Comments on "Observability of Switched Linear Systems: Characterization and Observer Design"

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Abstract

This technical note points out certain limitations of our results from the paper mentioned in the title and provides a modified approach to overcome these limitations. In particular, the observer design addressed in the aforementioned paper is, in general, only applicable to switched linear systems with *invertible* state reset maps and this note presents a modified algorithm for state estimation that can also handle non-invertible state reset maps. In the process, we also identify some equalities from that paper which may not hold in general for arbitrary state reset maps.

I. Introduction

In our recent papers [1], [2], we studied observability conditions and observer construction for switched linear systems described as

$$\dot{x}(t) = A_q x(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q), \tag{1a}$$

$$x(t_q) = E_q x(t_q^-) + F_q v_q, \tag{1b}$$

$$y(t) = C_q x(t) + D_q u(t), \qquad t \in [t_{q-1}, t_q), \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^{d_y}$ is the output, $v_q \in \mathbb{R}^{d_v}$ and $u(t) \in \mathbb{R}^{d_u}$ are the inputs, and $u(\cdot)$ is a locally bounded measurable function. The index $q \in \mathbb{N}$ determines the active subsystem over the interval $[t_{q-1}, t_q)$ and it is assumed that the switching times do not accumulate at any time instant.

In our work [2], we have derived geometric conditions for observability and used them in designing an observer where we consider a very general class of state reset maps so that E_q, $q \in$

$\mathbb{N}$, may be non-invertible. However, it turns out that certain equalities derived in [2, Section II.C] only hold for a certain class of state reset maps (specified later in (4)), and in particular the invertible matrices E_q, $q \in \mathbb{N}$. Because of that, the state estimator proposed in [2] mainly works for invertible state reset maps. The primary objective of this note is to present a modified observer design to cater for general state reset maps, where E_q may be non-invertible. This generality comes at the cost of complexity involved in designing the state estimators: The observer proposed in [2] is simpler to design, whereas the observer designed to handle general state reset maps in this paper is relatively more complex.

In order to make this note self-contained, we first recall the geometric tools for characterization of observability in Section II on which the observer design of Section III is based. In the process, we also point out the errors from [2], that is, which mathematical formulae may not hold for non-invertible state reset maps.

II. Observability Conditions

Our observer is built on the notion of *determinability* considered in [2, Definition 1] and in this section we recall some tools that are used in deriving determinability conditions and designing observers. Roughly speaking, the switched system (1) is determinable if there exists $m \in \mathbb{N}$ such that $x(t_m)$ could be determined from the knowledge of external signals (u, v, y) measured over the interval $[t_0, t_{m+1})$. Because $x(t_{m+1}^-) = e^{A_{m+1}(t_{m+1}-t_m)}x(t_m)$, the unknown information contained in $x(t_m)$ and $x(t_{m+1}^-)$ is the same, so that, recovering $x(t_m)$ is equivalent to recovering $x(t_{m+1}^-)$. We now proceed towards quantifying the unknown information about the state using the measurements of (u, v, y) over a certain interval. Since our notion of observability does not require individual subsystems to be observable, the basic idea in formulating the geometric conditions that quantify the unknown information is to characterize how much information could be extracted from each subsystem about the state by measuring the output over a certain interval. To do so, it is seen that system (1) is an LTI system between two consecutive switching times, so that its unobservable subspace on the interval $[t_{q-1}, t_q)$ is simply given by the largest A_q-invariant subspace contained in $\ker C_q$, i.e., $\ker G_q$ where

$$G_q := \mathrm{col}(C_q, C_qA_q, \cdots, C_qA_q^{n-1}).$$

For system (1), let $\mathcal{Q}_q^m$ be the subspace such that $x(t_m^-)$ is determined modulo $\mathcal{Q}_q^m$ using the knowledge of external signals (u, v, y) over the interval $[t_{q-1}, t_m)$. We call $\mathcal{Q}_q^m$ the *undeterminable* *subspace for* $[t_{q-1}, t_m)$ and compute it recursively as follows for $q \in \mathbb{N}$:

$$\begin{aligned} \mathcal{Q}_q^q &:= \ker G_q \\ \mathcal{Q}_q^k &:= \ker G_k \cap E_{k-1} e^{A_{k-1}\tau_{k-1}} \mathcal{Q}_q^{k-1}, \quad q+1 \le k \le m, \end{aligned} \tag{2}$$

where $\tau_k := t_k - t_{k-1}$. Alternatively, by computing the orthogonal complement of $\mathcal{Q}_q^m$ and denoting it by $\mathcal{M}_q^m$, we can quantify the information about the state trajectory that can be recovered using the signals (u, v, y). The recursive expression for $\mathcal{M}_q^m$ is thus given by

$$\begin{aligned} \mathcal{M}_q^q &= \mathcal{R}(G_q^\top) \\ \mathcal{M}_q^k &= E_{k-1}^{-\top} e^{-A_{k-1}^\top \tau_{k-1}} \mathcal{M}_q^{k-1} + \mathcal{R}(G_k^\top), \quad q+1 \le k \le m, \end{aligned} \tag{3}$$

where the notation $E^{-\top}\mathcal{M}$ denotes $(E^\top)^{-1}\mathcal{M} := \{x \in \mathbb{R}^n \mid E^\top x \in \mathcal{M}\}$ for a matrix $E \in \mathbb{R}^{n\times n}$ and a subspace $\mathcal{M} \subseteq \mathbb{R}^n$. From (3), it is observed that the dimension of $\mathcal{M}_q^k$ is non-decreasing when k increases and q is fixed. We now characterize the determinability of system (1) using these subspaces in the following result which follows directly from [2, Proposition 1 & Theorem 3]:

