Observer Design for Detectable Switched Differential-Algebraic Equations

Aneel Tanwani * **Stephan Trenn** **

* *LAAS – CNRS, University of Toulouse, 7 Ave. Colonel du Roche, 31400 Toulouse, France. Email:* aneel.tanwani@laas.fr
** *Department of Mathematics, TU Kaiserslautern, 48 Gottlieb-Daimler-Straße, 67663 Kaiserslautern, Germany. Email:* trenn@mathematik.uni-kl.de

Abstract: This paper studies detectability for switched linear differential-algebraic equations (DAEs) and its application in synthesis of observers. Equating detectability to asymptotic stability of zero-output-constrained state trajectories, and building on our work on interval-wise observability, we propose the notion of interval-wise detectability: If the output of the system is constrained to be identically zero over an interval, then the norm of the corresponding state trajectories scales down by a certain factor over that interval. Conditions are provided under which the interval-wise detectability leads to asymptotic stability of zero-output-constrained state trajectories. An application is demonstrated in designing state estimators. Decomposing the state into observable and unobservable components, we show that if the observable component in the estimator is reset appropriately and persistently, then the estimation error converges to zero asymptotically under the interval-wise detectability assumption.

1. INTRODUCTION

Contributing to the growing literature on structural properties of switched linear differential-algebraic equations (switched DAEs), this article proposes a detectability notion for this system class and its utility in observer design. Switched DAEs arise naturally when the system dynamics undergo sudden structural changes (switches) and the dynamics of each mode are algebraically constrained. A typical example are electrical circuits with switches where the constraints are induced by Kirchhoff's laws. We have already proposed an observer for switched DAEs in [Tanwani and Trenn, 2013, 2016] under the stronger assumption of determinability (which in the nonswitched case is equivalent to observability and roughly speaking means that the state at the end of the observation interval can be determined to any given accuracy). However, it is well known from the observer design techniques commonly employed for linear time-invariant systems that detectability is already sufficient for constructing state estimates; where one possible way to interpret detectability is that the system dynamics resulting from constraining the output to zero are asymptotically stable. In the literature on switched systems, we have results on a related notion of output-to-state stability in [Muller and Liberzon, 2012], where the focus is on characterizing a class of switching signals under which the growth of the state trajectory is bounded by some increasing function of the output norm. Conditions on the structure of the system ensuring detectability of switched systems are proposed in [De Santis et al., 2009].

In our work, we define a different detectability notion for switched systems (see Section 3 for the formal definition) and design state estimators for systems satisfying a detectability assumption in an appropriate sense. To the best of our knowledge, this result is also new for the case of switched ordinary differential equations (switched ODEs), as the previous works have only dealt with observable switched systems [Tanwani et al., 2013]. It turns out that an observer for the detectable case has to work fundamentally different to our observer proposed for the determinable case. We illustrate this by the following simple example.

Example 1. Consider the switched ODE on the interval $[0,3)$ given by

$$\left.\begin{aligned} \dot{x}_1(t) &= 0\\ \dot{x}_2(t) &= 0\\ \dot{x}_3(t) &= 0\\ y(t) &= x_1(t)\\ t &\in [0,1)\cup[2,3), \end{aligned}\;\right|\;\begin{aligned} \dot{x}_1(t) &= x_1(t)+x_2(t)\\ \dot{x}_2(t) &= 0\\ \dot{x}_3(t) &= x_2(t)-x_3(t)\\ y(t) &= 0\\ t &\in [1,2). \end{aligned}$$

Based on our previous work [Tanwani et al., 2013, Tanwani and Trenn, 2016], if we restrict our attention to the interval $[0,3)$, then $y(t)\equiv 0$ on this interval implies $x_1(t)\equiv x_2(t)\equiv 0$, and hence (x_1,x_2) is observable. Also, the identically zero output would imply that the magnitude of x_3 decreases, which is the notion of detectability we adopt in this paper (see Section 3). It is possible to design an impulsive estimator with states $\widehat{x}_1,\widehat{x}_2,\widehat{x}_3$ which copies the system dynamics over the interval $[0,3)$, and at $t=3$ we reset the estimator state as

$$\begin{pmatrix}\widehat{x}_1(3)\\ \widehat{x}_2(3)\end{pmatrix} := \mathcal{O}(y_{[0,3)})$$

for some map $\mathcal{O}$, so that, if $e = x-\widehat{x}$ denotes the state estimation error, we have

$$\left|\begin{pmatrix}e_1(3)\\ e_2(3)\end{pmatrix}\right| \le \alpha\begin{pmatrix}e_1(0)\\ e_2(0)\end{pmatrix}$$

for some desired $\alpha\in(0,1)$. However, for the unobservable error e_3, we get

$$\begin{aligned} \dot{e}_3(t) &= 0, && t \in [0,1) \cup [2,3) \\ \dot{e}_3(t) &= e_2(t) - e_3(t), && t \in [1,2) \end{aligned} \tag{1}$$

and hence

$$e_3(3) = \mathtt{e}^{-1} e_3(0) + (1 - \mathtt{e}^{-1}) e_2(0).$$

Thus, for a large initial value $e_2(0)$, the final error $e_3(3)$ may be significantly larger than $e_3(0)$ and therefore a direct application of our previous presented observer to detectable systems will not work. The underlying problem for this example is that it is not enough to have a good estimate of the observable states at the end of the considered interval, but the estimate must be available already when the observable states influence the unobservable states.

