Stabilization of Boundary Controlled Hyperbolic PDEs via Lyapunov-Based Event Triggered Sampling and Quantization

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***Abstract*— With the growing utility of hyperbolic systems in modeling physical and controlled systems, this paper considers the problem of stabilization of boundary controlled hyperbolic partial differential equations where the output measurements are communicated after being time-sampled and space-quantized. Static and dynamic controllers are designed, which establish stability in different norms with respect to measurement errors using Lyapunov-based techniques. For practical purposes, stability in the presence of event-based sampling and quantization errors is analyzed. The design of sampling algorithms ensures practical stability.**

I. Introduction

Hyperbolic partial differential equations (PDEs) have been useful in modeling physical networks of different nature: e.g. hydraulic, road traffic, gas networks ([2], [3], [13]) to mention a few. Stabilizing this class of infinite dimensional systems, when applying control action either on the domain or on the boundary, has also been well studied, see for example [6] for backstepping control, and [4], [5], [7] for other Lyapunov techniques in general. Several results on the modeling of physical systems in hyperbolic PDE setting, along with the stability and boundary stabilization of such systems can be found in a recent book [1]. For the most part of boundary controllers for hyperbolic systems, digital control without reducing the model has *not* been studied in general. In fact, for control of PDEs, digital control synthesis commonly relies on reducing the model by discretizing the space so that one gets ordinary differential equations. In that case, finite dimensional approaches for digital control can be applied. However, without reducing the model, it is not sufficiently clear how fast continuous-time boundary controllers of hyperbolic PDEs must be sampled in a periodic fashion so as to implement them into a digital platform. Besides this, in large scale scenarios where sensors and actuators are distributed, information is transmitted through digital communication channels. Therefore, the need to reduce energy consumption and save communication resources is also a central issue.

Motivated by all of this, a recent work [10] introduced event-based sampling algorithms for boundary control of linear hyperbolic systems of conservation laws. The proposed rigorous framework establishes well-posedness of the closed-loop system and uses Lyapunov techniques for sampling algorithms to ensure exponential stability of the system. Furthermore, in [22], [23], boundary control of linear hyperbolic systems is treated when the output measurements are quantized. Few approaches on sampled data and event-triggered control for another class of infinite dimensional systems, namely, parabolic PDEs, are considered e.g. in [11] and [18], [25]. The later combines event-triggered sampling with a logarithmic quantizer.

It is worth recalling that for finite dimensional networked control systems, several contributions have been developed in the field of event-triggered and quantized control. See for instance, [14], [16], [17], and [21]. Among triggering strategies, we point out a static rule based on a robustness notion, called input-to-state stability (ISS), as introduced in [20], a dynamic rule as introduced in [12], [24], and the strategies relying on the time-derivative of a Lyapunov function as developed for instance in [15], [19].

When considering sampling and quantization issues, measurement errors are introduced which in most cases can cause the hyperbolic system to become unstable. Therefore, ISS properties with respect to those errors must be properly addressed. In [10], a static boundary control yields ISS in L^2-norm by means of Lyapunov analysis. While in [22], [23], ISS in H^1-norm leading to practical stability is obtained by using a dynamic controller. The use of H^1-norm is motivated by the fact that the output function to be quantized, must remain within the range of the quantizer, which is considered to be bounded.

This work builds on the ISS notions developed in [10] and [22] to solve the stabilization problem of hyperbolic PDEs when the output is subjected to event-based sampling, and quantization. The main contribution lies on the fact that even under event-triggered sampling of the output, one can still obtain ISS stability in both L^2 and H^1 norms and the well-posedness of the system. In the first instance, assuming the quantizers do not have limitations on data rate, a static control is used and bounds on L^2-norm of the state are obtained. For finite data-rate quantizers, it turns out to be necessary to work with a dynamic controller, and stability in H^1-norm is established.

This paper is organized as follows. In Section II, we introduce the problem statement. In Section III, we present the static control and stability result in L^2-norm. Section IV provides the result on stability in H^1-norm for dynamic boundary control. Section V provides a numerical example to illustrate the main results. Due to space limitations, the proofs of several results are omitted. We sketch the outline describing the major steps involved in the proof of main results.

