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Analytical solutions for impulsive elliptic out-of-plane rendezvous problem via primer vector theory

R. Serra · D. Arzelier · A. Rondepierre

Abstract This paper focuses on the fixed-time minimum-fuel out-of-plane rendezvous between close elliptic orbits of an active spacecraft, with a passive target spacecraft, assuming a linear impulsive setting, and a elliptic relative motion. It is shown that the out-of-plane elliptic relative dynamics are simple enough to allow for an analytical solution of the problem reviewed. Indeed, the approach relies on the primer vector theory by writing down and directly solving the optimality necessary conditions. After analyzing the characteristics of the dynamics of the optimal primer vector candidates, the complete analytical optimal solution is obtained for arbitrary durations of the rendezvous and arbitrary boundary conditions.

Keywords Orbital rendezvous, fuel optimal space trajectories, primer vector theory, impulsive maneuvers, linear equations of motion

Nomenclature

- a = semi-major axis, km
- e = eccentricity
- θ = true anomaly, rad
- $\phi(\theta)$ = the Tschauner-Hempel fundamental matrix of relative motion
- $\Phi(\theta, \theta_0) = \phi(\theta)\phi^{-1}(\theta_0)$ = transition matrix of relative motion
- $R(\theta)$ = primer vector evolution matrix
- N = number of velocity increments
- θ_i, $\forall\ i = 1, \cdots, N$ = impulses application locations, rad
- $\delta(\theta - \theta_i)$ = Dirac impulse at ν_i ;
- $\Delta V(\theta_i)$ = velocity increment vector at θ_i, m/s
- $\{b_i\}_{i=1,\cdots,N}$ = sequence of variables b_i, $\forall\ i = 1, \cdots, N$;
- $\mathrm{sgn}(z)$ = sign of the variable z ;
- $|\cdot|$ = the absolute value or the Euclidean norm depending whether its argument is a scalar or a matrix.

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- $\theta \mapsto f(\theta)$ = the function $f(\theta)$;
- $\dot{X}(\theta)$ = differentiation of the variable X with respect to θ ;

1 Introduction

For the next years, there will be an increasing demand for the efficient execution of the autonomous rendezvous between an active chaser spacecraft and a passive target spacecraft. Therefore, new challenges are met when designing appropriate guidance schemes for achieving autonomous far range rendezvous on highly elliptical orbits. Even if the rendezvous phase is currently achieved via manual operations in most of the space missions, some recent experiments including Orbital Express, Engineering Test Satellite VII (ETS-VII) and Autonomous Transfer Vehicle (ATV) have demonstrated that autonomous rendezvous operations are not far from being routinely performed in a near future [8,21,14,9]. Most recent advances have been proposed in the literature for autonomous trajectory planning for rendezvous and proximity operations in [12] by considering a variety of operational constraints and using a second-order cone programming approach. Autonomy means also that the simplicity of onboard implementation while preserving optimality in terms of fuel consumption, is a fundamental feature of the proposed solution. Hence, obtaining an analytical closed-form solution for these types of problems is of paramount interest and results, in general, in a more efficient and rapid technological solution.

Here, the fixed-time linearized minimum-fuel impulsive rendezvous problem, as defined in [4], [7], is studied. The impulsive approximation for the thrust means that instantaneous velocity increments are applied to the chaser whereas its position is continuous. The focus of the paper is on the elliptic out-of-plane rendezvous problem for which no complete solution exists to the best of our knowledge. When the eccentricity of the reference orbit is equal to 0, the out-of-plane rendezvous problem amounts to solving an optimal impulsive control problem for a simple harmonic oscillator. A complete characterization of the optimal solutions for this problem has been given by Prussing in [17]. In [4], a complete solution for the circular case has been given for every possible boundary condition for a duration of rendezvous of 2π. As the relative motion between two vehicles in highly elliptic orbits differs significantly from the relative motion seen in circular rendezvous, the solution of the elliptic problem is much more complicated as will be seen in the sequel. Note also the work in [10] dedicated to the circular open-time minimum-fuel m-impulse transfer problem.