Paper outline: In Section 2, we formally introduce the system class of switched DAEs. We introduce in Section 3 the notion of detectability which is used for observer design in Section 4. The key result in Section 4 is Theorem 7, which shows how the ideal correction term decreases the estimation error. Convergence of the observer for non-ideal correction terms is shown in Theorem 9 in Section 5.

2. PRELIMINARIES AND NOTATION

We consider switched linear DAEs of the form

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \tag{2}$$

where x, u, y denote the state (with dimension $n \in \mathbb{N}$), input (with dimension $\mathtt{u} \in \mathbb{N}$) and output (with dimension $\mathtt{y} \in \mathbb{N}$) of the system, respectively. The switching signal $\sigma : [0, \infty) \to \mathbb{N}$ is a piecewise constant, right-continuous function of time and in our notation it changes its value at time instants $0 < t_1 < t_2 < \ldots$ called the switching times. We adopt the convention that over the interval $[t_p, t_{p+1})$ of length $\tau_p := t_{p+1} - t_p$, the active mode is defined by the quadruple $(E_p, A_p, B_p, C_p, D_p) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times \mathtt{u}} \times \mathbb{R}^{\mathtt{y} \times n} \times \mathbb{R}^{\mathtt{y} \times \mathtt{u}}$, $p \in \mathbb{N}$ and $t_0 := 0$. If $E_p = I$ for all $p \in \mathbb{N}$ we call (2) a switched ODE. In general, E_p is not assumed to be invertible, which means that in addition to differential equations the state x has to satisfy certain algebraic constraints. Because of this, the state variable may jump to satisfy different algebraic constraints before and after the switch. Another characteristic of switched DAES is the possible presence of Dirac impulses in the state variable x in response to a state jump. For this reason, we introduce the space of piecewise distributions, denoted $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ as solution space. We refer the reader to [Trenn, 2009] for formal details, but for this paper, it suffices to recall that $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ is written as

$$x = x^f_{\mathbb{D}} + x[\cdot], \tag{3a}$$

where $x^f_{\mathbb{D}}$ denotes the distribution induced by the piecewise smooth function $x^f : \mathbb{R} \to \mathbb{R}^n$ and $x[\cdot]$ denotes the impulsive part of x given by

$$x[\cdot] = \sum_{k \in \mathbb{Z}} x[t_k] = \sum_{k \in \mathbb{Z}} \sum_{i=0}^{n_k} a_k^i \delta_{t_k}^{(i)}, \tag{3b}$$

where $\{ \, t_k \in \mathbb{R} \mid k \in \mathbb{Z} \, \}$ is a strictly increasing set without finite accumulation and $\delta_{t_k}^{(i)}$ denotes the i-th derivative of the Dirac impulse with support at t_k. For $x = x^f_{\mathbb{D}} + x[\cdot] \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ the left- and right-evaluation at any $t \in \mathbb{R}$ are well defined:

$$x(t^-) := x^f(t^-) \quad \text{and} \quad x(t^+) := x^f(t^+).$$

Lemma 1. (cf. Trenn [2009]). Consider the switched DAE (2) and assume that each matrix pair (E_p, A_p) is regular, i.e. $\det(sE_p - A_p)$ is not the zero polynomial. Then for every $u \in \mathbb{D}^{\mathtt{u}}_{\mathrm{pw}\mathcal{C}^\infty}$, any $x_0 \in \mathbb{R}^n$ and any interval $[a, b) \subseteq [0, \infty)$ there exists $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ uniquely defined on $[a, b)$ such that $x(a^-) = x_0$ and (2) holds as an equation of piecewise-smooth distributions restricted to $[a, b)$.

This motivates the following solution definition of (2).

Definition 2. (Solution of switched DAE). A tuple (x, u, y) (or just x when u and y are clear) is called a solution of (2) on an interval $\mathcal{I}$ if $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^{\mathtt{u}}_{\mathrm{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^{\mathtt{y}}_{\mathrm{pw}\mathcal{C}^\infty}$ and (2) restricted to $\mathcal{I}$ holds in the distributional sense. If $\mathcal{I} = [0, \infty)$ we omit "on the interval $[0, \infty)$" in the following.

3. DETECTABILITY NOTIONS

Roughly speaking, in classical literature on nonswitched systems, a dynamical system is called detectable if, for a fixed input and an observed output, the trajectories starting from every pair of indistinguishable initial states converge to a common trajectory asymptotically. The formal definition for switched DAEs is as follows:

Definition 3. The switched DAE (2) is called *detectable* for a given switching signal σ, if there exists a class $\mathcal{KL}$ function $\beta : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for any two distributional solutions (x_1, u, y), (x_2, u, y) of (2) we have

$$|x_1(t^+) - x_2(t^+)| \leq \beta(|x_1(0^-) - x_2(0^-)|, t), \quad \forall\, t \geq 0. \tag{4}$$

Because of linearity the definition can be simplified to the case that $u = 0$ and $y = 0$, in particular, convergence to zero has only to be checked for the homogeneous system and the initial states in

$$\mathcal{N}^\sigma := \left\{ \, x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (x, u = 0, y = 0) \text{ solves } (2) \\ \wedge\ x(0^-) = x^0 \end{array} \right\}, \tag{5}$$

or in other words, detectability is the same as asymptotic stability of the switched DAE (2) with $u = 0$ and $y = 0$.

