TIME DELAY APPROACH TO THE MODELING OF FLUID NETWORKS

David Novella
Emmanuel Witrant
Olivier Sename

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Fluid Networks

Fluid network systems appear in different areas

Figure 1: Mine Ventilation Systems

Figure 2: Gas and Water Distribution Lines
Figure 3: Traffic Flow

Figure 4: Blood flow
The main difficulties of dealing with this class of systems are:

- High order nonlinear dynamics
- Complex interconnected flows
- Transport phenomena
Different Approaches

Lumped Parameter Model

- Modeling of pipes as lumped parameters
- Use of approximations of incompressible Navier-Stokes equation
- Network modeled using Kirchhoff’s laws
- Analogies with RL non-linear circuits
Different Approaches

### Lumped Parameter Model

- Modeling of pipes as lumped parameters
- Use of approximations of incompressible Navier-Stokes equation
- Network modeled using Kirchhoff’s laws
- Analogies with RL non-linear circuits

- [Petrov et al., 1992]
- [HL et al., 1997]
- [Hu et al., 2003]
- [Koroleva and Krstic, 2005]
Boundary Feedback Control

- Modeling by means of partial differential equations
- Riemann invariants transformation
- Boundary control techniques
Boundary Feedback Control

- Modeling by means of partial differential equations
- Riemann invariants transformation
- Boundary control techniques

- [de Halleux et al., 2003], [Halleux, 2004]
- [Prieur et al., 2008]
- [Bastin et al., 2008]
- [Gugat and M., 2011], [Gugat et al., 2011]
Time-delay modeling

- Model for large convective flows
- Transport properties involved in the flow model
- Parameter estimation of the transport coefficient
- Using some appropriate physical hypotheses
- A mathematical equivalence is then obtained between the distributed model and a time-delay system
Time-delay modeling

- Model for large convective flows
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- A mathematical equivalence is then obtained between the distributed model and a time-delay system

- [Witrant and Marchand, 2008]
- [Witrant and Niculescu, 2010]
- [Bradu et al., 2010]
Aims

- To improve the classical lumped parameter model.
- To obtain a dynamic model from the physics properties.
- To introduce the transport phenomena as a time-delay.

Figure 5: Fluid Flow Network
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Flow dynamics
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Objective

Physical Model
Isothermal Euler Equations
\[ \rho_t + q_x = 0 \]
\[ q_t + (\frac{q^2}{\rho} - a^2 \rho)_x = -G \left( \frac{\rho \mid q \mid}{2D \rho} \right) \]

Riemann Invariants

Hyperbolic Quasilinear System
\[ \xi_t + A(\xi)\xi_x = S(\xi) \]

Physical Hypotheses

Hyperbolic Decoupled System
\[ \xi^1_t + \lambda_1 \xi^1_x = s_1(\xi^1) \]
\[ \xi^2_t + \lambda_2 \xi^2_x = s_2(\xi^2) \]

Method of Characteristics

Delayed equations of the propagation waves

Conservation Laws

Fluid flow network
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Navier-Stokes Equations\(^1\)

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \vec{v} \\ \rho E \end{bmatrix} + \vec{\nabla} \cdot \begin{bmatrix} \rho \vec{v} \\ \rho \vec{v} \otimes \vec{v} + p \vec{I} - \mathbf{T} \\ \rho \vec{v} H - \mathbf{T} \cdot \vec{v} - k \vec{\nabla} T \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{f}_e \\ W_f + q_H \end{bmatrix} \tag{1}
\]

- \(\rho\) is the density,
- \(\vec{v}\) is the velocity vector,
- \(E\) is the total energy,
- \(p\) is the pressure,
- \(\mathbf{T}\) is the stress tensor,
- \(H\) is the total enthalpy,
- \(k\) is the coefficient of thermal conductivity,
- \(T\) is the temperature,
- \(\vec{f}_e\) is the external force vector,
- \(q_H\) is the heat source.

\(^1\)Hirsch, 2007
Euler Equations

\[
\frac{\partial}{\partial t} \begin{bmatrix}
\rho \\
\rho \mathbf{V} \\
\rho E
\end{bmatrix} + \nabla \cdot \begin{bmatrix}
\rho \mathbf{V}^T \otimes \mathbf{V} + p \mathbf{I} \\
\rho \mathbf{V} \mathbf{H}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\dot{q}
\end{bmatrix}
\] (2)

- \( \rho \) is the density,
- \( \mathbf{V} \) is the velocity,
- \( \rho \mathbf{V} \) is the moment,
- \( \dot{q} \) is the rate of heat addition,
- \( \mathbf{I} \) is the identity matrix,
- \( \mathbf{H} \) is the total enthalpy,
- \( \otimes \) is a tensor product.

\[\text{[Toro, 2009]}\]
Isothermal Euler Equations

- A common model for gas flow in pipes
- The temperature is considered constant
- [Gugat and M., 2011], [Gugat et al., 2011]
- Pressure is obtained from a equation of state:

\[ p = p(\rho) \equiv a^2 \rho, \]  \hspace{1cm} (3)

where \( a \) is a non zero constant propagation speed of sound, [Toro, 2009].
We define:

\[ a = \sqrt{\frac{ZRT}{M_g}} \]  

