Nonsmooth and Constrained Dynamical Systems: Stability, Estimation and Control – Lectures

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Project Context and Overview

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- Title: Control of Constrained Interconnected Systems Using Variational Analysis
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This Mini-Course: Stability and Control of Nonsmooth Systems

- Overview of Lyapunov stability for differential inclusions
- Nonsmooth systems as Lur'e Systems
- Stability results using passivity methods
- Lyapunov functions for constrained systems

Part I : Overview of Stability Analysis

A combined reference for the material in these lectures is:

B. Brogliato and A. Tanwani, *Dynamical systems coupled with monotone set-valued operators: Formalisms, applications, well-posedness, and stability*. Submitted for publication, 2019.

Useful references for related topics:

S. Adly. *A Variational Approach to Nonsmooth Dynamics: Applications in Unilateral Mechanics and Electronics*, Springer Briefs in Mathematics, Springer International Publishing, Cham, 2017.

R. Goebel, R. Sanfelice, and A. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton Press, 2012.

H.-K. Khalil. *Nonlinear Systems*, Prentice Hall, 3rd ed., 2002.

D. Liberzon. Switching in systems and control. Birkhaüser, 2003.

R.-I. Leine and N. van de Wouw, *Stability and Convergence of Mechanical Systems with Unilateral Constraints*, vol. 36 of Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, Berlin Heidelberg, 2008.

A Model Differential Inclusion

Consider the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad t \geq t_0, x(0) = x_0 \in \mathrm{dom}(F) \qquad \textbf{(DI)}$$

where it is assumed that

- $F : \mathrm{dom}(F) \rightrightarrows \mathbb{R}^n$ is closed, and convex valued.
- $\mathrm{dom}(F)$ is closed.
- The solution at time t, starting from $x(t_0) = x_0$ is denoted by $x(t, t_0, x_0)$, or simply $x(t, x_0)$ if $t_0 = 0$.
- For each $T > 0$, there exists a unique absolutely continuous solution $x : [0, T] \to \mathbb{R}^n$ that satisfies **(DI)** for almost every $t \geq 0$.
- If $x_0 = 0$, then $x(t, 0) \equiv 0$, for all $t \geq 0$, that is, $\{0\}$ is an equilibrium.
- Regularity of F is not being specified, which may be necessary for existence of solutions in the first place.
- Most of the discussion will revolve around stability of the origin.

Stability Notions

Definition

- **(Stability)** The origin is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_0 \in \mathsf{dom}(F), \|x_0\| \leq \delta \Rightarrow \|x(t, x_0)\| \leq \varepsilon, \forall t \geq 0.$$

- **(Attractivity)** The origin is attractive if there exists $\delta > 0$ such that

$$x_0 \in \mathsf{dom}(F), \|x_0\| \leq \delta \Rightarrow \lim_{t \to +\infty} \|x(t; x_0)\| = 0.$$

- **(Asymptotic Stability)** The origin is asymptotically stable if it is stable and attractive.
- **(Exponential Stability)** The origin is exponentially stable if there exists $c_0 > 0$ and $\alpha > 0$ such that $\|x(t; x_0)\| \leq c_0 e^{-\alpha t} x_0$, for every $x_0 \in \mathsf{dom}(F)$.

Exercise: Can you think of a system which is attractive but not stable?

Lyapunov Functions: Basic Idea

Stability is analyzed using a function $V : \mathbb{R}^n \to \mathbb{R}_+$.

Consider, for the moment, a single-valued system

$$\dot{x} = f(x)$$

then the derivative of V along the trajectories of this system is

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \langle \nabla V(x), f(x) \rangle$$

Also, if $x(0) = z$, we can write,

$$\dot{V}(z) = \frac{d}{dt} V(x(t;z)) \Big|_{t=0}$$

Therefore, if $\dot{V}$ is negative, V decreases along the solutions of the system.

Lyapunov Functions: Stability Conditions

Theorem (Lyapunov Conditions)

Consider the system (**DI**). *Suppose that there exists* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

- V *is continuously differentiable and positive definite on* $\text{dom}(F)$,
- *For each* $x \in \text{dom}(F)$

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq 0,$$

then $\{0\}$ *is Lyapunov stable.*

Furthermore, if there exists $W : \mathbb{R}^n \to \mathbb{R}$, *continuous and positive definite, such that*

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq -W(x)$$

then $\{0\}$ *is asymptotically stable.*

A function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite on $\text{dom}(F)$, if it is continuous on $\text{dom}(F)$, $V(0) = 0$, and $V(x) > 0$ for every $x \neq 0$, $x \in \text{dom}(F)$.
Proof: on the board in a while.

