Converse Lyapunov Theorem for Switched Nonlinear Systems with Constrained Switching Signals

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Abstract: This paper investigates converse Lyapunov theorems for switched nonlinear systems comprising both stable and unstable subsystems, uniformly over a constrained set of switching signals. A novel hybrid timer is introduced to quantify switching behavior, and the considered class of signals–characterized by a uniformly bounded hybrid timer–encompasses known signal classes defined by mixed average dwell-time and average activation-time conditions. The main result is a necessary and sufficient condition, expressed via the existence of multiple Lyapunov functions with prescribed decay or growth rates at flows and jumps, ensuring global uniform asymptotic stability uniformly over this set of switching signals. The proof employs extended classes of comparison functions and a constructive analysis tailored to hybrid-timer-based signal models. This work extends previous results, offering a deeper understanding of stability in switched systems with both stabilizing and destabilizing dynamics.

Keywords: Converse Lyapunov theorem; switched systems; multiple Lyapunov functions; average dwell-time; average activation time

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1. INTRODUCTION

Switched systems–whose dynamics evolve according to a family of subsystems and a switching signal–form a powerful modeling framework for complex engineering applications such as robotics, automotive control, and power electronics. The behavior and stability of such systems critically depend on the nature of the switching signal, whether time-driven, state-dependent, or stochastic (Liberzon, 2003; Boukas, 2006).

Lyapunov methods are among the most widely used tools for stability analysis in nonlinear and hybrid systems. Their appeal lies in their constructive nature and their suitability for both analytical reasoning and computational verification (Khalil, 2002). However, their practical usefulness hinges on the guarantee that an appropriate Lyapunov function *exists* whenever the system is stable–a guarantee provided by the *converse Lyapunov theorem.* From a theoretical standpoint, converse results ensure the completeness of Lyapunov's second method: stability can be characterized–not merely certified–by the existence of a Lyapunov function (Massera, 1956). Without such results, stability analysis might rely on ad hoc constructions whose existence cannot be formally assured. From an algorithmic perspective, converse theorems underpin many modern computational stability verification frameworks. For example, piecewise-quadratic or non-quadratic Lyapunov methods (Johansson and Rantzer, 1997; Blanchini, 1995) and sum-of-squares programming (Papachristodoulou and Prajna, 2002; Ahmadi and Parrilo, 2011) implicitly rely on converse-type guarantees to justify the search over specific function classes.

For switched systems, Lyapunov-based analysis takes several forms. If a *common Lyapunov function* exists, uniform stability is guaranteed under arbitrary switching (Liberzon, 2003), and converse results are available for this case (Dayawansa and Martin, 1999; Mancilla-Aguilar and García, 2000). However, it is well known that certain switched systems are uniformly stable only with respect to a restricted class of switching signals, and may fail to remain uniformly stable under arbitrary switching. In such cases, stability analysis often relies on the *multiple Lyapunov function* (MLF) approach, in which each subsystem is assigned its own Lyapunov function (Branicky, 1998). Extensive results have been obtained in this direction. Depending on whether the subsystems are stable or unstable, and whether switching acts to stabilize or destabilize the overall system, various constraints on the switching signals have been proposed–such as average dwell-time (ADT) or reverse ADT (Hespanha and Morse, 1999; Hespanha et al., 2008), average activation time (AAT) (Zhai et al., 2001; Müller and Liberzon, 2012), and mode-dependent conditions (Kundu and Chatterjee, 2015; Liu and Tanwani, 2025)–under which uniform stability can still be ensured. This naturally leads to a fundamental converse question: if a switched system is stable only with respect to a restricted class of switching signals, must there exist a set of MLFs with the properties required to certify that stability?

An implicit answer to such questions can be found by embedding switched systems as hybrid systems and looking at the construction of Lyapunov functions for this generic class of systems (Cai et al., 2008). However, there have been dedicated research studies on more explicit converse results for constrained switching scenarios. Building on the techniques introduced in the early work by Wirth (2005), converse Lyapunov results for systems with lower and upper bounds on switching intervals have been established by Protasov and Kamalov (2023); Della Rossa (2024). In Protasov and Kamalov (2023), converse Lyapunov theorems are obtained for linear systems under interval dwell-time constraints, proving that exponential stability is equivalent to the existence of Lyapunov multinorms adapted to the prescribed switching interval. On the other hand, Della Rossa (2024) extends the analysis to nonlinear switched systems under interval dwell-time constraints, providing two distinct converse multiple Lyapunov function characterizations of uniform stability. More recently, Della Rossa and Tanwani (2025) present converse Lyapunov results for switched nonlinear systems under ADT constraints. They prove that a tailored version of stability is equivalent to the existence of suitable multiple Lyapunov functions, and illustrate that the conditions differ fundamentally from the fixed minimum dwell-time case. Nevertheless, this result does not allow unstable subsystems.

