2) can be reformulated as follows:

$$J_o^*(x_0) = \inf_{\mu, \mu_T} \int_{\mathfrak{B}\times U} \mathcal{L}(\mathsf{x}, \mathsf{u})\, \mathrm{d}\mu(t, \mathsf{x}, \zeta, \mathsf{u}) \tag{4.24a}$$

subject to

$$\begin{aligned} \partial_t \mu(t, \mathsf{x}, \zeta, \mathsf{u}) + \nabla_x \cdot [(-\zeta + f(\mathsf{x}, \mathsf{u}))\mu(t, \mathsf{x}, \zeta, \mathsf{u})] = \delta_T \otimes \mu_T - \delta_0 \otimes \delta_{x_0}, \\ \mu \in \mathcal{M}_+(\mathfrak{B} \times U), \quad \mu_T \in \mathcal{M}_+(S(T)), \end{aligned} \tag{4.24b}$$

where $\nabla_x\cdot$ is the divergence operator, and the dynamics constraint in (4.24), has to be interpreted in the weak form, as in (4.22), i.e., when integrated against test functions $\phi \in \mathcal{C}^1([0, T] \times \Omega)$. Note that the search for the minimizer (μ, μ_T) in the above program is over the set $\mathfrak{O}$, defined in (4.23), which is weak-$\star$ compact. Further, the cost function is weak-$\star$ continuous (which can be checked using the fact that $\mathcal{L}(\cdot, \cdot)$ is continuous), so the existence of a minimizer follows from the direct method of calculus of variations. Before stating the main no-relaxation-gap theorem, we record a lemma used in the proof of Theorem 4.2.6.

Lemma 4.2.5. *Consider the sets $\mathfrak{R}$ and $\mathfrak{O}$ given in (4.20) and (4.23), respectively, and let $(\mu, \mu_T) \in \mathfrak{O}$ be a feasible occupation measure pair satisfying (4.22). Then there exists a probability measure $\theta \in \mathcal{P}(\mathfrak{R})$ such that*

$$\int_{\mathfrak{B}\times U} \mathcal{L}(x, u)\, \mathrm{d}\mu = \int_{\mathfrak{R}} \left[\int_{\mathfrak{B}\times U} \mathcal{L}(x, u)\, \mathrm{d}\hat{\mu}\right] \mathrm{d}\theta(\hat{\mu})$$

for every $\mathcal{L} : \mathbb{R}^n \times U \to \mathbb{R}$ satisfying Assumption 4.1.3.

The proof of this lemma is given in the Appendix 4.7.2.

Theorem 4.2.6. *For a fixed initial condition x_0, the optimal value $J_o^*(x_0)$ of occupation measure relaxed optimal control problem (4.24) and the optimal value $J_r^*(x_0)$ of Young's measure relaxed optimal control problem (4.16) are equal, i.e., there is no relaxation gap when relaxing to occupation measures.*

Proof. One direction of the inequality, namely $J_o^*(x_0) \leq J_r^*(x_0)$, holds trivially since the admissible set $\mathfrak{R}$ for (4.16) is a subset of $\mathfrak{O}$ which is an admissible set of (4.24).

Next, we show that there exists $\hat{\mu}$ such that the objective function value in both problems are equal. We consider an optimal pair $(\mu, \mu_T) \in \mathfrak{O}$ that solves (4.24). By applying the superposition principle as stated in Lemma 4.2.5, we can represent the objective value of (4.24) as

$$\int_{\mathfrak{B}\times U} \mathcal{L}(x,u)\,\mathrm{d}\mu = \int_{\mathfrak{R}} \left[\int_{\mathfrak{B}\times U} \mathcal{L}(x,u)\,\mathrm{d}\hat{\mu}\right] \mathrm{d}\theta(\hat{\mu}), \tag{4.25}$$

where $(\hat{\mu}, \delta_{x(T)}) \in \mathfrak{R}$ and $\theta \in \mathcal{P}(\mathfrak{R})$ is a probability measure over admissible relaxed trajectories.

If it were the case that $\int \mathcal{L}(x,u)\,\mathrm{d}\hat{\mu} > \int \mathcal{L}(x,u)\,\mathrm{d}\mu$ for all $\hat{\mu}$ in the support of θ, then the right-hand side of (4.25) would be strictly greater than the left-hand side, leading to a contradiction. Therefore, there must exist some $\hat{\mu} \in \mathfrak{R}$ such that

$$\int_{\mathfrak{B}\times U} \mathcal{L}(x,u)\,\mathrm{d}\hat{\mu} = \int_{\mathfrak{B}\times U} \mathcal{L}(x,u)\,\mathrm{d}\mu,$$

which implies that the optimal values of both problems are equal. □

The generalization of problem (4.24) to account for an initial measure other than a Dirac distribution concentrated at a point x_0 is straightforward. Specifically, when the initial condition is described by $\mu_0 \in \mathcal{P}(S(0))$, the objective and the dynamics in (4.24) are integrated with respect to μ_0. To generalize the solution of (4.24), which depends on a fixed initial condition x_0, we introduce average occupation measures by integrating over an initial distribution μ_0. Let us define the set

$$\mathfrak{C} := \{(t, \zeta, x, u) \mid t \in [0,T], x \in S(t), \zeta \in \mathcal{N}_{S(t)}(x), u \in U\}.$$

We then define the average occupation measure $\bar{\mu} \in \mathcal{P}(\mathfrak{C})$ as

$$\bar{\mu}(A) = \int_{S(0)} \mu(A \mid x_0)\,\mathrm{d}\mu_0(x_0), \tag{4.26}$$

for all measurable sets $A \subseteq \mathfrak{C}$, where $\mu(\cdot \mid x_0) \in \mathcal{P}(\mathfrak{C})$ is the occupation measure associated with the initial condition x_0. Similarly, the averaged terminal measure $\bar{\mu}_T$ is defined as

$$\bar{\mu}_T(B) = \int_{S(0)} \mu_T(B \mid x_0)\, \mathrm{d}\mu_0(x_0), \tag{4.27}$$

for all measurable sets $B \subseteq S(T)$.

Since (4.24) corresponds to the special case of a Dirac initial condition $\mu_0 = \delta_{x_0}$, we will use the same notation, μ and μ_T, to denote the occupation and terminal measures averaged over the initial distribution μ_0. With this notation, the optimal control problem of a nonsmooth dynamical system becomes the following LP:

$$\begin{aligned} J_o^*[\mu_0] \quad &= \quad \inf_{\mu,\, \mu_T} \int \mathcal{L}(\mathsf{x}, \mathsf{u})\, \mathrm{d}\mu(t, \mathsf{x}, \zeta, \mathsf{u}) \\ \text{s.t.} \quad & \partial_t \mu(t, \mathsf{x}, \zeta, \mathsf{u}) + \nabla_x \cdot [(\zeta + f(\mathsf{x}, \mathsf{u}))\mu(t, \mathsf{x}, \zeta, \mathsf{u})] \\ &\qquad = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 \\ & \mu \in \mathcal{M}_+(\mathfrak{B} \times U), \quad \mu_T \in \mathcal{M}_+(S(T)) \end{aligned} \tag{4.28}$$

where we recall $\mathfrak{B}$ from (4.18).

The dual program to (4.28) is described as follows:

$$\begin{aligned} D^*[\mu_0] = \quad & \sup_v \int_{S(0)} v(0, x) \mathrm{d}\mu_0(x) \\ \text{s.t. } & \mathcal{L} + \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(-\zeta + f)_i \in \mathcal{C}_+(\mathfrak{B} \times U) \\ & v(T, \cdot) \in \mathcal{C}_+(S(T)). \end{aligned} \tag{4.29}$$

Using the arguments in [100, Theorem 7.2, Chapter 4], it can be shown that strong duality holds between (4.28) and (4.29), that is, $J_o^*[\mu_0] = D^*[\mu_0]$.

4.3 Relaxed discrete-time optimal control

In this section, we relax the discrete-time optimal control problem (4.5) with dynamics (4.4) to the space of measures. We begin with the case where the constraint set varies with time. For this setting, we formulate an equivalent LP over measures, given in (4.38), that captures the evolution of the state distribution across time steps. We then specialize our construction to the static-set case. Under the assumption that S is time-independent, we derive in Lemma 4.3.1 a discrete-time counterpart of the continuity equation in (4.43). Theorem 4.3.2 then establishes that for any solution of (4.43), there exists a stochastic kernel whose propagation through (4.4) generates the corresponding sequence of state distributions. Together, these results enable a reformulation of the optimal control problem with a static constraint set, see (4.45), as a LP in the space of measures in which the only unknowns are the occupation measure μ^τ and the terminal measure μ_T^τ (with $\tau > 0$ the time step), in contrast to (4.38), which introduces a distinct measure variable for each time instant.

We consider the uniform partition of the time interval $[0, T]$

$$\{0 = t_0, t_1, \ldots, t_N = T\}, \quad t_k = k\tau, \quad N = T/\tau \in \mathbb{N}. \tag{4.30}$$

We know from (4.4) that the states at successive time instants $k\tau$ and $(k+1)\tau$ are related as

$$x_{k+1}^{\tau} = P_{S_{k+1}} \circ (\tau f(x_k^{\tau}, u_k^{\tau}) + x_k) := G_k^{\tau}(x_k, u_k) \tag{4.31}$$

where P_{S_k} is as before the projection mapping on the compact convex set $S_k := S(t_k)$. To relax the problem in (4.5) to the space of measures we will rely on probabilistic arguments since we are modeling the initial condition and the control as random variables[93]. For a fixed value of τ, we define the measures $\{\tilde{\mu}_k^{\tau}\}_{k\in\mathbb{N}}$, such that $\tilde{\mu}_k^{\tau} \in \mathcal{P}(S_k)$, at time instants t_k which are a result of recursive evolution of measures through dynamics (4.31). We introduce a time-varying stochastic kernel, also called Markov control policy, $\omega_k^{\tau}(u|x) \in \mathcal{P}(U)$, which is the probability measure on controls u at time instant k conditioned over the state x. Using this stochastic kernel we define a transition kernel which is the probability measure of states at time $k\tau$ given the state was at x_{k-1} at time $(k-1)\tau$:

$$Q_{\omega}^{\tau}(A|x_{k-1}, k-1) \triangleq \int_U \mathbb{I}_A(G_{k-1}^{\tau}(x_{k-1}, u))\mathrm{d}\omega_{k-1}^{\tau}(u|x_{k-1}) \tag{4.32}$$

where, $\mathbb{I}_A$ denotes the indicator function on set $A \subset S_k$. Kernel $Q_{\omega}^{\tau}(\cdot|\cdot,\cdot)$ captures the effect of feedback $\omega_k^{\tau}(\cdot|\cdot)$ at each time step. Now, given $\tilde{\mu}_0^{\tau}$ is the measure at time 0, we can compute the measure at the next time step, $\tilde{\mu}_1^{\tau}$, as its successor. This is given by:

$$\tilde{\mu}_1^{\tau}(A) = \int_{S_0}\int_U \mathbb{I}_A(G_0^{\tau}(x,u))\mathrm{d}\omega_0^{\tau}(u|x)\mathrm{d}\tilde{\mu}_0^{\tau}(x) \tag{4.33}$$

where $A \subset S_1$. We can recursively compute the measures at time k using the following equation:

$$\tilde{\mu}_k^{\tau}(A) = \int_{S_{k-1}}\int_U \mathbb{I}_A(G_{k-1}^{\tau}(x,u))\mathrm{d}\omega_{k-1}^{\tau}(u|x)\mathrm{d}\tilde{\mu}_{k-1}^{\tau}(x). \tag{4.34}$$

Next, we identify a measure $\mu_k^{\tau} \in \mathcal{P}(S_k, U)$

$$\mu_k^{\tau}(\mathrm{d}x, \mathrm{d}u) := \omega_k^{\tau}(\mathrm{d}u|x)\tilde{\mu}_k^{\tau}(\mathrm{d}x) \tag{4.35}$$

at each time instant. We also note that $\tilde{\mu}_k^{\tau}$ is the push-forward of μ_{k-1}^{τ} through G_k^{τ}. Therefore, the following relationships hold:

$$x_k^{\tau} \xleftarrow{G_{k-1}^{\tau}} (x_{k-1}^{\tau}, u_{k-1}^{\tau}), \tag{4.36}$$

$$\tilde{\mu}_k^{\tau}(\mathrm{d}x) \xleftarrow{G_{k-1\#}^{\tau}} \mu_{k-1}^{\tau}(\mathrm{d}x, \mathrm{d}u). \tag{4.37}$$

Let π_x be a projection mapping such that $\pi_x : \mathbb{R}^n \times U \to \mathbb{R}^n$. Using (4.37), we have the following LP in the space of Borel measures for (4.5):

$$J^{*\tau}[\mu_0] = \inf_{\mu_k^\tau} \sum_{k=1}^{N} \tau \int_{S_k \times U} \mathcal{L}(x,u) \mathrm{d}\mu_k^\tau(x,u) \tag{4.38}$$

$$\text{s.t. } \pi_{x\#}\mu_k^\tau = \Big[P_{S_k} \circ (\tau f(\cdot,\cdot) + \pi_x)\Big]_{\#} \mu_{k-1}^\tau, \tag{4.39}$$

$$\mu_k^\tau \in \mathcal{P}(S_k \times U), \quad \forall\, k = 1,..,N.$$

The dual problem to the above can be derived as follows:

$$\begin{aligned} J_d^{*\tau}[\mu_0] &= \max_{V_i(\cdot)} \int_{S_0} V_0(x) \mathrm{d}\mu_0(x) \\ \text{s.t.} \quad & V_i \leq \tau\mathcal{L} + V_{i+1}(G_k^\tau(x,u)) \quad \text{for } i = 0,.,.,N-1, \\ & V_N = 0 \quad \text{on } S_N \end{aligned} \tag{4.40}$$

where, $V_i \in \mathcal{C}_b(\mathbb{R}^n)$. Following the techniques used in [100, Theorem 7.2, Chapter 4], it can be shown that, for a given $\mu_0 \in \mathcal{P}(S_0)$, there is no duality gap between (4.38) and (4.40), i.e., $J_d^{*\tau}[\mu_0] = J^{*\tau}[\mu_0]$.
Next, we introduce a discrete-time continuity equation, which has a similar structure to (4.22), but applies when the set is static, i.e., $S(t) = S$. In this case, the update rule becomes

$$G^\tau(x_k,u_k) := P_S \circ (\tau f(x_k,u_k) + x_k). \tag{4.41}$$

We will later highlight that considering this specific case separately can be numerically advantageous, especially when the set defining the normal cone constraint remains static. For the following discussion, we define discrete-time occupation measure $\mu^\tau \in \mathcal{M}_+(S \times U)$ satisfying the following relationship

$$\int_{A\times B} \phi(x,u) \mathrm{d}\mu^\tau(x,u) := \sum_{k=0}^{N-1} \int_{A\times B} \phi(x,u) \mathrm{d}\mu_k^\tau(x,u) \tag{4.42}$$

for all $\phi(x,u)$ bounded measurable function and $A \subset S,\ B \subset U$. This measure captures the time spent by all possible trajectories in some subset of state space and admissible control values.

Lemma 4.3.1. *Let $\omega_k^\tau(\cdot|\cdot) \in \mathcal{P}(U)$ be a stochastic kernel and $\mu_k^\tau,\ \mu^\tau$ be as defined in (4.35), (4.42). Let $\tilde{\mu}_0^\tau,\ \tilde{\mu}_T^\tau$ be such that $x_0 \sim \tilde{\mu}_0^\tau$ and $x_N \sim \tilde{\mu}_T^\tau$ respectively. Then, $\mu^\tau,\ \tilde{\mu}_0^\tau,\ \tilde{\mu}_T^\tau$ satisfy the discrete-time continuity equation,*

$$\begin{aligned} &\int_{S\times U} v(x) \mathrm{d}\mu^\tau(x,u) + \int_S v(x) \mathrm{d}\tilde{\mu}_T^\tau(x) \\ &\qquad = \int_{S\times U} v(G^\tau(x,u)) \mathrm{d}\mu^\tau(x,u) + \int_S v(x) \mathrm{d}\tilde{\mu}_0^\tau(x) \end{aligned} \tag{4.43}$$

for all $v(\cdot) : \mathbb{R}^n \to \mathbb{R}$ bounded, Borel-measurable functions and G^τ defined in (4.41).

Proof. Let $v(x)$ be some bounded measurable function, then using (4.42) and (4.35),

$$\int_{S\times U} v(x)\mathrm{d}\mu^\tau(x,u) = \sum_{k=0}^{N-1}\int_{S\times U} v(x)\mathrm{d}\mu_k^\tau(x,u)$$
$$= \sum_{k=0}^{N-1}\int_{S\times U} v(x)\mathrm{d}\omega_k^\tau(u|x)\mathrm{d}\tilde{\mu}_k^\tau(x) = \sum_{k=0}^{N-1}\int_S v(x)\mathrm{d}\tilde{\mu}_k^\tau(x)$$
$$= \sum_{k=0}^{N-1}\int_S v(x)\,\mathrm{d}G^\tau_\# \mu_{k-1}^\tau(x,u)$$

where in the third and fourth equality we have used the fact that $\omega_k^\tau(\cdot|x) \in \mathcal{P}(U)$ and (4.37) respectively. Now using the change of variables formula for push-forward measures we get,

$$\int_{S\times U} v(x)\mathrm{d}\mu^\tau(x,u) = \sum_{k=0}^{N-1}\int_{S\times U} v(G^\tau(x,u))\mathrm{d}\mu_k^\tau(x,u)$$
$$+\int_S v(x)\mathrm{d}\tilde{\mu}_0^\tau(x) - \int_S v(x)\mathrm{d}\tilde{\mu}_T^\tau(x). \quad \square$$

Next, we prove that for any solution of (4.43), we have a stochastic kernel such that if we propagate this stochastic kernel through (4.31), we obtain a sequence of measures defining the probability of state at each time instant.

Theorem 4.3.2. *Let $\tilde{\mu}_0^\tau, \tilde{\mu}_T^\tau, \mu^\tau$ be the measures which satisfy discrete-time continuity equation* (4.43)*. Then there exists a stochastic kernel $\omega_k^\tau(\cdot|x) \in \mathcal{P}(U)$ at each time $k\tau$ which defines the evolution of measure $\tilde{\mu}_0^\tau$ through* (4.34) *to $\tilde{\mu}_T^\tau \in \mathcal{P}(S)$ and $\mu^\tau \in \mathcal{M}_+(S\times U)$ is the corresponding discrete-time occupation measure satisfying* (4.42)*.*

Proof. Let us consider nonnegative measurable functions $p_k \in L^1(\mu^\tau)$ such that

$$\sum_{k=0}^{N-1} p_k(x,u) = 1 \quad (\mu^\tau\text{-a.s.}), \quad \int_{S\times U} p_k(x,u)\mathrm{d}\mu^\tau(x,u) = 1, \quad \forall\ k = 0,\dots,N-1.$$

Define $\mu_k^\tau := p_k\mu^\tau$. Then, for any measurable $A\subset S$ and $B\subset U$,

$$\int_{S\times U} \mathbb{I}_{A\times B}(x,u)\,\mathrm{d}\mu^\tau(x,u) =$$
$$\int_{S\times U} \mathbb{I}_{A\times B}(x,u)\sum_{k=0}^{N-1} p_k(x,u)\,\mathrm{d}\mu^\tau(x,u) =$$
$$\sum_{k=0}^{N-1}\int_{S\times U} \mathbb{I}_{A\times B}(x,u)\mathrm{d}\mu_k^\tau(x,u)$$

where in the last equality we have used the definition of μ_k^τ. The set of such $\{p_k\}_k$ is nonempty as $p_k = \frac{1}{N}$ for $k = 0,\dots,N-1$ is a feasible point. Using [88, Corollary 10.4.13], each μ_k^τ can be disintegrated as $\mu_k^\tau(\mathrm{d}x,\mathrm{d}u) = \omega_k^\tau(\mathrm{d}u|x)\tilde{\mu}_k^\tau(\mathrm{d}x)$, then (4.43) becomes

$$\int_{S\times U} \mathbb{I}_A(x)\mathrm{d}\mu^\tau(x,u) + \tilde{\mu}_T^\tau(A) = \sum_{k=0}^{N-1} \int_{S\times U} \mathbb{I}_A(G^\tau(x,u))\mathrm{d}\omega_k^\tau(u|x)\mathrm{d}\tilde{\mu}_k^\tau(x) + \tilde{\mu}_0^\tau(A). \quad (4.44)$$

Using ω_k^τ as the time-varying stochastic kernel we can define $Q_\omega^\tau(A|x_k,k)$ using (4.32). Thus (4.44) can be equivalently written as,

$$\mu^\tau(A,U) + \tilde{\mu}_T^\tau(A) = \sum_{k=0}^{N-1} \int_S Q_\omega^\tau(A|x_k,k)\mathrm{d}\tilde{\mu}_k^\tau(x_k) + \tilde{\mu}_0^\tau(A).$$

So, $\mu^\tau(A,U) = \sum_{k=0}^{N} \tilde{\mu}_k^\tau(A) - \tilde{\mu}_T^\tau(A) = \sum_{k=0}^{N-1} \tilde{\mu}_k^\tau(A)$. □

Using Lemma 4.3.1 and Theorem 4.3.2, we define the following LP in the cone of nonnegative Borel measures,

$$\begin{aligned} J^{*\tau}[\mu_0] = \inf_{\mu^\tau,\tilde{\mu}_T^\tau} & \int \mathcal{L}(x,u)\mathrm{d}\mu^\tau(x,u) \\ \text{s.t. } & \pi_{x\#}\mu^\tau + \tilde{\mu}_T^\tau = \Big[P_S \circ (\tau f(\cdot,\cdot) + \pi_x)\Big]_\#\mu^\tau + \tilde{\mu}_0^\tau, \\ & \mu^\tau \in \mathcal{P}(S\times U), \quad \tilde{\mu}_T^\tau \in \mathcal{P}(S). \end{aligned} \quad (4.45)$$

Here, the dynamics constraint in (4.45) is interpreted in weak sense, as in (4.43). As we observe in the above program, we only solve for μ^τ and $\tilde{\mu}_T^\tau$, which are the aggregated occupation measure and the terminal measure respectively. The derivation in the proof of Lemma 4.3.1 requires the push-forward map G^τ to be time-invariant. For the case with time-varying set as (4.5), we cannot derive an equation analogous to (4.43) rather we directly use the relations defined in (4.36) to obtain (4.39).

4.4 Convergence of the discrete-time optimal control problem to the continuous time problem

Having defined the relaxation for the continuous-time problem in (4.28) and the discrete-time problem in (4.38), in this section, we study the convergence $J^{*\tau} \xrightarrow{\tau\to 0} J_o^*$, where $J^{*\tau}$ and J_o^* are defined in (4.38) and (4.28) respectively, and τ is the time step. There are three main ingredients of this convergence problem: (1) Construction of suitable interpolations for the admissible solutions to (4.38), (2) Limiting behavior of the interpolated curves as $\tau \to 0$, and (3) Convergence of the optimal value $J^{*\tau}$ of (4.38) towards the optimal value J_o^* of (4.28) as $\tau \to 0$.