Some Subtleties-I

Exercise: Consider a single-valued system in $\mathbb{R}^2$:

$$\dot{x}_1 = -\frac{2x_1}{1+x_1^2} + 2x_2$$
$$\dot{x}_2 = -2\frac{x_1 + x_2}{(1+x_1^2)^2}$$

Consider the Lyapunov function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2.$$

What can you conclude?

- $V(x) > 0$ and $\langle \nabla V(x), f \rangle < 0$, for $x \in \mathbb{R}^2 \setminus \{0\}$.
- The system is asymptotically stable, but not globally. Look at the curve $(\gamma) \in \mathbb{R}^2$ describe by $(1+x_1^2)(x_2-2) - 1 = 0$.

$$\dot{V}(x) = -\frac{4x_1^2}{(1+x_1^2)^4} - \frac{4x_2^2}{(1+x_1^2)^2}$$

Example of Unbounded Level Sets

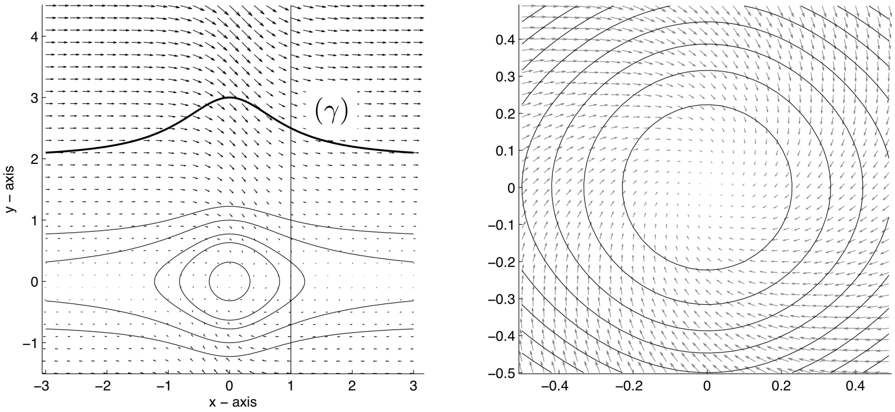


Figure: Unbounded level sets of a Lyapunov function.

Radially Unbounded Functions

- How can we conclude global asymptotic stability?
- In the proof, as we choose ε large, level sets of V may not be bounded, and δ-ball of initial condition stays bounded.
- To remedy this situation, we need V whose finite-level sets are compact.
- We say that V is radially unbounded if,

$$\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty$$

- Consequently, for each $c > 0$, there is $r > 0$, such that $\Omega_c \subset \mathbb{B}_r$.
- In the preceding theorem, if we add the condition that V is radially unbounded, then $\{0\}$ is globally asymptotically stable.

Some Subtleties-II

Exercise: Consider a single-valued system in $\mathbb{R}^2$:

$$\dot{x}_1 = -a\,x_1 - x_1 x_2, \quad a > 0$$
$$\dot{x}_2 = \gamma x_1^2$$

Consider the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}x_2^2$$

What can you conclude? $\dot{V}(x) = -a\,x_1^2 \leq 0.$

An Example

Consider the system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_2 - g(x_1)$$

where $a > 0$ (the damping coefficient) and g is such that $g(0) = 0$ and $a x_1^2 \leq x_1 g(x_1) \leq b x_1^2$.
Consider the Lyapunov function:

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(s)\, ds$$

then V is positive definite, and

$$\dot{V}(x) = -a\, x_2^2 \leq 0$$

What can you conclude? Can you justify that the origin is asymptotically stable?

Motivation for Invariance Principle:

The condition $\dot{V}(x) \leq 0$ guarantees stability, but in some cases, it is also possible to deduce asymptotic stability from such situations. These results are formalized under the notion of *LaSalle Invariance Principle*.

LaSalle's Invariance Principle

$$\dot{x} = f(x), \qquad x(0) \in \mathbb{R}^n. \qquad \textbf{(ODE)}$$

Theorem (Invariance Principle)

Consider system (**ODE**). *Suppose that there exists a positive definite $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\dot{V}(x) \leq 0$, for every x.*

Let M be the largest invariant set contained in the set $\{x \in \mathbb{R}^n \,|\, \dot{V}(x) = 0\}$. Then the origin of (**ODE**) *is stable. If, in addition, V is radially unbounded, then every solution approaches M as $t \to \infty$.*

- Radial unboundedness can be relaxed. If it can be established that a solution remains bounded, then that solution approaches M as $t \to \infty$.