Developing based on the work of Della Rossa and Tanwani (2025), we analyze the stability of switched systems comprising both stable and unstable subsystems in this paper. Our focus is on the set of switching signals introduced in our previous work (Liu et al., 2022)–a generalization of the combined ADT and AAT conditions–and on establishing a converse Lyapunov theorem that holds uniformly over this set. The main technical contributions are as follows:

1. By jointly counting the number of switches and accumulating the activation time of potentially unstable modes, we introduce a *hybrid timer* on the switching signal. This leads to the definition of *global uniform boundedness with hybrid timer characterization* (GUB w/HT).
2. We construct MLFs with prescribed growth/decay rates at flows and switches, and establish a converse Lyapunov theorem. In particular, we prove that the existence of such MLFs is both necessary and sufficient for a system to be GUB w/HT uniformly over switching signals with finite hybrid timer.
3. We show that GUB w/HT uniformly over switching signals with finite hybrid timer implies global uniform asymptotic stability (GUAS) over the set of switching signals in Liu et al. (2022). Hence, we validate that the sufficiency part of our main theorem is consistent with the stability criteria in the literature.

The rest of the paper is organized as follows. Section 2 introduces switching signals characterized by a hybrid timer, followed by the definitions of switched systems and stability-related notions. Section 3 presents the main stability theorem, and Section 4 contains the proof of the converse result. Finally, Section 6 concludes the paper with remarks on future work.

Notations. The sets of real and nonnegative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. For any $x \in \mathbb{R}^n$, let $|x|$ denote its Euclidean norm. By abuse of notation, $|\Omega|$ denotes the cardinality of a countable set Ω. For a mathematical relation $\mathcal{A}$, define

$$\mathbf{1}_{\mathcal{A}} := \begin{cases} 1 & \text{if } \mathcal{A} \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

We adopt some standard comparison function notations from Khalil (2002). A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

2. PRELIMINARIES ON SWITCHING SIGNALS AND SWITCHED SYSTEMS

The behavior of a switched system is determined by its governing switching signal. This section first recalls the class of admissible switching signals as studied in the work of Liu et al. (2022), and introduces a hybrid timer defined on such signals. We then formalize the switched system model under consideration and present the stability-related notions that will be used throughout the paper.

2.1 Switching signals with bounded hybrid timer

Let $\mathcal{P}$ be a countable set of modes, possibly finite or infinite, and let Σ denote the set of all right-continuous, piecewise-constant functions $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$. The elements of Σ are referred to as *switching signals*.

For any $\sigma \in \Sigma$, $t_2 > t_1 \geq 0$, and $\mathcal{P}_u \subset \mathcal{P}$, define

$$N_\sigma(t_1, t_2) := \left|\{t \in (t_1, t_2] : \lim_{s\to t^-} \sigma(s) \neq \sigma(t)\}\right|,$$

$$M_{\mathcal{P}_u,\sigma}(t_1, t_2) := \int_{t_1}^{t_2} \mathbf{1}_{\{\sigma(t)\in\mathcal{P}_u\}}\, \mathrm{d}t.$$

Here, $N_\sigma(t_1, t_2)$ counts the number of switches in $(t_1, t_2]$, while $M_{\mathcal{P}_u,\sigma}(t_1, t_2)$ measures the total time that modes in $\mathcal{P}_u$ are active over $(t_1, t_2]$.

Given constants $c > 1$ and $d > 0$, and a subset $\mathcal{P}_u \subseteq \mathcal{P}$, define for $\sigma \in \Sigma$ the function $\mathcal{T}_{c,d,\mathcal{P}_u,\sigma} : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ by

$$\mathcal{T}_{c,d,\mathcal{P}_u,\sigma}(t_1, t_2) := \begin{cases} d\, N_\sigma(t_1, t_2) + c\, M_{\mathcal{P}_u,\sigma}(t_1, t_2) + t_1 - t_2, & t_2 > t_1, \\ 0, & t_2 \leq t_1. \end{cases} \tag{1}$$

Since $c, d, \mathcal{P}_u$ are fixed throughout the paper, we abbreviate $\mathcal{T}_{c,d,\mathcal{P}_u,\sigma}$ as $\mathcal{T}_\sigma$. This is the *hybrid timer* defined for the signal σ, such that the value $\mathcal{T}_\sigma(t_1, t_2)$ reflects a weighted combination of switching activity and activation of modes in $\mathcal{P}_u$ over $(t_1, t_2]$: it increases with either more frequent switching or longer activation of modes in $\mathcal{P}_u$, and decreases linearly at rate -1 when σ dwells in modes in $\mathcal{P} \setminus \mathcal{P}_u$ without switching.

Two straightforward properties on the hybrid timer can be concluded from its definition:

1. $\mathcal{T}_\sigma(t_1, t_2)$ is continuous at $t_1 = t_2$ because switching signals are right-continuous.