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II. Problem Formulation

Consider the linear hyperbolic PDE:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \tag{1}$$

where $x \in [0,1]$, $t \in \mathbb{R}_+$ and $\Lambda = \text{diag}(\lambda_i)$ is a diagonal positive definite matrix. The boundary condition is given by

$$y(t,0) = Hy(t,1) + Bu(t) \tag{2}$$

and the initial condition is

$$y(0,x) = y^0(x), \quad x \in (0,1), \tag{3}$$

where $y : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, the input $u : \mathbb{R}_+ \to \mathbb{R}^m$, $H \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We consider the output of this system to be

$$z(t) = y(t,1). \tag{4}$$

The objective is to design the control input u in (2) as a function of the output measurements such that the resulting closed-loop system is asymptotically stable in appropriate sense. In our setup, we impose certain restrictions on the transmission of output to the controller. Motivated by the fact that the output is communicated to the controller via a communication channel, we determine the sampling instants, $t_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, such that $y(t_k,1)$ is transmitted to the controller for $t \in [t_k, t_{k+1})$. Additionally, after the sampling instants have been computed, a quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ encodes each output sample $y(t_k,1)$ to a discrete alphabet set $\mathcal{Q}$. We consider two cases:

- The domain of the quantizer is not necessarily compact, so that $\mathcal{Q}$ is countably infinite.
- The quantizers have a compact domain, and $\mathcal{Q}$ is finite.

In both cases, we solve the design problem where we compute the sampling instants t_k, $k \in \mathbb{N}$. In order to save communication resources, our objective is to employ event-based strategy for computing the sampling times. We provide the sampling algorithms for both aforementioned cases, and show that the closed-loop system is ISS with respect to the quantization error with appropriate norms.

III. Static Control with Infinite Data Rate

To highlight the fundamental ideas behind our approach, we first treat the case where quantization error is assumed to be bounded for all possible values of the output, for example, $q(y) = \lfloor y + 0.5 \rfloor$. In that case, we can talk about stability of $y(t,\cdot)$ in L^2-norm without requiring any bounds on $z(t)$. We first describe how ISS in L^2-norm is achieved via static output feedback, and then present the sampling algorithm.

A. ISS via static output feedback

We start by introducing perturbations in the output measurements by letting

$$z_d(t) = y(t,1) + d(t) \tag{5}$$

with $d \in L^\infty(\mathbb{R}_+;\mathbb{R}^n)$. We are interested in designing an output feedback which achieves ISS with respect to d in the following sense:

Definition 1 (L^2-ISS): *The system* (1)-(3),(5)*, with controller $u = \varphi(z_d)$ is input-to-state stable (ISS) in L^2-norm with respect to disturbance $d \in L^\infty(\mathbb{R}_+;\mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $y^0 \in L^2([0,1];\mathbb{R}^n)$, the solution satisfies, for all $t \in \mathbb{R}_+$,*

$$\|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^n)} \le C_1 e^{-\nu t}(\|y^0\|^2_{L^2([0,1];\mathbb{R}^n)}) + C_2 \|d_{[0,t]}\|^2_\infty. \tag{6}$$

In case there are no perturbation, i.e. $d \equiv 0$, a particular case of φ is a static output feedback control $u(t) = Kz(t)$, which renders the system globally exponential stable. Setting $G := H + BK$, the boundary condition (2) is

$$y(t,0) = Gz(t). \tag{7}$$

The design of $K \in \mathbb{R}^{m\times n}$ relies on the following assumption, which states a sufficient (*dissipative boundary*) condition for the global exponential stability of the system. Let us recall it here as follows [4]:

Assumption 1: *The following inequality holds:*

$$\rho_2(G) = \inf\left\{\|\Delta G \Delta^{-1}\|;\ \Delta \in \mathcal{D}_{n,+}\right\} < 1 \tag{8}$$

where $\|\cdot\|$ denotes the induced Euclidean-norm of matrices in $\mathbb{R}^{n\times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal positive definite matrices.

In case there are perturbations, and $u = Kz_d$, the resulting boundary condition can be expressed as

$$y(t,0) = Gz(t) + BKd(t). \tag{9}$$

Under Assumption 1, let us recall that the function defined for all $y \in L^2([0,1];\mathbb{R}^n)$ by

$$V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu x} dx \tag{10}$$

is a Lyapunov function for system (1)-(3), (4), (7) where Q is a diagonal positive definite matrix and $\mu > 0$ (see [8]). Even in the presence of perturbations, the system (1)-(3), (5) with control $u = Kz_d$, is ISS in L^2-norm. The result follows using (10) as well, see [10], [17].