Remark 4. In contrast to previous works on stability of switched DAEs [Liberzon and Trenn, 2009, 2012] we do not require impulse-freeness of solutions for asymptotic stability. The reason is that the presence of Dirac impulses may actually help to make certain states observable, hence the exclusion of Dirac impulses may exclude an important class of problems where Dirac impulses are needed for observability (or detectability). It should also be noted that the magnitude of the Dirac impulses is always proportional to the state value prior to the time the Dirac impulse occurs, i.e. when the state converges to zero as $t \to \infty$ the magnitude of the Dirac impulses also converges to zero (under an additional mild boundedness assumption on (E_p, A_p) as $p \to \infty$).

Computation of the set $\mathcal{N}^\sigma$ in general depends on all switching times and the data of all subsystems. For certain applications, such as state estimation which we discuss later, it may be desirable to work with system data available on finite intervals only, and in that case, Definition 3 may not be suitable. To overcome this problem, we consider the system behavior on finite intervals, and introduce the notion of interval-detectability:

Definition 5. (Interval-detectability). The switched DAE (2) is called $[t_p, t_q)$-detectable for a given switching signal σ, if there exists a class $\mathcal{KL}$ function $\beta : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}_{\geq 0}$ with

$$\beta(r, t_q - t_p) < r, \quad \forall\, r > 0 \tag{6a}$$

and for any local solution $(x, u = 0, y = 0)$ of (2) on $[t_p, t_q)$ we have

$$|x(t^+)| \leq \beta(|x(t_p^-)|, t - t_p), \quad \forall\, t \in [t_p, t_q). \tag{6b}$$

One should be aware, that a solution on some interval is not always a part of a solution on a larger interval. Consequently, detectability does not always imply interval-detectability: The switched system $0 = x$ on $[t_0, t_1)$ and $\dot{x} = 0$ on $[t_1, \infty)$ with zero output is obviously detectable (because zero is the only global solution), but it is not interval-detectable on $[t_1, s)$ for any $s > t_1$ because on $[t_1, s)$ there are nonzero solutions which do not converge towards zero.

Furthermore, we would like to emphasize that the interval $[t_p, t_q)$ in general contains multiple switches, i.e. it is *not assumed* that the individual modes of the switched systems are detectable. We need some uniformity assumption to conclude that interval-detectability on each interval of a partition of $[t_0, \infty)$ implies detectability:

Assumption 1. (Uniform interval-detectability). Consider the switched system (2) with switching signal σ and switching times t_k, $k \in \mathbb{N}$. Assume that there exists a strictly increasing sequence $(p_i)_{i=0}^{\infty}$ such that for $q_i := p_{i+1}$ the system is $[t_{p_i}, t_{q_i})$-detectable with $\mathcal{KL}$-function β_i for which additionally it holds that

$$\beta_i(r, t_{q_i} - t_{p_i}) \leq \alpha\, r, \quad \forall\, r > 0, \forall\, i \in \mathbb{N}, \tag{7a}$$

$$\beta_i(r, 0) \leq M\, r, \quad \forall\, r > 0, \forall\, i \in \mathbb{N}, \tag{7b}$$

for some uniform $\alpha \in (0, 1)$ and $M \geq 1$.

We can now show the following result:

Proposition 6. If the switched system (2) is uniformly interval-detectable in the sense of Assumption 1 then (2) is detectable.

The following examples show that we cannot drop any of the uniformity conditions (7a) and (7b).

Example 2. (1) *Nonuniformity of α.*
Consider a scalar switched ODE given by

$$\dot{x} = -\frac{1}{k^2} x \quad \text{on } [k, k+1),$$

then

$$|x(k+1)| = \mathsf{e}^{-1/k^2} |x(k)| < |x(k)|,$$

hence the switched system is $[t_k, t_{k+1})$-detectable for each $k \in \mathbb{N}$. However,

$$x(k) = \prod_{i=1}^{k} \mathsf{e}^{-1/i^2} x(0) = \mathsf{e}^{-\sum_{i=1}^{k} 1/i^2} x(0) \geq \mathsf{e}^{-C} x(0),$$

where $C := \sum_{i=1}^{\infty} 1/i^2 < \infty$, i.e. the solution does not converge to zero. The problem is that there is no uniform $\alpha < 1$ such that $|x(k+1)| \leq \alpha |x(k)|$.
(2) *Nonuniformity of M.*
Consider a scalar switched ODE given by

$$\begin{aligned} \dot{x} &= kx, && \text{on } [2k, 2k+1) \\ \dot{x} &= -(k+1)x, && \text{on } [2k+1, 2k+2) \end{aligned}$$

with zero output. This switched system is interval-detectable on each interval $[2k, 2k+2)$ because

$$x(2k+2) = \mathsf{e}^{-(k+1)} \mathsf{e}^{k} x(2k) = \mathsf{e}^{-1} x(2k)$$

and therefore $|x(2k+2)| \leq \alpha |x(2k)|$ with $\alpha = \mathsf{e}^{-1} < 1$. However,

$$\begin{aligned} x(2k+1) &= \mathsf{e}^{k} e(2k) = \mathsf{e}^{k} \mathsf{e}^{-1} e(2k-2) \\ &= \mathsf{e}^{k} \mathsf{e}^{-k} e(0) = e(0), \end{aligned}$$

hence $x(t)$ does not converge towards zero as $t \to \infty$. Note that in this example any $\mathcal{KL}$-function β_k for the interval $[2k, 2k+2)$ has to satisfy $\beta_k(r, 0) > \beta_k(r, 1) \geq e^k r$, hence for this example it is not possible to find a uniform M such that (7b) holds.

4. OBSERVER DESIGN

Our observer design is an extension of the algorithm proposed in [Tanwani and Trenn, 2016] for the determinable case (in particular, the interval-wise observer design), i.e. we propose an impulsive observer which consists of a system copy and a correction term which updates the state of the system copy at the end of the detectability interval.