The contribution of the paper is to give a complete analytical solution of the problem of fuel-optimal impulsive linearized rendezvous when the reference orbit is elliptic, whatever the duration of the rendezvous and for all possible initial and terminal conditions. These solutions are obtained via the analysis of the optimal conditions expressed in terms of the primer vector as in [4] and [17]. When eccentricity of the reference orbit is not equal to 0, the primer vector exhibits particular features that may be analyzed and which provide the basis for the derivation of the optimal solution. It is shown that the nature of the optimal solutions (number and locations of the optimal thrusts) strongly depends upon the duration of the rendezvous. Indeed, three ranges of duration are identified and the analytical closed-form solutions are given for each, with their conditions of validity expressed in terms of the eccentricity, the boundary conditions, initial and final anomaly of the rendezvous problem. Every result is rigorously proved by relying on the necessary and sufficient optimality conditions and working out these conditions to exhibit the optimal solutions. In addition, it is shown that the designer has extra degrees of freedom when the rendezvous lasts more than one revolution. Indeed, the optimal consumption may be spread over a maximum number of impulses that may be greater than the known upper-bound given by Neustadt [13] on the optimal number of impulses. By optimal number of impulses, it is meant the minimum number of impulses necessary to obtain the minimum fuel consumption for some instance of the linear fixed-time rendezvous problem. As a side result, it is also shown when considering some degenerate cases where the Lagrange multipliers involved in the computation of the primer vector are not unique and are underdetermined, that it is always possible to choose a particular solution for which an optimal primer vector will be completely defined. Finally, two numerical realistic examples illustrate these results.

The outline of the paper is as follows. In the next section, the problem formulation is given. The Section III. recalls the optimality conditions expressed in terms of conditions on the primer vector. The properties of primer vector candidates for optimality are analyzed in Section IV. The complete set of optimal solutions for the minimum-fuel out-of-plane linearized rendezvous problem is given in Section V. As there are many cases to be considered, two tables are given in this section. The first one summarizes the 8 different possible solutions with their velocity increments, the location of the optimal impulses

while the second gives the conditions to be met for each type of solution depending on the duration of the rendezvous. Section VI. is dedicated to the presentation of two different examples illustrating different cases. All the proofs of the different propositions are gathered in the Appendix.

2 Problem formulation

Under Keplerian and linearizing close-proximity assumptions, the complete rendezvous problem may be decoupled between the out-of-plane rendezvous problem for which an analytical solution may be found [17] when $e = 0$ and the coplanar problem for which numerous studies exist [16], [19], [4], [7], [6] and [1], [2] for more recent references. Note that when the fuel-optimal solutions are obtained independently for the coplanar case and the out-of-plane case, the complete optimal solution for the 3-dimensional, sixth-order fuel-optimal linearized rendezvous problem is simply obtained by gathering the two previous planning solutions in a single planning solution based on a 3-dimensional control vector.

If the relative equations of motion of the chaser are supposed to be linear and under Keplerian assumptions, the considered minimum-fuel out-of-plane rendezvous problem may be reformulated as the following optimal control problem:

$$\begin{aligned} \min_{N,\ \theta_i,\ \Delta V(\theta_i)} \quad & J = \sum_{i=1}^{N} |\Delta V(\theta_i)| \\ \text{s.t.} \quad & \dot{X}(\theta) = A(\theta)X(\theta) + B(\theta)\sum_{i=1}^{N} \Delta V(\theta_i)\delta(\theta - \theta_i) \\ & X(\theta_0) = X_0,\ X(\theta_f) = X_f,\ \text{fixed} \end{aligned} \tag{1}$$

where matrices $A(\theta)$ and $B(\theta)$ define the state-space model of relative dynamics given by Tschauner-Hempel [20]. $N \in \mathbb{N}$ is the number of impulses, $\theta_i \in [\theta_0, \theta_f]$ is the true anomaly where each applied impulse is located. θ_0 and θ_f respectively denote the fixed initial and final values of the true anomaly during the rendezvous. The state vector X is composed of the out-of-plane relative position and velocity $X = [y\ y']^T$ defined in the Local Vertical Local Horizontal (LVLH) frame as in [1]. $\Delta V(\theta_i)$ is the velocity increment applied at θ_i. Since a transition matrix may be computed in closed-form for the Tschauner-Hempel equations, it may be appropriate to replace the differential constraint on dynamics by the equivalent algebraic constraint involving this transition matrix. Problem (1) for a fixed number of impulses N may be reformulated as the following optimization problem:

