Observability of Switched Differential-Algebraic Equations

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***Abstract*— We study observability of switched differential-algebraic equations (DAEs) for arbitrary switching. We present a characterization of observability and, a related property called, determinability. These characterizations utilize the results for the single-switch case recently obtained by the authors. Furthermore, we study observability conditions when only the mode sequence of the switching signal (and not the switching times) are known. This leads to necessary and sufficient conditions for observability and determinability. We illustrate the results with illustrative examples.**

I. Introduction

In this paper, we study observability of a class of switched systems where the dynamical subsystems are modeled as *differential-algebraic equations* (DAEs):

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal, and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times d_u}$, $C_p \in \mathbb{R}^{d_y \times n}$, for $p \in \mathbb{N}$. Often ordinary differential equations are used to model the dynamical behavior of a system. However, the evolution of the states in a physical system may be constrained, e.g., current and voltage in electrical circuits due to Kirchoff's laws, or position variables in coupled mechanical systems. In the modeling of such physical systems, it is important to take into account the algebraic constraints imposed on the state variables alongside some differential equations that govern the evolution of these state variables and we therefore believe that a systems description as in (1) is important for modeling many phenomena.

This paper is a continuation of our work [1] where observability of (1) for a single switch was investigated. We are able to extend these results to the case of general switching signals and our main result is the characterization of observability of switched DAEs (1) with a fixed switching signal (Theorem 10). We also present a necessary and a sufficient condition for observability when only the mode sequence of the switching signal is known (and not the switching times). Alongside these results we also study the weaker property of determinability (which is called "forward observability" in [1]) which seems to be more suitable with respect to observer design (see [2] in a similar context).

For a more detailed literature review we refer the reader to the introduction of [1].

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II. Preliminaries

A. Properties and Definitions for Regular Matrix Pairs

In the following, we collect important properties and definitions for matrix pairs (E, A). We only consider *regular* matrix pairs, i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. A very useful characterization of regularity is the following well-known result.

Proposition 1 (Regularity and quasi-Weierstrass form): A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \le n_1 \le n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix. ◁

In view of [3], we call the decomposition (2) *quasi-Weierstrass form*. An easy way to calculate the transformation matrices S and T for (2) is to use the following so-called *Wong sequences* [4], [3]:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, \cdots \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i), & i &= 0, 1, \cdots \end{aligned}$$

The Wong sequences are nested and get stationary after finitely many steps. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \quad \mathcal{W}^* := \bigcup_i \mathcal{W}_i.$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2) holds.

Based on the Wong-sequences we define the following "projectors".

Definition 2 (Consistency, differential and impulse projectors): Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (2). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential projector* is given by

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the *impulse projector* is given by

$$\Pi^{\text{imp}}_{(E,A)} = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S,$$

where the block sizes correspond to the ones in (2). ◁

Note that only the consistency projector is a projector in the usual sense (i.e. $\Pi_{(E,A)}$ is an idempotent matrix); whereas $\Pi^{\text{diff}}_{(E,A)}$ and $\Pi^{\text{imp}}_{(E,A)}$ are not projectors because, in general, $\Pi^{\text{diff}}_{(E,A)}\Pi^{\text{diff}}_{(E,A)} \neq \Pi^{\text{diff}}_{(E,A)}$ and the same holds for $\Pi^{\text{imp}}_{(E,A)}$. Let

$$\mathfrak{C}_{(E,A)} := \left\{ \ x_0 \in \mathbb{R}^n \ \middle| \ \exists x \in \mathcal{C}^1 : E\dot{x} = Ax \wedge x(0) = x_0 \ \right\}$$

be the *consistency space* of the DAE $E\dot{x} = Ax$, where $\mathcal{C}^1$ is the space of differentiable functions $x : \mathbb{R} \to \mathbb{R}^n$. Then the following observations hold [3]:

1) All solutions $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ evolve within $\mathfrak{C}_{(E,A)}$,
2) $\mathfrak{C}_{(E,A)} = \mathcal{V}^*$, i.e. the first Wong-sequence converges to the consistency space,
3) $\operatorname{im}\Pi_{(E,A)} = \mathcal{V}^* = \mathfrak{C}_{(E,A)}$, hence the consistency projector maps onto the consistency space.

The following lemma motivates the name of the differential projector.

Lemma 3 ([1, Lem. 3]): Consider the DAE $E\dot{x} = Ax$ with regular matrix pair (E, A). Then any solution $x \in \mathcal{C}^1$ of $E\dot{x} = Ax$ fulfills

$$\dot{x} = \Pi^{\text{diff}}_{(E,A)}Ax =: A^{\text{diff}}x. \qquad \triangleleft$$

For understanding the role of the consistency projector and for studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [5] as the solution space; that is, we seek a solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ to the following initial-trajectory problem (ITP):

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax)_{[0,\infty)}, \end{aligned} \tag{3}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [5], [6] it is shown that the ITP (3) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. In particular, the following result concerning the consistency projector holds.