More formally, under the uniform interval-detectability (Assumption 1) with detectability intervals $[t_{p_i}, t_{q_i})$, $i \in \mathbb{N}$, the state estimator is given by $\widehat{x} := \sum_{i \in \mathbb{N}} (\widehat{x}_i)_{[t_{p_i}, t_{q_i})}$ with

$$\begin{aligned} \left.\begin{aligned} E_\sigma \dot{\widehat{x}}_i &= A_\sigma \widehat{x}_i + B_\sigma u, \\ \widehat{y} &= C_\sigma \widehat{x}_i + D_\sigma u, \end{aligned}\right\} &\text{ on } [t_{p_i}, t_{q_i}), \\ \widehat{x}_{i+1}(t_{q_i}^-) = \widehat{x}_i(t_{q_i}^-) - \xi_i. & \end{aligned} \tag{8}$$

where $\xi_i \in \mathbb{R}^n$ is a state estimation correction obtained from the available data on the interval $[t_{p_i}, t_{q_i})$ applied at the end of the corresponding interval. Similar to the technique adopted in [Tanwani and Trenn, 2016], the correction term ξ_i is obtained by collecting the local observability data for each mode. However, these local data is combined in a fundamentally different way compared to [Tanwani and Trenn, 2016], because ξ_i is obtained by composing the local observability data backward in time first and then propagating this forward in time under the error dynamics, c.f. Example 1.

In particular, a much more complicated algorithm is needed to obtain the correction term at the end of the interval. In fact, it consists of the three following steps which have to be carried out on each of the detectability intervals $[t_{p_i}, t_{q_i})$:

(1) Collect local observability data for each mode synchronous to the system dynamics from the measured input and output over the interval $[t_{p_i}, t_{q_i})$.
(2) Propagate back the collected information to obtain an estimation correction ξ_i^{left} at the beginning of the detectability interval.
(3) Propagate forward the correction term ξ_i^{left} to obtain the actual estimation correction ξ_i at the end of the interval.

We will now explain each of the steps in detail, for that we drop the index i and just consider the generic detectability interval $[t_p, t_q)$ for some $q > p \geq 0$. It is helpful to also introduce the estimation error error $e := \widehat{x} - x$ (which we don't know, because x is not known) and the corresponding output mismatch $y^e := \widehat{y} - y$ (which we know). It is easily seen that the error is governed by the following homogeneous switched DAE on $[t_p, t_q)$:

$$E_\sigma \dot{e} = A_\sigma e, \qquad y^e = C_\sigma e \tag{9}$$

and the idea of the observer is to estimate the error signal e from the measured output mismatch y^e.

4.1 Collecting local observability data for each mode

For each mode k with $p \leq k \leq q-1$ consider the local unobservable space:

$$\mathcal{W}_k := \left\{ e_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} e(t_k^-) = e_0, \text{where } (e, y^e = 0) \\ \text{solves (9) on } [t_k, t_{k+1}) \end{array} \right\} \tag{10}$$

Defining Π_k, O_k^{diff} and O_k^{imp} in terms of (E_k, A_k) as in the Appendix A, it can be shown (c.f. [Tanwani and Trenn, 2013, 2016]) that

$$\mathcal{W}_k = \Pi_k^{-1} \ker O_k^{\text{diff}} \;\cap\; \ker O_k^{\text{imp}}.$$

Note that in general Π_k is not invertible and Π_k^{-1} stands for the set-valued preimage.

If the output mismatch y^e is nonzero then the value of e in (9) prior to the switching time t_k can be decomposed as

$$e(t_k^-) = W_k w_k + Z_k z_k,$$

where $\operatorname{im} W_k = \mathcal{W}_k$ and $\operatorname{im} Z_k = \mathcal{W}_k^\perp$ and W_k, Z_k are orthonormal matrices. In particular, $z_k = Z_k^\top e(t_k^-)$ is the observable part of the error $e(t_k^-)$ based on the knowledge on the interval $[t_k, t_{k+1})$. It is possible to write the observable part z_k in terms of y^e:

$$z_k = \mathcal{O}_k(y^e_{[t_k, t_{k+1})}) \tag{11}$$

with some operator $\mathcal{O}_k$ which evaluates the impulsive part $y^e[t_k]$ as well as the smooth part $y^e_{(t_k, t_{k+1})}$ (possibly depending on the derivatives of y^e). The construction of this "ideal" observability operator $\mathcal{O}_k$ is provided in Appendix B.1. One may also refer to [Tanwani and Trenn, 2016, Section 5] for a detailed treatment. In practice, only an approximation $\widehat{\mathcal{O}}_k$ of $\mathcal{O}_k$ will be available, this will be discussed in Section 5.