$$\begin{aligned} \min_{N,\theta_i,\Delta V(\theta_i)} \quad & J = \sum_{i=1}^{N} |\Delta V(\theta_i)| \\ \text{s.t.} \quad & z_f = \begin{bmatrix} z_{f_1} \\ z_{f_2} \end{bmatrix} = \sum_{i=1}^{N} \frac{R(\theta_i)}{r(\theta_i)} \Delta V(\theta_i) \end{aligned} \tag{2}$$

where $z_f \in \mathbb{R}^2$ is given by,

$$z_f = n(1-e^2)^{-\frac{3}{2}}(\phi^{-1}(\theta_f)\tilde{X}_f - \phi^{-1}(\theta_0)\tilde{X}_0) \neq 0. \tag{3}$$

n and $0 \leq e < 1$ are respectively the mean motion and the eccentricity of the reference orbit. Note that the true anomaly θ has been chosen as the independent variable throughout in the paper. $\phi(\theta)$ is the Tschauner-Hempel fundamental matrix associated to the linearized relative free motion and $\Phi(\theta, \theta_0) = \phi(\theta)\phi^{-1}(\theta_0)$ denotes, therefore, the transition matrix of the linearized relative free motion [20]. The state vectors $\tilde{X}_f = \tilde{X}(\theta_f)$ and $\tilde{X}_0 = \tilde{X}(\theta_0)$ are composed of the relative positions and relative velocities vectors in the LVLH frame after the usual simplifying change of variables [20]. For the out-of-plane elliptic rendezvous problem, the different matrices defining the problem are defined as [20]:

$$\phi(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},\ R(\theta) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix},\ r(\theta) = 1 + e\cos(\theta). \tag{4}$$

The optimization decision variables are the number of impulses N, the sequence of thrust locations $\{\theta_i\}_{i=1,\cdots,N}$ and the sequence of thrusts $\{\Delta V(\theta_i)\}_{i=1,\cdots,N}$.

Due to the lack of *a priori* information about the optimal number of impulses to be considered, problem (2) is very hard to solve from both theoretical and numerical points of view. Therefore, the associated fixed-time minimum-fuel rendezvous problem for a fixed number N of impulses has been considered in the literature mainly via geometric methods near circular [16], [19], [6] or elliptic [7] orbits. These results are mainly based on the derivation of optimality conditions for the problem (2) when N is fixed *a priori*. These optimality conditions are now recalled in the framework of [7].

3 Optimality conditions

If the number of impulses is fixed *a priori* to N, problem (2) may be considered as a parametric nonlinear non convex transcendental optimization problem involving the N velocity increments $\Delta V(\theta_i)$ and N locations θ_i of maneuvers. By applying a Lagrange multiplier rule for the problem (2) as in [7], one can derive necessary conditions of optimality (5) to (8) in terms of the Lagrange multiplier vector $\lambda \in \mathbb{R}^n$, as is recalled in Theorem 1 below. These conditions are also sufficient in the case of linear relative motion when strengthening them by adding the semi-infinite constraint (9) that should be fulfilled on the continuum $[\theta_0, \theta_f]$ [18].

Theorem 1 ([11], [13])

$$(\theta_1, ..., \theta_N, \Delta V(\theta_1), ..., \Delta V(\theta_N))$$

is an optimal solution of problem (2) *if and only if there exists a non-zero vector* $\lambda \in \mathbb{R}^m,\ m = \dim(\phi)$ *that verifies the necessary and sufficient conditions:*

$$\Delta V(\theta_i) = -p(\theta_i)\,|\Delta V(\theta_i)|\,, \qquad \forall\ i = 1, \cdots, N, \tag{5}$$

$$|\Delta V(\theta_i)| = 0 \text{ or } |p(\theta_i)| = 1, \qquad \forall\ i = 1, \cdots, N, \tag{6}$$

$$|\Delta V(\theta_i)| = 0 \text{ or } \theta_i = \theta_0 \text{ or } \theta_i = \theta_f \text{ or } \frac{d\,|p|}{d\theta}(\theta_i) = 0, \forall\ i = 1, \cdots, N, \tag{7}$$

$$\sum_{i=1}^{N} \frac{R(\theta_i)}{r(\theta_i)} p(\theta_i)\,|\Delta V(\theta_i)| = -z_f, \tag{8}$$

$$|p(\theta)| \leq 1, \qquad \forall\ \theta \in [\theta_0, \theta_f]. \tag{9}$$

where $p(\theta)$ *is the so-called primer vector [5] and is defined, in the out-of-plane rendezvous problem as:*

$$p(\theta) = \frac{R(\theta)^T \lambda}{r(\theta)} = \frac{-\lambda_1 \sin(\theta) + \lambda_2 \cos(\theta)}{1 + e\cos(\theta)} \tag{10}$$

