On Asymptotic Stability of Hybrid Systems with Frequent Updates and Sampled-Data Observers

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***Abstract*— This paper investigates the stability of a class of hybrid systems featuring rapidly occurring discrete transitions, analyzed through the lens of singular perturbation theory. The considered model consists of the interconnection of two hybrid subsystems, a timer governing the jump instants, and discrete variables determining the indices of the jump maps. The evolution of these variables during flows is described by singularly perturbed differential equations, where smaller values of the perturbation parameter correspond to increased jump frequency. In the limiting case of this parameter, the system is decomposed into a quasi steady-state subsystem, modeled by a continuous differential equation without jumps, and a boundary-layer subsystem governed by purely discrete dynamics. Building upon our previous work that established practical stability, this paper derives sufficient conditions for the asymptotic stability of a compact attractor by imposing suitable assumptions on both the quasi steady-state and boundary-layer subsystems. As an application, we address the design of observers for nonlinear systems with time-sampled measurements and show that detectability of the system ensures asymptotic stability of the estimation error under an appropriate detectability condition.**

***Index Terms*— Hybrid system; Singular perturbation; Asymptotic stability; Sampled-data observers.**

I. Introduction

Analytical tools from nonlinear dynamical systems theory have long been instrumental in studying complex systems with multi-scale dynamics and interconnected subsystems. In particular, the hybrid systems framework [1] has provided a powerful foundation for analyzing control design problems over networks, where continuous-time evolution interacts with discrete events such as switching communication topologies or sampled-data updates. Extending this line of research, the present work employs tools from nonlinear and singular perturbation analysis within a hybrid systems setting to address the observer design problem for nonlinear systems with time-sampled measurements.

The theoretical setup adopted in this work is based on an interconnected hybrid system comprising two subsystems: one governed by stable continuous dynamics and another modeled as a discrete-time switching system, potentially with unstable continuous evolution. The overall system structure is analyzed using singular perturbation methods, which have classically been applied to ordinary differential equations with distinct slow and fast dynamics [2]–[4]. The key idea is to interpret the full system as a perturbation of a nominal composite system consisting of a quasi steady-state (QSS) subsystem, describing the slow evolution in the limit of vanishing fast dynamics, and a boundary-layer (BL) subsystem capturing the fast, discrete transitions. Similar decompositions have proven effective in analyzing robustness properties such as input-to-state stability under slowly varying parameters and rapidly varying signals [5].

Recent years have witnessed a growing interest in extending singular perturbation and multi-time-scale analysis techniques to hybrid and switching systems. Early efforts primarily addressed cases where fast evolution appears only in the continuous-time dynamics. In [6] and [7], averaging-based techniques were employed to show that solutions of the boundary-layer subsystem can be used to construct an averaged vector field approximating the slow dynamics, thereby establishing semiglobal practical stability for the overall system. Singular perturbation ideas have also been applied to switched and linear hybrid systems, as in [8], [9], where stability and performance bounds were derived under varying switching and perturbation regimes. More recent studies have expanded these approaches to broader classes of hybrid and networked systems. For instance, [10] provides the framework unifying averaging and singular perturbation methods for control and optimization problems across both ODEs and hybrid dynamical systems, while [11] presents generalized stability results for systems exhibiting both slow and fast switching and time variations.

In a related line of work, [12] introduced a class of singularly perturbed hybrid systems to analyze networks with frequently switching communication graphs. There, the perturbation parameter directly governed the rate of switching, and the stability analysis relied on decomposing the system into continuous and discrete components in a specific interconnection structure, leading to conditions for practical stability. The present paper builds upon and extends the framework of [12]. While our previous work established practical stability for such hybrid systems, the current work provides sufficient conditions for asymptotic stability of a compact attractor under appropriate assumptions on the QSS and boundary-layer subsystems.

As an application of the theoretical results, we study sampled-data observers for nonlinear systems and provide design criteria ensuring asymptotic convergence of the estimation error under a suitable detectability condition. Observer design for hybrid and switching systems has been an active area of research over the past decade, motivated by the need to reconstruct system states under discontinuous

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or event-driven dynamics. Certain studies in this direction, notably [13], [14], and [15], have developed rigorous frameworks for observer synthesis in switched and hybrid systems, addressing challenges such as switching observability, hybrid jump dynamics, and uncertainty in jump times. These works typically exploit structural properties of hybrid dynamics to derive sufficient conditions for convergence of the estimation error, often through Lyapunov-based or detectability arguments. In contrast, for sampled-data observers, where measurements are available intermittently at discrete instants, the analysis often seeks conditions expressed directly in terms of the underlying continuous-time system. For example, [16] proposed a self-triggered continuous-discrete observer with an adaptive sampling period, establishing stability results under suitable detectability and smoothness assumptions. Building upon these ideas, the present work extends this line of research by formulating the sampled-data observer design problem within the framework of singularly perturbed hybrid systems. The resulting observer dynamics can be expressed as an interconnection of continuous- and discrete-time subsystems, where the time elapsed between successive measurement updates acts as a singular perturbation parameter. Under a detectability assumption on the continuous-time system, we derive sufficient conditions for asymptotic stability of the estimation error. A key feature of this formulation is its conceptual simplicity: the stability conditions follow directly from the detectability property, without requiring additional structural constraints or time-scale separation assumptions beyond those inherent in the sampling process.

The remainder of the paper is organized as follows. Section II introduces the class of hybrid systems under consideration and presents the decomposition into the quasi steady-state and boundary-layer subsystems used for analysis. Section III states the main stability result, providing sufficient conditions for asymptotic stability of a compact attractor under appropriate assumptions on the subsystems. Section IV illustrates the applicability of the theoretical framework to the observer design problem for nonlinear systems with time-sampled measurements, where the sampling interval plays the role of the perturbation parameter. Section V contains detailed proofs of the main result, and Section VI concludes the paper with remarks and possible directions for future work.