4.2 Combining local information backwards in time

Next we want to combine the observable information z^k with $p \leq k \leq q-1$, to compute an expression for $e(t_p^-)$. To do so, we first quantify the information that can be extracted from the output over an interval $[t_k, t_q)$ by introducing the subspace

$$\mathcal{N}_k^q := \left\{ e_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} e(t_k^-) = e_0, \text{ where } (e, y^e = 0) \\ \text{solves (9) on } [t_k, t_q) \end{array} \right\} \tag{12}$$

which can be recursively calculated (backwards in time, i.e. for $k = q-1, q-2, \ldots, p$), see equation (B.2) in the Appendix. We then decompose the state estimation error just before the interval $[t_k, t_q)$ accordingly:

$$e(t_k^-) = M_k \mu_k + N_k \nu_k \tag{13}$$

for some vectors μ_k and ν_k of appropriate dimension. Here, M_k and N_k are the matrices with orthonormal columns such that $\operatorname{im} N_k = \mathcal{N}_k^q$ and $\operatorname{im} M_k = (\mathcal{N}_k^q)^\perp$. As shown in section B.2 of the Appendix, there exists a matrix $\mathcal{F}_k$ given in terms of M_{k+1}, N_{k+1}, (E_k, A_k) and the duration time $\tau_k = t_{k+1} - t_k$ such that for $p \leq k \leq q-2$

$$\mu_k = \mathcal{F}_k \begin{pmatrix} z_k \\ \mu_{k+1} \end{pmatrix},$$

and $\mu_{q-1} = z_{q-1}$. Note that by construction, for all $p \leq k \leq q-1$

$$e(t_k^-) - M_k \mu_k \in \mathcal{N}_k^q.$$

Now the ideal estimation error correction is

$$\begin{aligned} \xi^{\text{left}} &:= M_p \mu_p \\ &= M_p \mathcal{F}_p \begin{pmatrix} z_p \\ \mathcal{F}_{p+1} \begin{pmatrix} z_{p+1} \\ \mathcal{F}_{p+2} \begin{pmatrix} \ddots \\ \mathcal{F}_{q-2} \begin{pmatrix} z_{q-2} \\ z_{q-1} \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix} \\ &=: \mathcal{O}_p^{q-1} \mathbf{z}_p^{q-1}, \end{aligned} \tag{14}$$

where $\mathbf{z}_p^{q-1} = (z_p / z_{p+1} / \cdots / z_{q-1})$. In fact, the following is true:

$$\boxed{e(t_p^-) - \xi^{\text{left}} \in \mathcal{N}_p^q \text{ and } \xi^{\text{left}} \in \mathcal{N}_p^{q\perp},}$$

i.e. we are able to obtain the orthogonal projection of $e(t_p^-)$ onto $\mathcal{N}_p^q$ without actually knowing $e(t_p^-)$.

4.3 Propagating correction term forward in time

For the detectability interval $[t_p, t_q)$, let ξ^{left} be given as above, then let

$$\xi := \Phi_p^q \xi^{\text{left}},$$

where Φ_p^k, $k = p, p+1, \ldots, q$ is recursively given by

$$\Phi_p^{k+1} = \mathsf{e}^{A_k^{\text{diff}} \tau_k} \Pi_k \Phi_p^k \tag{15}$$

and $\Phi_p^p := I$, i.e. Φ_p^q is the transition matrix of the homogeneous error DAE (9) from $e(t_p^-)$ to $e(t_q^-)$. We then have the following result:

Theorem 7. Consider the switched DAE (2) which is detectable on $[t_p, t_q)$ with corresponding $\mathcal{KL}$-function β. Let $(\widehat{x}, \widehat{y})$ be the solution of the system copy

$$\begin{aligned} E_\sigma \dot{\widehat{x}} &= A_\sigma \widehat{x} + B_\sigma u, \\ \widehat{y} &= C_\sigma \widehat{x} + D_\sigma u \end{aligned} \tag{16}$$

on $[t_p, t_q)$. Based on the output mismatch $y^e = \widehat{y} - y$ let

$$\xi = \Phi_p^q \xi^{\text{left}} = \Phi_p^q \mathcal{O}_p^{q-1} \mathbf{z}_p^{q-1}$$

where Φ_p^q is given by (15), $\mathcal{O}_p^{q-1}$ is given (14) and $\mathbf{z}_p^{q-1} = (z_p / z_{p+1} / \cdots / z_{q-1})$ with $z_k = \mathcal{O}_k(y^e_{[t_k, t_k+1)})$, $k = p, p+1, \ldots, q-1$ is given by (11). Then

$$\begin{aligned} |\widehat{x}(t_q^-) - \xi - x(t_q^-)| &\leq \beta(|\widehat{x}(t_p^-) - x(t_p^-)|, t_q - t_p) \\ &< |\widehat{x}(t_p^-) - x(t_p^-)|, \end{aligned}$$

i.e. the correction term ξ indeed reduces the estimation error at the end of the interval in comparison to the estimation error at the beginning of the interval.

Remark 8. Note that by applying the correction ξ^{left} at the beginning of the interval, the output of the system copy is then identical to the output of the original system. However, for the observer design it is *not* necessary to rerun the system copy (in particular storing the whole input signal over the interval $[t_p, t_q)$), because we just propagate ξ^{left} via the homogenous error dynamics (9) which is independent of the input.

5. ESTIMATION ERRORS AND ASYMPTOTIC CONVERGENCE

In theory, it is possible to determine the observable part exactly from the output, however, in praxis one can only get approximations. Nevertheless, these approximations may be as accurate as desired (e.g. by chosing appropriate gains in an Luenberger observer). Similar as in [Tanwani and Trenn, 2016] we therefore make the following assumption about the ability to approximate the observable part to any given accuracy:

Assumption 2. For each mode k of the switched DAE (2) and a given $\varepsilon_k > 0$, there exists an estimator $\widehat{z}_k = \widehat{\mathcal{O}}_k(y^e_{[t_k,t_{k+1})})$ such that

$$|\widehat{z}_k - z_k| \leq \varepsilon_k |z_k|, \tag{17}$$

where $z_k = \mathcal{O}_k(y^e_{[t_k,t_{k+1})})$ is the ideal estimator of the observable part on $[t_k, t_{k+1})$ as given in Section 4.1.

