Module Edsys'18: Optimal Control

Course Title: Calculus of variations, optimal control, and optimization
Dates: 22nd – 25th January, 2018 (Total time: 20 hrs)
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Venue: ISAE – SUPAERO

1 Course Summary

This note is prepared in the spirit of summarizing the topics that we have covered during this course, and give some references for the material that was not covered with sufficient depth in the class due to time constraints. For a formal description of the material covered in this course, please go through to the references [15] and [9, 10].

1.1 Introduction and Calculus of Variations

We started this course with an overview of finite-dimensional optimization and studied first and second-order conditions for computing the minima of a function. Several cases for unconstrained, and constrained problems were studied. While these topics are a subject of many books, two standard references are [4, 17]. We then briefly discussed how these notions can be generalized for infinite-dimensional optimization. The conditions that we formulate are rather abstract, and it is seen that for the class of problems encountered in calculus of variations, one can compute explicit expressions. This way we derived the Euler-Lagrange equations which provide first oder necessary conditions for a local weak extremum. We saw several simplified cases of this equation and studied its relevance in the context of mechanics. One can also compute second order necessary conditions as well, although we did not cover it in the class. We refer the readers to article [21] for some historical references and see how a classical problem like *Brachristochrone* is solved using the tools we have studied. Several generalizations involving integral and non-integral constraints were also studied which allowed us to handle the constraints modeled by ordinary differential equations.

1.2 Pontryagin's Maximum Principle

In the context of finding optimal control for dynamical systems with cost functional in Bolza form, we extended the variational approach to find first order necessary conditions for optimal control inputs. This leads us to canonical equations for the state and the adjoint vector along the optimal trajectory, and we also see that the optimal control maximizes the Hamiltonian. These are the two key statements in the Pontryagin's Maximum Principle. One must be careful with the fact that our derivation of these equations cannot be seen as a proof of maximum principle which deals with local optimality in the stronger sense. Plus, certain assumptions (which were swept under the rug) in adopting the variational approach, are addressed very elegantly in the actual proof of Maximum principle. The two statements that we wrote down correspond to the cost functional in Lagrangian form and there was no time-dependence in the Lagrangian or the system vector fields [2]. These are not the most general results related to maximum principle. Interested reader may consult additional references such as [7, 19, 23].

We then saw the applications of maximum principle in the context of time-optimal control problems and in particular how the bang-bang optimal controls are obtained as a solution to such problems for controllable linear systems which are normal. For the discussion in class related to bounds on the number of switchings, please see [20, Theorem 8.1.2]. The same reference extends the treatment of bang bang optimal controls to nonlinear systems by discussing singular controls, and the role of Lie brackets in determining bang-bang optimal trajectories. A formal result appears in [20, Theorem 8.3.1], which provides conditions under which there exists a time optimal bang-bang control for nonlinear systems. One may also find a condensed version of these topics in [19, Handout 5]. However, an interesting observation in nonlinear systems is that, for certain problems, the optimal control may be bang bang but it may require infinitely many switches in finite-time, This is seen in *Fuller's* problem formulated in plane, but it may also happen in time-optimal control problems in higher dimensions. Again, [20, Section 8] provides a compact discussion on these issues.

For numerical aspects concerning the implementation of optimal candidates generated by maximum principle, the difficulty appears in finding algorithms for solving two-point boundary value problems, and the book [11] addresses some numerical techniques to address these issues. Another reference which is useful from implementation viewpoint is [18]. In the lab session, we implemented a simple instance of shooting method, which has its drawbacks. You may find some related discussion on numerical techniques in [22, Chapter 9].

1.3 Dynamic Programming and Hamilton-Jacobi-Bellman Equations

Another approach for optimal control originates from dynamic programming [3]. Using the principle of optimality, we saw that this approach leads to the formulation of Hamilton-Jacobi-Bellman (HJB) equation, which is a partial differential equation, and the big advantage here is that it provides sufficient conditions for global optimality. However, even for relatively simple problems, it is difficult to compute. One tool for solving HJB equations numerically is [5].

Also, we saw through examples that the value function may not be differentiable and hence the interpretation of partial derivatives in HJB equation becomes ambiguous if the value function is not differentiable. This issue was resolved in the seminal work [8], where the authors propose the notion of *viscosity solutions* by employing the notion of subdifferentials from nonsmooth analysis. You may consult the online lectures by second author [16].

1.4 Polynomial optimization and Moment-based techniques

This part of the course was based on a technique introduced in [12], see also [13], to solve globally nonconvex optimization problems on multivariate polynomials with the help of a hierarchy of convex semidefinite programming problems (linear matrix inequalities or LMI = linear programming problems in the cone of positive semidefinite matrices). Instrumental to the development of this technique is the duality between the cone of positive polynomials (real algebraic geometry) and the cone of moments (functional analysis). These basic objects were introduced, with a special focus on conic optimization duality, and some illustrative examples were presented. Then, we studied polynomial optimal control, which consists of minimizing a polynomial Lagrangian over a polynomial vector field subject to semi-algebraic constraints on control and state, typically a nonconvex problem for which there is no solution in classical Lebesgue spaces [14]. To overcome this, polynomial optimal control problems are first formulated as linear programming (LP) problems in the cone of occupation measures (standard objects in Markov decision processes and ergodic theory of dynamical systems), and infinite-dimensional convex duality is used to establish the link with subsolutions of the HJB equation satisfied by the value function. Then, the Lasserre hierarchy is applied to solve numerically these infinite-dimensional LP problems.

1.5 Applications

Using the aforementioned tools, we were able to address the *Linear Quadratic Regulator* problem in a more constructive manner. Exploiting the quadratic nature of the Lagrangian, and the linearity of the system dynamics, we saw that finding the optimal control (which is also a feedback law) boils down to solving Riccati differential equation for a matrix of size $n \times n$ backwards in time, along with the system equations. In the infinite horizon case, under the suitable controllability and observability assumption, this differential equation reduces to an algebraic equation of matrices. This theory can be applied to deal with more generalized problems which may come up, for example, in tracking a trajectory. The utility of Riccati equations is also seen in problems related to computing $\mathcal{L}^2$ gains of a system, and hence are relevant in the context of designing robust controllers which minimize $\mathcal{H}^\infty$ norm of the transfer function from the disturbance to regulated output. As a follow up to the preliminary discussions in the class on these topics, you may go through the books [1, 6], and [24] further details.

References

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