We refer to Section 2.3 for the basic notions of the 2-Wasserstein distance W_2, the space $\mathcal{P}_2(\mathbb{R}^n)$ of probability measures with finite second moment, and absolutely continuous curves in $(\mathcal{P}_2(\mathbb{R}^n), W_2)$, as well as other standard notions from optimal transport used throughout this chapter.

4.4.1 Interpolated curves

Consider the partition of the time interval $[0,T]$ introduced in (4.30). Recall from (4.37) and (4.34) that $\tilde{\mu}_k^\tau$ is the image measure of μ_{k-1}^τ under the push-forward map $P_{S_k} \circ \Big(\tau f(\cdot,\cdot) + \mathrm{id}_x\Big)$. Equivalently, for every Borel set $A \subset \mathbb{R}^n$,

$$\tilde{\mu}_k^\tau(A) = \int_{A,U} \Big[P_{S_k} \circ \Big(\tau f(y,u) + \pi_x\Big)\Big]_\# \, \mathrm{d}\mu_{k-1}^\tau(y,u).$$

Using the disintegration of μ_{k-1}^τ from (4.35), we obtain:

$$\tilde{\mu}_k^\tau = \Big[P_{S_k} \circ \Big(\tau \int_U f(\cdot,u)\,\mathrm{d}\widetilde{w}(u \mid k\tau,\cdot) + \pi_x(\cdot)\Big)\Big]_\# \tilde{\mu}_{k-1}^\tau.$$

In the following, we denote:

$$G_k^\tau(x) := P_{S_k}\Big(\tau \int_U f(x,u)\,\mathrm{d}\widetilde{w}(u \mid k\tau, x) + x\Big).$$

Thus, at each time $k\tau$ for $k \in \{0,1,..,\lceil \frac{T}{\tau} \rceil\}$, we have:

$$\tilde{\mu}_{k+1}^\tau = G_{k\#}^\tau \tilde{\mu}_k^\tau, \tag{4.46}$$

$$v_{k+1}^\tau = \frac{G_k^\tau(x) - x}{\tau}. \tag{4.47}$$

Given the discrete sequence $\{\mu_k^\tau\}_{k=0}^{\lceil T/\tau \rceil}$, we associate two interpolations that will be crucial in the convergence analysis.

(1) McCann (constant speed) interpolation: Fix $k \in \{0,\dots,\lceil T/\tau \rceil - 1\}$ and let

$$\theta \in \Pi_{\mathrm{opt}}(\mu_k^\tau, \mu_{k+1}^\tau)$$

be an optimal transport plan for the quadratic cost. For $t \in (k\tau,(k+1)\tau]$, set

$$s := \frac{t - k\tau}{\tau} \in (0,1].$$

We define the constant-speed (McCann) interpolation by

$$\tilde{\mu}_t^\tau := \big((1-s)\pi_1 + s\pi_2\big)_\# \theta, \qquad t \in (k\tau,(k+1)\tau], \tag{4.48}$$

where $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

One important characterization of this interpolated curve [83, Chapter 5] is that its metric derivative is constant on each interval, namely

$$|(\tilde{\mu}^\tau)'|(t) = \frac{W_2(\mu_k^\tau, \mu_{k+1}^\tau)}{\tau} \quad \text{for a.e. } t \in (k\tau,(k+1)\tau]. \tag{4.49}$$

Moreover, define the interpolated momentum $\tilde{E}_t^\tau$ by duality: for every test field $\varphi \in$

$\mathcal{C}_c(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi(z)\cdot d\tilde{E}_t^\tau(z) := \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi((1-s)x+sy)\cdot \frac{y-x}{\tau}\, d\theta(x,y).$$

Then $\tilde{E}_t^\tau \ll \tilde{\mu}_t^\tau$ and we set $\tilde{v}_t^\tau := \frac{d\tilde{E}_t^\tau}{d\tilde{\mu}_t^\tau} \in L^2(\tilde{\mu}_t^\tau;\mathbb{R}^n)$ so that $\tilde{E}_t^\tau = \tilde{v}_t^\tau\,\tilde{\mu}_t^\tau$. With this choice, the pair $(\tilde{\mu}_t^\tau, \tilde{E}_t^\tau)$ satisfies the continuity equation in the distributional sense

$$\partial_t \tilde{\mu}_t^\tau + \nabla\cdot(\tilde{E}_t^\tau) = 0,$$

and one can choose $\tilde{v}_t^\tau$ (e.g. the barycentric projection associated with θ) such that

$$\|\tilde{v}_t^\tau\|_{L^2(\tilde{\mu}_t^\tau)} = \frac{W_2(\mu_k^\tau,\mu_{k+1}^\tau)}{\tau} \quad \text{for a.e. } t\in(k\tau,(k+1)\tau].$$

(2) Piecewise constant interpolation curves such that

$$\hat{\mu}_t^\tau = \tilde{\mu}_{k+1}^\tau, \qquad \text{for } t\in(k\tau,(k+1)\tau] \tag{4.50}$$

and velocity $\hat{v}_t^\tau = v_{k+1}^\tau \quad \text{for } t\in(k\tau,(k+1)\tau]$.
We also define the corresponding momentum vector $\hat{E}_t := \hat{v}_t^\tau\,\hat{\mu}_t$.

In what follows, we will use the following notation

$$g^\tau(k\tau,x) = \int_U f(x,u)\ \mathrm{d}\omega^\tau(u|k\tau,x). \tag{4.51}$$

Remark 4.4.1. Two interpolations are used for distinct but complementary purposes. The convergence of the curve μ_t^τ ensures convergence to the limit μ_t, which solves the continuity equation. On the other hand, the convergence of velocities of the piecewise constant interpolated curve $\hat{\mu}_t^\tau$ ensures that the limiting velocity field of the continuity equation satisfies

$$v_t(x) \in -\mathcal{N}_{S(t)}(x) + \int_U f(x,u)\,\mathrm{d}\omega(u\mid t,x).$$

4.4.2 Convergence of dynamics constraint

Next, we will show that for $t\in[0,T]$, the interpolated trajectories μ_t^τ and $\hat{\mu}_t^\tau$ converge to μ_t which is the absolutely continuous solution to the continuity equation

$$\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0 \tag{4.52}$$

with vector field $v_t:\mathbb{R}^n\to\mathbb{R}^n$ defined in (4.9).

Theorem 4.4.2. *For system* (4.1) *and measure* $\mu_0\in\mathcal{P}(S(0))$, *suppose that Assumptions 4.1.1 and 4.1.3 hold. For* $t\in[0,T]$, *the curves* $\{\tilde{\mu}_t^\tau\}_t$, $\{\hat{\mu}_t^\tau\}_t$ *defined in* (4.48) *and* (4.50), *respectively, converge uniformly, with respect to the* W_2 *metric, to* $\{\mu_t\}_t\in\mathcal{P}(S(t))$ *which satisfies* (4.52). *The momentum vectors* $\tilde{E}_t^\tau = v_t^\tau\tilde{\mu}_t^\tau$ *and* $\hat{E}_t^\tau = \hat{v}_t^\tau\hat{\mu}_t^\tau$ *also weak-*$\star$ *converge to* $E_t = v_t\mu_t$ *where,* $v_t(x)\in -\mathcal{N}_{S(t)}(x) + \int_U f(x,u)\mathrm{d}\omega(u|t,x)$.

Proof. The proof is divided in four parts: (1) Proof of convergence of $\tilde{\mu}_t^\tau \to \mu_t$ and $\hat{\mu}_t^\tau \to \mu_t$; (2) Proof of convergence of $\tilde{E}_t^\tau$ and $\hat{E}_t^\tau$ to E_t; (3) Proof of absolute continuity of E_t w.r.t. μ_t such that $E_t = v_t\mu_t$; (4) Proof of convergence of $\hat{v}_t^\tau$ to v_t such that $v_t \in -\mathcal{N}_{S(t)}(x) + \int_U f(x,u)\mathrm{d}\omega(u|t,x)$.

In order to prove the first part, we need the following lemma; its proof can be found in [59, Lemma 2]. Let $C_{\max} := \sup_\Omega \|x_k\|$. By [29, Section 5], $C_{\max} < \infty$; it depends on the system data and the initial state x_0, and is independent of time instants k.

Lemma 4.4.3. *Consider the definitions of $\tilde{\mu}_k^\tau$ and $\tilde{\mu}_{k+1}^\tau$ as given in* (4.46)*. Then, the following inequality holds:*

$$W_2(\tilde{\mu}_k^\tau, \tilde{\mu}_{k+1}^\tau) \leq \tau(L_f C_{\max} + L_s) \tag{4.53}$$

where L_f is defined in Assumption 4.1.1, and L_s is defined in Assumption 4.1.2.

(1) *Proof of convergence of $\tilde{\mu}_t^\tau \to \mu_t$ and $\hat{\mu}_t^\tau \to \mu_t$*: Using Lemma 4.4.3, we can derive the following bound (see [59] for details),

$$W_2(\tilde{\mu}_t^\tau, \tilde{\mu}_s^\tau) \leq (t-s)^{1/2}(L_f C_{\max} + L_s)T^{1/2} \quad \forall s,t \in [0,T]. \tag{4.54}$$

The same estimate holds for the curve $\hat{\mu}_t$; specifically; $W_2(\hat{\mu}_t^\tau, \hat{\mu}_s^\tau)$ has the same bound as (4.54). Thus, the curves $\hat{\mu}_t^\tau$, and $\tilde{\mu}_t^\tau$ are $\frac{1}{2}-$Hölder continuous. Using the Arzelà-Ascoli theorem, we conclude that there exists a subsequence $\{\tau_j\}_j$ for which $\tilde{\mu}_t^{\tau_j} \to \tilde{\mu}_t$ uniformly in W_2 space and the limit curve $\tilde{\mu}_t$ is absolutely continuous. Same conclusion holds for curves $\hat{\mu}_t^\tau$ i.e., $\hat{\mu}_t^{\tau_j} \to \hat{\mu}_t$.

Moreover, observe that the curve $\hat{\mu}_t^\tau$ coincides with $\tilde{\mu}_t^\tau$ at times $k\tau$ and is constant on the interval $(k\tau,(k+1)\tau]$. Thus, we conclude from (4.54) that $W_2(\hat{\mu}_t^\tau, \tilde{\mu}_t^\tau) \leq \tau^{1/2}(L_f C_{\max} + L_s)T^{1/2}$. So, both curves converge to the same limit curve as $\tau \to 0$, which we denote by μ_t.

(2) *Convergence of $\tilde{E}_t^\tau$ to E_t*: We define $\tilde{m}^\tau \in \mathcal{M}^n([0,T]\times\mathbb{R}^n)$ as

$$\tilde{m}^\tau = \tilde{v}_t^\tau \tilde{\mu}_t^\tau \mathrm{d}t \tag{4.55}$$

which is a vector-valued measure. [3]

Lemma 4.4.4. *An estimate on the norm of $\tilde{m}^\tau$ defined in* (4.55) *is as follows:*

$$|\tilde{m}^\tau|([0,T]\times\Omega) \leq T^{\frac{3}{2}}(L_f C_{max} + L_s). \tag{4.56}$$

where $C_{\max}$ represents the uniform bound on $|x_k|$, L_f is defined in Assumption 4.1.1, and L_s is defined in Assumption 4.1.2.

The proof of the estimate (4.56) can again be found in [59, Lemma 3]. Using the same proof, one can obtain the same bounds on $|\hat{m}^\tau|$. The sequences $\tilde{m}^\tau$ and $\hat{m}^\tau$ are uniformly

[3]The space of vector-valued measures $\mathcal{M}^n(\Omega)$ is dual to $\mathcal{C}_0(\Omega;\mathbb{R}^n)$ and is endowed with weak-$\star$ convergence with respect to this duality.

bounded and therefore relatively compact in the weak-$\star$ topology. As a result, we have (up to a subsequence) $\tilde{m}^\tau \overset{*}{\rightharpoonup} \tilde{m}$, $\hat{m}^\tau \overset{*}{\rightharpoonup} \hat{m}$, $\tilde{E}_t^\tau \overset{*}{\rightharpoonup} \tilde{E}_t$ and $\hat{E}_t^\tau \overset{*}{\rightharpoonup} \hat{E}_t$. Finally, by applying [83, Lemma 8.9], we conclude that $\tilde{m} = \hat{m} =: m$ and thus $\hat{E}_t = \tilde{E}_t =: E_t$.

(3) Absolute continuity of E_t w.r.t. μ_t: To prove the absolute continuity of E_t we use the properties of Benamou-Brenier functional $\mathcal{B}(\mu \mathrm{d}t, E\mathrm{d}t)$ (for more details refer [83, Chapter 5]). A key property of $\mathcal{B}(\tilde{\mu}_t^\tau \mathrm{d}t, \tilde{E}_t^\tau \mathrm{d}t)$ which is of importance here is the following: $\tilde{E}_t^\tau \ll \tilde{\mu}_t^\tau$ such that $\tilde{E}_t^\tau = \tilde{v}_t^\tau \tilde{\mu}_t^\tau$, then $\mathcal{B}(\tilde{\mu}_t^\tau \mathrm{d}t, \tilde{E}_t^\tau \mathrm{d}t) = \int_0^T \int_\Omega |\tilde{v}_t^\tau|^2 \mathrm{d}\tilde{\mu}_t^\tau \mathrm{d}t$. Now the uniform bound on $|m^\tau|$ also implies an uniform bound on $\int_0^T \int_\Omega |v_t^\tau|^2 \mathrm{d}\tilde{\mu}_t^\tau dt)$ from (4.56). Moreover, nonnegativity, convexity and lower semi-continuity of $\mathcal{B}(\cdot,\cdot)$ [83, Proposition 5.18] imply

$$\mathcal{B}(\mu_t \mathrm{d}t, \mathrm{d}m) \leq \liminf_{k\to\infty} \mathcal{B}(\tilde{\mu}_t^\tau \mathrm{d}t, \mathrm{d}\tilde{m}^\tau) < \infty,$$

where μ_t and m are the weak-$\star$ limit of $\tilde{\mu}_t$ and $\tilde{m}^\tau$ as $\tau \to 0$. The finiteness of $\mathcal{B}(\mu_t \mathrm{d}t, m)$ implies that $E_t \ll \mu_t$ and $E_t = v_t\, \mu_t$ with $v_t \in L^2(\mu_t)$. Applying the same argument to the sequence $(\hat{\mu}_t^\tau, \hat{E}_t^\tau)$ gives $\hat{E}_t = \hat{v}_t\, \hat{\mu}_t$. From the previous step we already have $\hat{\mu}_t = \mu_t$ and $\hat{E}_t = E_t$; hence $\hat{v}_t = v_t$ μ_t-a.e.

(4) Proof that $v_t(x) \in -\mathcal{N}_{S(t)}(x) + \int_U f(x,u)\mathrm{d}\omega(u|t,x)$: To prove this statement, we consider the sequence of curves $\hat{v}_t^\tau$. Using (4.47) for $g^\tau(k\tau, x)$ (defined in (4.51)) and for every $x \in S_k$, we have

$$\left\langle y - P_{S_{k+1}}(\tau g^\tau(k\tau,x) + x)\,,\, \frac{(P_{S_{k+1}} \circ (\tau g^\tau(k\tau,\cdot) + \mathrm{id})x - x)}{\tau} \right\rangle \geq 0$$

which follows from the construction of the discrete-time velocities $\hat{v}_k^\tau$. Then

$$\langle y - P_{S_{k+1}} \circ (\tau g^\tau(k\tau,\cdot) + \mathrm{id})x)\,,\, v_{k+1}^\tau(x) - g^\tau(k\tau,x)\rangle \geq 0,$$

where the brackets denote the inner product between vectors in $\mathbb{R}^n$. This is the condition for $v_{k+1}^\tau(x) - g^\tau(k\tau,x)$ to be in the normal cone to set S_{k+1}. In the integral form the above condition is expressed as

$$\int_\Omega \Big\langle h(t_k,x)(y - P_{S_{k+1}} \circ (\tau g^\tau(k\tau,\cdot) + \mathrm{id})x)\ , \\ (v_{k+1}^\tau(x) - g^\tau(k\tau,x))\mathrm{d}\hat{\mu}_{k+1}^\tau(x)\Big\rangle \geq 0, \quad \forall y \in S_{k+1} \tag{4.57}$$

for any positive $h(t,x)$. We define the interpolation of $g^\tau(k\tau,x)$ as

$$g^\tau(t,x) = g^\tau(k\tau,x), \text{ for } t \in (k\tau,(k+1)\tau]. \tag{4.58}$$

Using the piecewise constant interpolations defined in (4.50), we have

$$\mathrm{d}\hat{\mu}_t^\tau = \mathrm{d}\hat{\mu}_{k+1}^\tau$$

$$\mathrm{d}\hat{E}_t^\tau = \hat{v}_t^\tau \mathrm{d}\hat{\mu}_t^\tau = v_{k+1}^\tau \mathrm{d}\hat{\mu}_{k+1}^\tau$$

for $t \in (k\tau,(k+1)\tau]$. In order to study the convergence of (4.57), we will need the following result.

Lemma 4.4.5. *For any positive measurable function $h(t,x)$ we have*

$$\lim_{\tau\to 0}\int h(t,x)(y-P_{S(t)}(x))g^\tau(t,x)\mathrm{d}\hat{\mu}_t^\tau(x)=\int h(t,x)(y-x)g(t,x)\mathrm{d}\mu_t(x)$$

where g^τ is defined in (4.58), $\hat{\mu}_t^\tau$ defined in (4.50) and $t\in(k\tau,(k+1)\tau]$.

The proof can be found in Appendix 4.7.3.

Using Lemma 4.4.5 and the convergence results for $\hat{\mu}_t^\tau\to\mu_t$, $\hat{E}_t^\tau\rightharpoonup E_t$, for any $h(t,x)$ positive function we have,

$$\int_0^T\int_\Omega\Big\langle h(t,x)(y-x),\mathrm{d}E_t(x)\mathrm{d}t\Big\rangle-\int_0^T\int_\Omega\Big\langle h(t,x)(y-x),g(t,x)\mathrm{d}\mu_t(x)\mathrm{d}t\Big\rangle\geq 0,\quad\forall y\in S(t).$$

Since $h(t,x)$ is an arbitrary positive function we get,

$$\langle y-x,\ v(t,x)-g(t,x)\rangle\geq 0,\quad\forall\, y\in S(t).$$

Thus, $v(t,x)-\int_U f(x,u)\mathrm{d}\omega(u|t,x)\in-\mathcal{N}_{S(t)}(x)$ which can be re-expressed in terms of some selection $\eta\in\mathcal{P}(\mathcal{N}_{S(t)}(x))$ as,

$$v(t,x)=-\int_{\mathcal{N}_{S(t)}(x)}\zeta\mathrm{d}\eta(\zeta|t,x)+\int_U f(x,u)\mathrm{d}\omega(u|t,x).$$

This completes the proof of Theorem 4.4.2. □

As a result of Theorem 4.4.2, we obtain the *continuity equation* in the following form:

$$\begin{aligned}\int_{S(T)}\varphi(T,x)\mathrm{d}\mu_T(x)-\int_{S(0)}\varphi(0,x)\mathrm{d}\mu_0(x)=\\ \int_{[0,T]\times S(t)}\int_U\Big(\partial_t\varphi(t,x)+\partial_x\varphi(t,x)\cdot\Big[-\int_{\mathcal{N}_{S(t)}(x)}\zeta\ \mathrm{d}\eta(\zeta|t,x)\\ +\int_U f(x,u)\mathrm{d}\omega(u|t,x)\Big]\Big)\mathrm{d}\tilde{\mu}_t(x)\mathrm{d}t,\end{aligned}\qquad(4.59)$$

for all test functions $\varphi\in C^1([0,T],\Omega)$. Moreover, Lemma 4.2.5 shows that any solution μ to the problem (4.24) admits a disintegration as $\mathrm{d}\mu(t,x,\zeta,u)=\mathrm{d}\nu(\zeta,u|t,x)\mathrm{d}\tilde{\mu}_t(x)\mathrm{d}t$ such that $\nu\in\mathcal{P}(\mathcal{N}_{S(t)}(x)\times U)$ has marginals $\pi_{\zeta\#}\nu=\eta$, $\pi_{u\#}\nu=\omega$ and the resulting continuity equation is (4.76) with the vector being (4.77) which is equivalent to (4.59).

4.4.3 Convergence of value function

In the previous subsection, we proved that for any sequence of feasible solutions μ^{τ_j}, to the family of problems (4.38), we obtain a feasible solution μ of (4.28). We now establish convergence of the optimal values as $\tau_j\to 0$. For point-mass initial data $\mu_0=\delta_{x_0}$, we write $J^{\tau_j*}(x_0):=J^{\tau_j*}[\mu_0]$ and $J_o^*(x_0):=J_o^*[\mu_0]$ for the optimal values of the discrete and continuous problems, respectively.

Theorem 4.4.6. *Let $\mu_0 = \delta_{x_0}$ with $x_0 \in S(0)$ be fixed. The optimal value of discrete-time LP (4.38) converges to the optimal value of continuous-time LP (4.28) as $\tau_j \to 0$, i.e., $J^{\tau_j *}(x_0) \to J^*_o(x_0)$. In other words, $J^{\tau_j *}$ converges pointwise to J^*_o for fixed x_0.*

Proof. The discrete-time objective function in (4.38) is

$$V^{\tau_j}[\omega^{\tau_j}, \mu_0] = \sum_{k=0}^{N} \tau_j \int_{S_k \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\omega^{\tau_j}(\mathsf{u}|k\tau, \mathsf{x}) \mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}).$$

We first prove that the objective function value V^{τ_j} for each admissible solution ω^{τ_j} converges to the objective function value of (4.28) obtained for $\omega = \lim_{\tau_j \to 0} \omega^{\tau_j}$. We study this convergence for $\mu_0 = \delta_{x_0}$. Using the convergence results proved in Theorem 4.4.2 we have

$$\begin{aligned}
&\sum_{k=0}^{N} \tau_j \int_{S_k \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\omega^{\tau_j}(\mathsf{u}|k\tau_j, \mathsf{x}) \mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}) \\
&\qquad = \sum_{k=0}^{N} \int_{S_k \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\omega^{\tau_j}(\mathsf{u}|k\tau_j, \mathsf{x}) \mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}) \tau_j \\
&\qquad \xrightarrow{\tau_j \to 0} \int_0^T \int_{S(t) \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\omega(\mathsf{u}|t, \mathsf{x}) \mathrm{d}\tilde{\mu}_t(\mathsf{x}) \mathrm{d}t \\
&\qquad\qquad = \int_0^T \int_{S(t) \times U} \int_{\mathcal{N}_{S(t)}(\mathsf{x})} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\mu(t, \mathsf{x}, \zeta, \mathsf{u})
\end{aligned}$$

where we have used the weak-$\star$ convergence results for $\omega^{\tau_j} \xrightharpoonup{*} \omega$ and $\mathrm{d}\tilde{\mu}_t^{\tau_j} \tau_j \xrightharpoonup{*} \mathrm{d}\tilde{\mu}_t \mathrm{d}t$. This establishes that for each admissible sequence $\{(\omega^{\tau_j}, \tilde{\mu}_k^{\tau_j})\}_{j \in \mathbb{N}}$, the objective function value converges to objective function value of (4.24).