II. System Class

In this paper, we consider hybrid dynamical systems described by ordinary differential equations on a flow set and difference equations on a jump set. A distinctive feature of the considered models is that they are *singularly perturbed*, meaning that a parameter $\varepsilon > 0$ appears in the differential equations, parameterizing the class of systems under study.

A. Overall Model

We are interested in modeling the evolution of two variables $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. The dynamics of the variable x are governed by an ordinary differential equation, whereas the evolution of y is defined by both a flow map and a collection of jump maps. The jump instants are determined by a timer variable τ, and the index of the jump map is given by a discrete variable $\mathsf{p} \in \mathcal{P}$.

The state variables of the overall system are $(x, y, \tau, \mathsf{p}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathcal{P}$. We consider the sets $C_x \subset \mathbb{R}^n$, $C_y \subset \mathbb{R}^m$, and the discrete set $\mathcal{P}$. The state evolves inside the flow set

$$\mathcal{C} := C_x \times C_y \times [0, N_0] \times \mathcal{P},$$

for some positive integer $N_0 \in \mathbb{N}$, and undergoes instantaneous changes inside the jump set

$$\mathcal{D} := C_x \times C_y \times [1, N_0] \times \mathcal{P}.$$

The evolution of the state variables is described by

$$\begin{cases} \dot{x} = f_x(x, y, \epsilon), \\ \dot{y} = f_y(x, y, \epsilon), \\ \epsilon\dot{\tau} \in [\sigma_1, \sigma_2], \\ \dot{\mathsf{p}} = 0, \end{cases} \quad \text{for } (x, y, \tau, \mathsf{p}) \in \mathcal{C}, \tag{1a}$$

$$\begin{cases} x^+ = x, \\ y^+ = g_{\mathsf{p}}(x, y), \\ \tau^+ = \tau - 1, \\ \mathsf{p}^+ \in \mathsf{S}(\mathsf{p}), \end{cases} \quad \text{for } (x, y, \tau, \mathsf{p}) \in \mathcal{D}. \tag{1b}$$

We assume that the sets $C_x \subset \mathbb{R}^n$ and $C_y \subset \mathbb{R}^m$ are closed and contained in open, connected domains $\mathscr{D}_x \subset \mathbb{R}^n$ and $\mathscr{D}_y \subset \mathbb{R}^m$, respectively. The vector fields $f_x : \mathscr{D}_x \times \mathscr{D}_y \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $f_y : \mathscr{D}_x \times \mathscr{D}_y \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are continuously differentiable in all arguments. For the timer dynamics, the constants σ_1 and σ_2 satisfy $\sigma_2 \geq \sigma_1 > 0$. For the jump maps, we assume that for each $\mathsf{p} \in \mathcal{P}$, the function $g_{\mathsf{p}} : \mathscr{D}_x \times \mathscr{D}_y \to \mathbb{R}^m$ is locally Lipschitz continuous.

In the problems considered in this paper, the parameter ϵ in (1) is assumed to be small, which corresponds to frequent jumps. Consequently, the evolution of y is predominantly governed by the jump dynamics. Since the jumps represent rapid evolution, we refer to x as the *slow variable* and y as the *fast variable*.

B. Auxiliary Dynamical Systems

As stated earlier, we are interested in analyzing the behavior of system (1) when the parameter ϵ is sufficiently small, i.e., $\epsilon \approx 0$. In this regime, the trajectories $(x(\cdot), y(\cdot))$ can be approximated by two auxiliary dynamical systems.

The first is the *boundary-layer system*, defined for a fixed value of $\mathsf{x} \in C_x$ as

$$\begin{aligned} \mathsf{y}^+ &= g_{\mathsf{p}}(\mathsf{x}, \mathsf{y}), \\ \mathsf{x}^+ &= \mathsf{x}, \\ \mathsf{p}^+ &\in \mathsf{S}(\mathsf{p}), \end{aligned} \tag{2}$$

for $\mathsf{y} \in C_y$, $\mathsf{p} \in \mathcal{P}$. A small value of ϵ implies that the timer τ evolves rapidly, leading to many jumps within a short continuous-time interval. Hence, starting from an initial condition $(x(0,0), y(0,0))$, while $x(\cdot)$ remains close to $x(0,0)$, the variable $y(\cdot)$ evolves according to the discrete-time switched system (2).

To define an equilibrium for the discrete system (2), we assume the existence of a continuously differentiable function $h:\mathbb{R}^n \to \mathbb{R}^m$ such that

$$g_{\mathsf{p}}(\mathsf{x}, h(\mathsf{x})) = h(\mathsf{x}), \quad \forall\, \mathsf{x} \in C_x,\ \forall\, \mathsf{p} \in \mathcal{P}. \tag{3}$$

Intuitively, $y(t,j)$ rapidly converges to $h(x(0,0))$ after several jumps and remains near $h(x(t,j))$ as time evolves. The evolution of $x(t,j)$ can thus be approximated by the *quasi steady-state (QSS) system*:

$$\dot{\mathsf{x}} = f_x(\mathsf{x}, h(\mathsf{x}), 0), \quad \mathsf{x} \in C_x, \tag{4}$$

which represents the dynamics on the *slow manifold* $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y = h(x)\}$. Finally, we assume that $h(\mathsf{x}) \in C_y$ for all $\mathsf{x} \in C_x$ to ensure well-posedness.

C. Stability Assumptions

We now impose stability assumptions on the boundary-layer and quasi steady-state systems. These properties will be used to establish the stability of the full system (1) for sufficiently small ϵ.