B. Static control with event-based sampling and quantizer

In this section, we analyze the stability of the closed-loop system when the output is subject to event-triggered sampling and quantization. Highly inspired by [21] and [10], we design the sampling algorithm so that L^2-norm of $y(t,\cdot)$ converges to a bound parameterized by quantization error. In the sequel, we use the boundary controller as $u = \varphi_s(z)$. The operator φ_s encloses the triggering condition, the quantizer and the control function. This requires us to first state a result from [10] that allows us to express V from (10) in terms of measured output.

Denoting $\underline{\lambda} = \min_{1\le i\le n}\{\lambda_i\}$, we define the function $\widetilde{V} : [\frac{1}{\underline{\lambda}}, \infty) \to \mathbb{R}_+$, at $t = \frac{1}{\underline{\lambda}}$, by

$$\widetilde{V}(t) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z\left(t - \tfrac{x}{\lambda_i}\right)\right)^2 e^{-2\mu x} dx, \tag{11}$$

and for all $t > \frac{1}{\underline{\lambda}}$, by

$$\widetilde{V}(t) = \sum_{i=1}^{n} Q_{ii} \int_0^1 \left(H_i z \left(t - \frac{x}{\lambda_i}\right) + B_i u \left(t - \frac{x}{\lambda_i}\right)\right)^2 e^{-2\mu x} dx \tag{12}$$

with $Q \in \mathbb{R}^{n\times n}$ a diagonal positive definite matrix.

Proposition 1 ([10]): *Let y be a solution to* (1)-(3), (4)*. It holds that for all $t \geq \frac{1}{\underline{\lambda}}$, $\widetilde{V}(t) = V(y(t,\cdot))$ with $\widetilde{V}$ given by* (12)*.*

Having stated the above issues, let us now characterize φ_s as follows:

Definition 2 (Defintion of φ_s): *Let $\sigma \in (0,1)$, γ_s, ξ, δ, μ, $\nu > 0$, and $K \in \mathbb{R}^{m\times n}$. Let $\varepsilon_s(t) = \varepsilon_s(0)e^{-\delta t}$, for all $t \geq \frac{1}{\underline{\lambda}}$, with $\varepsilon_s(0) \leq \xi \widetilde{V}(\frac{1}{\underline{\lambda}})$.*

To define the operator φ_s, which maps the output function z to u, we consider

- *The increasing sequence of time instants (t_k) that is defined iteratively by $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,*

$$\begin{aligned} t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \\ \gamma_s \|BK(-z(t)+z(t_k))\|^2 \geq 2\nu\sigma\widetilde{V}(t) + \varepsilon_s(t)\}. \end{aligned} \tag{13}$$

  If $\widetilde{V}(\frac{1}{\underline{\lambda}}) = 0$, the time instants are $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$ and $t_2 = \infty$.
- *The quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ having the property that $|q(x) - x| \leq \Delta_q$, for some countable set $\mathcal{Q}$, and a scalar $\Delta_q > 0$.*
- *The static control function φ_s is described by:*

$$\begin{aligned} &u(t) = 0 && \forall t \in [t_0, t_1), \\ &u(t) = Kq(z(t_k)) && \forall t \in [t_k, t_{k+1}), \quad k \geq 1. \end{aligned} \tag{14}$$

For each $t \geq \frac{1}{\underline{\lambda}}$, the boundary condition (2), with (4), under static boundary control, $u = \varphi_s(z)$ as $u(t) = Kq(z(t_k))$, $t \in [t_k, t_{k+1})$, can be rewritten as:

$$y(t, 0) = (H + BK)z(t) + d_q(t) + d_s(t) \tag{15}$$

where

$$\begin{cases} d_q(t) := BK(q(z(t_k)) - z(t_k)) \\ d_s(t) := BK(z(t_k) - z(t)) \quad \forall t \in [t_k, t_{k+1}) \end{cases} \tag{16}$$

can be seen as errors related to the quantization and to the sampling respectively.

1) Well-posedness of the closed-loop system: In this section, we use the notion of piecewise continuous solutions as in [10]. As a matter of fact, the controller φ_s introduced in this section has the same nature as the operator defined in [10]. Using $\mathcal{C}_{rpw}$ (resp. $\mathcal{C}_{lpw}$) to denote piecewise right (resp. left) continuous functions, it follows from the arguments presented in [10] that $u \in \mathcal{C}_{rpw}(\mathbb{R}_+, \mathbb{R}^m)$ provided $z \in \mathcal{C}_{rpw}(\mathbb{R}_+, \mathbb{R}^n)$. It allows us to state the following result on the existence of solutions [10, Proposition 1]:

Proposition 2 (Existence of solutions): *For any $y^0 \in \mathcal{C}_{lpw}([0,1], \mathbb{R}^n)$, there exists a unique solution to the closed-loop system* (1)-(3), (4) *with controller $u = \varphi_s(z)$.*

2) Stability result: Let us state the main result of this section on stability with static feedback.