Time-Varying Systems

$$\dot{x}(t) \in F(t, x(t)), \quad x(t_0) \in \mathrm{dom}(F(t_0, \cdot))$$

Definition

- The origin is **stable** if for every $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that
$$x(t_0) \in \mathrm{dom}(F), \|x(t_0)\| \leq \delta \Rightarrow \|x(t, x_0)\| \leq \varepsilon, \forall t \geq t_0.$$
- The origin is **uniformly stable** if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
$$x_0 \in \mathrm{dom}(F), \|x_0\| \leq \delta \Rightarrow \|x(t, x_0)\| \leq \varepsilon, \forall t \geq t_0.$$

- There is dependence on initial time t_0 in the definitions. We often consider uniform stability notion in applications.
- Lyapunov's stability theorem extends with straightforward generalizations.
- Invariance principle is not-so-straightforward. So, we often conclude that the trajectories to the set $\{\dot{V}(x) = 0\}$.

Part II : Lur'e Structures and Passivity

- B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control*, Communications and Control Engineering, Springer Nature Switzerland AG, London, third ed., 2020.
- M.-K. Camlibel and J.-M. Schumacher, *Linear passive systems and maximal monotone mappings*, Mathematical Programming B, 157 (2016), pp. 397–420.
- A. Tanwani, B. Brogliato, and C. Prieur, *Stability and observer design for Lur'e systems with multivalued, non-monotone, time-varying nonlinearities and state jumps*, SIAM Journal on Control and Optimization, 56 (2014), pp. 3639–3672.
- A. Tanwani, B. Brogliato, and C. Prieur, *Observer design for unilaterally constrained Lagrangian systems: A passivity-based approach*, IEEE Transactions on Automatic Control, 61 (2016), pp. 2386–2401.
- A. Tanwani, B. Brogliato, and C. Prieur, *Well-posedness and output regulation for implicit time-varying evolution variational inequalities*, SIAM Journal on Control and Optimization, 56 (2018), pp. 751–781.

Nonsmooth Systems as Lur'e System

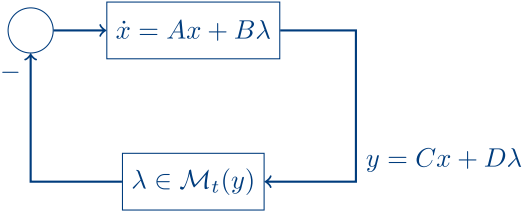
A nonsmooth system: For a given quadruple (A, B, C, D), consider the system

$$\begin{aligned}\dot{x} &= Ax + B\lambda \\ y &= Cx + D\lambda \\ \lambda &\in -\mathcal{M}_t(y)\end{aligned}$$

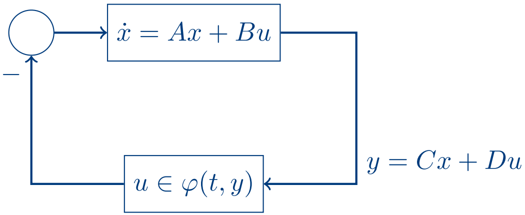
where $\mathcal{M}_t : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is a maximal monotone operator for each $t \geq 0$, so that

$$-\langle \lambda_1 - \lambda_2, y_1 - y_2 \rangle \geq 0,$$

Feedback perspective: A linear system with set-valued nonlinearities in feedback.



Lur'e structure



Lur'e system: A linear system with nonlinearities in the feedback

Definition (Sector bounded nonlinearities)

Consider a class of functions $\Phi_{[a,b]}$ such that each $\Phi \ni \varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}^p \to \mathbb{R}^p$ belongs to the sector $[a, b]$:

- For each $t \geq 0$, $\varphi(t, 0) = 0$.
- For each $t \geq 0$, $\langle \varphi(t, y) - ay, by - \varphi(t, y) \rangle \geq 0$, for each $y \in \mathbb{R}^p$

If $\varphi \in \Phi_{[0,\infty)}$, then $\langle \varphi(t, y), y \rangle \geq 0$, for each $y \in \mathbb{R}^p$ for each $t \geq 0$.

Absolute Stability Problem

Definition (Absolute Stability Problem)

Under what conditions on the quadruple (A, B, C, D), the dynamical system

$$\dot{x} = Ax + Bu, \quad u = \varphi(Cx + Du)$$

is globally asymptotically stable for all $\varphi \in \Phi_{[a,b]}$?

Definition (Aizerman's Conjecture with Linear Feedback)

Let $D = 0$, and $p = 1$, and $\Phi_{[a,b]}$ be time-invariant. If the matrix $(A - kBC)$, $k \in [a, b]$, is Hurwitz then the system

$$\dot{x} = Ax - B\varphi(Cx)$$

is asymptotically stable for each $\varphi \in \Phi_{[a,b]}$.