Lemma 4 (Role of consistency projector, [6, Thm. 4.2.8]): Consider the ITP (3) with regular matrix pair (E, A) and with arbitrary initial trajectory $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$. Let $\Pi_{(E,A)}$ be the consistency projector of (E, A), then there exists a unique solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and

$$x(0+) = \Pi_{(E,A)}x(0-). \qquad \triangleleft$$

Finally, the role of the impulsive projector becomes clear when expressing the impulsive part, denoted by $x[0]$, of the distributional solution x of the ITP (3).

Lemma 5 ([1, Cor. 5]): Consider the ITP (3) with regular matrix pair (E, A) and corresponding impulse and consistency projectors $\Pi^{\text{imp}}_{(E,A)}, \Pi_{(E,A)}$. Let $E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)}E$ then, for the unique solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$,

$$x[0] = \sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1}(\Pi_{(E,A)} - I)x(0-)\delta_0^{(i)},$$

where $\delta_0^{(i)}$ denotes the i-th (distributional) derivative of the Dirac-impulse δ_0 at $t = 0$. $\triangleleft$

III. Observability Conditions

The concepts introduced in the previous section are now utilized to obtain necessary and sufficient conditions for observability and determinability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals that are right continuous with a locally finite number of jumps; i.e., we exclude an accumulation of switching times.

A. Observability definitions

Definition 6 (Observability): The switched DAE (1) with some fixed switching signal σ, is called *(globally) observable* if for every pair of inputs and outputs $(y, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{d_y+d_u}$ there exists at most one $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ which solves (1). $\triangleleft$

The following result will be helpful for simplifying the analysis.

Lemma 7 (Observability of zero, [1, Prop. 7]): The switched DAE (1) is observable if, and only if, $y \equiv 0$ and $u \equiv 0$ implies $x \equiv 0$. $\triangleleft$

The above result justifies that we can ignore the input when studying observability of (1); hence in what follows, we can restrict our attention to the homogeneous switched DAE:

$$E_\sigma\dot{x} = A_\sigma x, \quad y = C_\sigma x. \tag{4}$$

The above observability definition aims at recovering the state also in the past, however in certain applications (e.g. observer design) one might only be interested in determining the state in the future. This motivates the following definition.

Definition 8 (Determinability): The switched DAE (1) is called *determinable* if for every pair of triplets $(x_1, u_1, y_1), (x_2, u_2, y_2) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+d_u+d_y}$ which solve (1) there exists $t \geq 0$ such that the implication $(u_1, y_1) = (u_2, y_2) \Rightarrow {x_1}_{(t,\infty)} = {x_2}_{(t,\infty)}$ holds. $\triangleleft$

Similar to observability we can restrict our attention to zero-determinability.

Proposition 9 (Determinability of zero): The switched DAE (1) is determinable if, and only if, $y \equiv 0$ and $u \equiv 0$ implies $x_{(t,\infty)} \equiv 0$ for some $t \geq 0$. $\triangleleft$

Proof: Necessity is obvious. Assume now (1) is not determinable, i.e. there exist external signals u and y and corresponding solutions x_1 and x_2 with $(x_1 - x_2)_{(t,\infty)} \neq 0$ for all $t \geq 0$. By linearity, $x := x_1 - x_2$ solves $E_\sigma\dot{x} = A_\sigma x$ and $C_\sigma x = C_\sigma x_1 - C_\sigma x_2 = y - y = 0$, hence $y \equiv 0$ and $u \equiv 0$ does not imply $x_{(t,\infty)} = 0$ for any $t \geq 0$. ∎

B. The single switch case

We recapitulate the result of [1] on the single switch case which are essential for the results on arbitrary switching. Therefore, we consider in this subsection the switching signal

$$\sigma(t) = 0 \text{ for } t < 0 \quad \text{and} \quad \sigma(t) = 1 \text{ for } t \geq 0. \tag{5}$$

That is, we only consider one switch from some initial subsystem given by $(C_-, E_-, A_-) := (C_0, E_0, A_0)$ – active before the switch – to some other subsystem given by $(C_+, E_+, A_+) := (C_1, E_1, A_1)$ that is active after the switch.

By regularity of the matrix pairs $(E_\pm, A_\pm)$ it is easily seen that for any solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ of the switched DAE (4) with the single switch switching signal (5) the following equivalence holds:

$$x \equiv 0 \quad \Leftrightarrow \quad x(0-) = 0.$$

For the characterization of observability and determinability for the single switch case, the following four subspaces play an essential role:

Consistency space. Independently of the observed output it holds that $\boxed{x(0-) \in \mathfrak{C}_-}$, where $\mathfrak{C}_- := \mathfrak{C}_{(E_-,A_-)}$ denotes the consistency space of the DAE corresponding to the matrix pair (E_-, A_-).

Left-unobservable space. If $y_{(-\infty,0)} \equiv 0$ then $y^{(i)}(0-) = 0$ for all $i \in \mathbb{N}$, hence, invoking Lemma 3, we have $\boxed{x(0-) \in \ker O_-}$, where

$$O_- := [C_-/C_-A_-^{\text{diff}}/\cdots/C_-(A_-^{\text{diff}})^{n-1}], \tag{6}$$

and $\ker O_-$ denotes the unobservable space of the ODE $\dot{x} = A_-^{\text{diff}}x$, $y = C_-x$.