Theorem 1 (L^2-stability): *Let $K \in \mathbb{R}^{m\times n}$ be such that Assumption 1 holds. Let $\mu > 0$, $Q \in \mathcal{D}_{n,+}$, $\nu = \mu\underline{\lambda}$, $\sigma \in (0,1)$, and $\delta > 2\nu(1-\sigma)$. Let $\varepsilon_s(t)$ be the decreasing function as in Definition 2 and assume that there exist γ_q and $\gamma_s > 0$ such that*

$$M_c = \begin{pmatrix} G^T Q\Lambda G - Q\Lambda e^{-2\mu} & G^T Q\Lambda & G^T Q\Lambda \\ \star & Q\Lambda - \gamma_q I & Q\Lambda \\ \star & Q\Lambda & Q\Lambda - \gamma_s I \end{pmatrix} \leq 0 \tag{17}$$

Then the closed-loop system (1)-(3), (4), (15) *with controller u in* (14) *is ISS in L^2-norm with respect to d_q.*

To prove that the system is ISS with respect to d_q, we use the Lyapunov function candidate (10). Computing the right-time derivative of the Lyapunov function as done in [10, Lemma 2], and using the definition of φ_s, one ends up with

$$V(y(t,\cdot)) \leq \widetilde{C}_1 e^{-2\nu(1-\sigma)t} V(y^0) + \frac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[0,t]} \|d_q(s)\|^2$$

for some $\widetilde{C}_1 > 0$.

IV. Dynamic control with finite data rate

In this section, we consider the case when the sampled-output is subject to a quantizer, which has constraints on the domain. We define a finite-rate uniform quantizer $q : \mathbb{R}^n \to \mathcal{Q}$ where $\mathcal{Q} := \{q_1, q_2, ..., q_N\}$ is a set of finite alphabets. It has the following property:

$$|q(x) - x| \leq \Delta_q \quad if \quad |x| \leq M_q \tag{18a}$$

and

$$|q(x)| \geq M_q - \Delta_q \quad if \quad |x| > M_q \tag{18b}$$

where $\Delta_q > 0$ is the sensitivity of the quantizer and M_q is the range of the quantizer. We refer to [14], or [21], [23] for further details. With the quantizer specified in (18), the sampled-output $y(t_k, 1)$ (for some t_k to be defined in the sequel), must be bounded in a proper sense. It turns out that it can be only bounded if the H^1-norm of $y(t,\cdot)$, defined as

$$\|y\|^2_{H^1([0,1];\mathbb{R}^n)} = \|y\|^2_{L^2([0,1];\mathbb{R}^n)} + \|\partial y\|^2_{L^2([0,1];\mathbb{R}^n)}$$

is bounded, as explained in Section IV-B. Thus, it is necessary for y to be absolutely continuous so that ∂y is well-defined. When dealing with quantized and sampled output, a static control would introduce discontinuous inputs at the boundary, which result in y being discontinuous. To overcome this problem, we use a dynamic controller as proposed in [22], [23], which helps in smoothing the discontinuities caused by the quantization and sampling. We introduce then the dynamic variable $\eta \in \mathbb{R}^n$ satisfying the following ordinary differential equation,

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z_d(t) \quad \eta(0) = \eta^0 \tag{19}$$

for some $\alpha > 0$ to be chosen later and $z_d(t)$ given by (5) where d will be characterized later on. Once again, we shall consider ISS issues with respect to d.

Definition 3 (H^1 input-to-state stability): *The system (1)-(3), (5), (19) with controller $u = \varphi_d(z, \eta)$ is ISS in H^1-norm with respect to disturbance $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $y^0 \in H^1([0, 1]; \mathbb{R}^n)$, $\eta^0 \in \mathbb{R}^n$, the solution satisfies, for all $t \in \mathbb{R}_+$,*

$$|\eta(t) - y(t, 1)|^2 + \|y(t, \cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} \leq C_1\|d_{[0,t]}\|^2_\infty + C_2 e^{-\nu t}(|\eta^0 - y(0, 1)|^2 + \|y^0\|^2_{H^1([0,1];\mathbb{R}^n)}) \quad (20)$$