These results date back to the seminal work of [11] in the early sixties, proved rigorously later by Neustadt in [13] and are based on the so-called primer vector theory. Obviously a primer vector candidate is completely defined by the choice of the Lagrange multipliers λ_1, λ_2. In the next section, the particular properties of the primer vector are analyzed such that these characteristics may be used for the derivation of the optimal solutions.

4 Primer vector candidate dynamics

By (10), $p(\theta)$ is obviously a 2π-periodic function. It is a harmonic oscillator divided by the positive function $r(\theta) = 1 + e\cos(\theta)$. As a result, its sign changes every π. Its derivative may be calculated as follows:

$$\frac{\mathrm{d}p}{\mathrm{d}\theta}(\theta) = -\frac{\lambda_1(e + \cos(\theta)) + \lambda_2 \sin(\theta)}{(1 + e\cos(\theta))^2}. \tag{11}$$

As $0 \leq e < 1$, it is easy to deduce that $p(\theta)$ reaches two global extrema of opposite sign at θ_{e_1} and θ_{e_2} modulo 2π.

4.1 Lagrange multipliers as functions of an extremum

If $p(\theta)$ has an extremum $p(\theta_e)$ at θ_e then it follows from (10) and (11) that:

$$\lambda_1 = -p(\theta_e)\sin(\theta_e),\ \lambda_2 = p(\theta_e)(e + \cos(\theta_e)). \tag{12}$$

A primer vector candidate can thus be rewritten as follows:

$$p(\theta) = p(\theta_e)\frac{\cos(\theta - \theta_e) + e\cos(\theta)}{1 + e\cos(\theta)}. \tag{13}$$

4.2 Extremum ratio

From (12), it comes that:

$$\frac{|p(\theta_{e_2})|}{|p(\theta_{e_1})|}\sin(\theta_{e_2}) = -\sin(\theta_{e_1}), \tag{14}$$

$$\frac{|p(\theta_{e_2})|}{|p(\theta_{e_1})|}(e + \cos(\theta_{e_2})) = -(e + \cos(\theta_{e_1})). \tag{15}$$

By combining equations (14) and (15), one can get a second order polynomial equation whom the ratio of the absolute values is a solution to:

$$X^2 - \frac{2e(e + \cos(\theta_{e_1}))}{1 - e^2}X - \frac{1 + 2e\cos(\theta_{e_1}) + e^2}{1 - e^2} = 0. \tag{16}$$

Only the positive one of the following roots

$$\{-1; \frac{1 + 2e\cos(\theta_{e_1}) + e^2}{1 - e^2}\}$$

corresponds to the ratio of absolute values, so that:

$$\frac{|p(\theta_{e_2})|}{|p(\theta_{e_1})|} = \frac{1 + 2e\cos(\theta_{e_1}) + e^2}{1 - e^2}, \tag{17}$$

which can be rearranged as:

$$\frac{|p(\theta_{e_2})|}{|p(\theta_{e_1})|} - 1 = 2e\frac{e + \cos(\theta_{e_1})}{1 - e^2}. \tag{18}$$

Note that $|p(\theta_{e_1})| \neq 0$ otherwise $p(\theta) \equiv 0$ by (12).

From (18), it is easily seen that the maximum norm extremum is such that $\cos(\theta_e) \leq -e$ whereas the minimum norm extremum is such that $\cos(\theta_e) \geq -e$. Indeed, suppose that the maximum norm extremum is given by θ_{e_2} then θ_{e_1} is the minimum norm extremum and it is deduced from (18) that

$$\frac{|p(\theta_{e_2})|}{|p(\theta_{e_1})|} - 1 = 2e\frac{e + \cos(\theta_{e_1})}{1 - e^2} \geq 0 \tag{19}$$

and therefore, $\cos(\theta_{e_1}) \geq -e$ for the minimum norm extremum. The condition for the maximum norm extremum is obtained the same way by assuming that θ_{e_2} is the minimum norm extremum. When $|p(\theta_{e_1})| = 1$ it comes that:

$$|p(\theta_{e_2})| = \frac{1 + 2e\cos(\theta_{e_1}) + e^2}{1 - e^2}. \tag{20}$$

Thus $|p(\theta_{e_2})| > 1$ if and only if $\cos(\theta_{e_1}) > -e$.