Projected right-unobservable space. Analogously as above, if $y_{(0,\infty)} \equiv 0$ then $x(0+) \in \ker O_+$, where O_+ is defined analogously as in (6). Due to Lemma 4, we obtain $\boxed{x(0-) \in \ker O_+^-}$, where

$$\ker O_+^- := \Pi_{(E_+,A_+)}^{-1} \ker O_+ = \ker O_+ \Pi_{(E_+,A_+)}.$$

Impulse unobservable space. Finally, due to Lemma 5, from $0 = y[0] = C_+x[0]$ it follows that $\boxed{x(0-) \in \ker O_+^{\text{imp}-}}$, where

$$O_+^{\text{imp}-} := O_+^{\text{imp}}(\Pi_{(E_+,A_+)} - I)$$

and

$$O_+^{\text{imp}} := [C_+E_+^{\text{imp}}/C_+(E_+^{\text{imp}})^2/\cdots/C_+(E_+^{\text{imp}})^{n_2-1}].$$

With these four subspaces, a complete characterization of observability is possible.

Theorem 10 (Single switch result [1, Cor. 13]): Consider the switched DAE (4) and the single switch switching signal (5). Then the unobservable subspace for $x(0-)$ is given by

$$\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-},$$

i.e., the following equivalence holds for all solutions $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ of (4):

$$y \equiv 0 \quad \Leftrightarrow \quad x(0-) \in \mathcal{M}.$$

In particular, the switched DAE (1) is observable if, and only if, $\mathcal{M} = \{0\}$. Furthermore, (1) is determinable if, and only if, $\Pi_{(E_+,A_+)}\mathcal{M} = \{0\}$. ◁

C. Multiple switching main results

So far, we have studied switched DAEs with a single switching instant. For switched DAEs (4) with more than two subsystems and multiple switchings, we build on the results of the previous section to obtain a characterization for the general case. For notational convenience, we assume that the switching signal $\sigma : \mathbb{R} \to \mathbb{N} \cup \{-1\}$ is given by:

$$\begin{aligned} \sigma(t) &= -1 \quad \text{for } t < t_0 := 0, \\ \sigma(t) &= k \quad \text{on } [t_k, t_{k+1}),\ k \in \mathbb{N}, \end{aligned} \tag{7}$$

where $t_k \in \mathbb{R}$, $k \in \mathbb{N}$, denote the (ordered) switching times of σ. In particular, we assume that σ has no switches before the initial time $t_0 := 0$. The latter is a slight restriction of generality as we do not allow accumulation of switching times towards $-\infty$. Otherwise this is not a restriction of generality as we do of course allow $(E_k, A_k, C_k) = (E_l, A_l, C_l)$ for any $k, l \in \mathbb{N}$.

Adopting the notation from the previous section we let, for $k \in \mathbb{N} \cup \{-1\}$,

$$\begin{aligned} \mathfrak{C}_k &:= \mathfrak{C}_{(E_k,A_k)}, \\ O_k &:= [C_k/C_kA_k^{\text{diff}}/\cdots/C_k(A_k^{\text{diff}})^{n-1}], \\ O_k^{\text{imp}} &:= [C_kE_k^{\text{imp}}/C_k(E_k^{\text{imp}})^2/\cdots/C_k(E_k^{\text{imp}})^{n_2-1}]. \end{aligned}$$

With $O_k^- := O_k\Pi_k$, and $O_k^{\text{imp}-} := O_k^{\text{imp}}(\Pi_k - I)$, where $\Pi_k := \Pi_{(E_k,A_k)}$ is the consistency projector of the k-th subsystem, we define $\mathcal{M}_k$, $k \in \mathbb{N}$, as:

$$\mathcal{M}_k := \mathfrak{C}_{k-1} \cap \ker O_{k-1} \cap \ker O_k^- \cap \ker O_k^{\text{imp}-},$$

According to Theorem 10 we call $\mathcal{M}_k$ the *locally unobservable subspace* at the k-th switching instance. The following example shows that the existence of $k \in \mathbb{N}$ with $\mathcal{M}_k = \{0\}$, i.e. local observability, is not necessary for (global) observability.

Example 11 ($\mathcal{M}_k = \{0\}$ not necessary, [1, Ex. 20]): Consider a switched DAE (4) excited by the switching signal

$$\sigma(t) = \begin{cases} -1, & t \in (-\infty, 0), \\ 0, & t \in [0, \frac{\pi}{2}), \\ 1, & t \in [\frac{\pi}{2}, \infty), \end{cases}$$

with modes given by the following matrices $E_{-1} = \left[\begin{smallmatrix} 0&0&0 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right]$, $A_{-1} = \left[\begin{smallmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{smallmatrix}\right]$, $C_{-1} = \left[\begin{smallmatrix} 0&0&1 \end{smallmatrix}\right]$, $E_0 = \left[\begin{smallmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{smallmatrix}\right]$, $A_0 = \left[\begin{smallmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&1 \end{smallmatrix}\right]$, $C_0 = \left[\begin{smallmatrix} 0&0&1 \end{smallmatrix}\right]$ and $(E_1, A_1, C_1) = (E_{-1}, A_{-1}, C_{-1})$.