Assumption 1. The quasi steady-state system (4) admits an asymptotically stable compact attractor $\mathcal{A}_x \subset C_x$, and there exists a continuously differentiable function $V_x : \mathscr{D}_x \to \mathbb{R}$ such that, for all $\mathsf{x} \in C_x$ and every $\epsilon \in [0,1]$,

$$\underline{a}|\mathsf{x}|^2_{\mathcal{A}_x} \le V_x(\mathsf{x}) \le \overline{a}|\mathsf{x}|^2_{\mathcal{A}_x}, \tag{5a}$$

$$\nabla_x V_x(\mathsf{x}) f_x(\mathsf{x}, h(\mathsf{x}), \epsilon) \le -a|\mathsf{x}|^2_{\mathcal{A}_x}, \tag{5b}$$

$$|\nabla_x V_x(\mathsf{x})| \le \theta|\mathsf{x}|_{\mathcal{A}_x}, \tag{5c}$$

where $\underline{a}, \overline{a}, a, \theta > 0$ are positive constants. ◁

Assumption 2. For every compact set $K_x \subset C_x$, the set, defined by,

$$\mathcal{A}_y := \{(\mathsf{x}, \mathsf{y}) \in K_x \times C_y \mid \mathsf{y} = h(\mathsf{x})\} \times \mathcal{P}, \tag{6}$$

is globally asymptotically stable (GAS) for the system (2). Furthermore, there exists a continuously differentiable function $V_y : \mathscr{D}_x \times \mathscr{D}_y \times \mathcal{P} \to \mathbb{R}$ satisfying:

$$\begin{gathered}\underline{b}|\mathsf{y} - h(\mathsf{x})|^2 \le V_y(\mathsf{x}, \mathsf{y}, \mathsf{p}) \le \overline{b}|\mathsf{y} - h(\mathsf{x})|^2,\\ V_y(\mathsf{x}, g_{\mathsf{p}}(\mathsf{x}, \mathsf{y}), \mathsf{q}) \le \gamma V_y(\mathsf{x}, \mathsf{y}, \mathsf{p}),\\ |\nabla_y V_y(\mathsf{x}, \mathsf{y}, \mathsf{p})| \le \ell_y|\mathsf{y} - h(\mathsf{x})|,\\ |\nabla_x V_y(\mathsf{x}, \mathsf{y}, \mathsf{p})| \le \ell_x|\mathsf{y} - h(\mathsf{x})|^2,\end{gathered}$$

for all $(\mathsf{x}, \mathsf{y}) \in K_x \times C_y$, $\mathsf{q} \in S(\mathsf{p})$, $\mathsf{p} \in \mathcal{P}$, where $\underline{b}, \overline{b}, \gamma, \ell_y, \ell_x > 0$ are constants and $0 < \gamma < 1$. ◁

In addition to the above stability assumptions on the auxiliary systems, we impose the following condition to ensure that the desired attractor corresponds to an equilibrium point.

Assumption 3. For every $x \in \mathcal{A}_x$ and $\epsilon \in [0,1]$, it holds that

$$f_y(\mathsf{x}, h(\mathsf{x}), \epsilon) - \frac{\partial h}{\partial x}(\mathsf{x}) f_x(\mathsf{x}, h(\mathsf{x}), \epsilon) = 0. \tag{7}$$

◁

Remark 1. Assumption 3 ensures that the y-dynamics admit an equilibrium for all values of $\epsilon \in [0,1]$. This condition is necessary because the properties imposed on the function V_y in Assumption 2 do not explicitly depend on the parameter ϵ. In contrast, the Lyapunov function V_x is required to decrease along the x-dynamics for all $\epsilon \in [0,1]$, thereby guaranteeing that $\mathcal{A}_x$ remains an attractor for the overall system (1). ◁

III. Lyapunov Function Certificate

We analyze the stability of system (1) by constructing a Lyapunov function that combines the functions V_x and V_y introduced in Assumptions 1 and 2, respectively.

We begin by introducing a compact set K_x, obtained from the function V_x satisfying Assumption 1,

$$K_x := \{x \in C_x \mid V_x(x) \le (e^{N_0}\mu + 1)\}, \tag{8}$$

for a given $\mu > 0$. For this choice of K_x, we select the function V_y satisfying Assumption 2 and define the set

$$K_y := \{y \in C_y \mid V_y(x, y, \mathsf{p}) \le e^{N_0}(\nu+1),\ x \in K_x,\ \mathsf{p} \in \mathcal{P}\},$$

for some $\nu > 0$.

Next, we introduce the functions W_x and W_y, defined as

$$W_x(x, \tau) := e^{c_x \epsilon \tau} V_x(x)$$

and

$$W_y(x, y, \tau, \mathsf{p}) := e^{-c_y \tau} V_y(x, y, \mathsf{p}),$$

where $c_x, c_y \in (0,1]$ are sufficiently small constants (to be specified later). Recall that V_y is associated with K_x as described above.

Our proposed Lyapunov function candidate, partly inspired by the constructions in [17], is

$$W(x, y, \tau, \mathsf{p}) = \frac{e^{N_0}\mu W_x(x, \tau)}{1 + e^{N_0}\mu - W_x(x, \tau)} + \frac{\nu W_y(x, y, \tau, \mathsf{p})}{1 + \nu - W_y(x, y, \tau, \mathsf{p})}. \tag{9}$$

An immediate consequence of this definition is that, for each $x \in \{x \in C_x \mid V_x(x) \le \mu\}$, $y \in \{y \in C_y \mid V_y(x, y, \mathsf{p}) \le \nu,\ x \in K_x,\ \mathsf{p} \in \mathcal{P}\}$, and $\tau \in [0, N_0]$, it follows that

$$W(x, y, \tau, \mathsf{p}) \le e^{2N_0}\mu^2 + \nu^2.$$

We are interested in analyzing the region of $(x, y) \in C_x \times C_y$, where

$$W(x, y, \tau, \mathsf{p}) \le e^{2N_0}\mu^2 + \nu^2 + 1.$$

In particular, this inequality implies

$$W_x(x, \tau) \le (e^{N_0}\mu + 1)\frac{e^{2N_0}\mu^2 + \nu^2 + 1}{e^{2N_0}\mu^2 + \nu^2 + 1 + e^{N_0}\mu} \tag{10}$$

and

$$W_y(x, y, \tau, \mathsf{p}) \le (\nu + 1)\frac{e^{2N_0}\mu^2 + \nu^2 + 1}{e^{2N_0}\mu^2 + \nu^2 + 1 + \nu}. \tag{11}$$