Aizerman's Conjecture holds for $n = 1, 2$. There is a counterexample for $n = 3$.

Another Solution

Definition (Kalman's Conjecture with Slope Restricted Nonlinearities)

Let $D = 0$, and $p = 1$, and $\Phi_{[a,b]}$ be time-invariant. If the matrix $(A - kBC)$, $k \in [a, b]$, is Hurwitz then the system

$$\dot{x} = Ax - B\varphi(Cx)$$

is asymptotically stable for each $\varphi \in \Phi_{[a,b]}$, $\varphi(0) = 0$, $a \leq \frac{d\varphi}{dy}(y) \leq b$.

Kalman's Conjecture holds for $n = 1, 2, 3$. There is a counterexample for $n = 4$.

How do we solve the problem in general?

- Circle criterion
- Popov criterion
- Positive Realness
- Passivity

Passivity and KYP Lemma

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Definition (Passivity)

System Σ is passive if there exists a positive semi-definite *storage function* V such that

$$V(x(t)) - V(x(0)) \leq \int_0^t \langle u(s), y(s) \rangle \, ds$$

holds along all solutions of Σ, for each $x(0) \in \mathbb{R}^n$, for each $t \geq 0$.

We say that Σ is strictly passive if there exists a storage function V, such that

$$V(x(t)) - V(x(0)) \leq \int_0^t \langle u(s), y(s) \rangle \, ds - \int_0^t \psi(x(s)) \, ds$$

for some positive definite function ψ.

PR and KYP Lemma

Lemma (Positive Real (PR) Lemma)

System Σ is passive with storage function $V(x) = x^\top Px$ if and only if there exist matrices $L \in \mathbb{R}^{n\times p}$ and $W \in \mathbb{R}^{p\times p}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that:

$$\begin{cases} A^\top P + PA = -LL^\top \\ B^\top P - C = -W^\top L^\top \\ D + D^\top = W^\top W. \end{cases}$$

Lemma (Kalman-Yakubovich-Popov (KYP) Lemma)

System Σ is strictly passive with storage function $V(x) = x^\top Px$ if and only if there exist matrices $L \in \mathbb{R}^{n\times p}$ and $W \in \mathbb{R}^{p\times p}$ and a symmetric positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$, such that:

$$\begin{cases} A^\top P + PA = -LL^\top - \varepsilon P \\ B^\top P - C = -W^\top L^\top \\ D + D^\top = W^\top W. \end{cases}$$

Absolute Stability Criterion for Nonsmooth Lur'e System

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ \lambda \in -\partial\varphi(y) \end{cases} \qquad \textbf{(EVI)}$$

Theorem (Stability of the Origin)

Consider the **(EVI)**, $\varphi(\cdot)$ *proper convex LSC,* $0 \in \partial\varphi(0)$, *and* (A, B, C, D) *strictly passive with LMI solution* $P = P^\top \succ 0$. *Then the origin is globally exponentially stable.*

Proof on the board. It follows from using the storage function $V(x) = x^\top P x$, passivity definition, and monotonicity of the subdifferential.

Invariance Principle for Nonsmooth Lur'e System

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ \lambda \in -\partial\varphi(y) \end{cases} \qquad \textbf{(EVI)}$$

Theorem (Invariance Result)

Consider the **(EVI)**, $\varphi(\cdot)$ *proper convex LSC,* $0 \in \partial\varphi(0)$, *and* (A, B, C, D) *strictly passive with LMI solution* $P = P^\top \succ 0$. *Let* $\mathcal{P}$ *be the largest invariant subset of* $E = \{z \in \mathbb{R}^n \mid z^\top(A^\top P + PA)z = 0\}$. *Then for each* $x_0 \in \mathrm{dom}(\mathcal{M})$, *one has* $\lim_{t\to+\infty} d_{\mathcal{P}}(x(t; x_0)) = 0$.

Part III : Conic Constraints, Convex Optimization, and Lyapunov Functions

D. Goeleven and B. Brogliato. *Stability and instability matrices for linear evolution variational inequalities*, IEEE Transactions on Automatic Control, 49 (2004), pp. 521–534.

M. Souaiby, A. Tanwani and D. Henrion. *Cone-copositive Lyapunov functions for complementarity systems: Converse result and polynomial approximation*. Submitted for publication.

Constrained Systems

What if the nonsmooth system does not satisfy the passivity assumption?