(2) $\mathcal{T}_\sigma$ satisfies a semigroup property: for any $t_3 \geq t_2 \geq t_1 \geq 0$,
$$\mathcal{T}_\sigma(t_1, t_3) = \mathcal{T}_\sigma(t_1, t_2) + \mathcal{T}_\sigma(t_2, t_3).$$

For each $T \in \mathbb{R}_{\geq 0}$, define
$$\Sigma(T) := \{\mathcal{T}_\sigma(t_1, t_2) \leq T \ \forall t_2 \geq t_1 \geq 0\}. \tag{2}$$
This is the set of switching signals for which the hybrid timer is uniformly bounded from above by T. Further define
$$\Sigma^\star := \bigcup_{T \in \mathbb{R}_{\geq 0}} \Sigma(T). \tag{3}$$
This set consists of all switching signals that have finite hybrid timers.

Remark 1. (On the domains of the parameters c, d). Note that if $c \leq 1$, then $cM_{\mathcal{P}_u,\sigma}(t_1, t_2) + t_1 - t_2 \leq 0$ for any $t_2 \geq t_1$. Because $\mathcal{P}_u$ in this work will be used to represent the set of potentially unstable modes, this leads to the situation where $\sigma_u \in \Sigma^\star$, with σ_u being a constant signal activating an unstable mode. Apparently, under such a switching signal, the system is unstable, and a switched system is not uniformly stable over any class of switching signals including σ_u. Therefore, we need to exclude this case for stability analysis. On the other hand, we require $d > 0$ to exclude Zeno behavior, as stated in the next lemma.

Lemma 2. $\Sigma^\star$ is Zeno-free. That is, for any $\sigma \in \Sigma^\star$ and $t_2 > t_1 \geq 0$, $N_\sigma(t_1, t_2) < \infty$.

Its proof is straightforward and omitted here.

Remark 3. (Comparison with literature). In the works by Müller and Liberzon (2012); Liu et al. (2022), a mixed condition on average dwell time (ADT) and average activation time (AAT) is studied. Specifically, ADT means the existence of $N_0, \tau_a > 0$ such that
$$N_\sigma(t_1, t_2) \leq N_0 + \frac{t_2 - t_1}{\tau_a} \tag{4}$$
for all $t_2 > t_1 \geq 0$, whereas AAT means the existence of $T_0 > 0$, $\eta \in [0, 1]$ such that
$$M_{\mathcal{P}_u,\sigma}(t_1, t_2) \leq T_0 + \eta(t_2 - t_1) \tag{5}$$
for all $t_2 > t_1 \geq 0$. These works study stability conditions for switched systems uniformly over a class of switching signals defined by
$$\begin{aligned}&\Sigma_{\mathrm{mix}}(N_0, T_0, \tau_a, \eta)\\ &\qquad := \{\sigma \in \Sigma : \ (4) \text{ and } (5) \text{ hold } \forall\, t_2 > t_1 \geq 0\}.\end{aligned}$$
In Liu et al. (2022), we can find conditions on the parameters N_0, T_0, τ_a, η in terms of the Lyapunov functions associated with each subsystem to obtain a class of switching signals in terms of system data for which asymptotic stability hold uniformly. We will revisit these conditions in the next section, but here we observe that, for a given N_0, T_0, τ_a, η, there exist c, d, T such that
$$\Sigma_{\mathrm{mix}}(N_0, T_0, \tau_a, \eta) \subset \Sigma(T). \tag{6}$$
To see this, we simply take c, d such that
$$c\eta + \frac{d}{\tau_a} = 1 \tag{7}$$
then, with $T := dN_0 + cT_0$, we have
$$\begin{aligned}&dN_\sigma(t_1, t_2) + cM_{\mathcal{P}_u,\sigma}(t_1, t_2) + t_1 - t_2\\ &\leq dN_0 + cT_0 + (c\eta + \frac{d}{\tau_a})(t_2 - t_1) + t_1 - t_2\\ &= dN_0 + cT_0 = T.\end{aligned} \tag{8}$$
Thus, under condition (7), for each $\sigma \in \Sigma_{\mathrm{mix}}(N_0, T_0, \tau_a, \eta)$, we have $\mathcal{T}_\sigma(t_1, t_2) \leq T$ for all $t_2 \geq t_1 \geq 0$, and hence $\Sigma_{\mathrm{mix}}(N_0, T_0, \tau_a, \eta) \subset \Sigma(T) \subset \Sigma^\star$.

2.2 System and stability definitions

Consider vector fields $f_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, $i \in \mathcal{P}$, such that each $f_i(t, x)$ is continuous in t and locally Lipschitz in x, uniformly over t. The dynamics of a switched system is given by
$$\dot{x} = f_{\sigma(t)}(t, x) \quad \text{for almost all } t \in \mathbb{R}_{\geq 0}, \tag{9}$$
where $x \in \mathbb{R}^n$ is the state, $t \in \mathbb{R}_{\geq 0}$ is time, and $\sigma \in \Sigma^\star$ is a switching signal. We assume that 0 is an equilibrium of the system, i.e., $f_i(t, 0) = 0$ for all $t \in \mathbb{R}_{\geq 0}$ and $i \in \mathcal{P}$. Let $\Phi_\sigma(t, t_0, \xi)$ denote the solution of (9) at time t, subject to the switching signal σ, with initial state ξ at initial time t_0, i.e., $\Phi_\sigma(t_0, t_0, \xi) = \xi$.