To study the convergence of value function, we consider two quantities: $\limsup_{\tau_j \to 0} J^{*\tau_j}$ and $\liminf_{\tau_j \to 0} J^{*\tau_j}$. We aim to prove the following inequality

$$\liminf_{\tau_j \to 0} J^{*\tau_j} \geq J^* \geq \limsup_{\tau_j \to 0} J^{*\tau_j}.$$

Proof of ($\liminf_{\tau_j \to 0} J^{*\tau_j} \geq J^*_o$): Let us consider a sequence of problems, as defined in (4.38), indexed by τ_j as $\tau_j \to 0$, and let ω^{τ_j} be the sequence of optimal stochastic kernels for each of the problems. As the sequence of $\omega^{\tau_j} \in \mathcal{P}(U)$ is bounded, the Banach Alaoglu theorem ensures the existence of a subsequence (without relabelling) which converges to some $\omega \in \mathcal{P}(U)$, i.e., $\omega^{\tau_j} \rightharpoonup \omega$. Furthermore from Theorem 4.4.2 we saw that the interpolated trajectories $\tilde{\mu}_t^{\tau_j}$, associated with each τ_j in the convergent subsequence, converge to a feasible trajectory of (4.24). Thus,

$$\begin{aligned}
\liminf_{\tau_j \to 0} J^{*\tau_j} &:= \liminf_{\tau_j \to 0} \sum_{k=0}^{N} \tau_j \int_{S(t) \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) d\omega^{\tau_j}(u|k\tau_j, \mathsf{x}) \mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}) \\
&= \int_0^T \int_{S(t) \times U} \mathcal{L}(\mathsf{x}, \mathsf{u}) \mathrm{d}\omega(\mathsf{u}|t, \mathsf{x}) \mathrm{d}\tilde{\mu}_t(\mathsf{x}) \mathrm{d}t \geq J^*_o
\end{aligned}$$

where for the last inequality we have used the fact that $\mathrm{d}\mu = \mathrm{d}\eta(\zeta|\mathsf{x},t)\mathrm{d}\omega(\mathsf{u}|t,\mathsf{x})\mathrm{d}\tilde{\mu}_t(\mathsf{x})\mathrm{d}t$ is a feasible solution to (4.24). So, we obtain the desired inequality $\liminf_{\tau_j\to 0} J^{*\tau_j} \geq J_o^*$.

Proof of ($\limsup_{\tau_j\to 0} J^{*\tau_j} \leq J_o^*$): Let μ be an optimal solution of (4.24) which can be decomposed as

$$\mathrm{d}\mu(t,\mathsf{x},\zeta,\mathsf{u}) = \mathrm{d}\eta(\zeta|\mathsf{x},t)\mathrm{d}\omega(\mathsf{u}|t,\mathsf{x})\mathrm{d}\tilde{\mu}_t(\mathsf{x})\mathrm{d}t. \tag{4.60}$$

Now, let us consider sequence of time steps $\tau_j \to 0$ and consider partitions of $[0,T]$ for these time steps. We then discretize the optimal feedback ω in (4.60), with time steps τ_j, to obtain a sequence of stochastic kernels $\{\omega^{\tau_j}\}_j$. We have

$$\begin{aligned} J_o^* &:= \int_0^T \int_{S(t)\times U} \mathcal{L}(\mathsf{x},\mathsf{u})\mathrm{d}\omega(\mathsf{u}|t,\mathsf{x})\mathrm{d}\mu_t(\mathsf{x})\mathrm{d}t \\ &= \lim_{\tau_j\to 0} \sum_{k=0}^{N} \tau_j \int_{S_k\times U} \mathcal{L}(\mathsf{x},\mathsf{u})\mathrm{d}\omega^{\tau_j}(\mathsf{u}|k\tau_j,\mathsf{x})\mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}). \end{aligned}$$

We note that $\omega^{\tau_j}, \tilde{\mu}^{\tau_j}$ are admissible for (4.38) with time step τ_j. So,

$$\sum_{k=0}^{N} \tau_j \int_{S_k\times U} \mathcal{L}(\mathsf{x},\mathsf{u})\mathrm{d}\omega^{\tau_j}(\mathsf{u}|k\tau,\mathsf{x})\mathrm{d}\tilde{\mu}_k^{\tau_j}(\mathsf{x}) \geq J^{\tau *}.$$

Thus we have the desired inequality $\limsup_{\tau_j\to 0} J^{*\tau_j} \leq J_o^*$.

So, $\lim_{\tau_j\to 0} J^{*\tau_j}$ exists, $\lim_{\tau_j\to 0} J^{*\tau_j} = J_o^*$ for each initial condition $x_0 \in S(0)$. □

Corollary 4.4.7. *Let $\mu_0 \in \mathcal{P}(S(0))$. Then, as $\tau_j \to 0$,*

$$\lim_{\tau_j\to 0} \int_{S(0)} |J^{*\tau_j}(x) - J_o^*(x)|\mathrm{d}\mu_0(x) = 0,$$

with convergence in the L^1-norm w.r.t. μ_0.

4.5 Computational aspects

To provide a numerical approximation of the optimal control problem (4.38) studied in this chapter, which is the measure relaxation of the discrete-time optimal control problem (4.5), we consider a finite-dimensional relaxation of the problem. This relaxation is obtained by reformulating the infinite-dimensional problem over measures as a finite-dimensional problem over their moments, truncated to a certain degree. The resulting truncated moment problem can then be relaxed into a family of semidefinite programs (SDP), which can be solved using some off-the-shelf solvers. The moment-SOS hierarchy is introduced in Section 2.5. In this section, we adapt these tools to the discrete-time optimal control formulation (4.38)–(4.39). We therefore only recall the basic definitions (moments, Riesz functional, moment and localizing matrices) and refer to Section 2.5 for background, definitions, and further discussion.

Let $\mathbb{R}[x,u]$ be the ring of real polynomials in the variables $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ where x and u are the state and the control vectors, respectively. Let $\mathbb{R}[x,u]_d \subset \mathbb{R}[x,u]$ be the subset of polynomials of degree up to d. We make the following assumptions:

Assumption 4.5.1. Each set $S_k \subset \mathbb{R}^n$ in equation (4.39) is a basic semialgebraic set, i.e.,

$$S_k := \left\{ x \in \mathbb{R}^n \;\middle|\; g_i^{(k)}(x) \geq 0,\ i = 1, \ldots, \ell \right\},$$

where each $g_i^{(k)}$ is a polynomial. For notational convenience, let $g_0^{(k)}(x) = 1$, and that one of the inequalities is of the form $R - \|x\|^2 \geq 0$ for some sufficiently large constant $R > 0$. Similarly, the control input set $U \subset \mathbb{R}^m$ is a basic semialgebraic set defined as

$$U := \{ u \in \mathbb{R}^m \mid h_j(u) \geq 0,\ j = 1, \ldots, r \},$$

where each h_j is a polynomial function, $h_0(u) = 1$, and one of the constraints is $R - \|u\|^2 \geq 0$. Furthermore, the cost function $\mathcal{L}(x,u)$ in (4.38) is a polynomial function.

For integers $n, m \geq 1$ and a fixed degree $d \in \mathbb{N}$, let

$$\mathcal{A}_d \;:=\; \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{n+m} \;:\; |\alpha_1| + |\alpha_2| \leq d\}.$$

Define the vector of monomial basis functions

$$b_d(x,u) \;:=\; \left[x^{\alpha_1} u^{\alpha_2} \right]_{\alpha \in \mathcal{A}_d} \in \mathbb{R}^{n_d}$$

where $n_d := |\mathcal{A}_d| = \binom{n+m+d}{d}$. Given a moment sequence $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^{n+m}}$, define the Riesz functional

$$\ell_y :\ \mathbb{R}[x,u]_d \;\to\; \mathbb{R}, \quad \ell_y\Big(\sum_{\alpha \in \mathcal{A}_d} f_\alpha\, x^{\alpha_1} u^{\alpha_2} \Big) \;=\; \sum_{\alpha \in \mathcal{A}_d} f_\alpha\, y_\alpha.$$

The *truncated moment matrix* of order d associated with y is defined by

$$M_d(y) \;:=\; \ell_y\big(b_d(x,u)\, b_d(x,u)^\top\big) \;=\; \Big[y_{\alpha+\beta} \Big]_{\alpha,\beta \in \mathcal{A}_d}$$

where $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. Given a nonnegative Borel measure $\nu \in \mathcal{M}_+(\mathbb{R}^{n+m})$, we can naturally associate its moment sequence y as $y = \int x^{\alpha_1} u^{\alpha_2} d\nu(x,u)$; the corresponding Riesz functional ℓ_y represents integration against ν.

Now let us consider the program in (4.38)-(4.39) and let y_k be the moment sequence associated to μ_k^τ at time step k. Then, the objective function in (4.38) can be written as

$$\sum_{k=1}^{N} \int \mathcal{L}(x,u) \mathrm{d}\mu_k^\tau(x,u) = \sum_k c_l^T y_k \tag{4.61}$$

where c_l is the coefficient vector for the cost function expressed in monomial basis. For the dynamics constraint in program (4.38) and any test function $v(x)$, we observe that

$$\begin{aligned}\int_{S_k} v(x)\mathrm{d}\mu_k^\tau(x,u) &= \int_{S_k} v(x)\mathrm{d}\omega^\tau(\mathrm{d}u|x)\mathrm{d}\tilde{\mu}_k^\tau(x) = \int_{S_{k-1}}\int_U v(x)\mathrm{d}G^\tau_{k\#}\mu^\tau_{k-1}(x,u)\\ &= \int_{S_{k-1}}\int_U v(G_k^\tau(x,u))\mathrm{d}\mu^\tau_{k-1}(x,u) \end{aligned}\tag{4.62}$$

where we have used (4.35) and (4.37) in the first and second equality. When $v(x)$ are monomials, we obtain

$$y_k - c_f^T y_{k-1} = 0 \tag{4.63}$$

where c_f is coefficient of $v(G_k^\tau(x,u))$ in the RHS of (4.62). For every $h \in \mathbb{R}[x,u]_d$ such that $\mathbf{h}$ is the vector of coefficients of h we have

$$\begin{aligned}\ell_{y_k}(h^2) &= \int \mathbf{h} b_d(x,u) b_d(x,u)^\top \mathbf{h}^\top \mathrm{d}\mu_k^\tau(x,u)\\ &= \mathbf{h}^T M_d(y_k)\mathbf{h} = \int h^2(x,u)\mathrm{d}\mu_k^\tau(x,u) \geq 0.\end{aligned}$$

Hence, for arbitrary $h(x)$ we have $\mathbf{h}^\top M_d(y_k)\,\mathbf{h} \geq 0$, and we conclude that

$$M_d(y_k) \succeq 0, \qquad \text{for each order } d \geq 0, \tag{4.64}$$

where the notation $A \succeq 0$ denotes that the matrix A is positive semidefinite. The localizing matrix associated with the polynomial $g_i^{(k)}$ and the moment sequence y_k is defined as

$$M_{d-\deg(g_i^{(k)})}(g_i^{(k)} y_k) = \int b(x,u)_{d-\deg(g_i^{(k)})}\, b(x,u)^\top_{d-\deg(g_i^{(k)})}\, g_i^{(k)}(x)\,\mathrm{d}\mu_k^\tau(x,u),$$

where $\deg(g_i^{(k)})$ denotes the degree of the polynomial $g_i^{(k)}$ introduced in Assumption 4.5.1. To enforce that the support of the measure μ_k^τ is contained in S, we require the associated localizing matrix to be positive semidefinite:

$$\begin{aligned} M_{d-\deg(g_i^{(k)})}(g_i^{(k)} y_k) \succeq 0 \quad & \text{for } d \geq \max_i \deg(g_i^{(k)})\\ M_{d-\deg(h_j^{(k)})}(h_j^{(k)} y_k) \succeq 0 \quad & \text{for } d \geq \max_j \deg(h_j^{(k)}). \end{aligned}\tag{4.65}$$

Furthermore, the following theorem provides necessary and sufficient condition for the existence of representing measure corresponding to an infinite moment sequence y_k.

Theorem 4.5.1. *Under Assumption 4.5.1, let y_k be a moment sequence which satisfies* (4.64) *and* (4.65)*, then y_k has a representing measure with support $S_k \times U$.*

Note, however, that for any fixed truncation order d, a truncated vector y_k that satisfies (4.64) and the localizing conditions (4.65) is not guaranteed to admit a representing measure. Consequently, the constraints (4.64) and (4.65) are only necessary conditions; imposing them therefore yields a truncated moment relaxation (an outer approximation) of the original problem.

Under Assumption 3, an order d truncation of the program in (4.38) is given by

$$J_d^* = \min_{\{y_k\}} \sum_{k=1}^{N} c_l^T y_k \tag{4.66}$$

$$\begin{aligned} &\text{s.t. } y_k - c_f^T y_{k-1} = 0 \quad \forall k = 1, .., N \\ &M_d(y) \succeq 0, \quad M_{d-\deg(g_i^{(k)})}(g_i^{(k)} y) \succeq 0, \forall i = 1, \dots, \ell \\ &M_{d-\deg(h_j)}(h_j y) \succeq 0, \ \forall j = 1, \dots, r \end{aligned} \tag{4.67}$$

where c_l and c_f are defined in (4.61)-(4.63), and y_{μ_0} is the moment of the initial distribution up to degree d.

Theorem 4.5.2. *Under Assumption 4.5.1, the sequence J_d^* (4.66) is a monotonically non-decreasing sequence and converges to $J^{\tau *}$(4.38) as $d \to \infty$.*

For the proof of Theorem 4.5.1, and Theorem 4.5.2 and a systematic treatment of such moment–SOS relaxations, see [68, 24]. For rate of convergence of the sequence, see[101].

4.6 Example

In this section, we illustrate the SDP relaxations introduced in (4.66) using an academic example of a two dimensional system. Consider the following example

$$\dot{\mathbf{x}}(t) \in (1 + u_x, u_y) - \mathcal{N}_S(\mathbf{x}(t)) \tag{4.68}$$

where, $S := \{(x, y) : x^2 + y^2 \leq 1\}$ and (u_x, u_y) is the control vector. We have the following discretization of (4.68):

$$(x_{k+1}, y_{k+1}) = P_S((x_k, y_k) + \tau(1 + u_x, u_y)). \tag{4.69}$$

To be able to apply the techniques described in the previous subsection and obtain moment relaxations of the form (4.66) for the problem defined in (4.5), we need to obtain a polynomial expression for the projection map onto the disc. This can be achieved by first noting that the expression for the projection map onto the set S in (4.69) is

$$P_S(w) = \begin{cases} w, & \text{if } \|w\| \leq 1, \\ \dfrac{w}{\|w\|}, & \text{if } \|w\| \geq 1. \end{cases}$$

Next, we introduce variable $z = 1/\sqrt{x^2 + y^2}$ to obtain an equivalent definition of projection map:

$$P_S(x, y) = \begin{cases} (x, y), & x^2 + y^2 \leq 1, \\ \{(zx, zy) : {}_{z^2(x^2+y^2)=1}^{z \geq 0}\}, & x^2 + y^2 \geq 1. \end{cases}$$

We decompose the measure μ_k^τ in (4.62) into two measures, $\mu_k^{\tau S}$, supported on the interior of the disc and $\mu_k^{\tau\bar{S}}$, supported on the exterior of the disc. Then,

$$\text{supp}(\mu_k^{\tau S}) = \{(x,y) \mid (x+\tau(u_x+1))^2 + (y+\tau u_y)^2 \leq 1\} \tag{4.70}$$

$$\text{supp}(\mu_k^{\tau\bar{S}}) = \{(x,y) \mid (x+\tau(u_x+1))^2 + (y+\tau u_y)^2 \geq 1\}. \tag{4.71}$$

In this case, the dynamics constraint in (4.39) becomes

$$\mu_k^{\tau S} + \mu_k^{\tau\bar{S}} = \Big[P_S \circ \Big(\tau f(\cdot,\cdot) + \text{id}(\cdot)\Big)\Big]_{\#}\mu_{k-1}^{\tau\bar{S}} + (\tau f(\cdot,\cdot) + \text{id}(\cdot))_{\#}\mu_{k-1}^{\tau S}. \tag{4.72}$$

We consider cost function as $\mathcal{L}(x,u) = (x - x_T)^2 + |u|^2$, the initial distribution is $\mu_0 = \delta_{(0.7,0.5)}$ and target distribution $\mu_T = \delta_{(0.7,0.1)}$. For this cost and dynamics (4.72), we obtain a SDP relaxation of (4.38) as described in (4.66). The convergence result of the values of the associated relaxations follow from Theorem 4.5.2.

In this example, we consider the case where the initial and the desired final distributions are Dirac measures. Consequently, the distribution is expected to evolve as a Dirac measure over time. In this setting, the first-order approximate moments y_k, also called *pseudo-moments*, of the distribution μ_k^τ serve as an approximation of the first-order moment of the measure at each discrete time step k. These moments provide directly information about the state and control trajectories across different time instants, as

$$\int x\mu_k^\tau(x,u) = \int x\delta_{(x_k^\tau,u_k^\tau)} = x_k^\tau, \tag{4.73}$$

$$\int u\mu_k^\tau(x,u) = \int u\delta_{(x_k^\tau,u_k^\tau)} = u_k^\tau. \tag{4.74}$$

For more general initial conditions, the extraction of state and control trajectories boil down to evaluating the support of the measure. This can be achieved using Christoffel-Darboux kernels, as discussed in [102], or [65] for a more specific exposition. Specifically, the method proposed in [102] employs a semialgebraic approximant based on the Christoffel polynomial, which is constructed using the moment information of the measure at each degree d. For the uncontrolled case, [59] proposed SDP relaxation to simulate the dynamical system (4.68) in the continuous time. And similar approach can be followed for the SDP relaxation optimal control problem in the continuous time as well.

The simulation results are displayed in Figure 4.3, where we plot the sum of first order moments of $\mu_k^{\tau S}$ and $\mu_k^{\tau\bar{S}}$ over the time horizon of the problem. The results are computed for a fixed relaxation order using `GloptiPoly` and `Mosek`[4] . We observe that the program finds an optimal solution which prefers to slide along the boundary of the set S before the control kicks the particle back into the set in order to reach the target configuration. Moreover, in Figure 4.5 we observe that during the sliding motion zero controls are applied.

[4]GloptiPoly is a MATLAB toolbox for modeling and solving generalized moment problems via the moment–SOS hierarchy, and SeDuMi is a solver for conic optimization problems (in particular semidefinite programs) used here to solve the resulting relaxations.

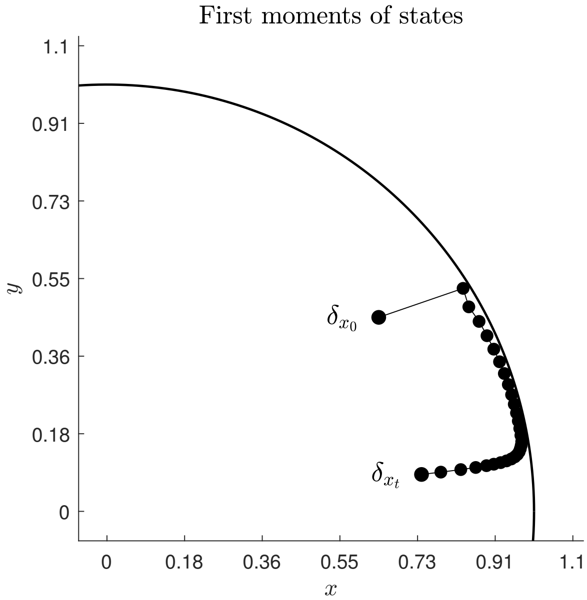
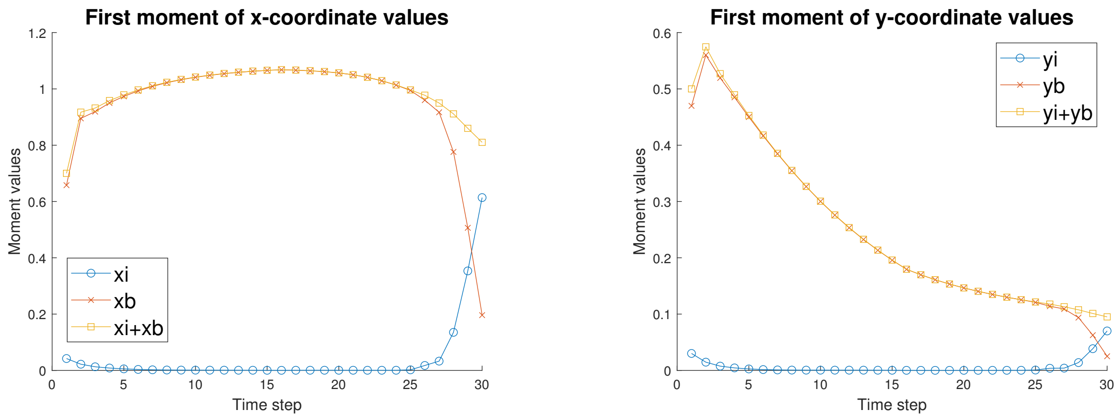


Figure 4.3: Approximate first order moments solving LP (4.66) for nonsmooth dynamical system (4.68).



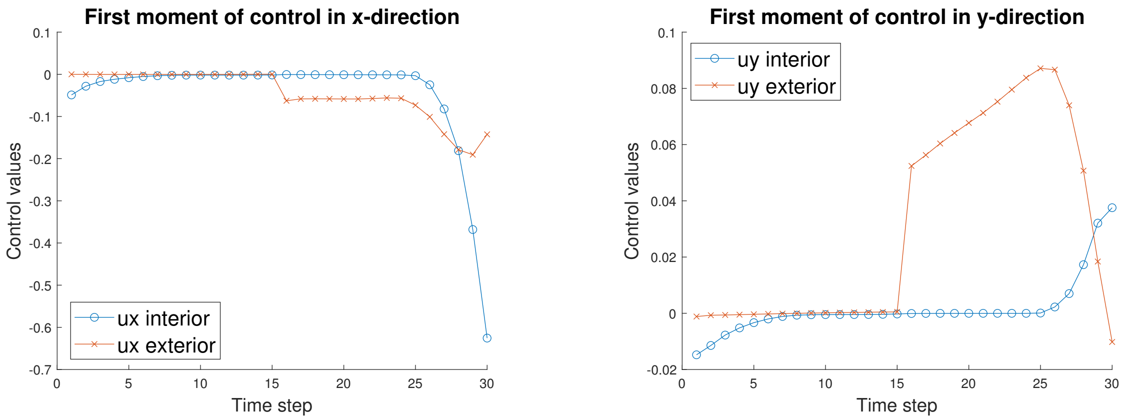
(a) First moment along x

(b) First moment along y

Figure 4.4: First moments along x and y coordinates. **xi/yi** come from the interior measures $\mu_k^{\tau S}$ (4.70); **xb/yb** from the exterior measures $\mu_k^{\tau \bar{S}}$ (4.71).