A. Event-based and quantized dynamic boundary control

Proceeding similarly as in Section III-B, we will call the dynamic boundary controller as φ_d, where this operator encloses the triggering condition, the quantizer and the dynamic control function. It is rigorously characterized as follows:

Definition 4 (Definition of φ_d): *Let $\sigma \in (0, 1)$, $\kappa_1 > 0$, γ_s, ξ, $\mu > 0$, $K \in \mathbb{R}^{m\times n}$ and P be a symmetric positive definite matrix. Let $\widetilde{V}$ be given, at $t = \frac{1}{\underline{\lambda}}$ by* (11) *and for all $t > \frac{1}{\underline{\lambda}}$, by* (12). *For each $t \geq \frac{1}{\underline{\lambda}}$, let*

$$\varepsilon_d(t) = \left(\widetilde{V}\left(\tfrac{1}{\underline{\lambda}}\right) + \left(\eta\left(\tfrac{1}{\underline{\lambda}}\right) - z\left(\tfrac{1}{\underline{\lambda}}\right)\right)^\top P\left(\eta\left(\tfrac{1}{\underline{\lambda}}\right) - z\left(\tfrac{1}{\underline{\lambda}}\right)\right)\right) e^{-\delta t}\xi.$$

To define the operator φ_d, which maps the output function z to u, we consider

- *The increasing sequence of time instants (t_k) that is defined iteratively by $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$, and for all $k \geq 1$,*

$$\begin{aligned} t_{k+1} = \inf\{&t \in \mathbb{R}^+ | t > t_k \wedge \\ &\gamma_s\|\alpha(-z(t) + z(t_k))\|^2 \geq \\ &\kappa_1(\eta(t) - z(t))^T P(\eta(t) - z(t)) + \kappa_1\widetilde{V}(t) \\ &+\varepsilon_d(t)\} \end{aligned} \quad (21)$$

  where η is obtained from (19) *by setting $z_d(t) = q(y(t_k, 1))$, with $t \in [t_k, t_{k+1})$, and q defined in* (18). *If $\widetilde{V}(\frac{1}{\underline{\lambda}}) = 0$ and $(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}}))^\top P(\eta(\frac{1}{\underline{\lambda}}) - z(\frac{1}{\underline{\lambda}})) = 0$, the time instants are $t_0 = 0$, $t_1 = \frac{1}{\underline{\lambda}}$ and $t_2 = \infty$.*

- *The dynamic control function is defined as*

$$\begin{aligned} u(t) &= \widetilde{u}(\eta^0, y^0) \quad \forall t \in [t_0, t_1) \\ u(t) &= K\eta(t) \quad \forall t \in [t_k, t_{k+1}), \quad k \geq 1 \end{aligned} \quad (22)$$

  where, for $i = 1, ..., m$,

$$\widetilde{u}_i(\eta^0, y^0) = \sum_{j=1}^n K_{ij} e^{-\alpha t_1}\left(\eta_j^0 + \int_0^{t_1} e^{\alpha s}\alpha y_j^0(1 - \lambda_j s)ds\right),$$

  with η^0, y^0 satisfying the compatibility condition $y^0(0) = Hy^0(1) + B\widetilde{u}(\eta^0, y^0)$ and $u(t_1) = \widetilde{u}(\eta^0, y^0) = K\eta(t_1)$.

With $u = \varphi_d(z)$, and (4), we can rewrite (19), for all $t > t_1$, as follows:

$$\dot{\eta}(t) = -\alpha\eta(t) + \alpha z(t) + d_q(t) + d_s(t) \quad (23)$$

where

$$\begin{cases} d_q(t) = \alpha(q(z(t_k)) - z(t_k)) \\ d_s(t) = \alpha(z(t_k) - z(t)) \quad \forall t \in [t_k, t_{k+1}) \end{cases} \quad (24)$$

can be seen as the measurement errors resulting from the quantization and sampling, respectively.

1) Well-posedness of the closed-loop system: The presence of dynamic controller makes it challenging to address the question of existence and uniqueness of solutions for the closed-loop systems. The authors of this paper have addressed this question in their other works. The work of [22], [23] proposes the framework where the closed-loop trajectories (y, η) are in $C^0([0, T], H^1([0, T], \mathbb{R}^n)) \times AC([0, T], \mathbb{R}^n)$, where AC denotes the space of absolutely continuous functions. Continuing along the lines of [10, Proposition 1], solutions y which are absolutely continuous in spatial variable, and whose derivatives are piecewise continuous functions, are studied in [9].