4.3 Extremum as a function of the Lagrange multipliers

For a given $\lambda \in \mathbb{R}^2$ such that $\lambda_2 \neq 0$, the extremum anomalies are given by the equation:

$$\frac{\sin(\theta_e)}{e + \cos(\theta_e)} = -\frac{\lambda_1}{\lambda_2}. \tag{21}$$

By defining $Y = \cos(\theta_e)$ and $Q = \frac{\lambda_1}{\lambda_2}$, it follows after taking the square of (21) that:

$$(1 + Q^2)Y^2 + 2eQ^2Y + e^2Q^2 - 1 = 0. \tag{22}$$

The roots of (22) are:

$$\frac{\pm\sqrt{1 + Q^2(1 - e^2)} - eQ^2}{1 + Q^2}.$$

So that:

$$\cos(\theta_e) = \frac{\pm\sqrt{1+Q^2(1-e^2)}-eQ^2}{1+Q^2} \text{ and } \sin(\theta_e) = -Q\frac{\pm\sqrt{1+Q^2(1-e^2)}+e}{1+Q^2}. \tag{23}$$

Thus, keeping in mind the restrictions on the maximum and minimum norm extremum, the maximum and minimum norm value of the primer vector can be expressed in terms of λ. Inserting the expressions of $\cos(\theta_e)$ and $\sin(\theta_e)$ of (23) in (10), the following expression is obtained after noting that $\lambda_1 = Q\lambda_2$:

$$p(\theta_e) = \frac{\pm\lambda_2(1 + Q^2)\sqrt{1 + Q^2(1 - e^2)}}{1 + (1 - e^2)Q^2 \pm e\sqrt{1 + Q^2(1 - e^2)}}. \tag{24}$$

Noting that (by completing the square):

$$\begin{aligned} 1 + (1 - e^2)Q^2 + e\sqrt{1 + Q^2(1 - e^2)} &= \sqrt{1 + Q^2(1 - e^2)}(\sqrt{1 + Q^2(1 - e^2)} + e) \\ 1 + (1 - e^2)Q^2 - e\sqrt{1 + Q^2(1 - e^2)} &= \sqrt{1 + Q^2(1 - e^2)}(\sqrt{1 + Q^2(1 - e^2)} - e) \end{aligned}$$

and taking the absolute value of (24), the following expressions are obtained:

$$\max_{\theta\in\mathbb{R}} |p(\theta)| = \frac{|\lambda_2|(1 + Q^2)}{\sqrt{1 + Q^2(1 - e^2)} - e}, \min_{\theta\in\mathbb{R}} |p(\theta)| = \frac{|\lambda_2|(1 + Q^2)}{\sqrt{1 + Q^2(1 - e^2)} + e}. \tag{25}$$

5 Minimum-fuel out-of-plane optimal solutions

The method used to derive the analytical solution of the problem mainly consists in exploiting the different features of an optimal primer vector and in discussing all the different possible configurations of an optimal primer vector. A solution of the minimum-fuel elliptic out-of-plane rendezvous problem is strongly dependent upon the duration of the rendezvous $d_\theta = \theta_f - \theta_0$, the initial and final anomalies θ_0 and θ_f and upon the vector z_f.

- When the duration of the rendezvous is longer than a period (i.e. when $d_\theta \geq 2\pi$), the primer vector $p(\theta)$ will reach its local extrema. Therefore, only two possibilities may occur for $p(\theta)$ to be optimal since it must verify (9): one extremum has a unit norm and the other has a norm strictly less than 1 (one impulse, at most, per 2π), or both extrema of $p(\theta)$ have a unit norm (two impulses, at most, per 2π).
- When the duration of the rendezvous is shorter than a period (i.e. when $d_\theta < 2\pi$), this dependency may be quite complicated since an extremum could be at an end point, as illustrated by the next subsections.