4.7 Conclusion

We addressed the problem of optimal control for nonsmooth dynamical systems by formulating it as a LP in the cone of nonnegative Borel measures. We showed that relaxing the problem from finite dimensional space to infinite dimensional does not produce any relaxation gap. We further showed the convergence of the discrete-time problem to continuous-time problem in the space of measures using tools from optimal transport theory. We defined interpolated curves and then showed that as the time step goes to zero, the interpolated curves converge to the solutions of the continuity equation where the corresponding velocity vector



(a) Control along x (b) Control along y

Figure 4.5: Approximate control in x and y directions, where **ux/uy interior** denote controls from the interior measure $\mu_k^{\tau S}$ (4.70) and **ux/uy exterior** from the exterior measure $\mu_k^{\tau \bar{S}}$ (4.71).

defines the nonsmooth dynamical system. We further establish the convergence of the value function of the discrete-time optimal control problem to the value function of continuous-time problem as the time step goes to zero. We observe that in comparison to the continuous-time problem, the discrete-time case has fewer variables as there is no extra variable corresponding to the selection of the vector field from the normal cone, which makes it computationally less demanding.

Appendix

4.7.1 Proof of Lemma 4.2.4

Proof. By Markov's inequality and the coercivity constraint $\int_{\mathfrak{B}\times U} |\zeta|\, d\mu \le C_\eta$,

$$\sup_{(\mu,\mu_T)\in\mathfrak{D}} \mu(\{|\zeta| > R\}) \;\le\; \frac{C_\eta}{R} \xrightarrow[R\to\infty]{} 0,$$

so $\{\mu\}$ is tight in the ζ–coordinate. Since $[0,T]\times\Omega\times U$ is compact, this implies tightness of $\{\mu\}$ on $\mathfrak{B}\times U$. Likewise, the compactness of $S(T)$ implies tightness of $\{\mu_T\}$. With the uniform mass bounds $\|\mu\| \le T$ and $\|\mu_T\| \le 1$, Prokhorov's theorem [99, Thm. 5.1] yields relative weak-$\star$ sequential compactness of $\mathfrak{D}$.

Now we show weak-$\star$ closedness. Let $(\mu_k, \mu_{T,k}) \in \mathfrak{D}$ and $(\mu_k, \mu_{T,k}) \xrightarrow{*} (\mu, \mu_T)$ (along a subsequence given by the relative weak-$\star$ compactness of $\mathfrak{D}$). Then: (i) positivity and the mass bounds are preserved under weak-$\star$ limits; (ii) the map $\nu \mapsto \int_{\mathfrak{B}\times U} |\zeta|\, d\nu$ is weak-$\star$ lower semicontinuous, hence $\int |\zeta|\, d\mu \le C_\eta$; (iii) the constraint (4.22) is stable under weak-$\star$ limits. Indeed, for $\varphi \in C^1([0,T]\times\Omega)$ set $g_1(t,x,\zeta,u) := \partial_t\varphi(t,x)$, $g_2(t,x,\zeta,u) := \nabla_x\varphi(t,x)\cdot(-\zeta + f(x,u))$. The terms with g_1 and with $\nabla_x\varphi\cdot f$ are bounded continuous on $\mathfrak{B}\times U$ and therefore *converge* by narrow convergence. For the $-\nabla_x\varphi\cdot\zeta$ term, fix $R>0$ and write

$$\int \nabla_x\varphi\cdot(-\zeta)\,\mathrm{d}\mu_k = \int \nabla_x\varphi\cdot(-\zeta)\,\mathbf{1}_{\{|\zeta|\le R\}}\,\mathrm{d}\mu_k \;+\; \int \nabla_x\varphi\cdot(-\zeta)\,\mathbf{1}_{\{|\zeta|>R\}}\,\mathrm{d}\mu_k.$$

The first part converges to the corresponding expression with μ because the truncated integrand is bounded and continuous; the second part is uniformly small, since

$$\sup_k\left|\int \nabla_x\varphi\cdot(-\zeta)\,\mathbf{1}_{\{|\zeta|>R\}}\,\mathrm{d}\mu_k\right| \le \|\nabla_x\varphi\|_\infty \sup_k \int_{|\zeta|>R} |\zeta|\,\mathrm{d}\mu_k \le \|\nabla_x\varphi\|_\infty\,\frac{C_\eta}{R}.$$

Let $k\to\infty$ and then $R\to\infty$ to conclude. The terminal term $\int_{S(T)}\varphi(T,x)\,\mathrm{d}\mu_{T,k}$ also converges to $\int_{S(T)}\varphi(T,x)\,\mathrm{d}\mu_T$. Hence (4.22) holds for (μ,μ_T).

Therefore $\mathfrak{O}$ is weak-$\star$ closed; combined with relative compactness, this proves weak-$\star$ sequential compactness. □

4.7.2 Proof of Lemma 4.2.5

Proof. The occupation measure μ satisfying (4.22) admits the following disintegration:

$$\mathrm{d}\mu(t,x,\zeta,u) = \mathrm{d}\nu(\zeta,u\mid t,x)\,\mathrm{d}\bar{\mu}_t(x)\,\mathrm{d}t, \tag{4.75}$$

where $\nu(\cdot\mid t,x)\in\mathcal{P}(\mathcal{N}_{S(t)}(x)\times U)$ and $\bar{\mu}_t$ is the marginal of μ at time t. Substituting the disintegration of μ from (4.75) in (4.22), we obtain

$$\int_{[0,T]\times S(t)}\Big[\partial_t\phi(t,x)\mathrm{d}\bar{\mu}_t(x)\mathrm{d}t + \partial_x\phi(t,x)\cdot b(t,x)\mathrm{d}\bar{\mu}_t(x)\mathrm{d}t\Big] = \int_{S(T)}\phi(T,x)\mathrm{d}\mu_T(x) - \phi(0,x_0). \tag{4.76}$$

where the averaged vector field $b(t,x)$ is

$$b(t,x) := \int_{\mathcal{N}_{S(t)}(x)\times U} (-\zeta + f(x,u))\;\mathrm{d}\nu(\zeta,u\mid t,x). \tag{4.77}$$

Let $\pi_{\zeta\#}\nu=\eta$, $\pi_{u\#}\nu=\omega$ (i.e. the marginals of $\nu\in\mathcal{P}(\mathcal{N}_{S(t)}(x)\times U)$), then the vector field in (4.77) can be written as,

$$\begin{aligned} b(t,x) = -\int_{\mathcal{N}_{S(t)}\times U}\zeta\mathrm{d}\nu(\zeta,u|t,x) + \int_{\mathcal{N}_{S(t)}\times U} f(x,u)\mathrm{d}\nu(\zeta,u|t,x) \\ = -\int_{\mathcal{N}_{S(t)}}\zeta\mathrm{d}\eta(\zeta|t,x) + \int_U f(x,u)\mathrm{d}\omega(u|t,x). \end{aligned}$$

Using Assumption 4.1.1, we have

$$\int_{[0,T]\times S(t)}\int_{\mathcal{N}_{S(t)}}\int_U |f(x,u)|\,\mathrm{d}\nu(\zeta,u|t,x)\mathrm{d}\bar{\mu}_t(x)\mathrm{d}t < \infty.$$

The uniform moment bound in ζ direction (which is part of the definition in set (4.23)) yields

$$\int_{[0,T]\times S(t)}\int_{\mathcal{N}_{S(t)}}\int_U |\zeta|\,\mathrm{d}\nu(\zeta,u|t,x)\mathrm{d}\bar{\mu}_t(x)dt<\infty.$$

Using triangle's inequality, we have

$$\int_0^T\int_{S(t)}|b(t,x)|\,\mathrm{d}\bar{\mu}_t(x)\,\mathrm{d}t<\infty.$$

So, the integrability of $b(t,x)$ allows the use of superposition principle [16, Theorem 8.2.1]; there exists a probability measure $\Pi\in\mathcal{P}(AC([0,T];\mathbb{R}^n))$ such that for each t, the marginal $\bar{\mu}_t=(e_t)_\#\Pi$, and Π-almost every path $x(\cdot)$ satisfies the ODE $\dot{x}(t)=b(t,x(t))$. We note that $\dot{x}(t)=b(t,x(t))$ is the same as the Young measure relaxed dynamics in (4.17) but driven by state dependent ω. For each triple $(x(\cdot),\eta(\cdot|t,x),\omega(\cdot|t,x))$ satisfying (4.17), define the pair $(\hat{\mu},\xi)$ as

$$\mathrm{d}\hat{\mu}(t,x,\zeta,u)=\mathrm{d}t\,\delta_{x(t)}(\mathrm{d}x)\,\eta(\mathrm{d}\zeta|t,x(t))\,\omega(\mathrm{d}u|t,x(t)),\xi:=\delta_{x(T)}.$$

Then $(\hat{\mu},\xi)\in\mathfrak{R}$. Also, define the push-forward measure $\theta:=\Phi_\#\Pi$, where $\Phi(x(\cdot)):=(\hat{\mu},\xi)$. For any admissible cost function $\mathcal{L}(x,u)$, we compute:

$$\begin{aligned}\int\mathcal{L}(x,u)\,\mathrm{d}\mu&=\int\mathcal{L}(x,u)\mathrm{d}\bar{\mu}_t(x)\mathrm{d}\eta(\zeta|t,x)\mathrm{d}\omega(u|t,x)\mathrm{d}t\\&=\int\mathcal{L}(x,u)\delta_{x(t)}\Pi(x(\cdot))\mathrm{d}\eta(\zeta|x)\mathrm{d}\omega(u|t,x)\mathrm{d}t\\&=\int_{\mathfrak{R}}\left[\int\mathcal{L}(x,u)\,\mathrm{d}\hat{\mu}\right]\mathrm{d}\tilde{\theta}(\hat{\mu}),\end{aligned}$$

where $\tilde{\theta}$ is the marginal of θ. □

4.7.3 Proof of Lemma 4.4.5

Proof. We already have shown that $\hat{\mu}_t^\tau\to\mu_t$ uniformly in the W_2 metric when $\tau\to0$. The term

$$(y-P_{S(t)}(x))\,g^\tau(t,x)\xrightarrow[\tau\to0]{}(y-x)\,g(t,x)$$

converges pointwise, and the integrand $(y-P_{S(t)}(x))\,g^\tau(t,x)$ is uniformly bounded (because $y-P_{S(t)}(x)$ lies in a compact set, while g^τ is bounded by Assumption 4.1.1). Hence, by the dominated convergence theorem we get

$$\Big|\lim_{\tau\to0}\int h(t,x)(y-P_{S(t)}(x))g^\tau(t,x)\mathrm{d}\hat{\mu}_t^\tau(x)-\int h(t,x)(y-x)g(t,x)\mathrm{d}\mu_t(x)\Big|\to0.$$

□

4.7.4 Proof of Corollary 4.4.7

Proof. We consider an initial condition $\mu_0\in\mathcal{P}(S(0))$ in (4.24) which are different from Dirac distribution δ_{x_0} just considered above. To prove L^1 convergence of $J^{*\tau}[\mu_0]$ as $\tau\to0$ for notational convenience, we write τ instead of the sequence τ_j), we first establish an uniform bound on $J^{*\tau} = \tau \sum_{k=0}^{N} \mathcal{L}(x_k^\tau, u_k^\tau)$. Using Assumption 4.1.3, we know that $\mathcal{L}(x_k^\tau, u_k^\tau) \leq l(x_k^\tau)(1+|u_k^\tau|^2)$ for $l(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Next, we recall from [29] that the following bound holds for x_k^τ in the discrete-time dynamics (4.31):

$$|x_k^\tau| \;\leq\; C_1|x_0| + C_2,$$

where C_1 and C_2 depend on the system data. Consequently, we have

$$\begin{aligned} J^{*\tau}(x_0) &= \tau \sum_{k=0}^{N} \mathcal{L}(x_k^\tau, u_k^\tau) \leq \tau \sum_{k=0}^{N} l(x_k^\tau)(1 + |u_k^\tau|^2) \\ &\leq \tau \sum_{k=0}^{N} l_b(C_1|x_0| + C_2)U_b \leq T h_b(C_1 \max_{x \in S(0)} |x| + C_2)U_b \end{aligned}$$

where, l_b is the bound on $l(\cdot)$ and U_b is the bound on the controls. Thus we have uniform bounds on the $J_o^*[\mu_0]$ for $\mu_0 \in \mathcal{P}(S(0))$. The uniform bounds and the pointwise convergence of $J_o^*(x_0)$, established in the previous part of the proof, allows us to use the dominated convergence theorem to obtain

$$\lim_{\tau \to 0} \int |J^{*\tau}(x) - J_o^*(x)| \mathrm{d}\mu_0(x) = 0. \qquad \square$$

Chapter 5

Evolution in Hilbert space

In this chapter, we prove that there is no relaxation gap between a quasi-dissipative nonlinear evolution equation in a Hilbert space and its linear Liouville equation reformulation on probability measures. In other words, strong and generalized solutions of such equations are unique in the class of measure-valued solutions. As a major consequence, non-convex numerical optimization over these non-linear partial differential equations can be carried out with the infinite-dimensional moment-SOS hierarchy with global convergence guarantees. This covers in particular all reaction-diffusion equations with polynomial nonlinearity.

5.1 Introduction

The moment–SOS hierarchy provides a numerical framework to solve broad classes of nonconvex optimization problems with global optimality guarantees by reformulating them as linear programs over measures and approximating these by semidefinite relaxations of increasing order. We refer to Chapters 1 and 2 (and the references therein, in particular [24]) for a detailed presentation of this methodology, its measure reformulations, and the associated literature. In this chapter we focus instead on adapting this framework to the quasi-dissipative infinite-dimensional evolution problems introduced earlier in the thesis.

For non-linear partial differential equations (PDEs) and their control, an early attempt to use the linear measure formulation was reported in [103] for semi-linear elliptic equations. More recent attempts using the moment-SOS approach can be found in [104, 105, 106] for linear PDEs, and more recently in [107, 27] for non-linear PDEs. A fully general non-linear setup including optimization, calculus of variations and PDEs was described in [52]. As far as we know, in these references, there is neither convergence guarantee of the proposed hierarchy, nor a proof of no relaxation gap.

In the context of scalar hyperbolic conservation laws, a particular class of nonlinear PDEs, no relaxation gap was ensured by introducing entropy inequalities [53]. In [26] it was shown that there is no relaxation gap for scalar problems (i.e. when then dimension of the domain or the dimension of the codomain is equal to one), both for calculus of variations and for optimal control problems. Conversely, an example of a variational problem with relaxation gap is provided when the dimensions of the domain and of the codomain are greater than one. It is also shown that in the presence of integral constraints, a relaxation gap may occur at any dimension of the domain and of the codomain. More recently, it was shown [28] that under convexity assumptions, there is no relaxation gap for a broad class of variational and optimal control problems on non-linear PDEs. The question of the absence of a relaxation gap for specific classes of non-linear PDEs remains however widely open.

These results were obtained for measures supported on finite-dimensional subsets of the time, space, function and function gradient domains. There is however the possibility of reformulating optimization problems over nonlinear PDEs as linear problems on measures supported on infinite-dimensional functional spaces. This is the point of view adopted e.g. in [108, Part III] for optimization with relaxed controls. Measures on infinite-dimensional spaces and statistical measure-valued solutions are also prominently appearing in [109] and in the subsequent work [110] on Navier-Stokes equations. This perspective was already discussed in Section 1.3 in the context of the Burgers equation, where we highlighted how measure formulations can encode PDE constraints while retaining a linear structure at the level of measures. This notion of statistical solutions allows for sound numerical implementations, see e.g. [111]. These solutions seem to be related with the measure-valued solutions studied in [112] and approximated numerically in the context of the moment-SOS hierarchy. This motivated us to focus on reformulations with measures on infinite-dimensional spaces and to address the question of relaxation gap in this infinite-dimensional setup.

In this chapter, we show that there is no relaxation gap for a broad class of non-linear PDEs defined by evolution equations on infinite-dimensional functional spaces, with an operator satisfying a quasi-dissipativity condition also known as quasi-monotonicity or m-accretive property, explained in details in [113, 114, 115]. The key ingredient in the proof is differential calculus with probability measures [116]. No relaxation gap for this class of PDEs implies that we can use the infinite-dimensional moment-SOS hierarchy to solve optimization and optimal control problems on these equations, with convergence guarantees. It is of interest to mention [117] which also provides an alternative proof in the case of Hamiltonian PDEs, but without providing a numerical scheme nor making a clear link with the semigroup literature. To the best of our knowledge, this link is disclosed in our chapter for the first time.

Existing numerical methods for non-linear PDEs focus on specific classes of PDEs. Except conservation laws for which sophisticated methods such as the wave front tracking is available [118], these methods focus on semi-linear equations and provide a numerical scheme for the linear part which is proved to work with the nonlinear terms of the equation by using some fixed-point arguments. Classical numerical schemes for PDEs are based on space(-time) discretizations include the finite-difference method [119], the finite-element method [120], the finite-volume method [121] or discontinuous Galerkin methods [122]. We refer the interested reader to [123] for a general overview of numerical schemes for nonlinear PDEs. Existing methods being based on fixed-point arguments and discretization (including implicit schemes), only local convergence can be guaranteed. Furthermore, when facing optimization problems involving PDEs, only local minima can be obtained with such methods. See e.g. [124] for a simple example of a non-convex PDE optimal control problem with several local (and global) optima.

In contrast with all these methods, the moment-SOS hierarchy is designed to approximately compute *globally optimal solutions*. The main difference with finite-difference, finite-element and finite-volume methods consists in discretizing measures via finitely many of their moments, instead of discretizing the time and the space. Therefore, one may interpret this as a Galerkin method. However, the difference relies on the equation which is solved: indeed, while the Galerkin method focuses on solving (possibly non-linear) equations with functions as unknowns, the moment-SOS hierarchy is used in the context of (always linear) equations with measures as unknowns.

More broadly speaking, the moment–SOS hierarchy has proved applicable in many other settings, as outlined in Chapters 1 and 2, and it is used in Chapters 3 and 4 to address measure evolution and optimal control problems for quasi-dissipative nonsmooth dynamical systems. We hope that the results presented in this chapter will pave the way for similar achievements for nonlinear PDEs, in particular for optimal (including boundary) control and the computation of regions of attraction.

The outline of the chapter is as follows. In Section 5.2 we introduce the non-linear PDEs under study, in the form of differential equations with quasi-dissipative operators evolving in a Hilbert space. A few examples are described in Section 5.3. In Section 5.4 we introduce the Liouville equation, a linear measure reformulation of the non-linear PDE, as well as its relationship with various notions of solutions. Section 5.5 contains our two main results: first we state in Theorem 5.5.2 that there exists a unique solution to the Liouville as soon as the generator is supposed to be quasi-dissipative; and second, as a consequence of the first result, we show a no relaxation gap result which states that for Dirac initial data, the unique solution to the Liouville equation is the Dirac at the solution of the non-linear PDE. In Section 5.6 we briefly describe the infinite-dimensional moment-SOS hierarchy. It is then illustrated numerically in Section 5.7. Further research directions and extensions are described in the concluding Section 5.8.

5.2 Quasi-dissipative evolution equations

Consider the evolution equation

$$\dot{y}(t) = f(y(t)), \quad y(0) = y_0, \quad t \in [0,1] \tag{5.1}$$

with the time-dependent function $y : [0,1] \to \mathcal{H}$, where $\mathcal{H}$ is a given real Hilbert space equipped with a norm $|\cdot|$ and a scalar product $\langle \cdot, \cdot \rangle$, the dot denotes the time derivative, $f : D(f) \subset \mathcal{H} \to \mathcal{H}$ is a given nonlinear operator, with domain $D(f)$ densely defined in $\mathcal{H}$, and $y_0 \in D(f)$ is a given initial condition. We further assume that $\mathcal{H}$ forms a rigged Hilbert space (or a Gelfand triple) $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$, where $\mathcal{H}_1$ is supposed to be equal to $D(f)$, which is equipped with a norm $|z|$ for all $z \in \mathcal{H}_1$, and where $\mathcal{H}_{-1}$ is defined as the topological dual to $\mathcal{H}_1$. A typical example is $\mathcal{H} := L^2(\mathbb{R})$ (square integrable functions), $\mathcal{H}_1 := H^1(\mathbb{R})$ (functions with square integrable weak derivatives) and $\mathcal{H}_{-1} := H^{-1}(\mathbb{R})$ (dual space including distributions). In the case where f is semilinear, i.e., $f(y) := Ay + g(y)$, with $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ the generator of a strongly continuous semigroup, and g a bounded operator from $\mathcal{H}$ to $\mathcal{H}$, the space $\mathcal{H}_1$ is defined as $\mathcal{H}_1 := D(A)$ (equal to $D(f)$ since g is bounded) and $\mathcal{H}_{-1}$ can be built [125] as the completion of $\mathcal{H}$ with respect to the norm $|(A - \rho I)^{-1} y|$ for all $y \in \mathcal{H}$, where ρ is an element of the resolvent of A. We suppose furthermore that these Hilbert spaces are separable. A solution of (5.1) is denoted $y(t)$ or $y(t|y_0)$ if we want to emphasize the dependence on initial condition. Finally, let $\mathcal{Y}$ denote a subset of $\mathcal{H}$.

Before considering further properties of the operator f, we explain now what we mean by a solution of evolution equation (5.1). Indeed, there are several non-equivalent notions of solutions for infinite-dimensional systems. The first one, perhaps the most straightforward, is referred as strong solution [113, Definition 2.2].

Definition 5.2.1 (Strong solution)**.** Let $y_0 \in D(f)$. A function $y \in \mathcal{C}([0,1],\mathcal{H})$ is a strong solution to (5.1) if it is Lipschitz continuous on compact subsets of $[0,1]$, differentiable almost everywhere, and equation (5.1) is satisfied for almost every $t \in [0,1]$.

When facing lower regularity (i.e., with $y_0 \in \mathcal{H}$), a more suitable notion of solution should be introduced [114, Definition, page 183].