Let $\Sigma' \subset \Sigma$ be a set of switching signals. We propose the following two properties on switched systems that are uniform over Σ'. The first property is adapted from (Della Rossa and Tanwani, 2025).

Definition 4. The switched system (9) is *globally uniformly bounded with hybrid timer characterization* (equivalently, the system is GUB w/HT), uniformly over Σ' if there exist $\theta_1, \theta_2 \in \mathcal{K}_\infty$, constant $\kappa > 0$ such that
$$\theta_1\big(|\Phi_\sigma(t, t_0, \xi)|\big) \leq \theta_2\big(|\xi|\big)e^{\mathcal{T}_\sigma(t_0,t) - \kappa(t - t_0)} \tag{10}$$
for all $t \geq t_0 \geq 0$, $\xi \in \mathbb{R}^n$, and $\sigma \in \Sigma'$.

The second is a stability notion.

Definition 5. The equilibrium 0 is *globally uniformly asymptotically stable* for the switched system (9) (equivalently, the system is GUAS), uniformly over Σ' if there exist $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{K}_\infty$, constant $\tilde{\kappa} > 0$ such that
$$\tilde{\theta}_1\big(|\Phi_\sigma(t, t_0, \xi)|\big) \leq \tilde{\theta}_2\big(|\xi|\big)e^{-\tilde{\kappa}(t - t_0)} \tag{11}$$
for all $t \geq t_0 \geq 0$, $\xi \in \mathbb{R}^n$, and $\sigma \in \Sigma'$.

Remark 6. (Connection between GUB w/HT and GUAS). Note that since $\mathcal{T}_\sigma(t_0, t)$ can be arbitrarily large when no assumption is imposed on the set Σ', the state of the system can diverge to infinity under the bound (10) when t approaches to infinity. Hence GUB w/HT is not a stability notion. Nevertheless, for any $T \in \mathbb{R}_{\geq 0}$ and $\sigma \in \Sigma(T)$, we have $\mathcal{T}_\sigma(t_0, t) \leq T$ for all $t \geq t_0 \geq 0$. Hence if a switched system is GUB w/HT uniformly over $\Sigma^\star$, then by defining $\tilde{\theta}_1(s) = \theta_1(s)$, $\tilde{\theta}_2(s) = \theta_1(s)e^T$ and $\tilde{\kappa} = \kappa$, (11) holds for all $t \geq t_0 \geq 0$, $\xi \in \mathbb{R}^n$, and $\sigma \in \Sigma(T)$. In other words, if a switched system (9) is GUB w/HT uniformly over $\Sigma^\star$, then the system is GUAS uniformly over $\Sigma(T)$ for any $T \in \mathbb{R}_{\geq 0}$. The converse–implication from GUAS uniformly over $\Sigma(T)$ for any $T \in \mathbb{R}_{\geq 0}$ to GUB w/HT uniformly over $\Sigma^\star$–does not hold in general. In order for it to hold, we additionally need that

- The function $\tilde{\theta}_1$ and constant $\tilde{\kappa}$ are the same for all sets $\Sigma(T)$,
- The function $\tilde{\theta}_2$ for all sets $\Sigma(T)$ can be expressed as $\tilde{\theta}_2(s) = \theta_1(s)e^T$ for some $\theta_1 \in \mathcal{K}_\infty$.

Remark 7. (On the parameter κ). Note that since $t \geq t_0$, for any $\kappa' \in (0, \kappa)$, we have
$$e^{\mathcal{T}_\sigma(t_0,t) - \kappa(t - t_0)} \leq e^{\mathcal{T}_\sigma(t_0,t) - \kappa'(t - t_0)}.$$

In other words, the parameter κ not only serves as a ratio between the equivalent effects of time and the hybrid timer, but it can also be chosen arbitrarily small as long as the system is GUB w/HT.

3. NECESSARY AND SUFFICIENT CONDITION FOR GUB W/HT

We begin this section with the main result stated below.

Theorem 8. Consider the switched system (9) with a countable set of modes $\mathcal{P}$. Let $c > 1$, $d > 0$, $\mathcal{P}_u \subset \mathcal{P}$, and let $\Sigma^\star$ be defined as in (3). Then, the system is GUB w/HT uniformly over $\Sigma^\star$ if and only if there exist $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous in first argument and locally Lipschitz in second argument, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and constants $\lambda_u < c-1$, $\lambda_s > 1$, $\mu = e^d$ such that, for all $t \geq 0$, $x \in \mathbb{R}^n$, and $i, j \in \mathcal{P}$ with $i \neq j$:

$$\alpha_1(|x|) \leq V_i(t,x) \leq \alpha_2(|x|), \tag{12}$$

$$D^+_{f_i} V_i(t,x) \leq \lambda_u V_i(t,x), \qquad \forall i \in \mathcal{P}_u, \tag{13a}$$

$$D^+_{f_i} V_i(t,x) \leq -\lambda_s V_i(t,x), \qquad \forall i \notin \mathcal{P}_u, \tag{13b}$$

$$V_j(t,x) \leq \mu V_i(t,x), \qquad \forall i \neq j. \tag{14}$$

Here, $D^+_{f_i}$ denotes the Dini derivative along the vector field f_i:

$$D^+_{f_i} V_i(t,x) := \limsup_{h \to 0^+} \frac{V_i(t+h, \Phi_i(t+h,t,x)) - V_i(t,x)}{h}.$$