Next, the different solutions and the associated conditions are summarized in the clearest way possible. For each type of optimal solution, the associated conditions involving d_θ, θ_0, θ_f and z_f are given. The optimal Lagrange multipliers and related primer vector are then presented. Let first define some notations that will be needed in the sequel:

$$\theta_\pm = \min\{\theta \geq \theta_0 \ / \ \cos(\theta) = -e, \ \sin(\theta) = \pm\sqrt{1 - e^2}\} \tag{26}$$

The notation $\theta_\pm$ defines two different locations θ_+ and θ_- belonging to the interval $[\theta_0,\ \theta_0+2\pi)$ and verifying $|p(\theta_\pm)|=1$ and $\dfrac{dp(\theta_\pm)}{d\theta}=0$. They are defined by an identical cosine and by a positive or a negative sine respectively, i.e. $\cos(\theta_+)=\cos(\theta_-)=-e$, $\sin(\theta_+)=\sqrt{1-e^2}$ and $\sin(\theta_-)=-\sqrt{1-e^2}$.

$$\begin{aligned}\varepsilon_1 &= \mathrm{sgn}(z_{f_1}),\ \varepsilon_2=\mathrm{sgn}(z_{f_2}),\\ \varepsilon_0 &= \mathrm{sgn}(\cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2}),\\ \varepsilon_f &= \mathrm{sgn}(\cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2}).\end{aligned} \tag{27}$$

Note that when $d_\theta<\pi$, due to (26), the condition $(\theta_-,\theta_+)\in[\theta_0,\theta_f]^2$ is equivalent to:

$$\sin(\theta_0)\geq\sqrt{1-e^2} \text{ and } \sin(\theta_f)\leq-\sqrt{1-e^2}. \tag{28}$$

When $\pi\leq d_\theta<2\pi$, the condition $(\theta_-,\theta_+)\in[\theta_0,\theta_f]^2$ is equivalent to:

$$\begin{cases}\sin(\theta_0)\geq\sqrt{1-e^2}\\ \text{or}\\ \sin(\theta_0)\leq-\sqrt{1-e^2} \text{ and } \sin(\theta_f)\leq-\sqrt{1-e^2}\\ \text{or}\\ |\sin(\theta_0)|<\sqrt{1-e^2} \text{ and } (e+\cos(\theta_0))(e+\cos(\theta_f))>0.\end{cases} \tag{29}$$

5.1 Two interior impulses solution

In this section, the case of optimal solutions with two interior impulses per period is investigated. Independently of the duration of the rendezvous, the optimal primer vector will reach its two extrema, and both extrema have a unit norm.

Proposition 1 *An optimal solution for the linearized impulsive out-of-plane rendezvous problem is a 2-impulse trajectory defined by the optimal locations $\theta_\pm$ and the optimal directions of thrust given by:*

$$\Delta V(\theta_\pm)=\frac{\sqrt{1-e^2}}{2e}(\mp e z_{f_1}-\sqrt{1-e^2}z_{f_2}), \tag{30}$$

if the following conditions are verified :

$$e|z_f|>|z_{f_2}| \text{ and } d_\theta\geq 2\pi \tag{31}$$

or

$$e|z_f|>|z_{f_2}| \text{ and } d_\theta<\pi \text{ and } \sin(\theta_0)\geq\sqrt{1-e^2} \text{ and } \sin(\theta_f)\leq-\sqrt{1-e^2} \tag{32}$$

or

$$\begin{aligned}&e|z_f|>|z_{f_2}| \text{ and } \pi\leq d_\theta<2\pi\\ &\text{and}\\ &\begin{cases}\sin(\theta_0)\geq\sqrt{1-e^2}\\ \textit{or}\\ \sin(\theta_0)\leq-\sqrt{1-e^2} \text{ and } \sin(\theta_f)\leq-\sqrt{1-e^2}\\ \textit{or}\\ |\sin(\theta_0)|<\sqrt{1-e^2} \text{ and } (e+\cos(\theta_0))(e+\cos(\theta_f))>0.\end{cases}\end{aligned} \tag{33}$$

Finally, the optimal Lagrange multipliers and the optimal primer vector are:

$$\lambda_1=-\varepsilon_1\sqrt{1-e^2},\ \lambda_2=0,\ p(\theta)=\frac{\varepsilon_1\sqrt{1-e^2}\sin(\theta)}{1+e\cos(\theta)}. \tag{34}$$

Remark 1 As indicated in the proof of Proposition 1 given in the Appendix, when $d_\theta\geq 2\pi$, the optimal solution of the planning may be chosen to be spread over $N^*=N_-+N_+$ impulses verifying (89) and (90), depending on the duration of the rendezvous and operational constraints while preserving the optimal consumption.