Definition 5.2.2 (Generalized solution)**.** Given $y_0 \in \mathcal{H}$, a generalized solution to (5.1) is a function $y \in \mathcal{C}([0,1],\mathcal{H})$ for which there exists a sequence of initial conditions $(y_0^n)_{n\in\mathbb{N}} \in \mathcal{H}_1$ converging to y_0 and strong solutions $(y^n)_{n\in\mathbb{N}}$ to (5.1) converging to y, in the topology of $\mathcal{H}$ and $\mathcal{C}([0,1],\mathcal{H})$ respectively.

Another notion of interest, particularly relevant for our purposes, is the positive invariance of a set of solutions.

Definition 5.2.3 (Positive invariance)**.** Set $\mathcal{Y} \subset \mathcal{H}$ is called positively invariant if $y_0 \in \mathcal{Y}$ implies $y(t) \in \mathcal{Y}$ for all $t \in [0,1]$.

In this chapter, we focus on evolution equations with operators satisfying a specific positivity condition.

Definition 5.2.4 (Quasi-dissipative)**.** Operator f is quasi-dissipative on $\mathcal{Y}$ if there exists a constant $a \geq 0$ such that

$$\langle y_1 - y_2, f(y_1) - f(y_2)\rangle \leq a|y_1 - y_2|^2 \tag{5.2}$$

for all $y_1, y_2 \in \mathcal{Y}$.

Definition 5.2.5 (Maximality)**.** Operator f is maximal on $\mathcal{Y}$ if

$$\forall h \in \mathcal{Y},\ \exists y \in D(f),\ y = f(y) + h. \tag{5.3}$$

Remark 5.2.1. These notions are the infinite-dimensional counterparts of the quasi-dissipativity and maximality assumptions introduced earlier for finite-dimensional nonsmooth dynamical systems; see Section 2.2.

In the literature, quasi-dissipative operators are also called (up to a change of sign) quasi-monotone or m-accretive, see [113, 114, 115]. If operator f in (5.1) is quasi-dissipative and maximal on $\mathcal{H}$, it follows from the Crandall-Liggett theorem [115, Theorem 5.6] or [114, Theorem 4.1 and Theorem 4.1a] that for any $y_0 \in \mathcal{H}$, resp. $y_0 \in D(f)$, there exists a unique generalized solution $y \in \mathcal{C}([0,1],\mathcal{H})$, resp. unique strong solution $y \in \mathcal{C}([0,1],\mathcal{H})$ to (5.1). Furthermore, the operator f generates a (nonlinear) strongly continuous semigroup denoted by $(\mathbb{S}(t))_{t\geq 0}$, meaning that the solution of (5.1) can be written as

$$y(t) := \mathbb{S}(t)y_0$$

for all $t \in [0,1]$, for all $y_0 \in \mathcal{H}$ for generalized solutions, and for all $y_0 \in D(f)$ for strong solutions. In particular, if f is quasi-dissipative and maximal, solutions exploding in finite-time cannot exist, meaning that the solution is always bounded on a finite interval of time. Therefore, there always exists a bounded solution set $\mathcal{Y}$ which is positively invariant.

5.3 Examples

5.3.1 Semilinear equations

Let

$$f(y) := Ay + g(y) \tag{5.4}$$

with $g : \mathcal{H} \to \mathcal{H}$ a bounded and quasi-dissipative operator and $A : D(A) \subset \mathcal{H}$ a linear operator generating a strongly continuous semigroup denoted by $(\mathbb{T}(t))_{t\geq 0}$. A function $y \in \mathcal{C}([0,1],\mathcal{H})$ is called a mild solution to evolution equation (5.1) with semilinear operator (5.4) if it satisfies the Duhamel formula

$$y(t) = \mathbb{T}(t)y_0 + \int_0^t \mathbb{T}(t-s)g(y(s))ds, \; y_0 \in \mathcal{H}. \tag{5.5}$$

In the semilinear case, generalized solutions and mild solutions coincide, as proved in [126, Lemma 4.3.4]. This notion of mild solution can be defined for semilinear operators as in (5.4), but it is difficult to apply it for quasi-linear dynamical systems (such as conservation laws) since operator f does not admit a linear part, which might therefore prevent the use of the Duhamel formula. Moreover, solving the implicit formulation (5.5), when it is possible, gives the semigroup $(\mathbb{S}(t))_{t\geq 0}$, generated by the operator f. In other words, the solution of (5.5) can be written as $y(t) := \mathbb{S}(t)y_0$ for a.e. $t \geq 0$.

5.3.2 The heat equation

As a canonical example, we may take $\mathcal{H} = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded C^∞ domain and

$$f(y) := \Delta y, \qquad D(f) := H^2(\Omega) \cap H^1_0(\Omega),$$

corresponding to the heat equation with homogeneous Dirichlet boundary conditions. The quasi-dissipativity and maximality of this operator on $\mathcal{H}$, and the generation of the associated contraction semigroup $(\mathbb{S}(t))_{t\geq 0}$, have already been established in Section 2.1.

5.3.3 Reaction-diffusion

Let $\mathcal{H} := L^2(0,1)$ and consider the following periodic reaction-diffusion operator

$$f(y) := \partial_{xx}y + g(y) \tag{5.6}$$

with domain

$$D(f) := \{y \in H^2(0,1) : y[0] = y[1], \partial_x y[0] = \partial_x y[1]\}$$

and Fréchet differentiable operator g from $\mathcal{H}$ to $\mathcal{H}$ (the operator g does not have to be polynomial) such that $g(y)[x] \leq 0$ if $y[x] \leq y_{\min}$ or $y[x] \geq y_{\max}$ for given bounds $y_{\min} \leq y_{\max}$. For $y \in H^2(0,1)$, the notation $y[x]$ refers to the evaluation of function y at the point $x \in [0,1]$.

Lemma 5.3.1. *The set* $\mathcal{Y} := \{y \in D(f) : y_{\min} \leq y[x] \leq y_{\max}, \forall x \in (0,1)\}$ *is positively invariant for evolution equation* (5.1) *with operator* (5.6)*.*

Lemma 5.3.2. *The reaction-diffusion operator* (5.6) *is quasi-dissipative on the positively invariant set* $\mathcal{Y}$ *of Lemma 5.3.1.*

Lemma 5.3.3. *The reaction-diffusion operator* (5.6) *is maximal on the positively invariant set* $\mathcal{Y}$ *of Lemma 5.3.1.*

The proofs of these results are provided in the Appendix.

5.4 Linear measure formulation and notions of solutions

In this section, we introduce the Liouville equation in an infinite-dimensional, separable real Hilbert space $\mathcal{H}$. (The finite-dimensional continuity/Liouville equation was already introduced in Chapters 1 and 2.) Our goal is to reformulate the nonlinear evolution equation (5.1) as a linear equation in which the unknown is a measure-valued curve. We then prove that this measure reformulation exhibits no relaxation gap with respect to the original dynamics. This property is instrumental for establishing convergence of the numerical scheme used in this thesis, namely the moment–SOS hierarchy, which relies on tools from convex optimization. As we will see, the Liouville equation must be understood in a weak sense; this requires introducing cylindrical test functions. For the reader's convenience, and to make the chapter as self-contained as possible, we recall their definition [116, Definition 5.1.11].

Definition 5.4.1 (Cylindrical function)**.** Given an integer d, we denote by $\Pi_d(\mathcal{H})$ the space composed by all projective maps $\pi : \mathcal{H} \to \mathbb{R}^d$ of the form:

$$\pi(z) = (\langle z, e_1 \rangle, \langle z, e_2 \rangle, \ldots, \langle z, e_d \rangle),\ z \in \mathcal{H}$$

where $\{e_1, \ldots, e_d\}$ is any orthonormal family of vectors in $\mathcal{H}$. We denote by $\mathrm{Cyl}(\mathcal{H})$ the space of cylindrical functions ϕ defined as $\phi := \psi \circ \pi$ with $\pi \in \Pi_d(\mathcal{H})$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, smooth functions with compact support.

By definition, cylindrical functions are Lipschitz continuous and everywhere Fréchet differentiable with respect to the weak topology of $\mathcal{H}$. In the sequel, the cylindrical test functions depend on time, i.e. $\phi := \psi(t, \pi(z))$, with $\psi \in \mathcal{C}_c^\infty([0,1] \times \mathbb{R}^d)$, which actually is equivalent to take $\phi \in \mathrm{Cyl}([0,1] \times \mathcal{H})$. Cylindric functions are necessary from a theoretical viewpoint, since our proofs are based on results of [116] relying on them. They are also crucial when developing the numerics because they can model polynomials in infinite dimensions.

Following [112], consider the time-dependent Dirac measure $\mu_t = \delta_{y(t)}$ supported on the strong solution $y(t)$ to evolution equation (5.1) for a given initial condition $y_0 \in \mathcal{H}$. Let $\phi \in \mathrm{Cyl}([0,1] \times \mathcal{H})$ be a cylindrical test function. It holds

$$\int_0^1 \dot{\phi}(t, y(t))dt = \phi(1, y(1)) - \phi(0, y(0)) = \int_{\mathcal{H}} \phi(1, z)d\mu_1(z) - \int_{\mathcal{H}} \phi(0, z)d\mu_0(z),$$

and using the chain rule

$$\begin{aligned}\int_0^1 \dot{\phi}(t, y(t))dt &= \int_0^1 \left(\partial_t \phi(t, y(t)) + \partial_y \phi(t, y(t))\dot{y}(t)\right) dt \\ &= \int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t, z) + \langle \partial_z \phi(t, z), f(z)\rangle_{\mathcal{H}}\right) d\mu_t(z)\end{aligned}$$

where $\partial_t \phi(t, z)$ is the partial derivative of ϕ w.r.t. time, and $\partial_z \phi(t, z)$ is the Fréchet derivative of ϕ w.r.t. $z \in \mathcal{H}$, a linear operator on $\mathcal{H}$, which exists due to the definition of cylindrical functions.

Equating both expressions, the Dirac measure $\mu_t = \delta_{y(t)}$ solves the Liouville equation:

$$\int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t, z) + \langle \partial_z \phi(t, z), f(z)\rangle\right) d\mu_t(z)dt = \int_{\mathcal{H}} \phi(1, z)d\mu_1(z) - \int_{\mathcal{H}} \phi(0, z)d\mu_0(z) \tag{5.7}$$

which is a linear transport equation in $\mu : [0,1] \to \mathcal{M}_+(\mathcal{H})$, where $\mathcal{M}_+(\mathcal{H})$ denotes the space of measures[1] on $\mathcal{H}$, identified with bounded linear functionals on the vector space $\mathcal{C}(\mathcal{H})$ of continuous functions on $\mathcal{H}$. More generally, let μ_t denote a solution of (5.7). Time-dependent measures μ_t are called parametrized measures or Young measures in the calculus of variations literature.

Let $\nu \mapsto -\partial_y \cdot (f\nu)$ denote the linear operator which is adjoint to the linear operator $\psi \mapsto \partial_y \psi f(y)$, i.e. such that for every cylindrical function $\psi \in \mathrm{Cyl}(\mathcal{H})$ and every measure $\nu \in \mathcal{M}_+(\mathcal{H})$, it holds $\int_{\mathcal{H}} \partial_y \psi(z) f(z) d\nu(z) = -\int_{\mathcal{H}} \psi(z)\, \partial_z \cdot (f\nu)(dz)$. The Liouville equation (5.7) can then be equivalently written as an evolution equation on measures:

$$\dot{\mu_t} + \partial_z \cdot (f\mu_t) = 0 \tag{5.8}$$

with $\mu_0 = \delta_{y_0}$, for given initial data $y_0 \in \mathcal{H}$.

Consider the semigroup $(\mathbb{S}(t))_{t\geq 0}$ generated by the operator f. Trajectories of (5.1) can be therefore written as $y(t, \cdot) = \mathbb{S}(t)y_0$ with $y_0 \in D(f)$ and for all $t \geq 0$. Observe that if the initial data is an arbitrary probability measure $\mu_0 \in \mathcal{M}_+(\mathcal{H}_1)$, then the push-forward measure $\mu_t = \mathbb{S}(t)_{\#}\mu_0$ through the flow map solves Liouville equation (5.8). Indeed, in this case (5.7)

[1]In this chapter by measure we mean a positive Radon measure, i.e. locally finite and tight.

writes

$$\begin{aligned}
&\int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t,z) + \langle \partial_z \phi(t,z), f(z)\rangle\right) d\mu_t(z)dt \\
&\qquad = \int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t,\mathbb{S}(t)z) + \langle \partial_z \phi(t,\mathbb{S}(t)z), f(\mathbb{S}(t)z)\rangle\right) d\mu_0(z)dt \\
&\qquad = \int_0^1 \int_{\mathcal{H}} \Big(\partial_t \phi(t,\mathbb{S}(t)z) + \langle \partial_z \phi(t,\mathbb{S}(t)z), (\dot{\mathbb{S}(t)z})\rangle\Big) d\mu_0(z)dt \\
&\qquad = \int_0^1 \int_{\mathcal{H}} \dot{\phi}(t,\mathbb{S}(t)z) d\mu_0(z)dt \\
&\qquad = \int_{\mathcal{H}} \phi(1,\mathbb{S}(1)z)d\mu_0(z) - \int_{\mathcal{H}} \phi(0,\mathbb{S}(0)z)d\mu_0(z) \\
&\qquad = \int_{\mathcal{H}} \phi(1,z)\mu_1(dz) - \int_{\mathcal{H}} \phi(0,z)\mu_0(dz).
\end{aligned}$$

It is worth mentioning that the (formal) computations performed all along this section require the solution to (5.1) to be strong. Otherwise, the operator $f(\mathbb{S}(t)z)$, for all $t \geq 0$, would not exist. In other words, the weak formulation (5.7) with test functions actually corresponds to the strong solution to (5.1). Therefore, as in the case of (5.1), we need different notions of solution.

Definition 5.4.2 (Strong measure-valued solution)**.** Given $\mu_0 \in \mathcal{M}_+(\mathcal{H}_1)$, every strongly narrowly continuous measure curve[a] μ_t solving (5.7) is called a strong measure-valued solution of the Liouville equation (5.8) associated with evolution equation (5.1).

[a]A measure curve μ_t is strongly narrowly continuous if the real-valued map $t \in [0,1] \mapsto \int_{\mathcal{H}} \phi(z)d\mu_t(z) \in \mathbb{R}$ is continuous for every continuous bounded function ϕ.

Remark 5.4.1. Note that the measure $\mu_t = \mathbb{S}(t)_{\#}\mu_0$ with $\mu_0 \in \mathcal{M}_+(\mathcal{H}_1)$ is a strong measure-valued solution. In other words, existence of solution is straightforward. However, uniqueness cannot be deduced easily. To prove uniqueness, we will use the Wasserstein distance, and this requires second order moments to be bounded. Using the notion of strongly narrowly continuous measure curve, we can deduce that the second order moments are bounded as follows. We first observe that $z \mapsto |z|$ is continuous and bounded on $\mathcal{H}$, by definition of the norm. It should then follow that if μ_t is a strongly narrowly continuous measure curve, then $t \in [0,1] \mapsto \int_{\mathcal{H}} |z|^2 d\mu_t(z)$ is bounded. A similar conclusion holds for any moment $\int_{\mathcal{H}} |z|^p d\mu_t(z)$, $p \in \mathbb{N}$.

Definition 5.4.3 (Generalized measure-valued solution)**.** Given $\mu_0 \in \mathcal{M}_+(\mathcal{H})$ and a sequence $(\mu_0^n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(\mathcal{H}_1)$, a generalized measure-valued solution to (5.8) is a strongly narrowly continuous measure curve $\mu_t \in \mathcal{M}_+(\mathcal{H})$ for which there exists a sequence of strong measure-valued solutions $(\mu_t^n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(\mathcal{H}_1)$ with μ_t^n converging narrowly[a] to μ_t when μ_0^n converges narrowly to μ_0.

[a]A sequence of measures $(\mu^n)_{n\in\mathbb{N}}$ converges narrowly to a measure μ if $(\int_{\mathcal{H}} \phi(z)d\mu^n(z))_{n\in\mathbb{N}}$ converges to $\int_{\mathcal{H}} \phi(z)d\mu(z)$ for every continuous bounded function ϕ.

Remark 5.4.2 (The semilinear case). Again, by specifying the semilinear case as in Section 5.3.1, one can reformulate (5.8) as follows:

$$\begin{aligned}\int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t,z) + \langle A^* \partial_z \phi(t,z), z\rangle + \langle \partial_z \phi(t,z), g(z)\rangle\right) \mu_t(dz)dt = \\ \int_{\mathcal{H}} \phi(1,z)\mu_1(dz) - \int_{\mathcal{H}} \phi(0,z)\mu_0(dz)\end{aligned} \tag{5.9}$$

where A^* is the adjoint operator of A. Taking cylindrical test function $\phi \in \mathrm{Cyl}([0,T] \times D(A^*))$ and recalling furthermore that $D(A^*)$ is a dense subset of $\mathcal{H}$ since A generates a strongly continuous semigroup, the latter equation makes even sense when considering generalized solutions.

5.5 Main result: no relaxation gap

In this section, we state and prove our main results: first, we show that strong (and generalized) solutions to the Liouville equation are uniquely defined; second, as a corollary, it can be deduced that, for any initial measure μ_0 equal to the Dirac measure δ_{y_0} on $\mathcal{H}_1$ (resp. $\mathcal{H}$), the solution μ_t to the Liouville equation which is the Dirac measure $\delta_{y(t|y_0)}$ on $\mathcal{H}_1$ (resp. $\mathcal{H}$) corresponding to the strong solution to (5.1) (resp. the generalized solution to (5.1)). Note that the existence of (either strong or generalized) solutions to the Liouville equation can be proved easily if one sets $\mu_t = \mathbb{S}(t)_{\#}\mu_0$ with $\mu_0 \in \mathcal{M}_+(\mathcal{H})$, meaning that the well-posedness of (5.7) reduces to proving the uniqueness. Our strategy relies on the Wasserstein distance, which requires the second moment of the (strong or generalized) measure-valued solution to be finite. For strong measure-valued solutions, strong narrow continuity implies uniform-in-time bounds on moments: in particular, the second moment is bounded, and in fact all higher-order moments are bounded as well (see Remark 5.4.1). This justifies the use of the Wasserstein distance. We refer to Chapter 1 and Section 2.3 for basic notions and definitions from optimal transport.

Theorem 5.5.1. *Suppose that operator f is quasi-dissipative and maximal. Consider two initial measures $\check{\mu}_0$ and $\hat{\mu}_0 \in \mathcal{M}_+(\mathcal{H})$, and any associated generalized measure-valued solutions $\check{\mu}_t$ and $\hat{\mu}_t$. Then, for any $t \in [0,1]$, we have*

$$W_2(\check{\mu}_t, \hat{\mu}_t) \leq W_2(\check{\mu}_0, \hat{\mu}_0)e^{at}. \tag{5.10}$$

Proof. The proof is divided in two parts. In the first part we consider $\check{\mu}_0$ resp. $\hat{\mu}_0 \in \mathcal{M}_+(\mathcal{H}_1)$ and the associated strong measure-valued solutions $\check{\mu}_t$ resp. $\hat{\mu}_t$ to (5.7). Let us consider $V(t) = W_2^2(\check{\mu}_t, \hat{\mu}_t)$. Its time derivative is given by

$$\frac{d}{dt}V(t) = \lim_{h\to 0} \frac{1}{h}(W_2^2(\check{\mu}_{t+h}, \hat{\mu}_{t+h}) - W_2^2(\check{\mu}_t, \hat{\mu}_t)).$$

where $W_2(\mu_1, \mu_2)$ is the Wasserstein distance between two probability measures $\mu_1, \mu_2 \in P(\mathcal{H})$, i.e.

$$W_2^2(\mu_1, \mu_2) := \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathcal{H}^2} |z_1 - z_2|^2 d\gamma(z_1, z_2)$$

where

$$\Gamma(\mu_1,\mu_2) := \{\gamma \in P(\mathcal{H}^2) : \pi^{z_1}_{\#}\gamma = \mu_1,\ \pi^{z_2}_{\#}\gamma = \mu_2\}$$

is the set of transport plans between μ_1 and μ_2. The notation $g_{\#}\gamma$ stands for the push-forward measure of γ through a map g. The map $\pi^{z_1} : \mathcal{H}^2 \to \mathcal{H}, (z_1,z_2) \mapsto z_1$ is the projection on the z_1 coordinate, so that $\pi^{z_1}_{\#}\gamma$ is the z_1 marginal of γ, and similarly for π^{z_2}. Let us also denote by

$$\Gamma^*(\mu_1,\mu_2) := \{\gamma \in \Gamma(\mu_1,\mu_2) : W_2^2(\mu_1,\mu_2) = \int_{\mathcal{H}^2} |z_1 - z_2|^2 d\gamma(z_1,z_2)\}$$

the set of optimal transport plans.