- \( Z \) is the natural gas compressibility factor
- \( R \) the universal gas constant
- \( T \) the absolute gas temperature
- \( M_g \) the gas molecular weight
The isothermal Euler equations for a single pipe are defined by:

- **Mass conservation**

\[
\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0
\]  

(5)

- **Momentum conservation**

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q^2}{\rho} + a^2 \rho \right) = -f_g \frac{q \left| q \right|}{2D \rho}
\]  

(6)

- \(f_g\) is the friction factor,
- \(D\) is the diameter of the pipe.
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Definition

Consider the hyperbolic systems described as follows

\[
\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = 0 \tag{7}
\]

\[x \in [0, L], \quad t \in [0, T]\]

The system (7) can be transformed into a system of coupled transport equations

\[
\frac{\partial \xi_i(x, t)}{\partial t} + \lambda_i(\xi(x, t)) \frac{\partial \xi_i(x, t)}{\partial x} = 0 \quad \text{for} \quad i = 1 \cdots , n. \tag{8}
\]
\[
\frac{dx}{dt} = \lambda_i(\xi(x, t)). \tag{9}
\]

Since \(d\xi_i/dt = 0\) along the characteristic curve, it follows that \(\xi_i\) is constant (or invariant) along the characteristic curve.
Riemann Invariants for the Isothermal Euler Equations

We can express the equations (5) and (6) as follows

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = D(U), \tag{10}
\]

with \( U(x, t) = [\rho, q] \). The Jacobian of the flux matrix \( F(U(x, t)) \) is

\[
A(U) = \begin{pmatrix}
0 & 1 \\
\frac{a^2}{\rho^2} - \frac{q^2}{\rho^2} & 2 \frac{q}{\rho}
\end{pmatrix}. \tag{11}
\]
Eigenvalues and Eigenvectors

The eigenvalues of the Jacobian matrix $A(U)$ are

$$\lambda_{1,2} = \frac{q}{\rho} \pm a. \quad (12)$$

And the right eigenvectors are

$$K_1 = \begin{bmatrix} 1 \\ q/\rho - a \end{bmatrix} \quad K_2 = \begin{bmatrix} 1 \\ q/\rho + a \end{bmatrix} \quad (13)$$
Diagonal System

Then we obtain the following transformation of the system (10):

$$\frac{\partial \xi}{\partial t} + \Lambda(\xi) \frac{\partial \xi}{\partial x} = S(\xi),$$

(14)

where

$$\Lambda(\xi) = \begin{bmatrix} -\frac{\xi_1 + \xi_2}{2} + a & 0 \\ 0 & -\frac{\xi_1 + \xi_2}{2} - a \end{bmatrix}$$

and the source term

$$S(\xi) = -\frac{fg}{8D}(\xi_1 + \xi_2)|\xi_1 + \xi_2| \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
Riemann Invariant

The respective Riemann invariant for this conservation system are

$$\xi_{1,2}(\rho, q) = -\frac{q}{\rho} \mp a \ln(\rho) \quad (15)$$

We can express the original variables $\rho$ and $q$ in terms of the Riemann invariant as

$$\rho = \exp\left(\frac{\xi_2 - \xi_1}{2a}\right), \quad (16)$$

$$q = \frac{\xi_1 + \xi_2}{2} \exp\left(\frac{\xi_2 - \xi_1}{2a}\right)$$
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Definition

Allows to solve linear, quasilinear and nonlinear first-order PDEs. E.g. for the first order linear equation:

\[ a(x, y)u_x + b(x, y)u_y = c(x, y) \]  \hspace{1cm} (17)

- Suppose we can find a solution \( u(xy) \). Consider the graph of this function given for

\[ S = \{(x, y, u(x, y))\} \]

- If \( u \) is a solution of (17), we know that at each point \((x, y)\), then

\[ (a(x, y), b(x, y), c(x, y)) \cdot (u_x(x, y), u_y(x, y), -1) = 0. \]  \hspace{1cm} (18)
Then the normal to the surface $S = \{(x, y, u(x, y))\}$ at the point $(x, y, u(x, y))$ is given by

$$N(x, y) = (u_x(x, y), u_y(x, y), -1).$$
Then the normal to the surface $S = \{(x, y, u(x, y))\}$ at the point $(x, y, u(x, y))$ is given by
\[ N(x, y) = (u_x(x, y), u_y(x, y), -1). \]

To construct a curve $C$ (the integral or characteristic curve) parameterized by $s$ such that it is tangent to
\[ (a(x(s), y(s)), b(x(s), y(s)), c(x(s), y(s))) \]
at each point $(x, y, z)$
Then the normal to the surface \( S = \{(x, y, u(x, y))\} \) at the point \((x, y, u(x, y))\) is given by
\[
N(x, y) = (u_x(x, y), u_y(x, y), -1).
\]