2) ISS stability result: Remark first that the boundary condition (2), with (4) under the dynamic boundary controller $u = \varphi_d(z, \eta)$ as $u = K\eta$ is rewritten as follows:

$$\begin{aligned} y(t, 0) &= Hz(t) + BK\eta(t) \\ &= Gz(t) + BK(\eta(t) - z(t)) \end{aligned} \quad (25)$$

with $G = H + BK$. In order to state the main result, let us introduce some notation. We denote

$$\begin{aligned} &F_0 := \Lambda^2 G^T \Lambda^{-1} Q_2 G \Lambda^2 - \Lambda^T Q_2 \Lambda^2 e^{-2\mu}; \\ &F_1 := K^T B^T \Lambda^{-1} Q_2 G \Lambda^2; \\ &F_2 := K^T B^T \Lambda^{-1} Q_2 BK, \end{aligned} \quad (26)$$

for scalars $\mu > 0, \alpha > 0$, $\underline{\lambda} = \min_{1\leq i\leq n}\{\lambda_i\}$, diagonal positive definite matrices Q_1, Q_2 and a symmetric positive definite matrix P. In addition, we denote by M_c the matrix in (27) (at the top of next page), and finally, define the matrix M_c^d, for some scalars γ_q and $\gamma_s > 0$, as

$$M_c^d = \left(\begin{array}{c|cc} M_c & \begin{matrix} 0 \\ F_1^T - F_2 \\ -\alpha F_2 \end{matrix} & \begin{matrix} 0 \\ F_1^T - F_2 \\ -\alpha F_2 \end{matrix} \\ \hline \star & \begin{matrix} F_2 - \gamma_q I \\ F_2 \end{matrix} & \begin{matrix} F_2 \\ F_2 - \gamma_s I \end{matrix} \end{array}\right). \quad (28)$$

Let us now present the main result of the second part of the paper.

Theorem 2 (H^1-stability): *Let $K \in \mathbb{R}^{m\times n}$ be such that Assumption 1 holds. Assume that there exist matrices $Q_1, Q_2 \in \mathcal{D}_{n,+}$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, K in $\mathbb{R}^{m\times n}$, $\alpha > 0$, $\mu > 0$, $\nu = \mu\underline{\lambda}$, $\sigma \in (0, 1)$, $\delta > 2\nu(1 - \sigma)$, and $\gamma_q, \gamma_s > 0$ such that*

$$M_c \leq 0 \quad (29)$$

and

$$M_c^d \leq 0. \quad (30)$$

Then, the closed-loop system (1)-(3), (4), (23) *with controller u in* (22) *is ISS in H^1-norm with respect to d_q.*

To prove that the system is ISS with respect to d_q, we use a Lyapunov function candidate $V : H^1([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}_+$ given by

$$V := V_1 + V_2 + V_3 \quad (31)$$

where $V_1 : H^1([0, 1], \mathbb{R}^n) \to \mathbb{R}_+$ is defined as,

$$V_1(y) = \int_0^1 y(x)^T Q_1 y(x) e^{-2\mu x} dx, \quad (32)$$

$$M_c = \begin{pmatrix} G^TQ_1\Lambda G - Q_1\Lambda e^{-2\mu} & 0 & G^TQ_1\Lambda BK \\ 0 & F_0 - F_1 - F_1^T + F_2 & -\alpha F_1^T + \alpha F_2 - P \\ \star & \star & \alpha^2 F_2 - 2\alpha P + 2\mu\underline{\lambda} P + K^TB^TQ_1\Lambda BK \end{pmatrix} \quad (27)$$

and $V_2 : H^1([0,1],\mathbb{R}^n) \to \mathbb{R}_+$ is defined as,

$$V_2(y) = \int_0^1 y_x(x)^T Q_2 y_x(x) e^{-2\mu x} dx \quad (33)$$

where $y_x := \frac{\partial y}{\partial x}$. Finally, $V_3 : H^1([0,1],\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}_+$ is defined as,

$$V_3(y,\eta) = (\eta - y(\cdot,1))^T P(\eta - y(\cdot,1)) \quad (34)$$

Computing the time-derivative of V, and using the definition of φ_d, we obtain

$$V(y(t,\cdot),\eta(t)) \leq \widetilde{C}_1 e^{-2\nu(1-\sigma)t} V(y^0,\eta^0) + \frac{\gamma_q}{2\nu(1-\sigma)} \sup_{s\in[0,t]} \|d_q(s)\|^2$$

for some $\widetilde{C}_1 > 0$, whence the desired claim follows.