Letting $\{e_1, e_2, e_3\}$ denote the natural basis vectors for $\mathbb{R}^3$, it can be verified that (for details see [1])

$$\mathcal{M}_0 = \mathcal{M}_1 = \text{span}\{e_2\}.$$

However, this switched DAE is observable as can be seen from the explicit solution given by, for some $a \in \mathbb{R}$,

$$\begin{aligned} x_1(t) &= a \sin t \cdot \mathbb{1}_{[0,\frac{\pi}{2})}, \\ x_2(t) &= ae^{2t} \cdot \mathbb{1}_{(-\infty,0)} + a \cos t \cdot \mathbb{1}_{[0,\frac{\pi}{2})}, \\ x_3(t) &= -a\delta_{\frac{\pi}{2}} = y(t), \end{aligned}$$

where $\mathbb{1}_{\mathcal{I}}$ denotes the indicator function of the interval $\mathcal{I}$. For an identically zero output, the impulsive part of the output at the second switching instant enforces $a = 0$ and this makes $x \equiv 0$. ◁

Concerning the sufficiency of $\mathcal{M}_k = \{0\}$ for observability, observe first that clearly $\mathcal{M}_0 = \{0\}$ makes the whole switched DAE observable as $x(0-)$ can be deduced from the information at the first switch alone. On the other hand $\mathcal{M}_k = \{0\}$, $k > 0$, only guarantees that $x(t_k-)$ can be deduced from the k-th switch but, in general, this does not allow for the deduction of $x(0-)$ (just consider $(E, A) = (0, I)$ as 0-th mode).

To use the information obtained by each individual switch given by $\mathcal{M}_k$, we will use the following iteration, $m \in \mathbb{N}$, $k = m-1, \ldots, 0$:

$$\begin{aligned} \mathcal{N}_m^m &:= \mathcal{M}_m, \\ \mathcal{N}_k^m &:= \mathcal{M}_k \cap \Pi_k^{-1}(e^{-A_k^{\text{diff}}\tau_k}\mathcal{N}_{k+1}^m), \end{aligned} \tag{8}$$

where $\tau_k := t_{k+1} - t_k$ is the duration of mode k. The intuition is that $\mathcal{N}_k^m$, $k < m$, is the unobservable subspace for $x(t_k-)$ based on the knowledge of the observed output at the switching time t_k, given by $\mathcal{M}_k$ and the information obtained from the future switching times up to t_m given by $\mathcal{N}_{k+1}^m$ together with the known flow on the interval $[t_k, t_{k+1})$ and the consistency projector Π_k at t_k. With this subspace iteration we can now characterize observability of switched DAEs.

Theorem 12 (Observability Characterization): Consider the switched DAE (1) with switching signal σ as in (7). For each positive integer $m \in \mathbb{N}$, define the sequence $\mathcal{N}_k^m$, for $0 \leq k \leq m$, according to (8). The switched system is globally observable if, and only if, there exists an $m \in \mathbb{N}$ such that

$$\mathcal{N}_0^m = \{0\}. \tag{9}$$

Proof: Sufficiency. We show that the identically zero output can only be produced by $x \equiv 0$. Fix m such that (9) holds. Assume that $y \equiv 0$ on $(-\infty, \infty)$; then according to Theorem 10, $x(t_m-) \in \mathcal{M}_m = \mathcal{N}_m^m$. We next apply the inductive argument to show that $x(t_k-) \in \mathcal{N}_k^m$ for $0 \leq k \leq m$. Assume that $x(t_k-) \in \mathcal{N}_k^m$; then, invoking Lemma 3, $x(t_{k-1}+) \in \exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m$. This implies that $x(t_{k-1}-) \in \Pi_k^{-1}\exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m$. Zero output on the interval (t_{k-2}, t_k) implies that $x(t_{k-1}-) \in \mathcal{M}_{k-1}$ and thus $x(t_{k-1}-) \in \mathcal{M}_{k-1} \cap \exp(-A_k^{\text{diff}}\tau_k)\mathcal{N}_k^m = \mathcal{N}_{k-1}^m$. As a result, $x(0-) \in \mathcal{N}_0^m = \{0\}$, i.e., $x(0-) = 0$; regularity of the matrix pairs (E_k, A_k), $k \in \mathbb{N} \cup \{-1\}$ implies that $x \equiv 0$.

Necessity. Assume that $\mathcal{N}_0^m \neq \{0\}$ for all $m \in \mathbb{N}$. Since $\mathcal{N}_m^{m+1} \subseteq \mathcal{N}_m^m$ it follows that $\mathcal{N}_k^{m+1} \subseteq \mathcal{N}_k^m$ for all $m \in \mathbb{N}$ and $0 \leq k \leq m$. Let $\mathcal{N}_k := \bigcap_{m \geq k} \mathcal{N}_k^m$, then, by finite dimensionality of $\mathbb{R}^n$,

$$\mathcal{N}_0 \neq \{0\}.$$

We will show that for all initial values $x^0 \in \mathcal{N}_0$ the unique non-zero solution $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ of the switched DAE