To construct a curve \( C \) (the integral or characteristic curve) parameterized by \( s \) such that it is tangent to
\[
(a(x(s), y(s)), b(x(s), y(s)), c(x(s), y(s)))
\]
at each point \((x, y, z)\)
\[
\Rightarrow \text{In particular, the curve } C = \{(x(s), y(s), u(x(s), y(s)))\} \text{ will satisfy the following system of ODEs:}
\[
\begin{align*}
\frac{dx}{ds} &= a(x(s), y(s)) \\
\frac{dy}{ds} &= b(x(s), y(s)) \\
\frac{dz}{ds} &= c(x(s), y(s))
\end{align*}
\]
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Assumptions

Remark 1

*Note that the method of characteristics can not be applied directly to the PDE system (14) due to the coupled term in the source*

\[
S(\xi) = -\frac{fg}{8D}(\xi_1 + \xi_2)|\xi_1 + \xi_2| \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Remark 2

*As a start point, let us consider the characteristic velocities of the hyperbolic system \( \lambda_1 \) and \( \lambda_2 \) as constant parameters.*
In order to handle with this term it is possible to approximate the PDE system (14) as follows

\[
\frac{\partial}{\partial t} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \alpha \begin{bmatrix} -\bar{\xi}_2 \\ -\bar{\xi}_1 \end{bmatrix}
\]

With

- \( \alpha = \frac{f_g}{4D} \)
- \( \bar{\xi}_i \) represents the averaged value of the respective wave.
Then we can solve separately each PDE, for $\xi_1$ we have

\[
\frac{dt}{ds_1} = 1 \quad t_0 = 0
\]

\[
\frac{dx}{ds_1} = \lambda_1 \quad x_0 = r
\]

\[
\frac{dz_1}{ds_1} = -\alpha(z_1 + \bar{\xi}_2) \quad z_0 = \phi_1(r)
\]

Solving the system of ODEs we obtain

\[
t = s
\]

\[
x = s\lambda_1 + r \iff r = x - t\lambda_1
\]

\[
z_1(s) = -\bar{\xi}_2 + e^{-\alpha s} \phi_1(r)
\]
Then, it is possible to obtain the following expression

$$\xi_1(L, t) = -\bar{\xi}_2 + e^{-\alpha t} \xi_1(0, t - \frac{L}{\lambda_1}), \quad (20)$$

Similarly for the second wave

$$\xi_2(0, t) = -\bar{\xi}_1 + e^{-\alpha t} \xi_1(L, t - \frac{L}{\lambda_2}), \quad (21)$$
Figure 7: Wave Propagation
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Network Model
**AIM**

- To consider each node as a finite control volume.
- To apply conservation fundamentals for each wave in each node.
- To obtain a time-delay model of the network in terms of the propagation waves.

*Figure 8: Network Example*
Dynamic Equations

In general, we can obtain the model from the following principle

\[ \dot{\xi}_i^N(t) = \sum \text{inflows} - \sum \text{outflows} \]  \hspace{1cm} (22)

Then, for the wave \( \xi_1 \) in the node \( N \) we have the following dynamics

\[ \dot{\xi}_1^N(t) = \sum_{i=\text{inflows}} \beta(X_i,N) \xi_1^i(t - h_1(X_i,N)) - \xi_2^i - \sum_{j=\text{outflow}} \xi_1^N_j(t) \]  \hspace{1cm} (23)
And for the wave $\xi_2$ we have

$$\dot{\xi}_2^N (t) = \sum_{j=\text{inflows}} \beta(Y_j, N) \xi_1^{Y_j} (t - h_2^{(Y_j, N)}) - \xi_1^{Y_j} - \sum_{i=\text{outflow}} \xi_1^{N_i} (t) \quad (24)$$

with

- $\beta = e^{-\alpha t}$
- The superscript $(X_i, N)$ points to the coefficient in the line between node $X_i$ and the node $N$
- The time delay $h_{1,2} = L^{(X_i,N)}/\lambda_{1,2}$. 
Conclusion

- We present a time delay model for the flow through a pipe based on the isothermal Euler equations.
- Some physical assumptions were done in order to simplify the solutions.
- Conservation laws yield to delayed differential equations model of the network system.
Further work

- **Validation of the model and comparison with different modeling approaches**
  - Lumped parameter model
  - Computational Fluid Dynamics (CFD)
- **Solution for the hyperbolic coupled quasilinear system**
  - Time varying characteristic velocities
  - Coupled nonlinear source term
- **Design of a feedback control strategy for the network system**
  - Decentralized control
  - LPV approach
Final Goal

Physical Model
Isothermal Euler Equations
\[ \rho_t + q_x = 0 \]
\[ q_t + \left( \frac{q^2}{p} - a^2 \rho \right)_x = -G \frac{q |q|}{2D \rho} \]

Riemann Invariants

Hyperbolic Quasilinear System
\[ \xi_t + A(\xi) \xi_x = S(\xi) \]

Method of Characteristics

Delayed equations of the propagation waves

Conservation Laws

Fluid flow network
Time-delay model

Desig of an appropriate Control Strategy
THANKS FOR YOUR ATTENTION


Flow control in gas networks: Exact controllability to a given demand.