B. Quantized control and practical stability

Let us consider now the study of practical stability of the system under quantization errors, (see [22], [23] for further details). As mentioned earlier, in order to apply quantized control with finite data rate, according to rule (18), we have to find a bound for $z(t) = y(t,1)$. It would hold also for $z(t_k)$ for the time instants defined in (21). The main reason of having used H^1-norm stability analysis, is because a suitable bound for $z(t)$ can be deduced as follows. Observe that

$$\begin{aligned} |z(t)|^2 &= \left(\left|\int_0^1 s y_x(t,s) + y(t,s) ds\right|\right)^2 \\ &\leq 2\left(\int_0^1 |s y_x(t,s)| ds\right)^2 + 2\left(\int_0^1 |y(t,s)| ds\right)^2 \\ &\leq 2\left(\int_0^1 |y_x(t,s)|^2 ds + \int_0^1 |y(t,s)|^2 ds\right) \\ &\leq 2\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)}. \end{aligned} \quad (35)$$

Moreover, from (35), it also holds that

$$|z(t)|^2 \leq 2(\|y_x(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^n)} + \|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^n)}) + 2|\eta(t) - z(t)|^2. \quad (36)$$

Next, for all $y \in H^1([0,1],\mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$, the Lyapunov function V given by (31) may be bounded as follows:

$$\begin{aligned} & c_1 e^{-2\mu}\Big(\|y(t,\cdot)\|^2_{L^2} + \|y_x(t,\cdot)\|^2_{L^2} + |\eta(t) - y(t,1)|^2\Big) \\ & \leq V(y(t,\cdot),\eta(t)) \leq c_2\Big(\|y(t,\cdot)\|^2_{L^2} \\ & \quad + \|y_x(t,\cdot)\|^2_{L^2} + |\eta(t) - y(t,1)|^2\Big) \end{aligned} \quad (37)$$

for some $c_1, c_2 > 0$. Therefore, using (37) with (36) we obtain that

$$|z(t)|^2 \leq \frac{2e^{2\mu}}{c_1} V(y(t,\cdot),\eta(t)). \quad (38)$$

Inequality (38) will be useful when determining the ultimate boundedness of the system. To that end, let us prove first that the output remains within the range of the quantizer. Following the same arguments provided in [23, Section 5.2], and assuming that the initial conditions y^0 and η^0 are such that

$$\frac{2e^{2\mu}}{c_1} V(y^0,\eta^0) \leq M_q^2 \quad (39)$$

where M_q is the range of the quantizer defined in (18), one can obtain that

$$V(y(t,\cdot),\eta(t)) \leq \frac{M_q^2 c_1}{2e^{2\mu}} \widetilde{C}_1 e^{-2\nu(1-\sigma)t} + \frac{\gamma_q}{2\nu(1-\sigma)} \Delta_q^2.$$

Using the bound on V from (37), we get also that

$$\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} + |\eta - y(t,1)|^2 \leq \frac{M_q^2}{2}\widetilde{C}_1 e^{-2\nu(1-\sigma)t} + \frac{\gamma_q e^{2\mu}}{2\nu(1-\sigma)c_1}\Delta_q^2.$$

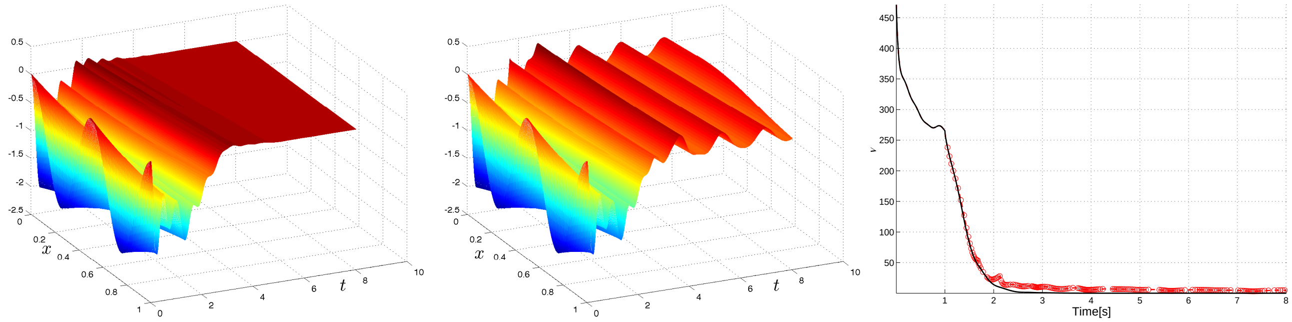
Considering the behavior for t sufficiently large, we obtain the practical stability with ultimate boundedness of the closed-loop system (1)-(3),(4) with controller φ_d, that is,