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0-) = x^0$$

fulfills $y = C_\sigma x = 0$, which implies unobservability. To this end, we first show the following implication, $0 \leq k \leq m$:

$$x(t_k-) \in \mathcal{N}_k \quad \Rightarrow \quad x(t_{k+1}-) \in \mathcal{N}_{k+1}. \tag{10}$$

Assume $x(t_k-) \in \mathcal{N}_k$. Since $x(t_k+) = \Pi_{k+1}x(t_k-)$ and $x(t_k-) \in \mathcal{N}_k^m$ for any $m \geq k+1$, it follows that

$$\begin{aligned} x(t_{k+1}-) &= e^{A_k^{\text{diff}}\tau_k}x(t_k+) = e^{A_k^{\text{diff}}\tau_k}\Pi_k x(t_k-) \\ &\in e^{A_k^{\text{diff}}\tau_k}\Pi_k\mathcal{N}_k^m \\ &\subseteq e^{A_k^{\text{diff}}\tau_k}\left(\Pi_k\mathcal{M}_k \cap e^{-A_k^{\text{diff}}\tau_k}\mathcal{N}_{k+1}^m\right) \\ &\subseteq \mathcal{N}_{k+1}^m\,. \end{aligned}$$

Therefore, implication (10) is shown and an inductive argument gives $x(t_k-) \in \mathcal{N}_k$ for all $k \in \mathbb{N}$.

For all $k \in \mathbb{N} \cup \{-1\}$, $x(t_{k+1}-) \in \mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1} \subseteq \ker O_k$, i.e. x evolves on (t_k, t_{k+1}) within the unobservable space of the k-th mode, where $t_{-1} := -\infty$. This implies $y_{(t_k,t_{k+1})} \equiv 0$. Finally, $y[t_k] = 0$ because $x(t_k-) \in \mathcal{N}_k \subseteq \mathcal{M}_k \subseteq \ker O_k^{\text{imp}-}$. Altogether, we have shown that $x^0 \in \mathcal{N}_0$ implies $y \equiv 0$ which concludes the proof. ■

Example 13 (Example 11 revisited): Consider again the switched DAE from Example 11. We calculate

$$\mathcal{N}_1^1 = \mathcal{M}_1 = \text{span}\{e_2\}$$

and, invoking $A_0^{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e^{-A_0^{\text{diff}}\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as well as $\Pi_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we get

$$\begin{aligned} \mathcal{N}_0^1 &= \mathcal{M}_0 \cap \Pi_0^{-1}(e^{-A_0^{\text{diff}}\frac{\pi}{2}})\mathcal{N}_1^1 \\ &= \text{span}\{e_2\} \cap \text{span}\{e_1, e_3\} = \{0\}. \end{aligned}$$

Hence, condition (9) holds and we can verify that the switched DAE is observable without analyzing the explicit solution formulas. ◁

Note that, although Theorem 12 gives a characterization of observability, it might not be so useful in practice as one does not know a priori how many switches are needed for observability. In particular, if the switched system is unobservable the check for unobservability runs indefinitely as $\mathcal{N}_0^m \neq \{0\}$ for all m. Furthermore, for large m the calculation of $\mathcal{N}_0^m$ via (8) might be very long as one always has to start with $\mathcal{N}_m^m = \mathcal{M}_m$ and has to iterate backwards from m to zero for each m. As already mentioned in Section III-A observability basically aims at reconstructing $x(0-)$ from the observed output which explains why at each new switching instance the obtained information must be iterated back to the initial switch at $t = 0$. If one only aims at determining the state in the future, then the situation improves significantly. Towards this end, consider the following sequence of subspaces:

$$\begin{aligned} \mathcal{Q}^0 &:= \Pi_0\mathcal{M}_0, \\ \mathcal{Q}^{k+1} &:= \Pi_{k+1}(\mathcal{M}_{k+1} \cap e^{A_k^{\text{diff}}\tau_k}\mathcal{Q}^k),\ \ k \in \mathbb{N}. \end{aligned} \tag{11}$$

The intuition behind this sequence of subspaces is as follows: The subspace $\mathcal{Q}^k$ contains all undeterminable states at the k-th switching instance where we use all the knowledge up to the k-th switching instance. At the next switching instance we propagate forward the information from $\mathcal{Q}^k$ and intersect this with the locally unobservable subspace $\mathcal{M}_{k+1}$. Using then the consistency projector Π_{k+1} gives the next undeterminable subspace $\mathcal{Q}^{k+1}$. This procedure is significantly different to the subspace iteration in (8) as it is not necessary to iterate back in time. We can now characterize determinability with the help of the subspace iteration (11).

Theorem 14 (Determinability Characterization): Consider the switched DAE (1) with switching signal σ as in (7). For each $m \in \mathbb{N}$ define $\mathcal{Q}^m$ according to (11). The switched system is determinable if, and only if, there exists an $m \in \mathbb{N}$ such that

$$\mathcal{Q}^m = \{0\}. \tag{12}$$

Proof: Due to Proposition 9, it suffices to consider (4) with a zero output.