We immediately provide the proof of sufficiency here, i.e., that the existence of multiple Lyapunov functions implies the switched system is GUB w/HT uniformly over $\Sigma^\star$. Fix $c > 1$, $d > 0$, and $\mathcal{P}_u \subset \mathcal{P}$. Assume there exist locally Lipschitz functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and constants $\lambda_u < c-1$, $\lambda_s > 1$ such that, for all $t \geq 0$, $x \in \mathbb{R}^n$, and $i, j \in \mathcal{P}$ with $i \neq j$, the inequalities (12), (13), and (14) hold. Choose $\delta \in (0, \min\{c-1-\lambda_u, \lambda_s - 1\})$; then we have $\lambda_u \leq c-1-\delta$ and $\lambda_s \geq 1+\delta$.

Following the standard approach for MLFs (Branicky, 1998), we combine the effects of flows (13) and jumps (14) to evaluate $V_{\sigma(t)}(t, \Phi_\sigma(t,t_0,\xi))$ for any $t \in \mathbb{R}_{\geq 0}$ and $\xi \in \mathbb{R}^n$, yielding

$$\begin{aligned} V_{\sigma(t)}&(t,\Phi(t,t_0,\xi)) \\ &\leq e^{dN_\sigma(t_0,t) - \lambda_s((t-t_0) - M_{\mathcal{P}_u,\sigma}(t_0,t))} \\ &\qquad \times e^{\lambda_u M_{\mathcal{P}_u,\sigma}(t_0,t)} V(t_0,\xi) \\ &\leq e^{dN_\sigma(t_0,t) - (1+\delta)((t-t_0) - M_{\mathcal{P}_u,\sigma}(t_0,t))} \\ &\qquad \times e^{(c-1-\delta) M_{\mathcal{P}_u,\sigma}(t_0,t)} V(t_0,\xi) \\ &= e^{dN_\sigma(t_0,t) - (1+\delta)(t-t_0) + cM_{\mathcal{P}_u,\sigma}(t_0,t)} V(t_0,\xi) \\ &= e^{\mathcal{T}_\sigma(t_0,t) - \delta(t-t_0)} V(t_0,\xi) \end{aligned}$$

By applying (12), we obtain

$$\alpha_1(|\Phi(t,t_0,\xi)|) \leq e^{\mathcal{T}_\sigma(t_0,t) - \delta(t-t_0)} \alpha_2(|\xi|),$$

which yields (10) with $\theta_1 = \alpha_1, \theta_2 = \alpha_2$ and $\kappa = \delta$.

The proof of necessity, corresponding to the converse result, is considerably more involved. We defer it to the next section. We remark that since $\lambda_s > 1$, the inequality (13b) guarantees that subsystems with indices $i \notin \mathcal{P}_u$ are asymptotically stable. Accordingly, the set $\mathcal{P}_u$ contains potentially unstable subsystems, whose states are allowed to diverge, as the condition $\lambda_u < c-1$ does not preclude $\lambda_u > 0$. At the same time, since $d > 0$, the gain at switches satisfies $\mu = e^d > 1$, so the inequality (14) reflects that all switches are considered to impose a destabilizing effect.

We also compare the sufficient MLF conditions with the stability criteria in (Liu et al., 2022). Suppose there exist MLFs V_i, class $\mathcal{K}_\infty$ functions α_1, α_2, and constants $\lambda_u > 0$, $\lambda_s > 1$, $\mu > 1$ such that (12), (13), and (14) hold. Then, the switched system can be shown to be GUAS uniformly over $\Sigma_{\text{mix}}(N_0, T_0, \tau_a, \eta)$ by (Liu et al., 2022, Theorem 1) under the following condition (15),

$$\left(1 + \frac{\lambda_u}{\lambda_s}\right)\eta + \frac{\ln \mu}{\lambda_s \tau_a} < 1, \tag{15}$$

On the other hand, let $\lambda'_s := 1 < \lambda_s$ and $\lambda'_u := \frac{1}{\eta}\big(1 - \frac{\ln\mu}{\tau_a}\big) - 1 > \lambda_u$, so that

$$\left(1 + \frac{\lambda'_u}{\lambda'_s}\right)\eta + \frac{\ln \mu}{\lambda'_s \tau_a} = 1. \tag{16}$$

As discussed in Remark 3, we then have the inclusion $\Sigma_{\text{mix}}(N_0, T_0, \tau_a, \eta) \subset \Sigma(T) \subset \Sigma^\star$, with the constants given by $c = 1 + \lambda'_u$, $d = \ln\mu$, and $T = N_0 \ln\mu + (1+\lambda'_u)T_0$. Additionally, since $\lambda_u < c-1$ and $\mu = e^d$, Theorem 8 implies that the switched system is GUB w/HT uniformly over $\Sigma^\star$, which in turn guarantees GUAS uniformly over $\Sigma_{\text{mix}}(N_0, T_0, \tau_a, \eta)$ by Remark 6. Therefore, the sufficiency part of Theorem 8 is consistent with the results in the literature.