Example: Consider the linear complementarity system

$$\dot{x} \in \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} x - \mathcal{N}_{\mathbb{R}^2_+}(x)$$

which is of the form Lur'e with quadruple $B = C = I_{2\times 2}$ and D $= 0$.

- There **does not** exist a positive definite matrix P such that the conditions of KYP Lemma hold. This is because A is not Hurwitz.
- The constrained, (or in this case complementarity) system is asymptotically stable.
- Constrained system may be unstable even if A is Hurwitz stable. In this case also, the passivity assumptions do not hold.
- How to modify the Lyapunov theory to handle constraints?

A Model for Constrained Systems

System Class:

$$\langle \dot{x} - f(x), v - x \rangle + \varphi(v) - \varphi(x) \geq 0, \quad \forall\, v \in \mathbb{R}^n, \forall\, x \in \mathrm{dom}(\partial\varphi) \qquad \textbf{(EVI)}$$

where

- $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex, proper, lower semicontinuous, and
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz.

Exercise: Recall the definition of subdifferential of a convex function, and write (**EVI**) using subdifferential of φ. Can you make connections with first order sweeping process for some choice of φ?

Recall: For a convex, lower semicontinuous function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, we say that $\eta \in \partial\varphi(x)$ if $\langle \eta, y - x \rangle + \varphi(x) - \varphi(y) \leq 0$ for all $y \in \mathbb{R}^n$.

Lyapunov Functions for Constrained Systems

Theorem (Sufficient Conditions with Constraints)

Consider the system (**EVI**)*. Assume that there exists a continuously differentiable, positive definite function* $V(\cdot)$ *such that*

- $V(0) = 0$, *and* $V(x) \geq c\|x\|^r$ *for* $x \in \mathrm{dom}(\varphi)$,
- *It holds that*

$$\langle f(x), \nabla V(x)\rangle + \varphi(x - \nabla V(x)) - \varphi(x) \leq -\lambda V(x), \quad \forall\, x \in \mathrm{dom}(\partial\varphi),$$

then the following hold:

- *If* $\lambda = 0$, *then* $\{0\}$ *is Lyapunov stable.*
- *If* $\lambda > 0$, *then* $\{0\}$ *is globally asymptotically stable.*

Proof on the board.

Copositive Lyapunov Functions

Cone-Complementarity System with Nonlinear Vector Fields:

$$\dot{x} = f(x) + \eta$$
$$K^\star \ni \eta \perp x \in K$$

where $f \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$, K is a closed convex cone, and $K^\star$ is its dual.

Proposition (Sufficient Conditions with Copositive Functions)

Consider the system (**EVI**). Assume that there exists a continuously differentiable, positive definite function $V(\cdot)$ such that

- $V(0) = 0$, and $V(x) \geq c\|x\|^r$ for $x \in \mathrm{dom}(\varphi)$,
- $x - \nabla V(x) \in K$, for every $x \in \mathrm{bd}(K)$
- $\langle f(x), \nabla V(x) \rangle \leq -\lambda V(x)$, for every $x \in K$.

then the following hold:

- If $\lambda = 0$, then $\{0\}$ is Lyapunov stable.
- If $\lambda > 0$, then $\{0\}$ is globally asymptotically stable.

Quadratic Forms with Copositive Matrices

Question: Can we still work with quadratic functions for linear vector fields?

Definition (Copositive Matrices)

- A matrix $P \in \mathbb{R}^{n \times n}$ is said to be copositive on K if $\langle Px, x \rangle \geq 0$, for every $x \in K$.
- A matrix $P \in \mathbb{R}^{n \times n}$ is said to be strictly copositive on K if there exists $c > 0$ such that
$$\langle Px, x \rangle \geq c \|x\|^2, \quad \text{for every} x \in K$$

Positive semidefinite matrices $\subset$ Copositive matrices

Stability with Copositive Matrices

Proposition (Cone-Membership Conditions for Matrices)

Consider the system (**EVI**). Assume that there exists a matrix $P = P^\top \in \mathbb{R}^{n\times n}$ such that

- P is strictly copositive.
- $x - Px \geq 0$, whenever $x_i = 0$.
- $-(A^\top P + PA)$ is (strictly) copositive,

then the origin is (asymptotically) stable.

Some Concluding Remarks

- Under passivity structure, we have to solve linear programs to compute a quadratic Lyapunov function with linear vector fields
- For complementarity systems, without the passivity assumption, we end up with copositive optimization problems.
- Copositive programming is still a convex optimization problem, but it is NP-hard.
- Several algorithms exist for solving such problems, and in this workshop, our paper talks about adapting those ideas for computing copositive Lyapunov functions.