Theorem 1 (Determinability Characterization): Consider the switched system (1) with $(u, v) \equiv 0$. Then $\mathcal{Q}_q^m$ for some $m \ge q \ge 1$ characterizes the undeterminable space in the following sense:

$$y_{[t_{q-1}, t_m)} \equiv 0 \quad \Leftrightarrow \quad x(t_m^-) \in e^{A_m \tau_m} \mathcal{Q}_q^m.$$

In particular, if there exists $m \ge q$ such that $\mathcal{Q}_q^m = \{0\}$, or equivalently $\mathcal{M}_q^m = \mathbb{R}^n$, then the state $x(t_{m-1})$, and hence the complete future trajectory $x_{[t_{m-1},\infty)}$, can be determined for system (1) (with possibly non-zero (u, v)) from the knowledge of (u, v, y) on the interval $[t_{q-1}, t_m)$.

Remark 1: We are often interested in deriving a direct formula for $\mathcal{Q}_q^m$ instead of the recursive one given in (2). For that, let us consider the matrix

$$\Psi_j^k := E_{k-1} e^{A_{k-1}\tau_{k-1}} \dots E_j e^{A_j \tau_j}, \quad k > j$$

which defines the flow of system (1) with zero inputs from t_{j-1} to t_{k-1}, and assume that the following condition holds for $k \ge q+2$, $i = 1, 2, \dots, k-q-1$, $q \in \mathbb{N}$:

$$\Psi_{k-i}^k (\ker G_{k-i} \cap \Psi_{k-i-1}^{k-i} \mathcal{Q}_q^{k-i-1}) = \Psi_{k-i}^k \ker G_{k-i} \cap \Psi_{k-i-1}^k \mathcal{Q}_q^{k-i-1}. \tag{4}$$

It is readily checked that, if (4) holds, then the sequential definition (2) leads to another equivalent expression for $\mathcal{Q}_q^m$, $m \ge q \ge 1$, that is,

$$\mathcal{Q}_q^m = \bigcap_{j=m,\dots,q} \Psi_j^m \ker G_j = \ker G_m \cap E_{m-1} \ker(G_{m-1}) \cap \left(\bigcap_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l e^{A_l \tau_l} E_i \ker G_i \right), \tag{5}$$

where Ψ_k^k denotes the identity matrix and we used the fact that $e^{A_k \tau_k} \ker G_k = \ker G_k$, for $k \in \mathbb{N}$. Condition (4) indeed holds when each of the matrix E_q, $q \in \mathbb{N}$, is invertible because in that case the mapping Ψ_j^k, for all $j, k \in \mathbb{N}$, $k > j$, is invertible. In [2, Section II.C], no such condition as (4) was specified and the equality (5) was claimed to hold without any constraints on the state reset maps E_q. We emphasize that [2, pg. 896, eq. (13)] holds if and only if (4) is satisfied.

Similarly, when (4) holds, one can obtain an equivalent expression for $\mathcal{M}_q^m$ from (3):

$$\mathcal{M}_q^m = (\mathcal{Q}_q^m)^\perp = \sum_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top} \mathcal{R}(G_i^\top) + E_{m-1}^{-\top} \mathcal{R}(G_{m-1}^\top) + \mathcal{R}(G_m^\top). \tag{6}$$

Once again, in [2, pg. 896, Remark 2], (6) was claimed to hold without specifying condition (4), and we emphasize that this may not be the case for arbitrary non-invertible state reset maps E_q, $q \in \mathbb{N}$. Equation (6) was used in the proof of convergence of state estimation error [2, Theorem 4], and thus for that result to be valid, condition (4), or (the simpler but stronger requirement of) invertibility of each matrix $E_q, q \in \mathbb{N}$, must be added to [2, Assumption 1].

III. Observer Design

Using the geometric conditions for determinability stated in the previous section, we now proceed to design an asymptotically convergent observer without requiring the matrices E_q, $q \in \mathbb{N}$, to satisfy (4). Our proposed observer is given by:

$$\dot{\hat{x}}(t) = A_q \hat{x}(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q), \tag{7a}$$

$$\hat{x}(t_q) = E_q(\hat{x}(t_q^-) - \xi_q) + F_q v_q, \tag{7b}$$

with an arbitrary initial condition $\hat{x}(t_0) \in \mathbb{R}^n$ and the expression for ξ_q will be computed in the sequel. The observer consists of a system copy and unlike classical methods where the continuous dynamics of the estimate are driven by an error injection term, *the observer* (7) *updates the state estimate only at discrete switching instants by an error correction vector* ξ_q. It is noted that the structure of the observer (7) is the same as one proposed in [1], [2]. However, the difference lies in the computation of ξ_q as the approach adopted in this note is different in several aspects which we highlight later.