Sufficiency. We will show that a zero output implies $x(t_m+) = 0$ hence, due to regularity of the involved matrix pairs, $x_{(t_m,\infty)} \equiv 0$. Therefore, let $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ be a solution of (4) with zero output. Then Theorem 10 ensures that $x(0+) \in \mathcal{Q}^0$. We will inductively show that $x(t_k+) \in \mathcal{Q}^k$ for all $k \in \mathbb{N}$, i.e. assume the latter for some k. As $x(t_{k+1}-) = e^{A_k^{\text{diff}}\tau_k} x(t_k+)$ it follows that $x(t_{k+1}-) \in e^{A_k^{\text{diff}}\tau_k}\mathcal{Q}^k$. Theorem 10 yields that $x(t_{k+1}-) \in \mathcal{M}_{k+1}$ hence we have shown that $x(t_{k+1}+) = \Pi_{k+1} x(t_{k+1}-) \in \mathcal{Q}^{k+1}$. In particular, $x(t_m+) \in \mathcal{Q}^m = \{0\}$.

Necessity. Determinability implies existence of $m \in \mathbb{N}$ such that $x(t_m+) = 0$ for all solution x of (4) with zero output. As the property of determinability does not depend on the switches after t_m, we can assume that there are no further switches after t_m. Seeking a contradiction, assume that $\mathcal{Q}^m \neq \{0\}$. We will now construct a solution x such that its corresponding output is zero and $x(t_m+) \neq 0$, contradicting determinability. Choose $x_+^m \in \mathcal{Q}^m \setminus \{0\}$, then by definition there exists $x_-^m \in \mathcal{M}_m \cap e^{A_{m-1}^{\text{diff}}\tau_{m-1}}\mathcal{Q}^{m-1}$ such that $x_+^m = \Pi_m x_-^m$. Let $x_+^{m-1} := e^{-A_{m-1}^{\text{diff}}\tau_{m-1}} x_-^m$ then $x_+^{m-1} \in \mathcal{Q}^{m-1}$. We can repeat this procedure inductively to obtain $x_\pm^0, x_\pm^1, \ldots, x_\pm^m \in \mathbb{R}^n$ such that $x_+^k \in \mathcal{Q}^k$, $x_-^k \in \mathcal{M}_k$, $x_+^k = \Pi_k x_-^k$ for $k = 0, 1, \ldots, m$ and $x_-^k = e^{A_{k-1}^{\text{diff}}\tau_{k-1}} x_+^{k-1}$ for $k = 1, 2, \ldots, m$. Let x be the unique solution of (4) with initial condition $x(0-) = x^{0-}$. By construction, $x(t_k\pm) = x_\pm^k$ for $k = 0, 1, \ldots, m$, in particular $x(t_k-) \in \mathcal{M}_k$ which, by Theorem 10, ensures zero output on $(-\infty, t_1)$, (t_{k-1}, t_{k+1}), $k = 1, 2, \ldots, m-1$, and (t_{m-1}, ∞); moreover, $y[t_k] = 0$, for $k = 0, 1, \ldots, m$. Hence we have obtained the sought contradiction as the constucted x produces a zero output but $x(t_m+) = x^m \neq 0$. ■

Remark 15 (Artificial switches): It is always possible to alter the switching signal σ as in (7) by adding an artificial switch at ${t_k}' \in (t_k, t_{k+1})$ between the k-th and $k+1$-th switching instants, resulting in σ' also given by (7) but now with different matrices describing the modes:

$$(E'_m, A'_m, C'_m) = \begin{cases} (E_m, A_m, C_m), & \text{for } m \leq k, \\ (E_{m-1}, A_{m-1}, C_{m-1}), & \text{for } m \geq k+1, \end{cases}$$

in particular, $(E'_k, A'_k, C'_k) = (E'_{k+1}, A'_{k+1}, C'_{k+1})$. Then, as both descriptions describe the same switched DAE, the observability conditions should not change. For $m \leq k$, this follows easily, and for, for $m \geq k+1$, this is verified by the following computation:

$$\begin{aligned} {\mathcal{N}'}_k^{m+1} &= \mathcal{M}'_k \cap \Pi_k^{'-1}(e^{-A_k^{'\text{diff}}\tau'_k} {\mathcal{N}'}_{k+1}^m) \\ &= \mathcal{M}'_k \cap \Pi_k^{'-1}(e^{-A_k^{'\text{diff}}\tau'_k}(\mathcal{M}'_{k+1} \cap \Pi_{k+1}^{'-1}(e^{-A_{k+1}^{'\text{diff}}\tau'_{k+1}} {\mathcal{N}'}_{k+2}^m))) \\ &= \mathcal{M}'_k \cap \Pi_k^{'-1}(\mathcal{M}'_{k+1} \cap e^{-A_k^{'\text{diff}}\tau'_k} \Pi_k^{'-1} e^{-A_k^{'\text{diff}}\tau'_{k+1}} {\mathcal{N}'}_{k+2}^m). \end{aligned}$$

We now us the fact that $\mathcal{M}'_{k+1} = \mathfrak{C}'_k \cap \ker O'_k$ is invariant under $A_k^{'\text{diff}}$, which gives $\Pi_k^{'-1}\mathcal{M}'_{k+1} = \ker O'_k \Pi'_k$, so that $\mathcal{M}'_k \cap \Pi_k^{'-1}\mathcal{M}'_{k+1} = \mathcal{M}'_k = \mathcal{M}_k$. Moreover, $e^{-A_k^{'\text{diff}}\tau'_k}\Pi_k^{'-1}(\mathcal{V}) = \Pi_k^{'-1}e^{-A_k^{'\text{diff}}\tau'_k}(\mathcal{V})$ for a subspace $\mathcal{V}$, which combined with ${\mathcal{N}'}_{k+2}^m = \mathcal{N}_{k+1}^m$, gives

$${\mathcal{N}'}_k^{m+1} = \mathcal{N}_k^m.$$

That is, (8) is "invariant" with respect to the addition of artificial switches. Similar calculations can be repeated for determinability using (11).