Under Assumption 2, the state estimation correction in (8) for the interval $[t_{p_i}, t_{q_i})$ is given by

$$\xi_i := \Phi^{q_i}_{p_i} \mathcal{O}^{q_i-1}_{p_i} \widehat{\mathbf{z}}^{q_i-1}_{p_i}, \tag{18}$$

where $\widehat{\mathbf{z}}^{q_i-1}_{p_i} = (\widehat{z}_{p_i}/\widehat{z}_{p_i+1}/\cdots/z_{q_i-1})$.

As detailed in Appendix B.1, the observable component z_k of the estimation error $e = \widehat{x} - x$ on the interval $[t_k, t_{k+1})$ is composed of the two components z^{diff}_k and z^{imp}_k, where the former is obtained from the continuous output mismatch y^e on (t_k, t_{k+1}) and the latter is obtained from the impulsive mismatch $y^e[t_k]$. The estimation of z^{diff}_k can be reduced to the classical state estimation problem for non-switched linear ODEs and there are many methods to do that. The only non-standard aspect here is that we have to obtain the state-estimation at the beginning of the interval (t_k, t_{k+1}) and not (as usual) at the end of the interval. This does not pose any serious problems, as we can use a standard Luenberger observer on the interval (t_k, t_{k+1}) to get an estimate at the end of the interval and then propagate this estimate back in time. Since the ODE dynamics are known as well as the length of the interval, we can ensure the desired estimation accuracy at the beginning of the interval by increasing the accuracy of the estimate at the end of the interval.Another (more sophisticated) way of obtaining such estimates is by the use of "back-and-forth observer" as presented in [Shim et al., 2012].

The estimation accuracy for z^{imp}_k is actually concerned with the measurement accuracy of the impulsive part $y^e[t_k]$, i.e. on how well Dirac impulses and their derivatives can be measured in practice, see [Tanwani and Trenn, 2016] for details.

Assumption 2, together with Assumption 1, provide all the ingredients we need for obtaining converging state estimates.

Theorem 9. Consider the switched DAE (2) satisfying the uniform local detectability Assumption 1, and the local estimation accuracy Assumption 2. For the α given in (7a), choose $\varepsilon_k = \varepsilon_k(\alpha)$, $k \in \mathbb{N}$, such that

$$\boxed{c_i \varepsilon^{\max}_i \leq \widehat{\alpha} - \alpha} \tag{19}$$

for some $\widehat{\alpha} \in (\alpha, 1)$, where

$$c_i := \|\Phi^{q_i}_{p_i} \mathcal{O}^{q_i-1}_{p_i}\| \left\| \begin{bmatrix} Z^\top_{p_i} \\ Z^\top_{p_i+1}\Phi^{p_i+1}_{p_i} \\ \vdots \\ Z^\top_{q_i-1}\Phi^{q_i-1}_{p_i} \end{bmatrix} \right\|$$

and

$$\varepsilon^{\max}_i := \max\{\, \varepsilon_k \mid p_i \leq k \leq q_i - 1 \,\}.$$

Then the observer given by the system copies (8), with error corrections ξ_i in (18) and the estimate $\widehat{z}_k$ chosen to satisfy (17) for ε_k specified in (19), results in

$$\widehat{x}(t^+) \to x(t^+) \text{ as } t \to \infty,$$

i.e. the observer converges asymptotically to the state.

Appendix A. PROPERTIES OF MATRIX PAIR (E, A)

A very useful characterization of regularity is the following well-known result (see e.g. Berger et al. [2012]).

Proposition 10. (Regularity and quasi-Weierstraß form). A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{A.1}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leq n_1 \leq n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix. $\lhd$

Definition 11. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstraß form (A.1). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Furthermore, let

$$A^{\text{diff}} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \qquad E^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}.$$

Finally, if also an output matrix C is considered let

$$C^{\text{diff}} := C\Pi_{(E,A)}.$$

We refer the reader to [Tanwani and Trenn, 2010] for utility of the matrices introduce in Defintion 11.

Appendix B. OUTPUT-TO-STATE MAPPINGS

B.1 Observable component of a subsystem

The local unobservable space (10) is given by

$$\mathcal{W}_k = \Pi^{-1}_k \ker O^{\text{diff}}_k \,\cap\, \ker O^{\text{imp}}_k,$$

where $O^{\text{diff}}_k := [C^{\text{diff}}_k / C^{\text{diff}}_k A^{\text{diff}}_k / \cdots / C^{\text{diff}}_k (A^{\text{diff}}_k)^{n-1}]$, and $O^{\text{imp}}_k := [C_k E^{\text{imp}}_k / C_k (E^{\text{imp}}_k)^2 / \cdots / C_k (E^{\text{imp}}_k)^{n-1}]$. In other words, $\ker O^{\text{diff}}_k$ denotes the unobservable space of the ODE $\dot{e} = A^{\text{diff}}_k e$, $y^e = C^{\text{diff}}_k e$, and $\ker O^{\text{imp}}_k$ denotes the impulse unobservable space in the sense that $y^e[t_k] = 0$ implies $e(t^-_k) \in \ker O^{\text{imp}}_k$.