Since c_x, c_y, and ϵ are chosen in $(0,1]$, it follows that

$$\begin{aligned}W_x(x, \tau) < (e^{N_0}\mu + 1) \Rightarrow V_x(x) &< e^{-c_x \epsilon \tau}(e^{N_0}\mu + 1)\\ &\le (e^{N_0}\mu + 1),\end{aligned}$$

for all $\tau \in [0, N_0]$ and $\epsilon \in (0, 1]$. Similarly,

$$W_y(x, y, \tau, \mathsf{p}) \le (\nu + 1) \Rightarrow V_y(x, y, \mathsf{p}) < e^{c_y \tau}(\nu + 1) \le e^{N_0}(\nu + 1),$$

where $\tau \in [0, N_0]$ and $\epsilon \in (0, 1]$.

Therefore, by showing that the set

$$\Omega_0 := \{(x, y, \tau, \mathsf{p}) \in C_x \times C_y \times [0, N_0] \times \mathcal{P} \text{ s.t. } W(x, y, \tau, \mathsf{p}) \le e^{2N_0}\mu^2 + \nu^2 + 1\} \quad (12)$$

is forward invariant under the dynamics (1), it follows that the sets K_x and K_y are forward invariant for the QSS and boundary-layer systems, respectively.

Theorem 1. *Consider the boundary-layer system* (2) *and the quasi steady-state system* (4) *associated with the hybrid system* (1)*. Suppose that Assumption 1, Assumption 2, and Assumption 3 hold. Let μ be chosen such that*

$$x(0, 0) \in \{x \in C_x \mid V_x(x) \le \mu\}.$$

With K_x defined in (8) *and $\mathcal{A}_y$ defined in* (6)*, let ν be such that*

$$y(0, 0) \in \{y \in C_y \mid V_y(x, y, \mathsf{p}) \le \nu,\ x \in K_x,\ \mathsf{p} \in \mathcal{P}\}.$$

Then, there exists $\epsilon^* \in (0, 1]$ such that, for every $0 < \epsilon \le \epsilon^*$, the solutions of (1) *satisfy*

$$\dot{W}(x, y, \tau, \mathsf{p}) < 0, \quad \Delta W(x, y, \tau, \mathsf{p}) < 0,$$

for all (x, y, τ, p) contained in the set Ω_0.

Theorem 1 establishes that the constructed Lyapunov function W decreases strictly along both the flow and jump dynamics of system (1), for sufficiently small ϵ. Consequently, all solutions starting in Ω_0 remain in Ω_0 and converge asymptotically to the set

$$\mathcal{A} := \{(x, y, \tau, \mathsf{p}) \in C_x \times C_y \times [0, N_0] \times \mathcal{P} \mid x \in \mathcal{A}_x,\ y = h(x)\}.$$

This provides a *Lyapunov function certificate* for the asymptotic stability of the coupled hybrid system, since W decreases everywhere on Ω_0 except on the attractor $\mathcal{A}$. In comparison with our earlier practical stability result in [12, Theorem 1], the present analysis employs stronger stability assumptions on the Lyapunov functions V_x and V_y. In certain applications, such as the observer design problem considered in the next section, these assumptions are naturally satisfied, allowing us to establish full asymptotic stability of the interconnected hybrid dynamics.

IV. Observer with Time-Sampled Measurements

As an application of the result proposed in the previous section, we consider the design of an observer for a continuous-time nonlinear system under the setup where output measurements are not continuously available, but occur at frequently spaced discrete time instants. Toward this end, consider a nonlinear dynamical system described by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad (13a)$$

and an output equation

$$\mathfrak{y} = x_1, \quad (13b)$$

so that $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^n$ is the state and $\mathfrak{y} \in \mathbb{R}^p$ is the measured output. We are interested in designing an estimator for the state x using only time-sampled measurements of the output $\mathfrak{y}$. In particular, the measurements used by the observer are described by the following model:

$$\begin{aligned} &\epsilon\dot{\tau} \in [\sigma_1, \sigma_2], \quad \tau \in [0, N_0], \\ &\dot{\widehat{y}} = 0, \\ &\tau^+ = \tau - 1, \quad \tau \in [1, N_0], \\ &\widehat{y}^+ = x_1, \end{aligned}$$

for some $\sigma_2 > \sigma_1 > 0$. Here, $\widehat{y}$ denotes the time-sampled measurements of the output $y = x_1$, and it is updated when $\tau \in [1, N_0]$. This formulation captures scenarios where the maximum allowable time between two updates, denoted by $\tau_{\max}$, satisfies

$$\tau_{\max} \le \frac{(1 + N_0)\epsilon}{\sigma_1}.$$

To proceed with the observer design, we first formalize the necessary assumptions on the system dynamics. We assume that the system (13) is *detectable* in the sense that there exists a matrix $L \in \mathbb{R}^{(n-p)\times p}$ such that, with a new variable $\xi := x_2 - Lx_1 \in \mathbb{R}^{n-p}$, the system

$$\begin{aligned} \dot{\xi} &= f_2(x_1, x_2) - Lf_1(x_1, x_2) \\ &= f_2(x_1, \xi + Lx_1) - Lf_1(x_1, \xi + Lx_1) =: F(\xi, x_1) \end{aligned}$$

is incrementally globally exponentially stable (GES) with respect to ξ. This is formalized in the following assumption.