IV. Observability conditions for partly unknown switching signal

In the previous section we presented characterization of observability and determinability under the assumption that the switching signal is known exactly a priori. In particular, the conditions depend on the switching times and how long each mode is active. In this section we want to weaken this assumption by only assuming knowledge of the mode sequence (and not of the switching times), i.e. we only know that σ is a member of the set

$$\Sigma_{\mathbb{N}} := \left\{ \sigma : \mathbb{R} \to \mathbb{N} \cup \{-1\} \;\middle|\; \begin{array}{l} \sigma \text{ is given by (7) for} \\ \text{some switching times} \\ 0 = t_0 < t_1 < t_2 < \ldots \end{array} \right\}$$

This will simplify the observability conditions as they will not depend on the switching times t_k. However, the price will be that we will only get a sufficient and a necessary condition for observability and we will show with examples that there is indeed a gap between these conditions. For the formulation of the results we will need the following notation.

Definition 16 (A-invariant subspaces): Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $\mathcal{V} \subseteq \mathbb{R}^n$ some subspace of $\mathbb{R}^n$. Let

$$\langle A \mid \mathcal{V} \rangle := \mathcal{V} + A\mathcal{V} + A^2\mathcal{V} \ldots + A^{n-1}\mathcal{V}$$

be the smallest A-invariant subspace containing $\mathcal{V}$ and let

$$\langle \mathcal{V} \mid A \rangle := \mathcal{V} \cap A^{-1}\mathcal{V} \cap A^{-2}\mathcal{V} \cap \ldots \cap A^{-(n-1)}\mathcal{V}$$

be the largest A-invariant subspace contained within $\mathcal{V}$. ◁

Corollary 17 (Sufficient condition for observability): Consider the DAE (1) with a switching signal $\sigma \in \Sigma_{\mathbb{N}}$. For each $m \in \mathbb{N}$, define the following sequence of subspaces, using the notation from Section III-A:

$$\begin{aligned} \overline{\mathcal{N}}_m^m &:= \mathcal{M}_m \\ \overline{\mathcal{N}}_{k-1}^m &:= \mathcal{M}_{k-1} \cap \Pi_{k-1}^{-1} \left\langle A_{k-1}^{\text{diff}} \;\middle|\; \overline{\mathcal{N}}_k^m \right\rangle, \; k = m, \cdots, 1. \end{aligned}$$

The switched DAE (1) is observable if there exists an $m \in \mathbb{N}$ such that

$$\overline{\mathcal{N}}_0^m = \{0\}. \qquad \lhd$$

Proof: Note that $e^{At}\mathcal{V} \subseteq \langle A \mid \mathcal{V} \rangle$ for any matrix A, any $t \in \mathbb{R}$ and any subspace $\mathcal{V} \subseteq \mathbb{R}^n$, hence $\overline{\mathcal{N}}_k^m \supseteq \mathcal{N}_k^m$ for all $m \geq k \geq 0$. In particular, $\overline{\mathcal{N}}_0^m = \{0\}$ implies $\mathcal{N}_0^m = \{0\}$ and Theorem 12 ensures observability. ∎

The condition in Corollary 17 is not necessary as the following (ODE) example shows:

Example 18: Consider a switched system with mode sequence indexed as follows:

$$\begin{aligned}
(E_{-1}, A_{-1}, C_{-1}) &= \left(I, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, (1, -1)\right), \\
(E_0, A_0, C_0) &= \left(I, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, (0, 0)\right), \\
(E_1, A_1, C_1) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (1, 0)\right) \\
&= (E_k, A_k, C_k), \quad k \geq 2.
\end{aligned}$$

Easy calculations show that $\mathcal{M}_0 = \operatorname{span}\{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$, $\mathcal{M}_1 = \operatorname{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\} = \mathcal{M}_m$, $m \geq 2$, and $\overline{\mathcal{N}}_0^m = \overline{\mathcal{N}}_0^1 = \mathcal{M}_0 \neq 0$. However, we will show that the switched system is observable. Therefore, consider any solution x with zero output. Then $x(0+) = x(0-) = (x_1^0, x_1^0)^\top$ for some $x_1^0 \in \mathbb{R}$. Furthermore, $x(t_1) = x(t_1-) = \begin{bmatrix} 1 & \tau_1 \\ 0 & 1 \end{bmatrix} x(0+) = (x_1^0(1 + \tau_1), x_1^0)^\top \in \mathcal{M}_1$. Hence, either $x_1^0 = 0$ or $\tau_1 = -1$. The latter is not possible, because we assumed that the switching times are in order, so $x_1^0 = 0$ must hold and $x \equiv 0$ is shown. $\lhd$

The above example, however, satisfies the following necessary condition obtained as a corollary to Theorem 12.