We may now write $e(t^-_k) = W_k w_k + Z_k z_k$, where $\operatorname{im} W_k = \mathcal{W}_k$ and $\operatorname{im} Z_k = \mathcal{W}^\perp_k$ and W_k, Z_k are orthonormal matrices. Here z_k determines the projection of $e(t^-_k)$ onto the subspace $\mathcal{W}^\perp_k$. The latter can further be decomposed as

$$\mathcal{W}^\perp_k = \operatorname{im}(O^{\text{diff}}_k \Pi_k)^\top + \operatorname{im} {O^{\text{imp}}_k}^\top$$

Let Z_k^{diff}, and Z_k^{imp} be the orthonormal matrices such that

$$\operatorname{im} Z_k^{\text{diff}} = \operatorname{im}\left(O_k^{\text{diff}\top}\right), \quad \operatorname{im} Z_k^{\text{imp}} = \operatorname{im}\left(O_k^{\text{imp}\top}\right).$$

Also, let $z_k^{\text{diff}} := Z_k^{\text{diff}\top}\Pi_k e(t_k^-) = Z_k^{\text{diff}\top} e(t_k^+)$, and $z_k^{\text{imp}} := Z_k^{\text{imp}\top} e(t_k^-)$. The motivation for introducing the components z_k^{diff} and z_k^{imp} is that they can be estimated using the output measurements on the interval $[t_k, t_{k+1})$. To express the vector z_k in terms of these components, we introduce the matrix U_k such that

$$Z_k = \left[\Pi_k^\top Z_k^{\text{diff}} \;\; Z_k^{\text{imp}}\right] U_k.$$

Such a matrix U_k always exists because

$$\begin{aligned}\operatorname{im} Z_k = \mathcal{W}_k^\perp &= (\Pi_k^{-1}(\ker O_k^{\text{diff}}))^\perp + (\ker O_k^{\text{imp}})^\perp \\ &= \Pi_k^\top \operatorname{im} Z_k^{\text{diff}} + \operatorname{im} Z_k^{\text{imp}} \\ &= \operatorname{im}\left[\Pi_k^\top Z_k^{\text{diff}} \;\; Z_k^{\text{imp}}\right].\end{aligned}$$

It then follows that

$$\begin{aligned} z_k = Z_k^\top e(t_k^-) &= U_k^\top \begin{bmatrix} Z_k^{\text{diff}\top}\Pi_k \\ Z_k^{\text{imp}\top}\end{bmatrix} e(t_k^-) \\ &= U_k^\top \begin{bmatrix} Z_k^{\text{diff}\top} e(t_k^+) \\ Z_k^{\text{imp}\top} e(t_k^-)\end{bmatrix} = U_k^\top \begin{bmatrix} z_k^{\text{diff}} \\ z_k^{\text{imp}}\end{bmatrix}.\end{aligned}$$

Next, we specify how to write z_k^{diff} and z_k^{imp} in terms of the output measured over the interval $[t_k, t_{k+1})$.

Mapping for the differentiable part z_k^{diff}: In order to define $z_k^{\text{diff}} \in \mathbb{R}^{r_k}$, where $r_k = \operatorname{rk} O_k^{\text{diff}}$, we first introduce the function $\mathbf{z}_k^{\text{diff}} : (t_k, t_{k+1}) \to \mathbb{R}^{r_k}, t \mapsto Z_k^{\text{diff}\top} e(t)$, which represents the observable component of the subsystem (E_k, A_k, C_k) that can be recovered from the smooth output measurements y^e over the interval (t_k, t_{k+1}). It follows (cf. [Tanwani and Trenn, 2016, Lem. 17]) that the evolution of $\mathbf{z}_k^{\text{diff}}$ is governed by an observable ODE

$$\dot{\mathbf{z}}_k^{\text{diff}} = S_k^{\text{diff}} \mathbf{z_k}^{\text{diff}}, \quad y^e = R_k^{\text{diff}} \mathbf{z}_k^{\text{diff}}, \tag{B.1}$$

where $S_k^{\text{diff}} := Z_k^{\text{diff}\top} A_k^{\text{diff}} Z_k^{\text{diff}}$ and $R_k^{\text{diff}} := C_k^{\text{diff}} Z_k^{\text{diff}}$. Because of the observability of the pair $(S_k^{\text{diff}}, R_k^{\text{diff}})$ in (B.1), there exists a (linear) map $\mathcal{O}^{\text{diff}}_{(t_k,t_{k+1})}$ such that

$$\mathbf{z}_k^{\text{diff}} = \mathcal{O}^{\text{diff}}_{(t_k,t_{k+1})}(y^e_{(t_k,t_{k+1})})$$

and we set $z_k^{\text{diff}} = \mathbf{z}_k^{\text{diff}}(t_k^+)$.

Mapping for the impulsive part z_k^{imp}: The impulsive part of the output at switching time t_k can be represented as $y^e[t_k] = \sum_{j=0}^{n-2} \eta_k^j \delta_{t_k}^{(j)}$, where the coefficients η_k^j satisfy the relation $\boldsymbol{\eta}_k = -O_k^{\text{imp}} e(t_k^-)$, with $\boldsymbol{\eta}_k := (\eta_k^0 / \cdots / \eta_k^{n-2}) \in \mathbb{R}^{(n-1)\mathtt{y}}$. We chose a matrix U_k^{imp} such that $O_k^{\text{imp}\top} U_k^{\text{imp}} = -Z_k^{\text{imp}}$, then

$$z_k^{\text{imp}} = Z_k^{\text{imp}\top} e(t_k^-) = -U_k^{\text{imp}\top} O_k^{\text{imp}} e(t_k^-) = U_k^{\text{imp}\top} \boldsymbol{\eta}_k.$$