To give an intuitive interpretation of how to calculate ξ_q, note that, if for some $q \in \mathbb{N}$, ξ_q equals the state estimation error $\hat{x}(t_q^-) - x(t_q^-)$, then the equation (7b) gives $\hat{x}(t_q) = x(t_q)$, and from there onwards we can recover the exact value of the trajectory x by setting $\xi_k = 0$ for

$k > q$. However, in practice, where we don't use the derivatives of the output, it is not easy to recover the exact value of the state estimation error. Thus, our goal is to compute ξ_q, for each $q \in \mathbb{N}$, such that it *approximates* the value of state estimation error at time t_q^- which will result in $\hat{x}(t)$ converging to $x(t)$ as t increases.

With this motivation, we introduce the state estimation error $\tilde{x} := \hat{x} - x$, and the error dynamics are given by

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t), \qquad t \in [t_{q-1}, t_q), \tag{8a}$$

$$\tilde{x}(t_q) = E_q(\tilde{x}(t_q^-) - \xi_q). \tag{8b}$$

The corresponding output error is defined as

$$\tilde{y}(t) := C_q \hat{x}(t) + D_q u(t) - y(t) = C_q \tilde{x}(t), \quad t \in [t_{q-1}, t_q).$$

The basic idea in computing ξ_q is to

- First identify the observable components of the individual subsystems that can be estimated using classical state-estimation techniques. For subsystem $p \in \mathbb{N}$, let $z_p : [t_{p-1}, t_p) \to \mathcal{R}(G_p^\top)$ denote its observable component.
- Secondly, derive an equation for $\tilde{x}(t_q^-)$ of the form

$$\tilde{x}(t_q^-) = \Xi_q(z_q(t_q^-), z_{q-1}(t_{q-1}^-), \cdots, z_{q-N}(t_{q-N}^-), \xi_{q-1}, \cdots, \xi_{q-N}) \tag{9}$$

  for some linear function $\Xi_q(\cdot)$ and $N \in \mathbb{N}$.
- Finally, letting $\hat{z}_p^q : [t_{p-1}, t_p) \to \mathcal{R}(G_p^\top)$ denote the estimate of z_p which we compute at t_q^-, $q > N$, $q - N \leq p \leq q$, we set

$$\xi_q = \Xi_q(\hat{z}_q^q(t_q^-), \hat{z}_{q-1}^q(t_{q-1}^-), \cdots, \hat{z}_{q-N}^q(t_{q-N}^-), \xi_{q-1}, \cdots, \xi_{q-N}). \tag{10}$$

We will develop calculations for each of the aforementioned steps in detail and arrive at a formal statement on error convergence that results from the observer. To do that, we need to introduce some assumptions that allow us to follow this proposed line of thought.

The identification of observable components in the first step could be achieved easily by Kalman-like decomposition. For the second step, however, where we want to write $\tilde{x}(t_q^-)$, for each $q \in \mathbb{N}$, in terms of the observable components of the currently active mode and some past modes, we need the following assumption on the switching signal and system dynamics:

Assumption 1: The switched system (1) is persistently determinable in the sense that there exists an $N \in \mathbb{N}$ such that

$$\dim \mathcal{M}^q_{q-N} = n, \qquad \forall\, q \geq N+1. \tag{11}$$

The integer N in Assumption 1 is interpreted as the minimal number of switches required to gain determinability.

For the third step, it is seen that if $\hat{z}^q_{q-k}(t^-_{q-k})$ "closely approximates" $z_{q-k}(t^-_{q-k})$, for $k = 0, \dots, N$, then (8) implies that the norm of the state estimation error at switching instants $\tilde{x}(t_q)$ is "close" to zero. Since the individual subsystems are not assumed to be observable, so that the error dynamics for a particular mode (between any two switching instants) cannot be stabilized by output injection, it is important to update the estimate repeatedly for asymptotic convergence and also make sure that the error doesn't get arbitrarily large between the two switching instants. This motivates us to introduce the following assumptions for our observer design:

Assumption 2: The switching is persistent in the sense that a switch occurs at least once in any time interval of length T_D; that is,

$$t_q - t_{q-1} < T_D, \qquad \forall\, q \in \mathbb{N}. \tag{12}$$

Assumption 3: The induced matrix norms $\|A_q\|$ are uniformly bounded for all $q \in \mathbb{N}$.

Note that Assumption 3 holds when A_q, $q \in \mathbb{N}$, belong to a set of finite elements. By placing a uniform bound on the time between two consecutive error updates in Assumption 2, we can get a bound on the maximum growth of the state estimation error between two consecutive switches which is eventually compensated by obtaining sufficiently close approximations of observable components.

In the sequel, the above thought process is formalized by setting up a machinery to compute the correction vector ξ_q. The explicit formula appears in equation (20) and we show in Theorem 2 that by choosing certain design parameters in the computation of ξ_q appropriately, the estimate indeed converges to the actual state. To keep the presentation simple, we will neglect the effect of computation time required in processing the stored information and computing ξ_q. In order to take into account the computation time, the idea developed in this note could be tailored within the framework of [2] to obtain similar results, albeit implemented differently.