Assumption 4. There exists a smooth function $V_\xi : \mathbb{R}^{n-p} \times \mathbb{R}^n \to \mathbb{R}$ such that, for some positive constants α_a and α_b,

$$\alpha_a |e_\xi|^2 \le V_\xi(e_\xi, x) \le \alpha_b |e_\xi|^2, \quad \text{for all } e_\xi \in \mathbb{R}^{n-p},\ x \in \mathbb{R}^n,$$

and, with $\xi = x_2 - Lx_1$ and some $\alpha_c > 0$,

$$\frac{\partial V_\xi}{\partial e_\xi}(e_\xi, x)\,[F(e_\xi + \xi, x_1) - F(\xi, x_1)] + \frac{\partial V_\xi}{\partial x}(e_\xi, x)\,\mathrm{col}(f_1(x), f_2(x)) \le -\alpha_c |e_\xi|^2, \quad (14)$$

for all $e_\xi \in \mathbb{R}^{n-p}$ and $x \in \mathbb{R}^n$. ◁

Assumption 4 has been used in the study of reduced-order observers for nonlinear systems in [18].

Example 1. The detectability notion in Assumption 4 extends the detectability concept for a linear system:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2, \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2, \\ y &= \begin{bmatrix} I_{p\times p} & 0_{p\times(n-p)} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

If the linear system is detectable, then by the PBH test,

$$\operatorname{rank} \begin{bmatrix} I_{p\times p} & 0_{p\times(n-p)} \\ A_{11} - sI & A_{12} \\ A_{21} & A_{22} - sI \end{bmatrix} = n, \qquad \forall\, \mathrm{Re}(s) \ge 0,$$

which implies that

$$\operatorname{rank}\begin{bmatrix} A_{12} \\ A_{22}-sI \end{bmatrix} = n-p, \qquad \forall\, \mathrm{Re}(s)\geq 0,$$

or equivalently, (A_{22}, A_{12}) is detectable. Hence, there exists $L\in\mathbb{R}^{(n-p)\times p}$ such that

$$\bar{A}_{22} := A_{22}-LA_{12} \quad \text{is Hurwitz.}$$

With $\xi := x_2 - Lx_1$, we obtain

$$\begin{aligned}\dot{\xi} &= A_{21}x_1 + A_{22}(\xi+Lx_1) - L(A_{11}x_1 + A_{12}(\xi+Lx_1)) \\ &= (A_{21}+A_{22}L-LA_{11}-LA_{12}L)x_1 + (A_{22}-LA_{12})\xi \\ &=: \bar{A}_{21}x_1 + \bar{A}_{22}\xi, \end{aligned} \tag{15}$$

which is incrementally uniformly globally exponentially stable with respect to ξ. Indeed, with $V_\xi(e_\xi)=e_\xi^\top P e_\xi$, where $P>0$ satisfies $P(A_{22}-LA_{12})+(A_{22}-LA_{12})^\top P = -I$, inequality (14) holds since

$$2e_\xi^\top P[(A_{22}-LA_{12})e_\xi] = -|e_\xi|^2.$$

Conversely, if there exists L such that $(A_{22}-LA_{12})$ is Hurwitz, then by letting

$$L_n := \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix},$$

we obtain

$$L_n\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} - L_n^{-1}\begin{bmatrix}\rho I \\ 0\end{bmatrix}\begin{bmatrix} I & 0\end{bmatrix}\right)L_n^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix} - \begin{bmatrix}\rho I \\ 0\end{bmatrix}\begin{bmatrix} I & 0\end{bmatrix}, \tag{16}$$

where $\bar{A}_{11}$, and $\bar{A}_{12}$ are suitably defined. Since $\bar{A}_{22}$ is Hurwitz, sufficiently large $\rho>0$ renders the matrix on the right-hand side Hurwitz, thereby implying detectability of the system. ◁

Using the detectability notion in Assumption 4, we propose the following hybrid observer, compatible with the (x_1,ξ)-coordinate representation. The flow dynamics are given by

$$\begin{aligned} \dot{\hat{x}}_1 &= f_1(\hat{x}_1, \hat{\xi}+L\hat{x}_1), \\ \dot{\hat{\xi}} &= F(\hat{\xi}, \hat{x}_1), \\ \dot{\widehat{y}} &= 0, \\ \epsilon\dot{\tau} &\in [\sigma_1,\sigma_2], \quad \tau\in[0,N_0], \end{aligned} \tag{17a}$$

and the jump dynamics are

$$\begin{aligned} \hat{x}_1^+ &= \hat{x}_1 + \beta(\widehat{y}-\hat{x}_1), \\ \hat{\xi}^+ &= \hat{\xi}, \\ \widehat{y}^+ &= x_1, \\ \tau^+ &= \tau-1, \quad \tau\in[1,N_0], \end{aligned} \tag{17b}$$

where $\beta\in(0,2)$ is a design parameter. The variable x_1 in the update equation for $\widehat{y}$ corresponds to the state x_1 of system (13).

For the analysis, we introduce the error variables

$$e_1 := \hat{\xi}-\xi, \qquad e_2 := \hat{x}_1 - x_1,$$

and we are interested in the stability of the equilibrium point $(e_1,e_2)=(0,0)$. In general, for nonlinear systems, the dynamics of (e_1,e_2) are coupled with the x-dynamics. Hence, we consider the following system to analyze the stability of the (e_1,e_2)-dynamics. The continuous dynamics are

$$\begin{aligned} \dot{x} &= \mathrm{col}(f_1(x), f_2(x)), \\ \dot{e}_1 &= F(e_1+x_2-Lx_1, e_2+x_1) - F(\xi, x_1), \\ \dot{e}_2 &= f_1(e_2+x_1, e_1+\xi+L(e_2+x_1)) - f_1(x_1,x_2), \\ \epsilon\dot{\tau} &\in [\sigma_1,\sigma_2], \quad \tau\in[0,N_0], \end{aligned} \tag{18a}$$

and the jump dynamics, triggered by $\tau\in[1,N_0]$, are

$$\begin{aligned} x^+ &= x, \\ e_1^+ &= e_1, \\ e_2^+ &= (1-\beta)e_2, \\ \tau^+ &= \tau-1, \quad \tau\in[1,N_0]. \end{aligned} \tag{18b}$$

The following result shows that the error variable $e := (e_1,e_2)$ in system (18) converges to the origin.