Corollary 19 (Necessary condition for observability): Consider the DAE (1) with a switching signal $\sigma \in \Sigma_{\mathbb{N}}$. For each $m \in \mathbb{N}$ define the following sequence of subspaces:

$$\begin{aligned}
\underline{\mathcal{N}}_m^m &:= \mathcal{M}_m \\
\underline{\mathcal{N}}_{k-1}^m &:= \mathcal{M}_{k-1} \cap \Pi_{k-1}^{-1} \left\langle \underline{\mathcal{N}}_k^m \;\middle|\; A_{k-1}^{\text{diff}} \right\rangle, \ k = m, \cdots, 1.
\end{aligned}$$

If the switched DAE (1) is observable then there exists an $m \in \mathbb{N}$ such that

$$\underline{\mathcal{N}}_0^m = \{0\}.$$

Proof: Note that $e^{At}\mathcal{V} \supseteq \langle \mathcal{V} \mid A \rangle$ for any matrix A, any $t \in \mathbb{R}$ and any subspace $\mathcal{V} \subseteq \mathbb{R}^n$, hence $\underline{\mathcal{N}}_k^m \subseteq \mathcal{N}_k^m$ for all $m \geq k \geq 0$. Now observability implies the existence of some $m \in \mathbb{N}$ such that $\{0\} = \mathcal{N}_0^m \supseteq \underline{\mathcal{N}}_0^m$. ∎

In order to further illustrate the gap between the necessary condition and the sufficient condition, consider the example where a system satisfies the necessary condition but not the sufficient condition and is unobservable.

Example 20: Consider a switched system with mode sequence indexed as follows:

$$\begin{aligned}
(E_{-1}, A_{-1}, C_{-1}) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (0, 0)\right), \\
(E_0, A_0, C_0) &= \left(I, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (0, 0)\right), \\
(E_1, A_1, C_1) &= \left(I, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, (1, 0)\right), \\
&= (E_k, A_k, C_k), \quad k \geq 2.
\end{aligned}$$

Easy calculations show that $\mathcal{M}_0 = \mathbb{R}^2$, $\mathcal{M}_1 = \operatorname{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ and $\underline{\mathcal{N}}_0^1 = \{0\}$. However, we will show that the switched system is unobservable. Any solution x with $x(0-) = x^0 \in \mathbb{R}^2$ is given by $x(t_1) = \begin{bmatrix} \cos t_1 & \sin t_1 \\ -\sin t_1 & \cos t_1 \end{bmatrix} x^0$ on $(0, t_1)$ and remains constant afterwards. As a zero output only constrains the first component of x on the interval (t_1, ∞), we have the one-dimensional unobservable subspace $\left\{ \ (x_1^0, x_2^0)^\top \ \middle| \ x_1^0 \cos t_1 + x_2^0 \sin t_1 = 0 \ \right\}$ for $x(0-)$. $\lhd$

We conclude this section with the corresponding sufficient and necessary conditions for determinability, whose proofs are analoge to the ones of Corollaries 17 and 19.

Corollary 21 (Conditions for Determinability): Consider the switched DAE (1) with a switching signal $\sigma \in \Sigma_{\mathbb{N}}$. For each $m \in \mathbb{N}$, define the following subspaces:

$$\begin{aligned}
\overline{\mathcal{Q}}^0 &:= \Pi_0 \mathcal{M}_0, \\
\overline{\mathcal{Q}}^{k+1} &:= \Pi_{k+1} \left(\mathcal{M}_{k+1} \cap \left\langle A_k^{\text{diff}} \;\middle|\; \overline{\mathcal{Q}}^k \right\rangle \right), \ k \in \mathbb{N},
\end{aligned}$$

and

$$\begin{aligned}
\underline{\mathcal{Q}}^0 &:= \Pi_0 \mathcal{M}_0, \\
\underline{\mathcal{Q}}^{k+1} &:= \Pi_{k+1} \left(\mathcal{M}_{k+1} \cap \left\langle \underline{\mathcal{Q}}^k \;\middle|\; A_k^{\text{diff}} \right\rangle \right), \ k \in \mathbb{N}.
\end{aligned}$$

The switched DAE (1) is determinable if there exists an $m \in \mathbb{N}$ such that

$$\overline{\mathcal{Q}}_0^m = \{0\}.$$

On the other hand, if the switched DAE (1) is determinable then there exists an $m \in \mathbb{N}$ such that

$$\underline{\mathcal{Q}}_0^m = \{0\}. \qquad \lhd$$

V. Conlusions

We have presented a characterization of observability and determinability of switched DAEs with known but arbitrary switching signals. We also present a sufficient and a necessary condition for observability and determinability when only the mode sequence of the switching signal (and not the switching times) is known. We have illustrated with examples that there is a gap between these conditions.

As a future direction of research, the construction of observers for switched DAEs is a topic that has not been discussed so far and could be a potential application of the results derived in this paper.

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