A. Observability decomposition of error dynamics

As a first step in computing ξ_q, $q \in \mathbb{N}$, we want to write $\tilde{x}$ in terms of observable components of individual subsystems. To do that, we first find a coordinate change for each mode, similar to the Kalman decomposition. For each $p \in \mathbb{N}$, choose a matrix Z_p such that its columns are an orthonormal basis of $\mathcal{R}(G_p^\top)$, so that $\mathcal{R}(Z_p) = \mathcal{R}(G_p^\top)$. Similarly, choose a matrix W_p such that its columns are an orthonormal basis of $\ker G_p$. From the construction, there are matrices $S_p \in \mathbb{R}^{r_p \times r_p}$ and $R_p \in \mathbb{R}^{d_y \times r_p}$, where $r_p = \operatorname{rank} G_p$, such that $Z_p^\top A_p = S_p Z_p^\top$ and $C_p = R_p Z_p^\top$, and that the pair (S_p, R_p) is observable. Let $z_p := Z_p^\top \tilde{x} \in \mathbb{R}^{r_p}$ and $w_p := W_p^\top \tilde{x} \in \mathbb{R}^{n-r_p}$. Then, for the interval $[t_{p-1}, t_p)$, we obtain,

$$\dot{z}_p = Z_p^\top A_p \tilde{x} = S_p z_p, \quad \tilde{y} = C_p \tilde{x} = R_p z_p, \tag{13a}$$

$$z_p(t_{p-1}) = Z_p^\top \tilde{x}(t_{p-1}), \tag{13b}$$

which denotes observable components of the error dynamics (8), for mode $p \in \mathbb{N}$ during the interval $[t_{p-1}, t_p)$. Since z_p is observable over the interval $[t_{p-1}, t_p)$, a standard Luenberger observer is designed as

$$\dot{\hat{z}}_p^q(t) = S_p \hat{z}_p^q(t) + L_p^q(\tilde{y}(t) - R_p \hat{z}_p^q(t)), \quad t \in [t_{p-1}, t_p), \tag{14a}$$

$$\hat{z}_p^q(t_{p-1}) = 0, \tag{14b}$$

whose role is to estimate $z_p(t_p^-)$ at the end of the interval. This observer parses the data from $\tilde{y}$ over the interval $[t_{p-1}, t_p)$, and $\hat{z}_p^q(t_p^-)$ is used in the computation of ξ_q, $\max\{p, N+1\} \le q \le p+N$. Note that we have fixed the initial condition of the estimator to be zero for each interval. Since $\tilde{x}(t_q^-)$ can be written as,

$$\tilde{x}(t_q^-) = \begin{bmatrix} Z_q^\top \\ W_q^\top \end{bmatrix}^{-1} \begin{bmatrix} z_q(t_q^-) \\ w_q(t_q^-) \end{bmatrix} = Z_q z_q(t_q^-) + W_q w_q(t_q^-), \tag{15}$$

we obtain partial information of $\tilde{x}(t_q^-)$ in the sense that $Z_q z_q(t_q^-)$ can be recovered, but the value of $\tilde{x}(t_q^-)$ remains unknown because it is corrupted by the unobservable state $w_q(t_q^-)$.

B. Computing the vector ξ_q

The differences between the current observer and the observers treated in [1], [2] start at this stage as we will see that the calculations for the error correction vector ξ_q, and the gain criteria for asymptotic convergence are entirely different.

For $p, q \in \mathbb{N}$ with $p \le q$ let M_p^q and Q_p^q be matrices such that their columns are an orthonormal basis of $e^{-A_q^\top \tau_q}\mathcal{M}_p^q$ and $e^{A_q\tau_q}\mathcal{Q}_p^q$, respectively. The corresponding projections of $\tilde{x}(t_q^-)$ onto these subspaces can be expressed by letting $\mu_p^q := M_p^{q\top}\tilde{x}(t_q^-)$ and $\chi_p^q := Q_p^{q\top}\tilde{x}(t_q^-)$. Thus, it is seen that in addition to (15), another way of expressing $\tilde{x}(t_q^-)$ is:

$$\tilde{x}(t_q^-) = \begin{bmatrix} M_p^{q\top} \\ Q_p^{q\top} \end{bmatrix}^{-1} \begin{bmatrix} \mu_p^q \\ \chi_p^q \end{bmatrix} = M_p^q \mu_p^q + Q_p^q \chi_p^q. \tag{16}$$

The definition of μ_p^q implies that it contains the information of the error $\tilde{x}(t_q^-)$ which we are able to extract from the output on the interval $[t_{p-1}, t_q)$ as given by the observability space $\mathcal{M}_p^q$. For $q > N$, determinability of the system (Assumption 1) ensures that μ_{q-N}^q contains all the information of $\tilde{x}(t_q^-)$; in fact M_{q-N}^q is then an invertible matrix and hence the equation $\mu_{q-N}^q = M_{q-N}^{q\top}\tilde{x}(t_q^-)$ is uniquely solvable for $\tilde{x}(t_q^-)$.