Proposition 1. *Consider system* (18) *under Assumption 4, and suppose that*

(O1) *The state x evolves in a compact set $\Omega_x\subset\mathbb{R}^n$.*

Then, for every compact set $\Omega_e\subset\mathbb{R}^n$, there exists $\epsilon^*>0$ such that, for every $\epsilon\in(0,\epsilon^*]$ and every $(x(0,0),e(0,0))\in\Omega_x\times\Omega_e$, the solutions converge asymptotically to the set $\{(x,e)\in\mathbb{R}^{2n} \mid e=0\}$.

Proof. The proof is an application of Theorem 1. We take the QSS system to be

$$\begin{aligned} \dot{x} &= \mathrm{col}(f_1(x), f_2(x)), \\ \dot{e}_1 &= F(e_1+x_2-Lx_1, x_1) - F(\xi, x_1), \end{aligned}$$

and note that the set $\mathcal{A}_x = \Omega_x\times\{0\}\subset\mathbb{R}^{n+p}$ is asymptotically stable. Hence, Assumption 1 follows directly from Assumption 4. The boundary-layer system in this case is

$$e_2^+ = (1-\beta)e_2,$$

where $\beta\in(0,2)$. It admits a Lyapunov function $V(e_2)=e_2^\top e_2$, so that Assumption 2 also holds. Furthermore, the equilibrium condition (7) in Assumption 3 is also satisfied, and we therefore obtain asymptotic stability of the equilibrium $e=0$ by Theorem 1. □

Remark 2. In Proposition 1, condition **(O1)** ensures that the set $\{(x,e_1)\in\mathbb{R}^{n+p}\mid x\in\Omega_x, e_1=0\}$ is compact, which, under Assumption 4, serves as the asymptotically stable attractor for the QSS system. In certain cases, when the e-dynamics are completely decoupled from the x-dynamics, the estimation error e can be analyzed independently of x. This is particularly true when F and f_1 are linear. In such cases, compactness of Ω_x in **(O1)** is not required, and one may take $\Omega_x=\mathbb{R}^n$. ◁

To conclude this section, this case study illustrates how the strengthened stability conditions introduced in Section III can be naturally satisfied in practical applications. Under the detectability condition stated in Assumption 4, the quasi steady-state and boundary-layer subsystems associated with the hybrid observer admit Lyapunov functions that directly fulfill the hypotheses of Theorem 1. Consequently, the proposed sampled-data observer ensures asymptotic convergence of the estimation error for sufficiently small sampling intervals, with the sampling parameter ϵ playing the role of the singular perturbation parameter. This formulation unifies the analysis of continuous-discrete observer dynamics under Lyapunov framework.

V. Proof of Main Result

For the proof of Theorem 1, it is convenient to introduce the change of variable $z := y - h(x)$ and define the closed set

$$C_X := \{(x, y - h(x)) : (x, y) \in C_x \times C_y\}.$$

With the state $(x, z, \tau, \mathsf{p}) \in C_X \times [0, N_0] \times \mathcal{P}$, the flow dynamics of (1) can be rewritten as

$$\begin{aligned} \dot{x} &= f_x(x, h(x), \epsilon) + [f_x(x, z + h(x), \epsilon) - f_x(x, h(x), \epsilon)] \\ &=: f_x(x, h(x), \epsilon) + \widehat{f}_x(x, z, \epsilon), \\ \dot{z} &= f_y(x, z + h(x), \epsilon) - \frac{\partial h}{\partial x}(x) f_x(x, z + h(x), \epsilon) \\ &=: f_z(x, z, \epsilon), \\ \epsilon\dot{\tau} &\in [\sigma_1, \sigma_2], \\ \dot{\mathsf{p}} &= 0, \end{aligned} \tag{19a}$$

and the corresponding jump dynamics, for $(x, z, \tau, \mathsf{p}) \in C_X \times [1, N_0] \times \mathcal{P}$, as

$$\begin{aligned} x^+ &= x, \\ z^+ &= g_{\mathsf{p}}(x, z + h(x)) - h(x) =: \widetilde{g}_{\mathsf{p}}(x, z), \\ \tau^+ &= \tau - 1, \\ \mathsf{p}^+ &\in \mathsf{S}(\mathsf{p}). \end{aligned} \tag{19b}$$

The coordinate transformation isolates the deviation z of the fast variable from the slow manifold $y = h(x)$, which simplifies the analysis of convergence toward this manifold.

We next rewrite the Lyapunov candidate W from (9) in these new coordinates. Within an open neighborhood of $C_X \times [0, N_0] \times \mathcal{P}$, define

$$V(x, z, \tau, \mathsf{p}) := \frac{e^{N_0}\mu W_x(x, \tau)}{1 + e^{N_0}\mu - W_x(x, \tau)} + \frac{\nu W_z(x, z, \tau, \mathsf{p})}{1 + \nu - W_z(x, z, \tau, \mathsf{p})}, \tag{20}$$

where $W_z(x, z, \tau, \mathsf{p}) := e^{-c_y\tau} V_z(x, z, \mathsf{p})$, $V_z(x, z, \mathsf{p}) = V_y(x, z + h(x), \mathsf{p})$, and the positive scalars c_x and c_y are chosen sufficiently small (less than 1) and will be specified later. The upper bound ϵ^* of ϵ is also selected to satisfy $\epsilon^* \leq 1$.