4. TECHNICAL DETAILS FOR THE CONVERSE LYAPUNOV THEOREM

In this section, we present the proof of the necessity part of Theorem 8. The proof relies on a novel construction of MLFs that incorporates interval-replacement techniques for switching signals.

For any $\sigma \in \Sigma^\star$, $t \geq 0, h > 0, i \in \mathcal{P}$, define a new switching signal $\sigma^i_{t,h} \in \Sigma^\star$ by

$$\sigma^i_{t,h}(s) = \begin{cases} i, & s \in [t, t+h), \\ \sigma(s), & \text{otherwise.} \end{cases} \tag{17}$$

In other words, $\sigma^i_{t,h}$ is obtained by replacing the signal σ with a constant signal i over the interval $[t, t+h)$. This immediately implies that

$$\mathcal{T}_{\sigma^i_{t,h}}(t, t+h) = (c\mathbf{1}_{\{i \in \mathcal{P}_u\}} - 1)h + d\mathbf{1}_{\{i \neq \sigma(t+h)\}}. \tag{18}$$

Proof. [Proof of the necessity part of Theorem 8] When the switched system (9) is GUB w/HT uniformly over $\Sigma^\star$, there exist $\theta_1, \theta_2 \in \mathcal{K}_\infty$ and $\kappa > 0$ such that (10) holds for all $t \geq t_0 \geq 0$, $\xi \in \mathbb{R}^n$, and $\sigma \in \Sigma^\star$. Define sets

$$\Sigma^\star_{i,t} := \{\sigma \in \Sigma^\star : \sigma(t) = i\}.$$

In other words, $\Sigma^\star_{i,t}$ is a set of switching signals whose modes are i at time t. Construct the functions $V_i(t,x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for all $i \in \mathcal{P}$ as

$$V_i(t,x) := \sup_{\sigma \in \Sigma^\star_{i,t}} \sup_{s \geq 0} e^{-\mathcal{T}_\sigma(t,t+s) + \kappa s} \theta_1\big(|\Phi_\sigma(t+s,t,x)|\big). \tag{19}$$

We first check the condition (12). Note that the choice of $s = 0$ immediate leads to the inequality $V_i(t,x) \geq \theta_1(|x|)$.

On the other hand, it follows from (10) that $V_i(t,x) \leq \theta_2(|x|)$ for all $i \in \mathcal{P}$. Hence, the sandwich condition (12) is proven with $\alpha_1 = \theta_1, \alpha_2 = \theta_2$.

The proof for Lipschitz continuity of V_i is omitted here. It can be proven similarly as the ones for (Della Rossa and Tanwani, 2025, Lemma B.1) or (Mironchenko, 2023, Theorem B.31).

We next show (13) and (14). For each pair $i, j \in \mathcal{P}$ and $t \geq 0$, we pick arbitrary $h > 0$ and $\sigma \in \Sigma^\star_{j,t+h}$. Consider the switching signal $\sigma^i_{t,h}$. It follows from (18) and the semigroup property of $\mathcal{T}_\sigma$ that for any $s \geq 0$,

$$\begin{aligned}
&\mathcal{T}_{\sigma^i_{t,h}}(t, t+h+s) \\
&\qquad = \mathcal{T}_{\sigma^i_{t,h}}(t, t+h) + \mathcal{T}_{\sigma^i_{t,h}}(t+h, t+h+s) \\
&\qquad = (c\mathbf{1}_{\{i\in\mathcal{P}_u\}} - 1)h + d\mathbf{1}_{\{i\neq j\}} + \mathcal{T}_{\sigma^i_{t,h}}(t+h, t+h+s) \\
&\qquad = (c\mathbf{1}_{\{i\in\mathcal{P}_u\}} - 1)h + d\mathbf{1}_{\{i\neq j\}} + \mathcal{T}_{\sigma}(t+h, t+h+s).
\end{aligned}$$

where the fact that $\sigma^i_{t,h}(\tau) = \sigma(\tau)$ for all $\tau \geq t+h$ is used for the last equality. Meanwhile, it holds that

$$\Phi_{\sigma^i_{t,h}}(t+h+s, t, x) = \Phi_\sigma(t+h+s, t+h, \Phi_i(t+h, t, x)).$$