We are interested in representing $\tilde{x}(t_q^-)$ only in terms of the known vectors μ_p^q, and eliminate its dependency over the terms involving χ_p^q, $p = q, q-1, \dots, q-N$. For that, we introduce the matrix Θ_p^q whose columns form the basis of the subspace $\mathcal{R}(e^{A_{q+1}\tau_{q+1}} E_q Q_p^q)^\perp$; that is,

$$\Theta_p^{q\top} e^{A_{q+1}\tau_{q+1}} E_q Q_p^q = 0. \tag{17}$$

Compared to the case treated in [1], [2], the key difference is that we do not transport the observable components of the individual subsystems to one time instant through the state-transition matrix. Instead, we gather all the observable information for $\tilde{x}(t_{q-1}^-)$ over the interval $[t_{p-1}, t_{q-1})$ into the vector μ_p^{q-1}, $p < q$, and combine it with the local observability information $z_q(t_q^-)$ of $\tilde{x}(t_q^-)$ obtained on the interval $[t_{q-1}, t_q)$ in order to recover more information for $\tilde{x}(t_q^-)$, represented by μ_p^q. For that, the following relationship between $\tilde{x}(t_q^-)$ and μ_p^{q-1}, $p < q$, is crucial:

$$\begin{aligned} \tilde{x}(t_q^-) &= e^{A_q\tau_q} E_{q-1}(\tilde{x}(t_{q-1}^-) - \xi_{q-1}) \\ &= e^{A_q\tau_q} E_{q-1}\left(M_p^{q-1}\mu_p^{q-1} + Q_p^{q-1}\chi_p^{q-1} - \xi_{q-1}\right). \end{aligned} \tag{18}$$

Combining this with (15) and (17), we obtain

$$\begin{bmatrix} Z_q^\top \\ \Theta_p^{q-1\top} \end{bmatrix} \tilde{x}(t_q^-) = \begin{pmatrix} z_q(t_q^-) \\ \Theta_p^{q-1\top} e^{A_q\tau_q} E_{q-1}\left(M_p^{q-1}\mu_p^{q-1} - \xi_{q-1}\right) \end{pmatrix},$$

where the right-hand side consists of observable, or known terms only, which allow us to accumulate more information about $\tilde{x}(t_q^-)$ by combining $z_q(t_q^-)$, μ_p^{q-1}, and ξ_{q-1} accordingly.

Consider a full column rank matrix U_p^q such that

$$[Z_q, \Theta_p^{q-1}]U_p^q = M_p^q.$$

This matrix always exists because from the definition of $\mathcal{M}_p^q$ and Z_q it follows that

$$\mathcal{R}(M_p^q) = \mathcal{R}([Z_q, \Theta_p^{q-1}]).$$

Indeed, note that $\mathcal{R}(\Theta_p^{q-1}) = e^{-A_q^\top \tau_q} E_{q-1}^{-\top} e^{-A_{q-1}^\top \tau_{q-1}} \mathcal{Q}_p^{q-1\perp}$, so that

$$\begin{aligned}\mathcal{R}(M_p^q) &= e^{-A_q^\top \tau_q} \mathcal{M}_p^q \\ &= e^{-A_q^\top \tau_q} (E_{q-1}^{-\top} e^{-A_{q-1}^\top \tau_{q-1}} \mathcal{M}_p^{q-1} + \mathcal{R}(G_q^\top)) \\ &= \mathcal{R}(\Theta_p^{q-1}) + \mathcal{R}(Z_q),\end{aligned}$$

where the last equality was obtained using the fact that $\mathcal{R}(G_q^\top)$ is invariant under $A_q^\top$, and $\mathcal{M}_p^{q-1} = \mathcal{Q}_p^{q-1\perp}$. From $\mu_p^q = M_p^{q\top} \tilde{x}(t_q^-)$, it now follows that

$$\begin{aligned}\mu_p^q &= U_p^{q\top} \begin{bmatrix} Z_q^\top \\ \Theta_p^{q-1\top} \end{bmatrix} \tilde{x}(t_q^-) \\ &= U_p^{q\top} \begin{pmatrix} z_q(t_q^-) \\ \Theta_p^{q-1\top} e^{A_q \tau_q} E_{q-1} \left(M_p^{q-1} \mu_p^{q-1} - \xi_{q-1} \right) \end{pmatrix} \\ &= U_p^{q\top} \begin{bmatrix} Z_q^\top & 0 \\ 0 & \Theta_p^{q-1\top} e^{A_q \tau_q} E_{q-1} \end{bmatrix} \begin{pmatrix} Z_q z_q(t_q^-) \\ M_p^{q-1} \mu_p^{q-1} - \xi_{q-1} \end{pmatrix} \\ &\triangleq J_p^q Z_q z_q(t_q^-) + K_p^q \left(M_p^{q-1} \mu_p^{q-1} - \xi_{q-1} \right). \end{aligned} \tag{19}$$

Note that (19) expresses the vector μ_p^q recursively in terms of μ_p^{q-1}. Recall that $\mathcal{M}_p^p = \mathcal{R}(G_p^\top) = \mathcal{R}(Z_p)$, hence we can assume $M_p^p = Z_p$ and we have the "initial value" for the recursion (19) given by $\mu_p^p = z_p(t_p^-)$.

If $z_{q-N}, \cdots, z_q$ were known, then we would be able to compute μ_{q-N}^q, and hence the error $\tilde{x}(t_q^-)$ exactly, and would pick $\xi_q = \tilde{x}(t_q^-)$. Since this is not the case, we work with the estimates $\hat{z}_{q-N}^q, \cdots, \hat{z}_q^q$ to compute ξ_q.