For some small real h, we introduce the map $F_h\colon z \mapsto z + hf(z)$, with $z \in \mathcal{H}_1$. Using Lemma 5.8.1 in the Appendix, the following property holds:

$$W_2(\mu_{t+h}, F_{h\#}\mu_t) = o(h). \tag{5.11}$$

As a consequence, it is sufficient to consider the limit of

$$\frac{1}{h}(W_2^2(F_{h\#}\check{\mu}_t, F_{h\#}\hat{\mu}_t)) - W_2^2(\check{\mu}_t,\hat{\mu}_t))$$

when $h \to 0$ in order to deduce the time derivative of V. Let us now use first order differential calculus, as in the proof of [116, Thm. 8.4.7]. Select $\gamma \in \Gamma^*(\check{\mu}_t,\hat{\mu}_t)$, and define

$$\gamma_h := (F_h \circ \pi^{z_1}, F_h \circ \pi^{z_2})_{\#}\,\gamma$$

where $\circ$ is the composition operator, and note that $\gamma_h \in \Gamma(F_{h\#}\check{\mu}_t, F_{h\#}\hat{\mu}_t)$. By definition of the Wasserstein distance:

$$W_2^2(F_{h\#}\check{\mu}_t, F_{h\#}\hat{\mu}_t)) \le \int_{\mathcal{H}^2} |z_1 - z_2|^2 d\gamma_h(z_1,z_2). \tag{5.12}$$

On the other hand,

$$\begin{aligned}
\int_{\mathcal{H}^2} |z_1 - z_2|^2 d\gamma_h(z_1,z_2) &= \int_{\mathcal{H}^2} |z_1 + hf(z_1) - z_2 - hf(z_2)|^2 d\gamma(z_1,z_2)\\
&= \int_{\mathcal{H}^2} |z_1 - z_2|^2 d\gamma(z_1,z_2)\\
&\quad + 2h\int_{\mathcal{H}^2} \langle z_1 - z_2, f(z_1) - f(z_2)\rangle d\gamma(z_1,z_2) + O(h^2)\\
&= W_2^2(\check{\mu}_t,\hat{\mu}_t) + 2h\int_{\mathcal{H}^2} \langle z_1 - z_2, f(z_1) - f(z_2)\rangle d\gamma(z_1,z_2) + O(h^2).
\end{aligned}$$

Combining this equality with in the previous inequality (5.12), and letting $h \to 0^-$ and $h \to 0^+$ successively, one obtains

$$\frac{d}{dt}V(t) = 2\int_{\mathcal{H}^2} \langle z_1 - z_2, f(z_1) - f(z_2)\rangle d\gamma(z_1,z_2).$$

Using the quasi-dissipativity of operator f and the Grönwall lemma we have

$$W_2(\check{\mu}_t, \hat{\mu}_t) \leq W_2(\check{\mu}_0, \hat{\mu}_0)e^{at}. \tag{5.13}$$

For the second part of the proof, let $\check{\mu}_0$ resp. $\hat{\mu}_0 \in \mathcal{M}_+(\mathcal{H})$ with its corresponding narrowly converging sequence $(\check{\mu}_0^n)_{n\in\mathbb{N}}$ resp. $(\hat{\mu}_0^n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(\mathcal{H}_1)$. Let us consider the associated generalized measured-valued solution $\check{\mu}_t$ resp. $\hat{\mu}_t \in \mathcal{M}_+(\mathcal{H})$ to (5.7) with its corresponding narrowly converging sequence of strong solutions $(\check{\mu}^n)_{n\in\mathbb{N}}$ resp. $(\hat{\mu}^n)_{n\in\mathbb{N}} \subset \mathcal{M}_+(\mathcal{H}_1)$. The narrow convergence and the convergence of the second order moments implies convergence in Wasserstein metric [116, Rmk. 7.1.11]. Therefore, one has, for every $n \in \mathbb{N}$,

$$W_2^2(\check{\mu}_t, \hat{\mu}_t) \leq W_2^2(\check{\mu}_t, \check{\mu}_t^n) + W_2^2(\check{\mu}_t^n, \hat{\mu}_t^n) + W_2^2(\hat{\mu}_t^n, \hat{\mu}_t).$$

For large $n \in \mathbb{N}$ we have $W_2^2(\check{\mu}_t, \check{\mu}_t^n) \leq \epsilon/3$ and $W_2^2(\hat{\mu}_t^n, \hat{\mu}_t) \leq \epsilon/3$. For the second term in the above inequality we observe that

$$W_2^2(\check{\mu}_t^n, \hat{\mu}_t^n) \leq W_2(\check{\mu}_0^n, \hat{\mu}_0^n)e^{at} \leq W_2(\check{\mu}_0, \hat{\mu}_0)e^{at} + \epsilon/3.$$

which follows from (5.25) and the convergence of $\check{\mu}_0^n$ (resp. $\hat{\mu}_0^n$) to $\check{\mu}_0$ (resp. $\hat{\mu}_0$) w.r.t. Wasserstein metric. We deduce that, as n goes to infinity, we have, for a.e. $t \geq 0$

$$W_2(\check{\mu}_t, \hat{\mu}_t) \leq W_2(\check{\mu}_0, \hat{\mu}_0)e^{at},$$

concluding the proof of the result. □

Corollary 5.5.2 (No relaxation gap). *Suppose that f is quasi-dissipative and maximal. If $\mu_0 = \delta_{y_0}$ with $y_0 \in \mathcal{H}_1$ resp. $\mathcal{H}$ then $\mu_t = \delta_{\mathbb{S}(t)y_0}$ is the only strong resp. generalized measure-valued solution of* (5.7) *associated with μ_0.*

Proof. Let $\mu_0 = \delta_{y_0}$ with $y_0 \in \mathcal{H}_1$ then $\delta_{\mathbb{S}(t)y_0}$ is a solution to (5.7). We assume there exists another solution μ_t to (5.7) with the same initial condition $\mu_0 = \delta_{y_0}$. Let us consider the time-dependent function $V(t) := W_2^2(\mu_t, \delta_{y(t)}) : [0,1] \to \mathbb{R}$. Then using the results from Theorem 5.5.1 we get $W_2(\mu_t, \delta_{y(t)}) = 0$ for all time $t \in [0,1]$ as $\frac{dV(t)}{dt} \leq e^{at}V(0)$ and $V(0) = 0$. Similar result holds for $\mu_0 = \delta_{y_0}$ with $y_0 \in \mathcal{H}$. □

5.6 Infinite-dimensional moment-SOS hierarchy

In Chapter 2.5 we introduced the finite-dimensional moment-SOS hierarchy for polynomial optimization and for measure relaxations of optimal control problems. In the present chapter, our goal is to adapt these tools to an infinite-dimensional Hilbert space setting. This requires suitable notions of polynomials, moments, and sums of squares on Hilbert spaces, which we now introduce.

5.6.1 Polynomials and moments

Polynomials can be expressed as linear combinations of monomials. In the infinite-dimensional setting, polynomials are called Wiener polynomials or chaos polynomials [127], because they have been originally introduced for stochastic differential equations. Let $c_0(\mathbb{N})$ the set of integers with finitely many non-zero elements, i.e. if $a = (a_1, a_2, \ldots) \in c_0(\mathbb{N})$, then $\mathrm{card}\{i \in \mathbb{N} \mid a_i \neq 0\} < \infty$. In this setting, and considering $z \in \mathcal{H}$, a *monomial* of degree $a \in c_0(\mathbb{N})$ is defined as

$$z^a := \prod_{i=1}^{\infty} \langle z, \phi_i \rangle^{a_i}.$$

This is a product of finitely many powers of linear functionals $\langle z, \phi_i \rangle$ with $i = 1, 2, \ldots$ and given functions ϕ_i in $\mathcal{H}'$, the topological dual of $\mathcal{H}$. Polynomials in $\mathcal{H}$ are then defined as linear combinations of monomials, i.e.

$$\begin{aligned} p :\ \mathcal{H} &\to \mathbb{R} \\ z &\mapsto \sum_{a \in \mathrm{spt}(p)} p_a z^a \end{aligned}$$

where the sum runs over $\mathrm{spt}(p)$, the support of p, a (possibly infinite) countable subset of $c_0(\mathbb{N})$. Let $\mathbb{R}[z]$ denote the ring of polynomials. These polynomials have two types of degrees. To follow the terminology introduced in [112, 128], the *algebraic degree* is defined as

$$d := \max_{a \in \mathrm{spt}(p)} \sum_{i=1}^{\infty} a_i.$$

which corresponds to the total degree in the finite-dimensional setting. The second notion of degree, namely the *harmonic degree*, is defined as

$$n := \max_{a \in \mathrm{spt}(p)} \{i \in \mathbb{N} \mid a_i \neq 0\},$$

which corresponds to the number of variables in the finite-dimensional setting.

Given a measure ν on $\mathcal{H}$ and an index $a \in c_0(\mathbb{N})$ the quantity

$$m_a := \int_{\mathcal{H}} z^a d\nu(z) \tag{5.14}$$

is called the *moment* of order a of measure ν.

Given a sequence $m := (m_a)_{a \in c_0(\mathbb{N})}$, let us define the Riesz functional

$$\begin{aligned} \ell_m :\ \mathbb{R}[z] &\to \mathbb{R} \\ p &\mapsto \sum_{a \in \mathrm{spt}(p)} p_a m_a. \end{aligned}$$

If the sequence m has a representing measure ν, i.e. if (5.14) holds for $a \in c_0(\mathbb{N})$, then

$$\ell_m(p) = \int_{\mathcal{H}} p(z) d\nu(z).$$

5.6.2 Moment and localizing matrices

Let us derive conditions satisfied by the moments of a measure μ supported on a subset $\mathcal{Y}$ of $\mathcal{H}$. Let $\mathcal{Y} := \{z \in \mathcal{H} : p(z) \geq 0\}$ be defined as the compact superlevel set of a given polynomial p, see [129] for examples.

Since μ is positive, the Riesz functional corresponding to the sequence m of moments of μ must be positive on squares, i.e. $\ell_m(q_0^2) \geq 0$ for all $q_0 \in \mathbb{R}[z]$. It must also be positive on $\mathcal{Y}$, i.e. $\ell_m(pq_1^2) \geq 0$ for all $q_1 \in \mathbb{R}[z]$. It turns out that these necessary conditions are also sufficient for sequence m to have a representing measure, this is the dual moment formulation of Jacobi's Positivstellensatz that can be found in [130, Theorem 3.9]. See [131, Theorem 2.1] for a reformulation of Jacobi's Positivstellensatz – whose original statement is [132, Theorem 4] – which is valid for sums of squares (SOS) representations of positive polynomials in an Archimedean quadratic module of any unital commutative algebra, and in particular for the algebra generated by elements of $\mathbb{R}[z]$.

Numerically, the sequence m must be truncated up to a given algebraic and harmonic degree, i.e. the above positivity conditions are enforced for bounded degree polynomials. The positivity condition $\ell_m(q_0^2) \geq 0$ resp. $\ell_m(pq_1^2) \geq 0$ for bounded degree q_0 resp. q_1 is formulated as positive semidefiniteness of a symmetric matrix depending linearly on m, the so-called *moment matrix* resp. *localizing matrix*. Positive semidefiniteness of the moment and localizing matrices results in finite-dimensional convex linear matrix inequality (LMI) in the truncated moment sequence. These conditions are necessary for the entries of the truncated sequence to be moments of a measure on $\mathcal{Y}$. They are called moment relaxations, and the truncated sequence entries are called pseudo-moments. The LMI conditions grow in size with the truncation degree, and they become sufficient asymptotically, i.e. for infinitely many constraints. This is the essence of the infinite-dimensional moment-SOS hierarchy, as described in [112, Section 5] and [128]. See also [24] and [133] for the finite-dimensional moment-SOS hierarchy and its applications.

5.6.3 Moment formulation of the Liouville equation

The weak formulation of the Liouville equation (5.7) becomes a linear equation in the moments of measure $\mu_t(dz)dt$, provided with use monomials of both t and z in the test functions

$$\phi_a(t,z) = t^{a_0} \prod_{i=1}^{\infty} \langle \phi_i, z \rangle^{a_i} \tag{5.15}$$

for each given index $a = (a_0, a_1, a_2, \ldots) \in c_0(\mathbb{N})$. Let us define resp.

$$\begin{array}{rcl} m_a^{0,1} & := & \int_0^1 \int_{\mathcal{Y}} \phi_a(t,z)\mu_t(dz)dt \\ m_a^0 & := & \int_{\mathcal{Y}} \phi_a(0,z)\mu_0(dz) \\ m_a^1 & := & \int_{\mathcal{Y}} \phi_a(1,z)\mu_1(dz) \end{array} \tag{5.16}$$

the occupation resp. initial and terminal moments of order a.

For each given index $a \in c_0(\mathbb{N})$, the Liouville equation (5.7) corresponds to a linear equation

$$L_a(m^{0,1}) + m_a^1 = m_a^0. \tag{5.17}$$

where L_a is a given linear functional of the sequence of occupation moments $m^{0,1} = (m_a^{0,1})_{a\in c_0(\mathbb{N})}$, and $m^0 := (m_a^0)_{a\in c_0(\mathbb{N})}$ resp. $m^1 := (m_a^1)_{a\in c_0(\mathbb{N})}$ is the sequence of initial resp. terminal moments, as defined in (5.16).

By enumerating all index sequences, we generate countably infinitely many linear moment equations (5.17), resulting in an infinite-dimensional linear system of equations in the moment sequences $m^{0,1}$, m^0 and m^1. Each equation involves infinitely many moments, so for computational purposes we have to truncate the infinite sums in the expression of linear functional L_a to finitely many moments. Let us denote by L_a^h the corresponding linear functional truncated to harmonic degree h. The remaining terms are absorbed by a error residual denoted e_a^h, so that linear equation (5.17) becomes

$$L_a^h(m^{0,1}) + m_a^1 = m_a^0 + e_a^h(m^{0,1}). \tag{5.18}$$

It is possible to get estimates of the error residual, but for computational purposes it may suffice to minimize its quadratic norm.

5.6.4 Moment-SOS hierarchy

The moment-SOS hierarchy then consists of minimizing the quadratic norm of the error residual subject to the linear equations (5.18) for all $a \in c_0(\mathbb{N})$ such that $|a|_1 \leq r$, for increasing values of r, and the LMI conditions on the moment and localizing matrices described in Section 5.6.2. Since there is a unique measure solution to the Liouville equation, and a measure on compact set is uniquely determined by its moments, the overall numerical procedure generates approximations to the moments of the solutions that converge pointwise. Note also that more general optimization problems can be formulated in the same framework e.g. optimization over initial conditions, introduction of control parameters etc. Precise statements and convergence proofs lie however outside of the scope of this chapter.

Finally, at a given relaxation order h, we can approximate the solution of the original equation by using an infinite-dimensional extension of the Christoffel-Darboux polynomial [129], in analogy with what was achieved in the finite-dimensional case [134]. This approximation strategy, together with its convergence guarantees, are also outside of the scope of this chapter.

5.7 Numerical results for polynomial reaction-diffusion

Now let us follow the approach of Section 5.6 and formulate the Liouville equation for a quadratic diffusion operator

$$f(y) := \partial_{xx}y + \epsilon y(1-y)$$

for given $\epsilon \geq 0$, with domain

$$D(f) := \{y \in H^2(0,1) : y[0] = y[1], \partial_x y[0] = \partial_x y[1]\}$$

the periodic functions with square integrable weak derivatives on $[y_{\min}, y_{\max}] := [0,1]$. Let $T := 1$ be the terminal time.

In the weak formulation (5.7) of the Liouville equation, consider monomial test functions

(5.15) with dual functions $\phi_i \in \mathcal{H}'$. In our case, for periodic functions of $\mathcal{H} = \mathcal{H}' = L^2(0,1)$, the natural choice would be the complex exponentials $\phi_i(x) = e_{k_i}(x) := \exp(2\pi\sqrt{-1}k_i x)$ for a given $k_i \in \mathbb{Z}$. Test function (5.15) has algebraic degree $|a|_1 := \sum_{i=0}^{\infty} a_i$ and harmonic degree $|k|_\infty := \max_{i=1}^{\infty} |k_i|$.

Define the Fourier transform $F : L^2(0,1) \to \ell_2(\mathbb{Z}),\ z \mapsto c = (c_k)_{k\in\mathbb{Z}}$ where $c_k := \langle e_k, z\rangle$ is the k-th Fourier coefficient of $z \in \mathcal{H}$. The adjoint of F is the inverse Fourier transform $F^* : \ell_2(\mathbb{Z}) \to L^2(0,1),\ c \mapsto z = \sum_{a\in\mathbb{Z}} c_a e_a$. Given a measure μ on $\mathcal{H}$, let $\nu := F_\# \mu$ denote its push-forward measure through F, so that for our choice of monomials test functions (5.15) it holds

$$\int_0^1 \int_{\mathcal{H}} \phi(t,z)\mu_t(dz)dt = \int_0^1 \int_{\ell^2} t^{a_0} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_t(dc)dt$$

i.e. moments of μ become standard algebraic moments of ν. Consistently with (5.16), let us define resp.

$$\begin{array}{rcl} m_a^{0,1} & := & \int_0^1 \int_{\ell^2} t^{a_0} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_t(dc)dt \\ m_a^0 & := & 0^{a_0} \int_{\ell^2} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_0(dc) \\ m_a^1 & := & 1^{a_0} \int_{\ell^2} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_1(dc) \end{array}$$

the occupation resp. initial and terminal moments of order a.

Now observe that reporting these test functions in linear equation (5.7) we can express the following terms with our moments:

$$\begin{array}{rcccl} \int_{\mathcal{H}} \phi(0,z)\mu_0(dz) & = & 0^{a_0} \int_{\ell^2} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_0(dc) & = & m_a^0 \\ \int_{\mathcal{H}} \phi(1,z)\mu_1(dz) & = & 1^{a_0} \int_{\ell^2} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_1(dc) & = & m_a^1 \\ \int_0^1 \int_{\mathcal{H}} \partial_t \phi(t,z)\mu_t(dz)dt & = & a_0 \int_0^1 \int_{\ell^2} t^{a_0-1} c_{k_1}^{a_1} \dots c_{k_d}^{a_d} \nu_t(dc)dt & = & a_0 m_{a_0-1,a_1,\dots,a_d}^{0,1}. \end{array}$$

The Fréchet derivative of ϕ with respect to $y \in \mathcal{H}$ at $f \in \mathcal{H}$ is given by

$$\partial_z \phi(t,z)(f) = \sum_{j=1}^{d} a_j \frac{\langle \phi_j, f\rangle}{\langle \phi_j, z\rangle} \phi(t,z)$$

i.e.

$$\begin{aligned} \partial_z \phi(t,z)(f) = {} & a_1 \langle \phi_1, f\rangle t^{a_0} \langle \phi_1, z\rangle^{a_1-1} \langle \phi_2, z\rangle^{a_2} \cdots \langle \phi_d, z\rangle^{a_d} \\ & + a_2 \langle \phi_2, f\rangle t^{a_0} \langle \phi_1, z\rangle^{a_1} \langle \phi_2, z\rangle^{a_2-1} \cdots \langle \phi_d, z\rangle^{a_d} + \cdots \end{aligned}$$

The Fourier coefficients of f are given by

$$\begin{aligned} \langle \phi_i, f\rangle = \langle e_{k_i}, f\rangle & = -\langle e_{k_i}, \partial_{xx} z\rangle - \langle e_{k_i}, z\rangle + \langle e_{k_i}, z^2\rangle \\ & = (2\pi k_i)^2 \langle e_{k_i}, z\rangle - \langle e_{k_i}, z\rangle + \langle e_{k_i}, \sum_{k\in\mathbb{Z}} (\sum_{l\in\mathbb{Z}} c_l c_{k-l}) e_k\rangle \\ & = ((2\pi k_i)^2 - 1) c_{k_i} + \sum_{l\in\mathbb{Z}} c_l c_{k_i-l} \end{aligned}$$

where the non-linear term

$$z^2 = (\sum_{k\in\mathbb{Z}} c_k e_k)(\sum_{l\in\mathbb{Z}} c_l e_l) = \sum_{k,l\in\mathbb{Z}} c_k c_l e_{k+l} = \sum_{k\in\mathbb{Z}} (\sum_{l\in\mathbb{Z}} c_l c_{k-l}) e_k \tag{5.19}$$

depends on infinitely many Fourier coefficients. Equation (5.7) can then be written as the linear moment equation

$$\begin{aligned}
&\int_{\mathcal{H}} \phi(1,z)\mu_1(dz) - \int_{\mathcal{H}} \phi(0,z)\mu_0(dz) = \int_0^1 \int_{\mathcal{H}} \left(\partial_t \phi(t,z) + \partial_z \phi(t,z)(f)\right) \mu_t(dz)dt \\
&= 1^{a_0} \int_{\ell^2} c_1^{a_1} \cdots c_d^{a_d} \nu_1(dc) - 0^{a_0} \int_{\ell^2} c_1^{a_1} \cdots c_d^{a_d} \nu_0(dc) = a_0 \int_0^1 \int_{\ell^2} t^{a_0-1} c_1^{a_1} \dots c_d^{a_d} \nu_t(dc)dt \\
&+ a_1 \int_0^1 \int_{\ell^2} t^{a_0} \left(((2\pi k_1)^2 - 1)c_{k_1} + \sum_{l\in\mathbb{Z}} c_l c_{k_1-l} \right) c_{k_1}^{a_1-1} c_{k_2}^{a_2} \cdots c_{k_d}^{a_d} \nu_t(dc)dt \\
&+ a_2 \int_0^1 \int_{\ell^2} t^{a_0} \left(((2\pi k_2)^2 - 1)c_{k_2} + \sum_{l\in\mathbb{Z}} c_l c_{k_2-l} \right) c_{k_1}^{a_1} c_{k_2}^{a_2-1} \cdots c_{k_d}^{a_d} \nu_t(dc)dt + \cdots
\end{aligned}$$

which has the linear form (5.17).

Let $\epsilon = 0.1$ and $h = 4$. The initial moment sequence m^0 is given, corresponding to a Gaussian distribution with mean 0 and standard deviation $10^{-1/2}$.

At relaxation order $r = 4$, the resulting semidefinite optimization problem has moment matrices of size 165, with moment vectors of size 3575 subject to 1100 linear equations. We solve this optimization problem with MOSEK, and we obtain approximate occupation moments $\tilde{m}^{0,1}$ and terminal moments $\tilde{m}^1$, to be compared with the occupation moments $m^{0,1}$ and terminal moments m^1 computed by a finite difference scheme as described in [135].

The percentage of entries of the occupation resp. terminal moments that match within relative accuracy less than 10^{-3} is equal to 91% resp. 93%.

5.8 Conclusion

Nonlinear nonconvex optimization over PDEs can be reformulated as a linear optimization problem in a measure space, but it may happen that the optimal value on measures differ from the optimal value of the original problem: measures satisfying the transport equation may not correspond to solutions of the nonlinear PDE. In this case we say that there is a relaxation gap. In this chapter we prove that this does not happen for a broad class of nonlinear PDEs, namely evolution equation on Hilbert space with a nonlinear operator satisfying quasi-monotonicity conditions.

The important practical consequence of no relaxation gap is that we can guarantee the convergence of numerical approximation schemes based on an infinite-dimensional version of the moment-SOS hierarchy.

Our approach is illustrated numerically on a simple reaction-diffusion equation, but our setup allows readily extensions to optimization problems over PDEs, such as approximation of the region of attraction (defined as the largest set of initial data compatible with the equation and the constraints), or optimal control [136].

Further works in this line might be followed by considering the generator to be *locally quasi-dissipative*, which is a concept explained in details in [115, Chapter 6]. In particular, quasi-linear equations (including for instance conservation laws, the Korteweg-de Vries equation or the Kuramoto-Sivashinsky equation) can be studied through this framework, see [115, Chapter 6.9]. Indeed, in an earlier work [53], concentration of the measure-valued solution has been proved thanks to some contraction inequality (deduced from entropy inequalities) similar to the one obtained in the present chapter and that follows from the quasi-dissipative property of the generator.

Techniques from functional analysis distinct from dissipative arguments could also be used to prove the absence of a relaxation gap in the measure formulation. For example, the absence of relaxation gap for the problem of approximating the region of attraction of controlled ordinary differential equations was proved in [58] with the help of Ambrosio's superposition principle. In [137, Section 7.2], the authors first recall the superposition principle in finite-dimensional Euclidean spaces (for ordinary differential equations), then in $\mathbb{R}^\infty$ (e.g. for stochastic differential equations) and then in abstract metric spaces (e.g., evolutionary PDE in Hilbert space or Banach space). In all these setups, it is fundamental to have a deeper understanding of the moment problem for measures supported in such infinite-dimensional spaces. First attempts along these lines are reported in [138, 112, 128].