$$\limsup_{t\to\infty}\{\|y(t,\cdot)\|^2_{H^1([0,1],\mathbb{R}^n)} + |\eta - y(t,1)|^2\} \leq \frac{\gamma_q e^{2\mu}\Delta_q^2}{2\nu(1-\sigma)c_1}.$$

V. Simulations

We illustrate the results of Section IV by considering the following example of a linear system of 2×2 hyperbolic conservation laws of the form (1) with $y = [y_1 \quad y_2]^T$, $\Lambda = \mathrm{diag}(1\,;\,\sqrt{2})$, initial condition $y(0,x) = \begin{bmatrix} \cos(4\pi x)-1 & \cos(2\pi x)-1 \end{bmatrix}^T$ for all $x \in [0,1]$ and dynamic boundary condition given by $y(t,0) = Hz(t) + Bu(t)$ where $H = \begin{pmatrix} 0 & 1.1 \\ 1 & 0 \end{pmatrix}$, $B = I_2$ and $u(t) = K\eta(t)$. Let us consider first the case when stabilization is carried out using a dynamic controller without any measurement error, that is, we set $d_q \equiv 0$ and $d_s \equiv 0$ in (24). Therefore, η just satisfies $\dot{\eta}(t) = -\alpha\eta(t) + \alpha z(t)$, where we choose $\alpha = 10$. Furthermore, the gain of the dynamic controller K has been chosen such that $\rho_2(G) < 1$ with $G = H + BK$. Indeed with $K = \begin{pmatrix} 0 & -0.7 \\ -1 & 0 \end{pmatrix}$, $\rho_2(G) = 3.82 \times 10^{-1}$ and $\Delta = \begin{pmatrix} 9.96\times10^{-1} & 0 \\ 0 & 1.04 \end{pmatrix}$. Hence, the dissipativity condition holds, which is a necessary condition for the result in Theorem 2. In addition, condition (29) is verified for suitable Q_1, Q_2 and P. In fact, by choosing properly K and α and performing a line search on μ, one leads to several LMIs (linear in variables P, Q_1, and Q_2). With K and α given as before, we obtain $P = \begin{pmatrix} 3.64\times10^{1} & 0 \\ 0 & 2.51\times10^{1} \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1.24\times10^{2} & 0 \\ 0 & 1.34\times10^{2} \end{pmatrix}$, $Q_2 = \begin{pmatrix} 4.41\times10^{-2} & 0 \\ 0 & 4.67\times10^{-2} \end{pmatrix}$ and scalars $\mu = \nu = 1.4 \times 10^{-1}$.

Consider now the case of event-triggered sampling and quantized output. We use the following uniform quantizer



(a) Evolution of y_1 with $d_s = d_q = 0$.   (b) Evolution of y_1 with quantized samples.   (c) Evolution of V and discrete samples.

Fig. 1. Simulation results with dynamic controller.

$q(x) = \Delta_q \left\lfloor \frac{x+0.5}{\Delta_q} \right\rfloor$ whose sensitivity is given by Δ_q. We choose $\Delta_q = 1$ and, for the sake of simplicity, we assume the range of the quantizer to be large. The parameters for the triggering condition are $\sigma = 0.9$, $\varepsilon(0) = 0.1$ and $\delta = 0.28$. Condition $M_c^d \leq 0$ in (30) is verified with $\gamma_s = 22.4$ and $\gamma_q = 67.4$. Then, Theorem 2 applies. Figure 1(a) shows the plot of y_1 with a dynamic controller without measurement errors, and Fig. 1(b) shows evolution of y_1 with the same controller in the presence of sampling and quantization errors. Figure 1(c) shows the time-evolution of function V given by (31) with $d_s = d_q = 0$ (black line) and with φ_d (red dashed line with circle markers) using sampled and quantized measurements.

VI. Conclusion

In this paper, the problem of stabilization of boundary controlled hyperbolic PDEs has been considered, where the output measurements are event-triggered and quantized. We have studied ISS in L^2- and H^1- norms. It could be fruitful to consider also sampling algorithms for the control input in order to keep it constant until an update is necessary.

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