In summary, having introduced the matrices Z_q and as in (13), M_p^q as in (16), and Θ_p^q as in (17), for $q \in \mathbb{N}$, we let

$$\boxed{\xi_q = \begin{cases} 0, & 1 \le q \le N, \\ M_{q-N}^q \hat{\mu}_{q-N}^q, & q > N, \end{cases}} \tag{20a}$$

where $\hat{\mu}_{q-N}^{q-k}$, for $k = N-1, \cdots, 0$, is computed recursively as follows:

$$\boxed{\begin{aligned}&\hat{\mu}_{q-N}^{q-N} = \hat{z}_{q-N}^{q}(t_{q-N}^{-})\\ &\hat{\mu}_{q-N}^{q-k} = J_{q-N}^{q-k} Z_{q-k} \hat{z}_{q-k}^{q}(t_{q-k}^{-}) + K_{q-N}^{q-k}\left(M_{q-N}^{q-k-1}\hat{\mu}_{q-N}^{q-k-1} - \xi_{q-k-1}\right),\end{aligned}} \tag{20b}$$

and

$$[J_{q-N}^{q-k}, K_{q-N}^{q-k}] := U_{q-N}^{q-k\,\top} \begin{bmatrix} Z_{q-k}^{\top} & 0 \\ 0 & \Theta_{q-N}^{q-k-1\,\top} e^{A_{q-k}\tau_{q-k}} E_{q-k-1} \end{bmatrix}. \tag{20c}$$

C. Error Convergence and Gain Criterion

The only design parameters in the computation of ξ_q, $q > N$, are the gain matrices L_p^q, $p = q-N, \ldots, q$ which were introduced in obtaining the estimates $\hat{z}_p^q$ in (14). It is not true that every choice of L_p^q, that makes $(S_p - L_p^q R_p)$ Hurwitz, would actually result in asymptotic convergence of the state estimation error. In order to state the criteria for choosing the gain matrix that guarantees the convergence of the state estimation error to zero, for each $p \in \mathbb{N}$, and $\max\{p, N+1\} \leq q \leq p+N$, we introduce the matrices

$$\Lambda_p^q := e^{(S_p - L_p^q R_p)\tau_p}. \tag{21}$$

Since the pair (S_p, R_p), $p \in \mathbb{N}$ is observable in the classical sense, the norm of Λ_p^q can be made arbitrarily small by choosing L_p^q appropriately. In order to make precise statements about the "smallness" of Λ_p^q we need to define the following matrices for $q > N$, $k = N-2, \ldots, 0$ and $i = 0, \ldots, N-k-1$

$$V_{q-N,q-N}^{q-N+1} := K_{q-N}^{q-N+1} \tag{22a}$$

$$V_{q-N,q-N+1}^{q-N+1} := J_{q-N}^{q-N+1} \tag{22b}$$

$$V_{q-N,q-N+i}^{q-k} := K_{q-N}^{q-k} M_{q-N}^{q-k-1} V_{q-N,q-N+i}^{q-k-1} \tag{22c}$$

$$V_{q-N,q-k}^{q-k} := J_{q-N}^{q-k}. \tag{22d}$$

The main result on observer convergence now follows:

Theorem 2: Consider the observer (7) under Assumptions 1 – 3, with ξ_q given in (20). If, for each $q > N$, and $k = N, \ldots, 0$, the output injection matrices L_{q-k}^q are chosen to reduce the norm of Λ_{q-k}^q such that, for some $0 < c < \frac{1}{N+1}$,

$$\|E_q M_{q-N}^q V_{q-N,q-k}^q Z_{q-k} \Lambda_{q-k}^q Z_{q-k}^{\top}\| \leq c, \tag{23}$$

then, it holds that $\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0$.

Remark 2: An important thing to note is that $\hat{z}_p^q$ (with q varying) represent the estimate of the same variable z_p, and the computation of ξ_q depends on $\hat{z}_p^q(t_p^-)$, $\max\{q-N, 1\} \le p \le q$. For a fixed $p \in \mathbb{N}$, condition (23) requires that, the gains L_p^q, $\max\{p, N+1\} \le q \le p+N$, used to generate the estimate $\hat{z}_p^q$ must satisfy (at most) $N+1$ different inequalities, each corresponding to a different value of q. Hence, even for the estimates of a single mode $p \in \mathbb{N}$, we have $N+1$ different gain criteria (given by L_p^q), because the estimates of that mode are used for (at most) $N+1$ subsequent error correction updates ξ_q, $\max\{p, N+1\} \le q \le p+N$. If the knowledge of switching times is available offline, then the gains can be computed offline, else verifying (23) for each $q > N$, would require the knowledge of τ_{q-k}, $k = N, \dots, 0$. Choosing different gains for the estimates of the observable components of a single mode is in contrast to the strategy adopted in [2], which only relied on recycling the single estimate (and choosing single gain matrix) for every single mode.