Appendix

Proof of Lemma 5.3.1

We use Stampacchia's truncation method as in the proof of [139, Theorem 10.3]. Let $G : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $G(s) = 0$ if $s \leq 0$ and $G'(s)$ is strictly positive and bounded for $s > 0$, and hence $G(s) \geq 0$ for all $s \in \mathbb{R}$. Let

$$H(s) := \int_0^s G(\sigma)d\sigma.$$

Let $y(t)$ denote a solution to (5.1) with operator (5.6), and define the function

$$V(t) := \int_0^1 H(y(t)[x] - y_{\max})dx. \tag{5.20}$$

For $t > 0$, it holds

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 G(y(t)[x] - y_{\max})\partial_t y(t)[x]dx \\
&= \int_0^1 G(y(t)[x] - y_{\max})\left(\partial_{xx} y(t)[x] + g(y(t)[x])\right)dx \\
&= [G(y(t)[x] - y_{\max})\partial_x y(t)[x]]_{x=0}^1 - \int_0^1 G'(y(t)[x] - y_{\max})(\partial_x y(t)[x])^2 dx \\
&\quad + \int_0^1 G(y(t)[x] - y_{\max})g(y(t)[x])dx.
\end{aligned}$$

Since $y(t)[0] = y(t)[1]$ and $\partial_x y(t)[0] = \partial_x y(t)[1]$, the first term on the right-hand size is zero. Since G' is positive and bounded, it follows that

$$\dot{V}(t) \leq \int_0^1 G(y(t)[x] - y_{\max})g(y(t)[x])dx.$$

Now observe that if $y[x] \leq y_{\max}$ then $G(y[x] - y_{\max}) = 0$, and if $y[x] \geq y_{\max}$ then $G(y[x] -$

$y_{\max}) \geq 0$ and $g(y)[x] \leq 0$. Hence $\dot{V}(t) \leq 0$. By construction, $V(t) \geq 0$ for all $t \geq 0$. If $y(0)[x] \leq y_{\max}$ for all $x \in [0,1]$ then $V(0) = 0$, which implies $V(t) = 0$ and hence from (5.20) it holds $y(t)[x] \leq y_{\max}$ for all $t \geq 0$ and $x \in [0,1]$.

Invariance of the lower bound follows along the same lines by replacing (5.20) with

$$V(t) := \int_0^1 H(y_{\min} - y(t)[x])dx,$$

ending therefore the proof.

Proof of Lemma 5.3.2

It follows readily from Definition 5.2.4 that the sum of quasi-dissipative operators is a quasi-dissipative operator. From Section 5.3.2, if g is quasi-dissipative then f in (5.6) is quasi-dissipative. So let us prove that g is quasi-dissipative on $\mathcal{Y}$.

Given $y_1, y_2 \in \mathcal{H}$ define the maps

$$y(\tau) := y_2 + \tau(y_1 - y_2), \quad G(\tau) := \langle y_1 - y_2, g(y(\tau))\rangle$$

for $\tau \in [0,1]$. Since G is continuous, there exists $\bar{\tau} \in [0,1]$ such that

$$G(1) - G(0) = \langle y_1 - y_2, g(y_1) - g(y_2)\rangle = \frac{dG}{d\tau}(\bar{\tau}). \tag{5.21}$$

Notice that

$$\frac{dG}{d\tau} = \langle y_1 - y_2, D_y g(y)\frac{dy}{d\tau}\rangle = \langle y_1 - y_2, D_y g(y)(y_1 - y_2)\rangle \tag{5.22}$$

where $D_y g(y)$ is the Fréchet derivative of g with respect to y. Note that, by definition, any Fréchet derivative is a linear and bounded operator. Therefore, defining

$$a := \max_{y \in \mathcal{Y}} \|D_y g(y)\|_{\mathfrak{L}(\mathcal{H})}$$

where $\mathfrak{L}(\mathcal{H})$ stands for the space of bounded operators having their domains and range equal to $\mathcal{H}$, and $\|.\|_{\mathfrak{L}(\mathcal{H})}$ is the associated operator norm, one has

$$\langle z, D_y g(y) z\rangle \leq a|z|^2 \tag{5.23}$$

for all $y \in \mathcal{Y}$ and $z \in \mathcal{H}$. Combining (5.22) and (5.23) and letting $z = y_1 - y_2$, it follows that

$$\frac{dG}{d\tau}(\tau) \leq a|y_1 - y_2|^2 \tag{5.24}$$

for all $y_1, y_2 \in \mathcal{Y}$ and all $\tau \in [0,1]$. Then for each given pair $y_1, y_2 \in \mathcal{Y}$, there is a value of $\bar{\tau} \in [0,1]$ such that (5.21) holds, and quasi-dissipativity inequality (5.2) then follows by plugging (5.21) into (5.24).

Statement and proof of Lemma 5.8.1

Lemma 5.8.1. *Suppose that f is quasi-dissipative and maximal. Consider an initial measure $\mu_0 \in \mathcal{M}_+(\mathcal{H}_1)$, and the associated strong measure-valued solution μ_t. Then, for any $t \in (0,1)$*

$$W_2(\mu_{t+h}, F_{h\#}\mu_t) = o(h) \tag{5.25}$$

where $F_h\colon z \mapsto z + hf(z)$, with $z \in \mathcal{H}_1$.

Proof. We need to prove that $f \in \mathrm{Tan}_{\mu_t} P_2(\mathcal{H})$, i.e., $f \in L^2(\mu_t; \mathcal{H})$ belongs to the tangent bundle at μ_t for almost every $t \in [0,1]$. Using [116, Prop. 8.4.5], this boils down to proving that

$$\Big(\int_{\mathcal{H}} |f(x)|^2 d\mu_t(x)\Big)^{1/2} \leq |\mu'|(t) \text{ for almost every } t \in [0,1]$$

where $|\mu'|(t) = \lim_{s\to t} \frac{W_2(\mu_s,\mu_t)}{|s-t|}$. Let $\mu_{t+h,t} \in \Gamma_o(\mu_t, \mu_{t+h})$ be the optimal transport plan between μ_t and μ_{t+h}. Then

$$\frac{|\mu_{t+h}(\phi) - \mu_t(\phi)|}{|h|} = \int_{\mathcal{H}\times\mathcal{H}} (\phi(x) - \phi(y)) d\mu_{t+h,t}(x)$$

$\forall \phi \in \mathrm{Cyl}([0,1] \times \mathcal{H})$. We note that $\mu_{t+h,t} \to (x,x)_{\#}\mu_t$ narrowly as $h \to 0$ and by letting $t \in [0,1]$ be the point where μ_t is metrically differentiable w.r.t. to t we obtain

$$\begin{aligned}\limsup_{h\to 0} \frac{|\mu_{t+h}(\phi) - \mu_t(\phi)|}{|h|} \leq \lim_{h\to 0} \frac{W_2(\mu_{t+h}, \mu_t)}{|h|} \Big(\int_{\mathcal{H}} (\nabla\phi(x))^2 d\mu_t(x)\Big)^{1/2} \\ = |\mu'|(t)\ ||\nabla\phi||_{L^2(\mu_t,\mathcal{H})}\end{aligned}$$

where we have used the Hölder inequality. Moreover

$$\begin{aligned}\Big|\int_0^1 \int_{\mathcal{H}} \partial_s \phi(t,x) d\mu_t(x) dt\Big| &= \Big|\lim_{h\to 0} \int_0^1 \int_{\mathcal{H}} \frac{\phi(t,x) - \phi(t-h,x)}{h} d\mu_t(x) dt\Big| \\ &\leq \int \lim_{h\to 0} \frac{|\mu_{t+h}(\phi) - \mu_t(\phi)|}{|h|} dt \\ &\leq \int |\mu'|(t)\ |\nabla\phi|_{L^2(\mu_t,\mathcal{H})} dt \\ &\leq \Big(\int_0^1 |\mu'|(t)^2 dt\Big)^{1/2} \Big(\int_0^1 |\nabla\phi(t,x)|^2 d\mu_t(x) dt\Big)^{1/2}\end{aligned}$$

where for the first inequality we used Fatou's lemma and for the last we used Cauchy Schwarz. Next we use the fact that μ_t satisfies the Liouville equation (5.7), so

$$\Big|\int_0^1 \int_{\mathcal{H}} \nabla\phi(t,x) \cdot f(t,x) d\mu_t(x) dt\Big| \leq \Big(\int_0^1 |\mu'|(t)^2 dt\Big)^{1/2} \Big(\int_0^1 |\nabla\phi(t,x)|^2 d\mu_t(x) dt\Big)^{1/2}$$

Dividing both sides by $|\nabla \phi|_{L^2(\mu_t dt)}$ we obtain

$$\frac{\langle \nabla \phi, f \rangle_{L^2(\mu_t dt)}}{|\nabla \phi|_{L^2(\mu_t dt)}} \leq |\mu'|(t)$$

which is the desired inequality as the left hand side is the $L^2(\mu_t dt)$ norm of f. □

Proof of Lemma 5.3.3

The maximality of (5.6) is equivalent to the existence of a positive value λ such that

$$\mathrm{Ran}(-f - \lambda \mathrm{I}_{\mathcal{H}}) = \mathcal{H}.$$

Given that the inclusion $\mathcal{H} \subset \mathrm{Ran}(-f - \lambda \mathrm{I}_{\mathcal{H}})$ is straigtforward, it suffices to prove $\mathcal{H} \subset \mathrm{Ran}(-f - \lambda \mathrm{I}_{\mathcal{H}})$. In other words, for $\bar{y} \in \mathcal{H}$, one must show that there exists $\tilde{y} \in D(f)$ such that

$$(\lambda \mathrm{I}_{\mathcal{H}} - A)\tilde{y} = \bar{y} + g(\tilde{y}),$$

where $A = \partial_{xx}$ with $D(A) = D(f)$. Note that any $\lambda > 0$ belongs to the resolvent of A since the latter is maximal dissipative. To prove the maximality of $-f$, we will use a fixed-point strategy, based on the following mapping:

$$\begin{aligned} \mathcal{T} : \ & \mathcal{H} \to D(A) \\ & y \mapsto (\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1} \left[\bar{y} + g(y) \right]. \end{aligned}$$

The operator A being closed and having a compact resolvent, then, invoking [140, Proposition 4.24], the injection from $D(A)$ to $\mathcal{H}$ is compact. Therefore, the set

$$\mathcal{B} := \{ y \in D(A) \mid |y|_{D(A)} \leq N \},$$

with N a positive constant to be defined later, is compact and convex, as a ball of radius N and centered at 0. Denoting by L_g the Lipschitz constant of g, one has

$$\begin{aligned} |\mathcal{T}(y)|_{D(A)} \leq & \|(\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1}\|_{\mathfrak{L}(\mathcal{H})} (|\bar{y}| + L_g |y|) \\ \leq & \|(\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1}\|_{\mathfrak{L}(\mathcal{H})} (|\bar{y}| + L_g C |y|_{D(A)}) \\ \leq & \|(\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1}\|_{\mathfrak{L}(\mathcal{H})} (|\bar{y}| + L_g C N) \end{aligned}$$

where C is the constant describing the compact injection of $D(A)$ in $\mathcal{H}$. According to [125, Corollary 2.3.3], one has $\|(\lambda \mathrm{I}_{\mathcal{H}} - A)^{-1}\|_{\mathfrak{L}(\mathcal{H})} \leq \frac{M}{\lambda - \omega}$ with M and ω given positive numbers and $\lambda > \omega$. Therefore, one can choose λ sufficiently large such that

$$|\mathcal{T}(y)|_{D(A)} \leq \frac{N}{2}.$$

Using the Schauder fixed-point theorem [141, Theorem B.17], one can easily deduce that the operator (5.6) is maximal.

Chapter 6

Conclusions and future perspectives

This thesis advances a measure-theoretic program for nonsmooth ODEs and nonlinear PDEs: evolution equations are lifted to linear transport dynamics on spaces of probability measures, and the resulting infinite-dimensional problems are tackled numerically with the moment-SOS hierarchy. Beyond algorithmic tractability, a central theme has been *exactness*: showing that the measure relaxations faithfully represent the original dynamics and control problems (i.e., no relaxation gap), and that discretizations converge to their continuous-time limits. We summarize the contributions chapterwise below.

Chapter 3: Evolution of measures for nonsmooth dynamical systems. We established a rigorous description of measure evolution in systems driven by evolution variational inequalities (EVIs), despite the underlying vector field being set-valued and maximally monotone. Three complementary constructions were developed. First, a superposition-based formulation represented measure solutions to the continuity equation (governing continuous-time measure evolution) as probability mixtures of admissible trajectories generated by measurable selections of the set-valued dynamics, yielding existence and uniqueness for the associated continuity equation. Second, a smoothing route via Moreau–Yosida regularization provided quantitative stability: solutions of the continuity equations with regularized vector field converged to measure-valued solutions to continuity equation with nonsmooth vector field in the Wasserstein distance W_2, with explicit rates derived from properties of the resolvent. Third, a time-discrete scheme addressed the measure evolution by pushing forward measures through the resolvent map; piecewise-constant and geodesic interpolations of these iterates converged (in W_2) to the continuous-time measure solution, and the limiting velocity belonged, in a weak sense, to the set-valued map. Each construction was illustrated numerically: moment–SOS relaxations approximated moments; explicit Wasserstein convergence bounds for the regularized flows were validated on a test case with a closed-form distance; and the resolvent-based time-stepping scheme was demonstrated, highlighting measure concentration effects.

Chapter 4: Optimal control of nonsmooth dynamical systems. In Chapter 4, we addressed the optimal control problem for the class of nonsmooth dynamical systems considered in Chapter 3. Following the classical relaxation program [25], we lifted admissible controls to Young measures [98], recast the problem as a linear program over occupation measures, and asked whether the passage to measures creates a relaxation gap in the optimal value [56]. The key results for the continuous–time problem are:

(i) Under maximal monotonicity of the differential inclusion and convexity of the running cost in the control, any bounded sequence of controls that generates a Young measure has the property that the associated state trajectories converge strongly to the trajectory of the Young–measure–relaxed dynamics induced by the limit Young measure.

(ii) The continuous–time occupation–measure formulation has no relaxation gap relative to the Young–measure formulation, by an application of the superposition principle.

We also established convergence of discrete–time optimal–control problems to their continuous counterpart. Adapting the stability estimates derived earlier in Chapter 3 for the uncontrolled dynamical system, we proved that every feasible sequence for the discrete–time problem admits an interpolated trajectory that converges, as the time step $h \to 0$, to a feasible trajectory of the continuous–time problem. We derived matching $\liminf$ and $\limsup$ inequalities for the cost functionals, ensuring convergence of optimal values. Using the moment–SOS hierarchy, the discrete–time problem was reformulated as a sequence of finite–dimensional convex programs, yielding certified lower bounds. Finally, we illustrated the numerical behavior of the approach on an academic example.

Chapter 5: Nonlinear PDEs and Liouville reformulation without relaxation gap. For a broad class of nonlinear evolution equations on Hilbert spaces whose generators are quasi-dissipative and maximal (including many reaction–diffusion equations with polynomial nonlinearities), we develop a rigorous framework for measure relaxation. We show that, for Dirac initial data, there is no relaxation gap between the nonlinear PDE and its linear Liouville equation reformulation: strong and generalized measure-valued solutions of the Liouville equation are unique in the class of measure-valued solutions and coincide with the law of the PDE solution. This equivalence legitimizes replacing nonconvex optimization over PDE trajectories by linear optimization over measures without changing the optimal value, and it underpins the use of an infinite-dimensional moment–SOS hierarchy with global convergence guarantees.

In this last part of the chapter, we outline three natural directions for future work: (i) developing a framework for measure evolution for sweeping processes with time evolution of sets having bounded variation regularity, together with the associated optimal control problems via measure relaxation; (ii) deriving optimality conditions for nonsmooth dynamics, through Pontryagin's Maximum Principle and Hamilton–Jacobi–Bellman equations; and (iii) extending the methodology to controlled reflected Brownian motion.

6.1 Measure evolution for sweeping processes with RCBV data

In many real systems the admissible set is not updated smoothly in time: constraints can change abruptly when a contact is created or lost, when safety regions are redefined after a sensor update, when operating modes switch, or when constraints are enforced only at sampling instants. We propose to extend the measure-evolution framework developed in this thesis to first-order sweeping processes whose data vary in time with right-continuous bounded variation (RCBV). The goal is to develop both (i) a discrete-time evolution and (ii) a Liouville-type (continuity) description that account for continuous transport *and* the jump events induced by time-discontinuous state constraints.

RCBV setting and trajectory notion

Let $S : [0,T] \rightrightarrows \mathbb{R}^n$ be time-varying set valued mapping with nonempty closed sets at each time and let $\kappa_S : [0,T] \to [0,\infty)$ be a right-continuous function of bounded variation such that

$$d_H\big(S(t_1), S(t_2)\big) \;\le\; \kappa_S\big([t_1,t_2)\big) \qquad \forall\, 0 \le t_1 \le t_2 \le T. \tag{6.1}$$

Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be Carathéodory with integrable linear growth: there exists $\alpha \in L^1(0,T)$ such that

$$|f(t,x)| \;\le\; \alpha(t)\,(1+|x|) \qquad \text{for a.e. } t \in [0,T],\ \forall x \in \mathbb{R}^n. \tag{6.2}$$

Definition 6.1.1 (RCBV solution)**.** A function $x : [0,T] \to \mathbb{R}^n$ is an *RCBV solution* of the sweeping process

$$\dot{x}(t) \;\in\; -\mathcal{N}_{S(t)}\big(x(t)\big) \;+\; f\big(t,x(t)\big)$$

if x has right limits, bounded variation on $[0,T]$, and there exists a nonnegative Radon measure ν on $[0,T]$ such that

$$d\nu \;\gg\; dt + d\kappa_S, \qquad dx \;\ll\; d\nu,$$

and for $d\nu$-a.e. $t \in [0,T]$ one has the differential inclusion in density form

$$\frac{dx}{d\nu}(t) \;\in\; f\big(t,x(t)\big)\,\frac{dt}{d\nu}(t) \;-\; \mathcal{N}_{S(t)}\big(x(t^+)\big). \tag{6.3}$$

Denote by $\mathrm{j}_S := \{\tau \in [0,T] :\ d\kappa_S(\{\tau\}) > 0\}$ the (at most countable) set of jump times of κ_S. On each maximal open interval $I \subset [0,T] \setminus \mathrm{j}_S$, the dynamics is pathwise well-posed and generates a measurable flow map

$$\mathbb{S}(s,t) : S(s) \to S(t), \qquad s \le t, \quad s,t \in I.$$

At each jump $\tau \in \mathrm{j}_S$, we postulate a *jump operator* given by a projection to $S(\tau)$,

$$J_\tau :\ S(\tau^-) \to S(\tau), \qquad J_\tau(x) \;\in\; \operatorname*{arg\,min}_{y \in S(\tau)} |y - x|. \tag{6.4}$$

(When $S(\tau)$ is closed and uniformly prox-regular or convex, J_τ is single-valued.)

Returning to Example 2.2, if $a(\cdot)$ is RCBV and

$$|a(t_2) - a(t_1)| \;\le\; \kappa_S\big((t_1,t_2]\big) \qquad \forall\, 0 \le t_1 < t_2 \le T,$$

then κ_S is admissible for the definition above. On continuity intervals I the evolution is governed by the flow $\mathbb{S}(s,t)$; at $\tau \in \mathrm{j}_S$ the state updates by the metric projection

$$x(\tau^+) = J_\tau\big(x(\tau^-)\big) = \arg\min_{y \in S(\tau)} |y - x(\tau^-)|,$$

which is exactly the jump rule encoded by the density inclusion (6.3).

Measure evolution with jumps

Let $\mu_0 \in \mathcal{P}(S(0))$ be a Borel probability measure. We define a right-continuous curve of measures $\{\mu_t\}_{t\in[0,T]} \subset \mathcal{P}(\mathbb{R}^n)$ by

$$\text{Continuous evolution on } I: \quad \mu_t \;=\; \mathbb{S}(s,t)_\# \, \mu_s, \qquad s \le t, \quad s,t \in I, \tag{6.5}$$

$$\text{Jump update at } \tau \in \mathrm{j}_S: \quad \mu_{\tau^+} \;=\; (J_\tau)_\# \, \mu_{\tau^-}. \tag{6.6}$$

On each continuity interval I, μ_t satisfies the standard Liouville equation driven by f; at jump times it is instantaneously pushed forward by J_τ.

Weak formulation with jump contributions. Let $\gamma \in \mathcal{M}_+\big((0,T)\times\mathbb{R}^n\big)$ be a measure with first marginal dt such that $d\gamma(t,x) = dt\,\mu_t(dx)$ for dt-a.e. t away from j_S. Then, for every $\varphi \in C_c^1\big([0,T]\times\mathbb{R}^n\big)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(T,x)\,d\mu_T(x) \;-\; \int_{\mathbb{R}^n} \varphi(0,x)\,d\mu_0(x) = \; & \int_{(0,T)\times\mathbb{R}^n} \Big(\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\cdot f(t,x)\Big)\,d\gamma(t,x) \\ & + \sum_{\tau\in\mathrm{j}_S} \int_{\mathbb{R}^n} \Big(\varphi(\tau, J_\tau(x)) - \varphi(\tau,x)\Big)\,d\mu_{\tau^-}(x). \end{aligned} \tag{6.7}$$

Proposed research problems

- **Well-posedness of measure evolution with RCBV data.** Prove existence, uniqueness, and stability (in Wasserstein or narrow topology) of $\{\mu_t\}_{t\in[0,T]}$ satisfying (6.5)–(6.6) and the weak formulation (6.7), under (6.1)–(6.2). Clarify minimal regularity ensuring a superposition principle linking μ_t to RCBV solutions $x(\cdot)$ of (6.3).

- **Numerical schemes.** Design time-discrete measure evolution schemes, and prove convergence to (6.7). Numerical experiments either via the moment-SOS approach from Chapter 3, or via the time-discretization methods of Section 3.6.3, with jump updates handled by $(J_\tau)_\#$, should also be investigated.

- **Optimal control in the space of measures with jumps.** Formulate continuous-time measure optimal control with dynamics (6.5)–(6.6), prove absence of relaxation gap, and derive first-order optimality conditions. Develop a discrete-time measure-control problem and prove convergence of optimal values and minimizers as the time step vanishes.

6.2 Optimality conditions using PMP and HJB equations

The PMP and the HJB equation provide two complementary, widely used routes to optimality conditions: PMP yields a trajectory-wise necessary condition in terms of an adjoint arc and a maximization rule, while HJB characterizes optimality through a value function and a dynamic-programming principle. In nonsmooth constrained dynamics such as sweeping processes, these tools are particularly valuable because they (i) guide the design and validation of numerical solvers [41], (ii) help detect structural properties of optimal controls (e.g., bang–bang behavior, boundary arcs), and (iii) provide verifiable certificates of (local) optimality beyond existence results. Given a first–order sweeping process with time–varying closed constraint sets $S(t) \subset \mathbb{R}^n$, Carathéodory drift $f(t,x,u)$ with integrable linear growth, and compact control set U, we aim to study the optimal control problem via the Moreau–Yosida (MY) regularization and pass to the limit $\lambda \to 0$.