we therefore obtain that

$$\begin{aligned}
&\sup_{s\geq 0}\Big(e^{-\mathcal{T}_\sigma(t+h,t+h+s)+\kappa s} \\
&\qquad\quad \times \theta_1\big(|\Phi_\sigma(t+h+s, t+h, \Phi_i(t+h,t,x))\big)\Big) \\
&\leq \sup_{s\geq 0}\Big(e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1)h + d\mathbf{1}_{\{i\neq j\}} - \mathcal{T}_{\sigma^i_{t,h}}(t,t+h+s)+\kappa s} \\
&\qquad\quad \times \theta_1\big(|\Phi_{\sigma^i_{t,h}}(t+s+h, t, x)|\big)\Big) \\
&= e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1-\kappa)h + d\mathbf{1}_{\{i\neq j\}}} \sup_{s^*\geq h}\Big(e^{-\mathcal{T}_{\sigma^i_{t,h}}(t,t+s^*)+\kappa s^*} \\
&\qquad\quad \times \theta_1\big(|\Phi_{\sigma^i_{t,h}}(t+s^*, t, x)|\big)\Big) \\
&\leq e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1-\kappa)h + d\mathbf{1}_{\{i\neq j\}}} \sup_{\sigma^*\in\Sigma^\star_{i,t}} \sup_{s^*\geq 0}\Big(e^{-\mathcal{T}_{\sigma^*}(t,t+s^*)+\kappa s^*} \\
&\qquad\quad \times \theta_1\big(|\Phi_{\sigma^*}(t+s^*, t, x)|\big)\Big) \\
&= e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1-\kappa)h + d\mathbf{1}_{\{i\neq j\}}} V_i(t,x),
\end{aligned}$$

where the fact $\sigma^i_{t,h} \in \Sigma^\star_{i,t}$ and a change of variable $s^* := h + s$ are used. Since $\sigma \in \Sigma^\star_{j,t+h}$ is arbitrary, the above inequality also holds when its left-hand side is replaced by the supremum over $\Sigma^\star_{j,t+h}$; i.e.,

$$\begin{aligned}
&V_j(t+h, \Phi_i(t+h,t,x)) \\
&\quad = \sup_{\sigma\in\Sigma^\star_{j,t+h}} \sup_{s\geq 0}\Big(e^{-\mathcal{T}^\star_\sigma(t+h)+\kappa s} \\
&\qquad\quad \times \theta_1\big(|\Phi_\sigma(t+h+s, t+h, \Phi_i(t+h,t,x))|\big)\Big) \\
&\quad \leq e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1-\kappa)h + d\mathbf{1}_{\{i\neq j\}}} V_i(t,x).
\end{aligned} \tag{20}$$

To show the inequalities on the flow (13), set $j = i$. We conclude from (20) that

$$\begin{aligned}
D^+_{f_i} V_i(t,x) &= \limsup_{h\to 0^+} \frac{V_i(t+h, \Phi_i(t+h,t,x)) - V_i(t,x)}{h} \\
&\leq \limsup_{h\to 0^+} \frac{e^{(c\mathbf{1}_{\{i\in\mathcal{P}_u\}}-1-\kappa)h} - 1}{h} V_i(t,x) \\
&= (c\mathbf{1}_{\{i\in\mathcal{P}_u\}} - 1 - \kappa) V_i(t,x)
\end{aligned}$$

Let $\lambda_u := c - 1 - \kappa$ and $\lambda_s = 1 + \kappa$. Since κ can be arbitrarily small as discussed in Remark 7, the inequalities in (13) are proven with any $\lambda_u < c - 1$ and $\lambda_s > 1$.

To show the inequality on the jumps (14), set $j \neq i$. We conclude from (20) that

$$V_j(t+h, \Phi_i(t+h,t,x)) \leq e^{(c-1-\kappa)h+d} V_i(t,x),$$

regardless whether $i \in \mathcal{P}_u$ or not. By continuity of V_j, the above inequality also holds by taking $h \to 0^+$. In other words,

$$\begin{aligned}
&V_j(t,x) - e^d V_i(t,x) \\
&\qquad = \lim_{h\to 0^+}\big(V_j(t+h, \Phi_i(t+h,t,x)) - e^d V_i(t,x)\big) \\
&\qquad \leq \lim_{h\to 0^+}\Big(e^{(c-1-\kappa)h} - 1\Big) e^d V_i(t,x) \\
&\qquad = 0.
\end{aligned}$$

Thus (14) is proven with $\mu = e^d$.

5. EXISTENCE OF NORMS FOR LINEAR SUBSYSTEMS

As a special case of our converse result, we now consider the linear autonomous systems and show that the Lyapunov functions that we associate with each subsystem are actually norms in that case. To see this, we consider the system class

$$\dot{x} = A_{\sigma(t)} x \tag{21}$$

where $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$.

Definition 9. The switched system (21) is *globally uniformly exponentially bounded with hybrid timer characterization* (equivalently, the system is GUEB w/HT), uniformly over Σ' if there exist constants $M > 1, \kappa > 0$ such that

$$|\Phi_\sigma(t, 0, \xi)| \leq M e^{\mathcal{T}_\sigma(0,t) - \kappa t} |\xi|, \tag{22}$$

for all $t \geq 0$, $\xi \in \mathbb{R}^n$, and $\sigma \in \Sigma'$.