Proof of Theorem 2: Using (8), it follows from Assumptions 2 and 3 that the estimation error $\tilde{x}(t)$ for the interval $[t_q, t_{q+1})$ is bounded by

$$|\tilde{x}(t)| = |e^{A_{q+1}(t-t_q)}\tilde{x}(t_q)| \le e^{b_A(t-t_q)}|\tilde{x}(t_q)|, \quad t \in [t_q, t_{q+1}),$$

with a constant b_A such that $\|A_q\| \le b_A$, for all $q \in \mathbb{N}$, and thus,

$$|\tilde{x}(t)| \le e^{b_A T_D}|\tilde{x}(t_q)|, \quad t \in [t_q, t_{q+1}).$$

Therefore, if $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, then convergence of $\hat{x}(t)$ towards $x(t)$ as $t \to \infty$ follows. It is noted that, for $q > N$, $\tilde{x}(t_q^-) = M_{q-N}^q \mu_{q-N}^q$ by definition, and $\xi_q = M_{q-N}^q \hat{\mu}_{q-N}^q$ using (20), so that,

$$\tilde{x}(t_q) = E_q(\tilde{x}(t_q^-) - \xi_q) \tag{24a}$$

$$= E_q M_{q-N}^q(\mu_{q-N}^q - \hat{\mu}_{q-N}^q) \tag{24b}$$

$$= -E_q M_{q-N}^q \tilde{\mu}_{q-N}^q, \tag{24c}$$

where $\tilde{\mu}_{q-N}^q := \hat{\mu}_{q-N}^q - \mu_{q-N}^q$. In the sequel, we will derive an expression for $\tilde{\mu}_{q-N}^q$ for a fixed $q > N$ and plug it in (24c) to show that $|\tilde{x}(t_q)|$ converges to zero as q increases.

Towards this end, we first compute the difference $\tilde{z}_p^q := \hat{z}_p^q - z_p$, for $q-N \leq p \leq q$, as follows:

$$\begin{aligned}\tilde{z}_p^q(t_p^-) &= \hat{z}_p^q(t_p^-) - z_p(t_p^-)\\ &= e^{(S_p - L_p^q R_p)\tau_p}\tilde{z}_p^q(t_{p-1})\\ &= -e^{(S_p - L_p^q R_p)\tau_p} {Z_p}^\top \tilde{x}(t_{p-1}).\end{aligned}$$

As a first step in arriving at the expression for $\tilde{\mu}_{q-N}^q$, we observe that $\tilde{\mu}_{q-N}^{q-N} = \tilde{z}_{q-N}^q(t_{q-N}^-)$ and we compute $\tilde{\mu}_{q-N}^{q-N+1}$ as follows:

$$\begin{aligned}\tilde{\mu}_{q-N}^{q-N+1} &= \hat{\mu}_{q-N}^{q-N+1} - \mu_{q-N}^{q-N+1}\\ &= J_{q-N}^{q-N+1} Z_{q-N+1}\tilde{z}_{q-N+1}^q(t_{q-N+1}^-) + K_{q-N}^{q-N+1} Z_{q-N}\tilde{z}_{q-N}^q(t_{q-N}^-)\\ &= -\sum_{i=0}^{1} V_{q-N,q-N+i}^{q-N+1} Z_{q-N+i}\Lambda_{q-N+i}^q Z_{q-N+i}^\top \tilde{x}(t_{q-N+i-1}).\end{aligned}$$

Finally, with these calculations, the expression for $\tilde{\mu}_{q-N}^{q-k}$, $k = N-2, \ldots, 0$, is derived recursively below:

$$\begin{aligned}\tilde{\mu}_{q-N}^{q-k} &= \hat{\mu}_{q-N}^{q-k} - \mu_{q-N}^{q-k}\\ &= J_{q-N}^{q-k} Z_{q-k}\tilde{z}_{q-k}^q(t_{q-k}^-) + K_{q-N}^{q-k} M_{q-N}^{q-k-1}\tilde{\mu}_{q-N}^{q-k-1}\\ &= -\sum_{i=0}^{N-k} V_{q-N,q-N+i}^{q-k} Z_{q-N+i}\Lambda_{q-N+i}^q Z_{q-N+i}^\top \tilde{x}(t_{q-N+i-1}).\end{aligned}$$

Plugging this expression for $\tilde{\mu}_{q-N}^q$ in (24c), we now obtain

$$\tilde{x}(t_q) = E_q M_{q-N}^q \sum_{i=q-N}^{q} V_{q-N,i}^q Z_i \Lambda_i^q Z_i^\top \tilde{x}(t_{i-1}). \tag{25}$$

From condition (23), it now follows that

$$|\tilde{x}(t_q)| \leq c \sum_{i=q-N}^{q} |\tilde{x}(t_{i-1})|$$

for some $0 < c < \frac{1}{N+1}$. Using Lemma 1 in [1], we obtain $\lim_{q\to\infty} |\tilde{x}(t_q)| = 0$, which proves the desired result. ■

IV. Simulations

To illustrate our observer design, we consider a simple academic example of a third order ($n = 3$) switched system with three modes where A_q, B_q, F_q, D_q, $q \in \mathbb{N}$, are zero matrices of appropriate dimensions. The output measurements are given by:

$$C_{3k-2} = [1\ 0\ 0], \quad C_{3k-1} = [0\ 1\ 0], \quad C_{3k} = [0\ 0\ 1], \quad k \geq 1$$

and the state reset maps are:

$$E_{3k-2} = E_{3k} = I_{3\times 3}, \quad E_{3k-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad k \geq 1.$$