Moreau-Yosida regularisation and optimality conditions

The Moreau-Yosida (MY) regularization of the first–order sweeping process leads to

$$\dot{x}^\lambda(t) = g_t^\lambda(x^\lambda(t), u(t)) := f(t, x^\lambda(t), u(t)) \; - \; \frac{1}{\lambda}\Big(x^\lambda(t) - \mathrm{proj}(x^\lambda(t), S(t))\Big), \tag{6.8}$$

which is globally Lipschitz in x (uniformly in t and u) and hence admits absolutely continuous solutions.

For a Bolza problem with running cost $\mathcal{L} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and terminal cost $\mathcal{L}_T : \mathbb{R}^n \to \mathbb{R}$, we define the Hamiltonian as

$$H_\lambda(t,x,u,p) := \langle p,\; f(t,x,u) \; - \; \tfrac{1}{\lambda}(x - \mathrm{proj}(x, S(t)))\rangle \; - \; \mathcal{L}(t,x,u). \tag{6.9}$$

PMP equation. Since the vector field is Lipschitz, one obtains the following (nonsmooth) Pontryagin-type optimality conditions for an optimal pair $(x_\lambda(\cdot), u_\lambda(\cdot))$:

$$\begin{aligned}
\dot{x}_\lambda(t) &= \partial_p H_\lambda(t, x_\lambda(t), u_\lambda(t), p_\lambda(t)),\\
-\dot{p}_\lambda(t) &\in \partial_x H_\lambda(t, x_\lambda(t), u_\lambda(t), p_\lambda(t))\\
&\quad = \; \partial_x \Big\langle p_\lambda(t),\; f(t, x_\lambda(t), u_\lambda(t)) - \tfrac{1}{\lambda}(x_\lambda(t) - \mathrm{proj}(x_\lambda(t), S(t)))\Big\rangle\\
&\qquad\qquad - \; \partial_x \mathcal{L}(t, x_\lambda(t), u_\lambda(t)),\\
u_\lambda(t) &\in \operatorname*{argmax}_{v \in U} \; H_\lambda(t, x_\lambda(t), v, p_\lambda(t)),\\
p_\lambda(T) &\in \partial \mathcal{L}_T(x_\lambda(T)).
\end{aligned} \tag{6.10}$$

(If the data are C^1 in x these become equalities with classical gradients.)

HJB equation. Let V_λ be the value function for the regularized problem. Then V_λ is a viscosity solution of

$$\partial_t V_\lambda(t,x) \; + \; \sup_{u \in U} \Big\{ \langle \nabla_x V_\lambda(t,x),\; f(t,x,u) - \tfrac{1}{\lambda}(x - \mathrm{proj}(x, S(t)))\rangle - \mathcal{L}(t,x,u) \Big\} = 0,$$
$$V_\lambda(T,x) = \mathcal{L}_T(x). \tag{6.11}$$

Proposed research problems

(P1) **Limit PMP.** *Limit* $\lambda \to 0$. To obtain optimality conditions for the unregularized sweeping process it suffices to show: (i) $x_\lambda(\cdot)$ is uniformly Lipschitz (or, more generally, has a uniform BV bound), and (ii) $p_\lambda(\cdot)$ has uniformly bounded variation. These yield (up to subsequences)

$$x_\lambda \to x \text{ uniformly on compact subintervals}, \qquad p_\lambda \rightharpoonup p \text{ weakly}^* \text{ in } BV([0,T];\mathbb{R}^n),$$

so that the MY term $\frac{1}{\lambda}(x_\lambda - \operatorname{proj}(x_\lambda, S(t)))$ converges to a normal-cone selection and the limiting pair (x,p) satisfies the PMP/HJB system for the original sweeping dynamics (with jump contributions when $t \mapsto S(t)$ has discontinuities). To summarize we would like to show that (along subsequences) $x_\lambda \to x$, $p_\lambda \rightharpoonup^* p$, and $\frac{1}{\lambda}(x_\lambda - \operatorname{proj}(x_\lambda, S(t))) \rightharpoonup \eta(t) \in \mathcal{N}_{S(t)}(x(t))$, so that (x,p) satisfies the PMP for the unregularized sweeping process, including the appropriate jump/transversality conditions when $t \mapsto S(t)$ has discontinuities.

(P2) **Limit HJB.** For the HJB conditions we aim to prove $V_\lambda \to V$ uniformly and identify V as a viscosity solution of the HJB associated with the sweeping dynamics (with jump contributions if $S(\cdot)$ is discontinuous in time).

6.3 Controlled reflected Brownian motion via measure relaxation

Reflected diffusions are a standard model for systems evolving under random perturbations while being constrained to remain feasible. Typical examples include robots navigating with uncertainty while avoiding obstacles, inventory or portfolio processes constrained by risk or regulatory limits, and queueing networks whose buffer contents cannot become negative. In all these settings, feasibility is enforced by an instantaneous "push" at the boundary, naturally modeled by reflection. Allowing $S(t)$ to vary in time is also practically motivated: safe regions change when obstacles move, constraints are tightened after state-estimation updates, and operational limits vary with external conditions. These considerations motivate extending the measure–relaxation program to controlled reflected diffusions with time-dependent constraints. We therefore consider controlled reflected diffusions constrained to $S(t) \subset \mathbb{R}^n$:

$$dX_t \in -\mathcal{N}_{S(t)}(X_t)\,dt + f(X_t,u_t)\,dt + dB_t, \tag{6.12}$$

where B_t is an n-dimensional Brownian motion, $u_t \in U$ is a progressively measurable control, and $\mathcal{N}_{S(t)}(x)$ is the normal cone to $S(t)$ at x. This generalizes reflected Brownian motion to time-varying constraints and controlled drift.

Constrained JKO scheme with explicit drift

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^n)$ with $\mathrm{supp}(\mu_0) \subset S(0)$. Following the constrained minimizing-movement approach, we consider the entropy-based JKO step for the *uncontrolled* case ($f \equiv 0$):

$$\mu_{k+1}^\tau \in \operatorname*{arg\,min}_{\mu\in\mathcal{P}_2(\mathbb{R}^n)} \Big\{\mathcal{E}(\mu) + \tfrac{1}{2\tau} W_2^2(\mu,\mu_k^\tau)\Big\} \quad \text{subject to} \quad \mathrm{supp}(\mu) \subset S\big((k+1)\tau\big), \tag{6.13}$$

with logarithmic entropy $\mathcal{E}(\mu) = \int \rho \log \rho \, dx$ if $\mu = \rho\, dx$ and $+\infty$ otherwise. To incorporate a *controlled* drift, we propose an explicit transport of the previous iterate:

$$\mu_{k+1}^\tau \in \operatorname*{arg\,min}_{\mathrm{supp}(\mu)\subset S((k+1)\tau)} \Big\{\mathcal{E}(\mu) \;+\; \tfrac{1}{2\tau}\, W_2^2\big(\mu,\; (\mathrm{Id} + \tau f(\cdot,u_k))_{\#}\mu_k^\tau\big)\Big\}. \tag{6.14}$$

Continuum limit: reflected Fokker–Planck with control

Assuming standard a priori estimates (uniform second moments and discrete metric-derivative bound $\sum_k \frac{1}{\tau} W_2^2(\mu_{k+1}^\tau, \mu_k^\tau)$), compactness in W_2 yields a limit curve $t \mapsto \mu_t$ as $\tau \to 0$. We expect $\{\mu_t\}$ to solve the controlled Fokker–Planck equation with reflection:

$$\partial_t \mu_t \;+\; \nabla\cdot\big(f(\cdot,u(t))\,\mu_t\big) \;-\; \tfrac{1}{2}\,\Delta\mu_t \;=\; 0 \tag{6.15}$$

supplemented with a zero-flux (Neumann) condition on $\partial S(t)$ induced by the reflection term $-\mathcal{N}_{S(t)}$ and with feasibility $\mathrm{supp}(\mu_t) \subset S(t)$. When $t \mapsto S(t)$ has jumps, the constraint acts as an instantaneous projection at the measure level (pushforward by the metric projection onto $S(t)$).

Proposed research problems

- **Well-posedness of the constrained JKO limit.** Establish existence of minimizers in (6.13)–(6.14) under mild assumptions on $S(\cdot)$ (e.g., closed, prox-regular, right-continuous BV in Hausdorff distance) and f (Carathéodory with linear growth). Prove tightness and W_2-compactness of $\{\mu^\tau\}$ and identify any limit as a distributional solution of (6.15) with reflective boundary condition and feasibility $\mathrm{supp}(\mu_t) \subset S(t)$.
- **Controlled problem and dynamic programming.** Formulate an infinite-dimensional optimal control on $\{\mu_t\}$ with dynamics given by (6.15), and establish a dynamic programming principle and a Hamilton–Jacobi equation on the Wasserstein space (or a dual HJB on test functions), including reflection effects.
- **Numerical schemes with certificates.** Develop implementable algorithms via (i) moment–SOS relaxations (when data are polynomial and $S(t)$ semialgebraic) to approximate low-order moments and certify feasibility, and (ii) particle or grid-based JKO solvers with projection onto $S((k+1)\tau)$. Compare accuracy and cost; test scenarios with moving/abrupt constraints.

Appendix A

Background on functional analysis

A.1 Weak convergence

Let $\mathcal{X}$ be a Banach space, endowed with a norm $\|\cdot\|_{\mathcal{X}}$. The topological dual of $\mathcal{X}$, denoted by $\mathcal{X}^*$, consists of all bounded linear functionals $\ell : \mathcal{X} \to \mathbb{R}$. The duality pairing is written as

$$\langle \ell, x\rangle_{\mathcal{X}^*,\mathcal{X}} = \ell(x) \quad \forall\, \ell \in \mathcal{X}^*,\ x \in \mathcal{X}.$$

Let $\mathcal{Y}$ be another Banach space. The operator norm of a bounded linear mapping $f : \mathcal{X} \to \mathcal{Y}$ is defined as

$$\|f\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup_{x\in\mathcal{X}\setminus\{0\}} \frac{\|f(x)\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

For example:

- If $\mathcal{X} = C([0,1])$, the space of continuous functions endowed with the supremum norm, then its dual is $\mathcal{X}^* = \mathcal{M}([0,1])$, the space of finite signed Borel measures endowed with the total variation norm.
- If $\mathcal{Y} = L^1([0,1])$, the space of integrable functions, then its dual is $\mathcal{Y}^* = L^\infty([0,1])$.

A key difference between finite-dimensional and infinite-dimensional Banach spaces is highlighted by the Heine–Borel theorem: in finite dimensions every closed and bounded set is compact, whereas this fails in infinite dimensions.

For instance, let $e_n = (0,\dots,0,1,0,\dots)$ be the standard basis of ℓ^2. Then $\|e_n\|_2 = 1$ for all n, but for $n \neq m$,

$$\|e_n - e_m\|_2 = \sqrt{2}.$$

Hence no subsequence is Cauchy, so the closed unit ball of ℓ^2 is not compact. On the other hand, if $y \in \ell^2$, i.e., $\sum_{k=0}^{\infty} |y_k|^2 < \infty$, then $\langle e_n, y\rangle = y_n \to 0$. Thus (e_n) converges weakly to 0, which illustrates the notion of weak convergence in ℓ^2.

Definition A.1.1 (Sequential weak convergence)**.** A sequence $\{x_n\} \subseteq \mathcal{X}$ is said to converge weakly to $x \in \mathcal{X}$ if

$$x_n \rightharpoonup x \quad \iff \quad \ell(x_n) \to \ell(x) \quad \forall \ell \in \mathcal{X}^*.$$

The following theorem characterizes the dual of continuous functions on a compact space.

Theorem A.1.1 (Riesz–Markov–Kakutani)**.** *Let K be a compact Hausdorff space. Then*

$$C(K)^* \cong \mathcal{M}(K),$$

where $\mathcal{M}(K)$ is the space of finite signed Borel measures on K, via the pairing

$$\langle \phi, \mu \rangle := \int_K \phi \, d\mu, \quad \phi \in C(K), \ \mu \in \mathcal{M}(K).$$

For example, let $K = [0, 1]$ and $g \in L^1(K)$. Then

$$\ell_g(\phi) := \int_0^1 \phi(x)\, g(x)\, dx, \qquad \forall \phi \in C(K),$$

defines a bounded linear functional on $C(K)$. By the Riesz–Markov–Kakutani theorem, ℓ_g corresponds to the unique finite signed Borel measure μ_g with $d\mu_g = g\, dx$ (i.e., $\mu_g \ll dx$ and $\frac{d\mu_g}{dx} = g$). For comprehensive treatment of related notions from functional analysis and measure theory, see [99, 139].

Definition A.1.2 (Weak derivatives)**.** Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{\text{loc}}(\Omega)$. A function $v \in L^1_{\text{loc}}(\Omega)$ is called the weak derivative of u in the i–th direction if

$$\int_\Omega u(x)\, \partial_i \phi(x)\, dx = -\int_\Omega v(x)\, \phi(x)\, dx \quad \forall \phi \in C_c^\infty(\Omega).$$

We then write $v = \partial_i u$ in the weak sense.

For example, let $u(x) = |x|$ on $\Omega = \mathbb{R}$. Then u is not C^1 at the origin, but its weak derivative is given by $v(x) = \text{sign}(x)$ (with $\text{sign}(0)$ defined arbitrarily, say 0).

Definition A.1.3 (Sobolev space)**.** Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) := \Big\{ u \in L^p(\Omega) \,:\, D^\alpha u \in L^p(\Omega) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \le k \Big\},$$

where $D^\alpha u$ denotes the weak derivative of order α.

For example, let $u(x) = |x|$ on $\mathbb{R}$. Then $u \notin L^p(\mathbb{R})$ and its weak derivative $Du(x) = \text{sign}(x) \notin L^p(\mathbb{R})$ for any $p \ge 1$. However, on every bounded interval $(-r, r)$, $r > 0$, we have u, $Du \in L^p((-r, r))$, so

$$u \in W^{1,p}_{\text{loc}}(\mathbb{R}) \quad \text{but} \quad u \notin W^{1,p}(\mathbb{R}).$$

In the special case $p = 2$, the Sobolev space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$; these spaces are Hilbert spaces when equipped with the norm

$$\|u\|^2_{H^k(\Omega)} = \sum_{\alpha \le k} \|D^\alpha u\|_{L^2(\Omega)}.$$

For detailed exposition of topics related to Sobolev spaces, we refer to [142].

A.2 C_0-semigroups

To analyze the abstract Cauchy problem

$$\dot{y}(t) = Ay(t), \quad y(0) = y_0 \in D(A),$$

it is natural to encode the time evolution by a family of bounded operators $(\mathbb{S}(t))_{t\geq 0}$ acting on $\mathcal{X}$. When such a family exists with $y(t) = \mathbb{S}(t)y_0$, the differential relation is recovered in the sense that

$$\frac{d}{dt}\,\mathbb{S}(t)x\Big|_{t=0} = Ax \quad \text{for } x \in D(A).$$

Semigroup theory provides the right framework for this passage: it replaces solving the differential equation directly (which is delicate when A is unbounded) by studying properties of $(\mathbb{S}(t))_{t\geq 0}$ and its generator; see, e.g., [51, 142]. Characterization theorems (such as those of Hille–Yosida and Lumer–Phillips) link analytic properties of A with the existence of an associated C_0-semigroup.

Definition A.2.1. A C_0*-semigroup* of bounded linear operators on $\mathcal{X}$ is a one-parameter family $(\mathbb{S}(t))_{t\geq 0}$ of bounded linear mappings $\mathbb{S}(t) : \mathcal{X} \to \mathcal{X}$ such that

- $\mathbb{S}(0) = \mathrm{id}_{\mathcal{X}}$,
- $\mathbb{S}(t+s) = \mathbb{S}(t)\mathbb{S}(s)$ for all $s, t \geq 0$,
- $\lim_{t\to 0^+} \|\mathbb{S}(t)x - x\|_{\mathcal{X}} = 0$ for every $x \in \mathcal{X}$.

A C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ is called a *contraction semigroup* if

$$\|\mathbb{S}(t)\|_{\mathcal{L}(\mathcal{X})} \leq 1, \quad \forall t \geq 0.$$

The (infinitesimal) generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ is the (in general unbounded) operator A defined by

$$Ax := \lim_{t\to 0^+} \frac{\mathbb{S}(t)x - x}{t}, \tag{A.1}$$

whenever this limit exists in $\mathcal{X}$. The domain $D(A)$ of A is thus

$$D(A) := \Big\{ x \in \mathcal{X} \,\Big|\, \text{the limit in (A.1) exists} \Big\}.$$

If $(\mathbb{S}(t))_{t\geq 0}$ is a contraction semigroup on a Hilbert space $\mathcal{H}$ with generator A, then:

- $(0, \infty) \subset \rho(A)$, i.e., every $\lambda > 0$ belongs to the resolvent set of A;
- for all $\lambda > 0$,

  $$\|R(\lambda, A)\| \leq \frac{1}{\lambda},$$

  where $R(\lambda, A) := (\lambda I - A)^{-1}$ denotes the resolvent of A;

- more generally, for all $\lambda > 0$ and $n \in \mathbb{N}$,

$$\|R(\lambda, A)^n\| \leq \lambda^{-n}.$$

Moreover, if $(\mathbb{S}(t))$ is contractive, then for every λ with $\operatorname{Re}\lambda > 0$ and every $w \in \mathcal{X}$, the resolvent admits the Laplace representation

$$R(\lambda, A)w = \int_0^\infty e^{-\lambda t}\,\mathbb{S}(t)w\,\mathrm{d}t, \qquad \|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re}\lambda}.$$

For comprehensive treatment of semigroups, we refer to [51].

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Titre : Évolution des mesures et contrôle optimal de systèmes dynamiques quasi-dissipatifs à l'aide de relaxations Moment-SOS

Mots clés : Évolution des mesures, contrôle optimal, système dynamique non lisse, optimisation semi-définie, relaxations moments–SOS, optimisation semidéfinie

Résumé : L'optimisation des systèmes dynamiques non linéaires est difficile, et l'on exploite généralement des propriétés structurelles particulières des dynamiques pour analyser leur évolution et concevoir des lois de commande. Une classe de systèmes au cœur de cette thèse est celle des systèmes dynamiques emph{quasi-dissipatifs}, que l'on peut voir comme des systèmes comportant une composante dissipative perturbée par un terme additionnel. Compte tenu des limites des méthodes classiques d'optimisation pour de tels systèmes, cette thèse développe un cadre unifié pour leur analyse et leur optimisation au moyen de techniques de relaxation par mesures. La hiérarchie moments–sommes de carrés (SOS) constitue la principale motivation computationnelle de cette approche, car elle fournit des solutions globalement optimales avec des garanties de convergence.

Les méthodes classiques de commande et d'optimisation deviennent difficiles à appliquer lorsque les dynamiques comportent des discontinuités, des contraintes unilatérales ou des applications multivoques.
Nous étendons le formalisme de relaxation mesure-théorique à ces contextes non lisses. La première contribution établit une formulation rigoureuse de l'évolution de mesures pour les processus de balayage du premier ordre, en démontrant l'existence, l'unicité et une représentation de superposition des solutions à valeurs mesures. Une approche de régularisation fonctionnelle et un schéma de transport optimal discrétisé en temps sont développés pour approximer ces solutions, avec des garanties de convergence pour des métriques de Wasserstein. Dans ce cadre, nous formulons une relaxation semi-définie basée sur la hiérarchie moments–SOS pour résoudre le problème d'évolution de mesures.

La deuxième contribution traite de la commande optimale des processus de balayage via des techniques de relaxation par mesures. Nous montrons que la relaxation en un programme linéaire dans l'espace des mesures n'introduit aucun écart de relaxation, en temps continu comme en temps discret. En utilisant des outils de la théorie du transport optimal, nous prouvons la convergence du problème discrétisé vers le problème continu lorsque le pas d'échantillonnage tend vers zéro. Une relaxation semi-définie fondée sur moments–SOS est proposée pour résoudre le problème de commande optimale à valeurs mesures.

Enfin, le cadre est étendu aux équations d'évolution non linéaires quasi-dissipatives sur des espaces de Hilbert, incluant des EDP semilinéaires et de type réaction–diffusion. Nous montrons que la formulation par mesures de ces problèmes de dimension infinie reste exacte, sans écart de relaxation, et nous proposons une hiérarchie moments–SOS convergente afin d'obtenir des approximations numériques certifiées.

Dans l'ensemble, cette thèse combine l'analyse variationnelle, le transport optimal et la programmation semi-définie pour fournir des formulations convexes globalement convergentes pour une large classe de problèmes de commande optimale non lisses et de problèmes d'EDP.

Title: Measure Evolution and Optimal Control of Quasi-Dissipative Dynamical Systems using Moment-SOS Relaxations

Key words: Measure evolution, optimal control, non-smooth dynamical system, quasi-dissipative evolution equation, moment-SOS relaxations, semidefinite optimization

Abstract: Optimization of nonlinear dynamical systems is challenging, and one typically exploits specific structural properties of the dynamics to analyze their evolution and design control laws. A particular class of systems that is central to this thesis is that of {quasi-dissipative} dynamical systems, which can be viewed as systems with a dissipative component perturbed by an additional term. Given the limitations of classical optimization methods for such dynamical systems, this thesis develops a unified framework for their analysis and optimization using measure relaxation techniques. The moment–sums-of-squares (SOS) hierarchy is the main computational motivation for this approach, as it provides globally optimal solutions with convergence guarantees.
Classical control and optimization methods become difficult to apply when system dynamics involve discontinuities, unilateral constraints, or set-valued mappings.
We extend the measure-theoretic relaxation formalism to these nonsmooth settings. The first contribution establishes a rigorous formulation of measure evolution for first-order sweeping processes, proving existence, uniqueness, and a superposition representation of measure-valued solutions. A functional regularization approach and a time-discretized optimal transport scheme are developed to approximate these solutions, with convergence guarantees in Wasserstein metrics. Within this framework, we formulate a moment–SOS based semidefinite relaxation to solve the measure evolution problem.
The second contribution addresses optimal control of sweeping processes using measure relaxation techniques. We show that relaxing to a linear program in the space of measures introduces no relaxation gap in both continuous and discrete time. Using tools from optimal transport theory, we prove convergence of the discretized problem to the continuous one as the sampling interval tends to zero. A moment-SOS based semidefinite relaxation is proposed to solve the measure-valued optimal control problem.

Finally, the framework is extended to quasi-dissipative nonlinear evolution equations on Hilbert spaces, including semilinear and reaction diffusion partial differential equations. We show that the measure formulation of these infinite dimensional problems remains exact, with no relaxation gap, and we propose a convergent moment-SOS hierarchy to obtain certified numerical approximations.

Overall, the thesis combines variational analysis, optimal transport, and semidefinite programming to provide globally convergent convex formulations for a broad class of nonsmooth optimal control and PDE problems.