Corollary 10. System (21) is GUEB w/HT over $\Sigma^\star$ if and only if there exist norms $v_j : \mathbb{R}^n \to \mathbb{R}$, $j \in \mathcal{P}$, and constants $\lambda_u < c - 1$, $\lambda_s > 1$, $\mu = e^d$ such that, for all $t \geq 0$, $x \in \mathbb{R}^n$, and $i, j \in \mathcal{P}$ with $i \neq j$:

$$D^+_{A_i} v_i(x) \leq \lambda_u\, v_i(x), \qquad \forall i \in \mathcal{P}_u, \tag{23a}$$

$$D^+_{A_i} v_i(x) \leq -\lambda_s\, v_i(x), \qquad \forall i \notin \mathcal{P}_u, \tag{23b}$$

$$v_j(x) \leq \mu\, v_i(x), \qquad \forall i \neq j. \tag{24}$$

Proof. The proof basically follows from Theorem 8. The sufficiency is trivial. For the the necessity, recalling Definition 9, there exist $\kappa > 0$ and $M \geq 1$ such that (22) holds for every $\sigma \in \Sigma^\star$, every $x \in \mathbb{R}^n$ and for all $t \geq 0$.

Recalling that $\Sigma^\star_{i,0} = \{\sigma \in \Sigma^\star \mid \sigma(0) = i\}$, we define, for every $i \in \mathcal{P}$,

$$V_i(x) := \sup_{\sigma\in\Sigma^\star_{i,0}} \sup_{s\geq 0} e^{-\mathcal{T}_\sigma(0,s)+\kappa s} |\Phi_\sigma(s, 0, x)|.$$

$$v_i(x) := \sup_{\sigma\in\mathcal{S}_i} \sup_{s\geq 0} \frac{e^{(\rho+\alpha)t}}{e^{\alpha\tau N_\sigma(0,s)}} |\Phi_\sigma(s, x)|.$$

By (22), it holds that

$$|x| \leq v_i(x) \leq M|x|, \quad \forall\, i \in \mathcal{P},\ \forall x \in \mathbb{R}^n.$$

The inequalities in (23) and (24) can now be proved with the arguments presented in proof of Theorem 8. It remains to show that v_i are norms. For this, we recall that, by linearity, the flows maps of (21) are linear, i.e.,

$$x \mapsto \Phi_\sigma(t, 0, x) \text{ is linear}, \ \forall\, \sigma \in \Sigma^\star, \forall\, t \geq 0.$$

For this reason, for any $\gamma \in \mathbb{R}$, we have

$$\begin{aligned} v_i(\gamma x) &= \sup_{\sigma \in \Sigma^\star_{i,0}} \sup_{s \geq 0} e^{-\mathcal{T}_\sigma(0,s)+\kappa s} |\Phi_\sigma(s, 0, \gamma x)| \\ &= |\gamma| \sup_{\sigma \in \Sigma^\star_{i,0}} \sup_{s \geq 0} e^{-\mathcal{T}_\sigma(0,s)+\kappa s} |\Phi_\sigma(s, 0, x)| = |\gamma| v_i(x), \end{aligned}$$

while for the triangular inequality, considering $x_1, x_2 \in \mathbb{R}^n$ we have

$$\begin{aligned} v_i(x_1 + x_2) &= \sup_{\sigma \in \Sigma^\star_{i,0}} \sup_{s \geq 0} |\Phi_\sigma(s, 0, x_1 + x_2)| \\ &\leq \sup_{\sigma \in \Sigma^\star_{i,0}} \sup_{s \geq 0} |\Phi_\sigma(s, 0, x_1)| \\ &\quad + \sup_{\sigma \in \Sigma^\star_{i,0}} \sup_{s \geq 0} |\Phi_\sigma(s, 0, x_2)| \\ &= v_i(x_1) + v_i(x_2), \end{aligned}$$

concluding the proof.

6. CONCLUSION

This work investigates stability characterization of switched systems comprising both stable and unstable subsystems over a constrained set of switching signals, in terms of MLFs. By modeling switching signals using a hybrid timer, we propose a novel property for switched systems, termed GUB w/HT, which implies GUAS when switching signals are restricted to a smaller set where the hybrid timer is uniformly bounded. Our main result establishes the equivalence between GUB w/HT and the existence of MLFs with prescribed growth and decay rates during flows and jumps. In particular, the sufficiency part aligns with known results in the literature, while the necessity part extends our understanding of converse Lyapunov theorems for switched systems.

Future research could extend this study to switched systems with inputs. Specifically, converse results on the existence of MLFs for input-to-state stability (Sontag, 1989) and integral input-to-state stability (Angeli et al., 2000) could be investigated. Moreover, it is known that switched systems with only unstable subsystems can achieve asymptotic stability uniformly over switching signals whose dwell times are bounded in an interval Xiang and Xiao (2014); Della Rossa (2024). By introducing multiple hybrid timers and considering only switching signals with all finite hybrid timers, one could impose both upper and lower bounds on the average number of switches and the average activation time of unstable modes. Studying converse results over such sets of switching signals may reveal the underlying mechanisms that allow stability for switched systems composed entirely of unstable subsystems.

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