B.2 Observable component over an interval

For $p \le k \le q-1$, the $[t_k, t_q)$-unobservable subspace (12) can be computed recursively as follows

$$\mathcal{N}_{q-1}^q = \mathcal{W}_{q-1} \tag{B.2a}$$

$$\mathcal{N}_k^q = \mathcal{W}_k \cap \Pi_k^{-1} \mathsf{e}^{-A_k^{\text{diff}}\tau_k} \mathcal{N}_{k+1}^q, \quad q > k+1. \tag{B.2b}$$

The objective is to compute the observable part $\mu_k = M_k^\top e(t_k^-)$ in (13) recursively for $k = q-1, q-2, \ldots, p$. We choose $\mu_{q-1} = z_{k-1}$. By construction, we know that

$$\begin{aligned}\operatorname{im} M_k &= \left(\mathcal{W}_k \cap \Pi_k^{-1}(\mathsf{e}^{-A_k^{\text{diff}}\tau_k}\mathcal{N}_{k+1}^q)\right)^\perp \\ &= \mathcal{W}_k^\perp + \Pi_k^\top (\mathsf{e}^{-A_k^{\text{diff}}\tau_k}\mathcal{N}_{k+1}^q)^\perp.\end{aligned}$$

Recalling that $\operatorname{im} Z_k = (\mathcal{W}_k)^\perp$, and introducing the matrix Θ_k such that

$$\operatorname{im}\Theta_k = (\mathsf{e}^{-A_k^{\text{diff}}\tau_k}\mathcal{N}_{k+1}^q)^\perp$$

we obtain, $\operatorname{im} M_k = \operatorname{im}\left[Z_k,\ \Pi_k^\top \Theta_{k+1}\right]$. Hence there exists a matrix U_k such that

$$M_k = \left[Z_k,\ \Pi_k^\top \Theta_{k+1}\right] U_k.$$

Noting that

$$\begin{aligned}\Pi_k e(t_k^-) = e(t_k^+) &= \mathsf{e}^{-A_k^{\text{diff}}\tau_k} e(t_{k+1}^-) \\ &= \mathsf{e}^{-A_k^{\text{diff}}\tau_k}(M_{k+1}\mu_{k+1} + N_{k+1}\nu_{k+1})\end{aligned}$$

and multiplication on both sides from left by $\Theta_k^\top$ gives

$$\begin{aligned}\Theta_k^\top \Pi_k e(t_k^-) = {}& \Theta_k^\top \mathsf{e}^{-A_k^{\text{diff}}\tau_k} M_{k+1}\mu_{k+1} \\ &+ \underbrace{\Theta_k^\top \mathsf{e}^{-A_k^{\text{diff}}\tau_k} N_{k+1}}_{=0} \nu_{k+1}.\end{aligned}$$

This allows us to compute μ_k as follows:

$$\begin{aligned}\mu_k = M_k^\top e(t_k^-) &= U_k^\top \begin{bmatrix} Z_k^\top \\ \Theta_k^\top \Pi_k\end{bmatrix} e(t_k^-) \\ &= U_k{}^\top \begin{pmatrix} z_k \\ \Theta_k^\top \mathsf{e}^{-A_k^{\text{diff}}\tau_k} M_{k+1}\mu_{k+1}\end{pmatrix} =: \mathcal{F}_k \begin{pmatrix} z_k \\ \mu_{k+1}\end{pmatrix}.\end{aligned}$$

REFERENCES

Thomas Berger, Achim Ilchmann, and Stephan Trenn. The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl.*, 436(10):4052–4069, 2012. doi: 10.1016/j.laa.2009.12.036.

E. De Santis, M.D. Di Benedetto, and G. Pola. A structural approach to detectability for a class of hybrid systems. *Automatica*, 45(5): 1202–1206, 2009.

Daniel Liberzon and Stephan Trenn. On stability of linear switched differential algebraic equations. In *Proc. IEEE 48th Conf. on Decision and Control*, pages 2156–2161, December 2009. doi: 10.1109/CDC.2009.5400076.

Daniel Liberzon and Stephan Trenn. Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability. *Automatica*, 48(5):954–963, May 2012. doi: 10.1016/j.automatica.2012.02.041.

M.A. Muller and D. Liberzon. Input/output-to-state stability and state-norm estimators for switched nonlinear systems. *Automatica*, 48(9):2029–2039, 2012.

Hyungbo Shim, Aneel Tanwani, and Zhaowu Ping. Back-and-forth operation of state observers and norm estimation of estimation error. In *Proc. 51st IEEE Conf. Decis. Control, Maui, USA*, pages 3221 – 3226. 2012. doi: 10.1109/CDC.2012.6426816.

Aneel Tanwani and Stephan Trenn. On observability of switched differential-algebraic equations. In *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA*, pages 5656–5661, 2010. doi: 10.1109/CDC.2010.5717685.

Aneel Tanwani and Stephan Trenn. An observer for switched differential-algebraic equations based on geometric characterization of observability. In *Proc. 52nd IEEE Conf. Decis. Control, Florence, Italy*, pages 5981–5986, 2013. doi: 10.1109/CDC.2013.6760833.

Aneel Tanwani and Stephan Trenn. Determinability and state estimation for switched differential-algebraic equations. *Automatica*, 2016. doi: 10.1016/j.automatica.2016.10.024. in press.

Aneel Tanwani, Hyunngbo Shim, and Daniel Liberzon. Observability for switched linear systems: Characterization and observer design. *IEEE Trans. Autom. Control*, 58(4):891–904, 2013. ISSN 0018-9286. doi: 10.1109/TAC.2012.2224257.

Stephan Trenn. *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009. URL `http://www.db-thueringen.de/servlets/DocumentServlet?id=13581`.