To characterize the growth of V, note that for $\tau \in [0, N_0]$ and using (10)–(11), the function V satisfies

$$\begin{aligned} V(x, z, \tau, \mathsf{p}) &\geq \underline{C}_V \underline{\alpha}_x(|x|_{\mathcal{A}_x}) + \underline{C}_V e^{-N_0} \underline{\alpha}_y(|z|), \\ V(x, z, \tau, \mathsf{p}) &\leq e^{N_0} \overline{C}_V \overline{\alpha}_x(|x|_{\mathcal{A}_x}) + \overline{C}_V \overline{\alpha}_y(|z|), \end{aligned} \tag{21}$$

for some positive constants $\underline{C}_V, \overline{C}_V$. Hence V provides class-$\mathcal{K}_\infty$ bounds with respect to the distance from the set $\mathcal{A}_x \times \{0\} \times [0, N_0] \times \mathcal{P}$.

Define

$$\begin{aligned} \overline{\Omega}_0 := \{&(x, z, \tau, \mathsf{p}) \in C_X \times [0, N_0] \times \mathcal{P} \text{ such that} \\ &V(x, z, \tau, \mathsf{p}) \leq e^{2N_0}\mu^2 + \nu^2 + 1\}, \end{aligned}$$

and let $\Pi_x(\overline{\Omega}_0)$, $\Pi_z(\overline{\Omega}_0)$, and $\Pi_X(\overline{\Omega}_0)$ denote the projections of $\overline{\Omega}_0$ onto $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^n \times \mathbb{R}^m$, respectively.

To establish asymptotic stability, we first verify that system trajectories remain inside $\overline{\Omega}_0$ and that V decreases both along flows and at jumps. By compactness of $\overline{\Omega}_0$, there exists $M_x > 0$ such that

$$|f_x(x, z + h(x), \epsilon)| \leq M_x, \quad \forall \epsilon \in [0, 1]. \tag{22}$$

Furthermore, by the bounds in Appendix I, there exists $N_{x\epsilon}$ such that

$$|\widehat{f}_x(x, z, \epsilon)| \leq N_{x\epsilon}|z|, \tag{23}$$

for all $(x, z) \in \Pi_X(\overline{\Omega}_0)$ and $\epsilon \in [0, 1]$. Under (7), there also exist $N_{zz}, N_{zx} > 0$ satisfying

$$|f_z(x, z, \epsilon)| \leq N_{zz}|z| + N_{zx}|x|, \quad \forall (x, z) \in \Pi_X(\overline{\Omega}_0). \tag{24}$$

We now compute the derivative of V along the flow dynamics (19). Using the chain rule,

$$\dot{V}_{(19)} \leq \frac{e^{N_0}\mu(e^{N_0}\mu + 1)}{(1 + e^{N_0}\mu - W_x(x, \tau))^2} \dot{W}_x(x, \tau) + \frac{\nu(\nu + 1)}{(1 + \nu - W_z(x, z, \tau, \mathsf{p}))^2} \dot{W}_z(x, z, \tau, \mathsf{p}).$$

From (10)–(11), we have

$$\frac{e^{N_0}\mu}{e^{N_0}\mu + 1} \leq \frac{e^{N_0}\mu(e^{N_0}\mu + 1)}{(1 + e^{N_0}\mu - W_x(x, \tau))^2} \leq M_\mu,$$

and similarly,

$$\frac{\nu}{\nu + 1} \leq \frac{\nu(\nu + 1)}{(1 + \nu - W_z(x, z, \tau, \mathsf{p}))^2} \leq M_\nu,$$

where M_μ and M_ν are finite positive constants. Thus,

$$\dot{V}_{(19)} \leq M_\mu \dot{W}_x(x, \tau) + M_\nu \dot{W}_z(x, z, \tau, \mathsf{p}). \tag{25}$$

We now bound each term. For the first term,

$$\begin{aligned} \dot{W}_x(x, \tau) &\leq c_x \epsilon\dot{\tau} e^{c_x\epsilon\tau} V_x(x) + e^{c_x\epsilon\tau} \nabla_x V_x f_x(x, h(x), \epsilon) \\ &\quad + e^{c_x\epsilon\tau} \nabla_x V_x \widehat{f}_x(x, z, \epsilon) \\ &\leq c_x \sigma_2 e^{c_x\epsilon N_0} \overline{a}|x|^2_{\mathcal{A}_x} - a|x|^2_{\mathcal{A}_x} \\ &\quad + e^{c_x\epsilon N_0} \theta N_{x\epsilon} |x|_{\mathcal{A}_x} |z|. \end{aligned}$$

For the second term in (25), using (24),

$$\begin{aligned} \dot{W}_z &\leq -c_y \dot{\tau} e^{-c_y\tau} V_z(x, z, \mathsf{p}) + e^{-c_y\tau} (\nabla_z V_z f_z + \nabla_x V_z f_x) \\ &\leq -\frac{c_y\sigma_1}{\epsilon} e^{-N_0 c_y} \underline{b}|z|^2 + N_{zx}\ell_y |x|_{\mathcal{A}_x}|z| \\ &\quad + (N_{zz}\ell_y + M_x\ell_x)|z|^2. \end{aligned}$$