For this system, it can be checked that Assumption 1 indeed holds, that is, $\dim \mathcal{M}_{q-N}^q = 3$, for each $q > 2$, where we take $N = 2$. The observer (7) is now implemented to obtain the state estimate in which we let $\xi_1 = \xi_2 = 0$. For $q \geq 3$, the following expressions are obtained for the vector ξ_q using the calculations in the previous section:

$$\xi_{3k} = \begin{pmatrix} \hat{z}_{3k-2}^{3k} + \hat{z}_{3k-1}^{3k} \\ \hat{z}_{3k-2}^{3k} + \hat{z}_{3k-1}^{3k} \\ \hat{z}_{3k} \end{pmatrix} - \begin{pmatrix} \xi_{3k-2}(1) + \xi_{3k-1}(1) + \xi_{3k-1}(2) \\ \xi_{3k-2}(1) + \xi_{3k-1}(1) + \xi_{3k-1}(2) \\ 0 \end{pmatrix}, \quad k \geq 1,$$

$$\xi_{3k+1} = \begin{pmatrix} \hat{z}_{3k+1}^{3k} \\ 0 \\ \hat{z}_{3k}^{3k} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}}\xi_{3k}(1) - \frac{1}{\sqrt{2}}\xi_{3k}(2) \\ \frac{1}{\sqrt{2}}\xi_{3k}(2) - \frac{1}{\sqrt{2}}\xi_{3k}(1) \\ \xi_{3k}(3) \end{pmatrix}, \quad k \geq 1,$$

$$\xi_{3k+2} = \begin{pmatrix} \hat{z}_{3k+1}^{3k} \\ \hat{z}_{3k+2}^{3k} \\ \hat{z}_{3k}^{3k} \end{pmatrix} - \begin{pmatrix} \xi_{3k+1}(1) \\ 0 \\ \xi_{3k+1}(3) + \xi_{3k}(3) \end{pmatrix}, \quad k \geq 1,$$

where we use the notation $\xi_q(j)$ to denote the j-th component of the vector ξ_q and the short-hand $\hat{z}_p^q$ to denote $\hat{z}_p^q(t_p^-)$, which for each $p \in \mathbb{N}$, and $\max\{p, 3\} \leq q \leq p + 2$, is obtained from the following equation:

$$\dot{\hat{z}}_p^q(t) = -l_p^q \tilde{y}(t), \quad t \in [t_{p-1}, t_p), \qquad \hat{z}_p^q(t_{p-1}) = 0.$$

For simplicity, if we let $l_p^q = l$, and $\tau_p = \tau$ for some $l, \tau > 0$ and each $p \in \mathbb{N}$, then the condition (23) boils down to:

$$\sqrt{2} \cdot e^{-l\tau} < \frac{1}{3} \quad \Leftrightarrow \quad l > \frac{\log 3\sqrt{2}}{\tau}.$$

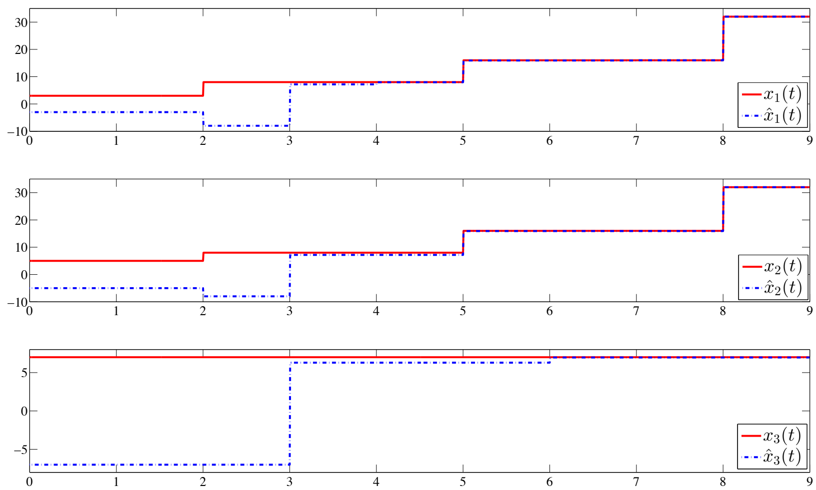

Fig. 1. The plot shows the state estimates $\hat{x}_i$, $i = 1, 2, 3$ (dashed lines in blue) converging to the actual states of the plant x_i, $i = 1, 2, 3$ (solid lines in red).

For $\tau = 1$, the simulation results are shown in Figure 1. The plots show the continuous and discrete nature of the error dynamics where the estimate doesn't improve between the two switching instants and only when the correction ξ_q is applied, the estimate gets closer to the actual state value.

References

[1] A. Tanwani, H. Shim, and D. Liberzon. Observability implies observer design for switched linear systems. In *Proc. ACM Conf. Hybrid Systems: Computation and Control*, pages 3 – 12, 2011.

[2] A. Tanwani, H. Shim, and D. Liberzon. Observability for switched linear systems: Characterization and observer design. *IEEE Trans. Autom. Control*, 58(4):891–904, 2013.