Combining the two bounds and taking $c_x \leq 1/N_0$ (so that $e^{N_0 c_x \epsilon} \leq e$), we obtain

$$\dot{V}_{(19)} \leq -\begin{pmatrix} |x|_{\mathcal{A}_x} \\ |z| \end{pmatrix}^\top Q_\epsilon \begin{pmatrix} |x|_{\mathcal{A}_x} \\ |z| \end{pmatrix},$$

where the entries of the symmetric matrix $Q_\epsilon \in \mathbb{R}^{2\times 2}$ are,

$$\begin{aligned} (Q_\epsilon)_{11} &= M_\mu(a - c_x\sigma_2 e\overline{a}) \\ (Q_\epsilon)_{12} &= (Q_\epsilon)_{21} = -\frac{1}{2}(M_\mu e\theta N_{x\epsilon} + N_{zx}\ell_y M_\nu) \\ (Q_\epsilon)_{22} &= \Big(\frac{c_y\sigma_1}{\epsilon} e^{-N_0 c_y}\underline{b} - N_{zz}\ell_y - M_x\ell_x\Big) M_\nu. \end{aligned}$$

By selecting $c_x \leq \min\{1/N_0,\, a/(\overline{a}e\sigma_2)\}$ and $\epsilon \in (0,1]$ sufficiently small, the matrix Q_ϵ becomes positive definite. Hence, for all $(x,z) \in \Pi_X(\overline{\Omega}_0) \setminus (\mathcal{A}_x \times \{0\})$ and $(\tau, \mathsf{p}) \in [0, N_0] \times \mathcal{P}$,

$$\dot{V}_{(19)} < 0.$$

For the jump analysis, we have

$$W_x^+(x^+, \tau^+) = e^{c_x\epsilon(\tau-1)} V_x(x) = e^{-c_x\epsilon} W_x(x,\tau),$$

and

$$\begin{aligned} W_z^+(x^+, z^+, \tau^+, \mathsf{p}^+) &= e^{-c_y(\tau-1)} V_z(x, \widetilde{g}_{\mathsf{p}}(x,z), \mathsf{q}) \\ &\leq e^{c_y}\gamma e^{-c_y\tau} V_z(x,z,\mathsf{p}) \\ &= e^{c_y}\gamma W_z(x,z,\tau,\mathsf{p}), \end{aligned}$$

where c_y is chosen small enough to satisfy $c_y < \ln(1/\gamma)$. Let $\widetilde{\gamma} := \max\{e^{-c_x\epsilon},\, e^{c_y}\gamma\} < 1$. Then

$$\begin{aligned} V_{(19)}^+ &\leq \frac{e^{N_0}\mu W_x^+}{1 + e^{N_0}\mu - W_x^+} + \frac{\nu W_z^+}{1 + \nu - W_z^+} \\ &\leq \widetilde{\gamma}\frac{e^{N_0}\mu W_x}{1 + e^{N_0}\mu - W_x} + \widetilde{\gamma}\frac{\nu W_z}{1 + \nu - W_z} \\ &\leq \widetilde{\gamma} V_{(19)}. \end{aligned}$$

Finally, since V decreases strictly during flows and across jumps within $\overline{\Omega}_0$, the standard result on asymptotic stability of hybrid systems, for example [1, Theorem 3.18], implies that all solutions of (1) converge asymptotically to the set $\mathcal{A}_x \times \{0\} \times [0, N_0] \times \mathcal{P}$, completing the proof.

VI. Conclusions

In contrast to our earlier work, which established practical stability results for singularly perturbed hybrid systems, this paper addressed the problem of *asymptotic stability* for the same class of systems. The main conditions rely on the assumption that the auxiliary dynamical systems describing the decoupled (quasi steady-state and boundary-layer) dynamics are asymptotically stable. These requirements introduce additional constraints on the growth and gradient properties of the corresponding Lyapunov functions–constraints that were not imposed in the analysis of practical stability in [12].

A natural direction for future research is to generalize the proposed sufficient conditions to identify the essential ingredients that enable the transition from practical to asymptotic stability. Another interesting extension would be to explicitly derive convergence rate estimates for system trajectories approaching the equilibrium set. Furthermore, it would be worthwhile to investigate stability analysis using multiple Lyapunov functions for the discrete-time boundary-layer subsystem, particularly in cases where the jump dynamics exhibit switching behavior.

In the context of the observer design problem, the singular perturbation framework developed here provides only a qualitative characterization of convergence of the estimation error as the sampling interval decreases. A valuable next step would be to establish quantitative lower bounds on the sampling period guaranteeing convergence. Beyond observer design, the framework could be applied to a broader class of networked and hybrid control problems, where convergence properties need to be analyzed with respect to a limiting design or scheduling parameter.

Appendix I
Derivation of the Constants $N_{x\epsilon}$, N_{zz}, and N_{zx}

Let $(x,z) \in \Pi_X(\overline{\Omega}_0)$ and $\epsilon \in [0,1]$, and define

$$\mathfrak{g}(\sigma) := f_x(x, h(x) + \sigma z, \epsilon).$$

Then,

$$\begin{aligned} \widehat{f}_x(x,z,\epsilon) &= f_x(x, z + h(x), \epsilon) - f_x(x, h(x), \epsilon) \\ &= \mathfrak{g}(1) - \mathfrak{g}(0) = \int_0^1 \mathfrak{g}'(\sigma)\, d\sigma \\ &= \int_0^1 \frac{\partial f_x}{\partial y}(x, h(x) + \sigma z, \epsilon)\, z\, d\sigma, \end{aligned}$$

where the integrand is well-defined due to the continuous differentiability of f_x.

Hence, the constant $N_{x\epsilon}$ can be expressed as

$$N_{x\epsilon} := \max_{\substack{(x,z)\in\Pi_X(\overline{\Omega}_0) \\ \epsilon\in[0,1]}} \int_0^1 \left\| \frac{\partial f_x}{\partial y}(x, h(x) + \sigma z, \epsilon) \right\| d\sigma,$$

which ensures that the bound in (23) holds.

Analogous arguments can be used to establish the existence of the constants N_{zz} and N_{zx} appearing in (24), obtained from suitable partial derivatives of f_z with respect to z and x, respectively.

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