5.2 One interior impulse solutions

Consider now the case of optimal solutions with only one interior impulse and no boundary impulse. In this case two geometrical configurations of the primer vector may occur. When the rendezvous lasts more than a period ($d_\theta \geq 2\pi$), the associated optimal primer vector necessarily reaches one extremum of unit norm while the other has a norm strictly less than 1. When the rendezvous lasts less than a period ($d_\theta < 2\pi$), there is no condition on the other extremum norm and need additional conditions will be needed to ensure $|p(\theta)| \leq 1,\ \theta \in [\theta_0, \theta_f]$.

Proposition 2 *Provided that* $\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2} \neq 0$ *and* $\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2} \neq 0$*, an optimal solution for the linearized impulsive out-of-plane rendezvous problem is a* 1*-impulse trajectory defined by the optimal locations* $\theta_{i\flat}$*:*

$$\cos(\theta_{i\flat}) = -\varepsilon_\flat \frac{z_{f_2}}{|z_f|},\ \sin(\theta_{i\flat}) = \varepsilon_\flat \frac{z_{f_1}}{|z_f|}, \tag{35}$$

and optimal thrusts are defined by:

$$\Delta V(\theta_{i\flat}) = -\varepsilon_\flat |z_f| + e z_{f_2}, \tag{36}$$

with $\flat = 0$*, if the following conditions are verified:*

$$d_\theta < 2\pi \text{ and } e|z_f| \leq \varepsilon_0 z_{f_2} \text{ and } \varepsilon_f = -\varepsilon_0 \tag{37}$$

or

$$\begin{array}{l} d_\theta < 2\pi \text{ and } e|z_f| > \varepsilon_0 z_{f_2} \text{ and } \varepsilon_f = -\varepsilon_0 \\ \text{and} \\ \left\{ \begin{array}{l} |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_0) + \varepsilon_0 z_{f_1}\sin(\theta_0) > 0 \\ \text{and} \\ \varepsilon_0(e + \cos(\theta_0))z_{f_1} + (\varepsilon_0 z_{f_2} - e|z_f|)\sin(\theta_0) > 0 \\ \text{and} \\ |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_f) + \varepsilon_0 z_{f_1}\sin(\theta_f) > 0 \\ \text{and} \\ \varepsilon_0(e + \cos(\theta_f))z_{f_1} + (\varepsilon_0 z_{f_2} - e|z_f|)\sin(\theta_f) < 0 \end{array} \right. \end{array} \tag{38}$$

with $\flat = 2$*, if the following conditions are verified:*

$$d_\theta \geq 2\pi \text{ and } |z_{f_2}| > e|z_f| \tag{39}$$

or

$$\pi \leq d_\theta < 2\pi \text{ and } e|z_f| \leq |z_{f_2}| \text{ and } \varepsilon_f = \varepsilon_0, \tag{40}$$

or

$$\begin{array}{l} \pi \leq d_\theta < 2\pi \text{ and } e|z_f| > |z_{f_2}| \text{ and } \varepsilon_f = \varepsilon_0 \\ \text{and} \\ \left\{ \begin{array}{l} |z_f| + (2e|z_f| - \varepsilon_2 z_{f_2})\cos(\theta_0) + \varepsilon_2 z_{f_1}\sin(\theta_0) > 0 \\ \text{and} \\ \varepsilon_2 z_{f_1}(e + \cos(\theta_0)) - (e|z_f| - \varepsilon_2 z_{f_2})\sin(\theta_0) > 0 \\ \text{and} \\ |z_f| + (2e|z_f| - \varepsilon_2 z_{f_2})\cos(\theta_f) + \varepsilon_2 z_{f_1}\sin(\theta_f) > 0 \\ \text{and} \\ -\varepsilon_2 z_{f_1}(e + \cos(\theta_f)) + (e|z_f| - \varepsilon_2 z_{f_2})\sin(\theta_f) > 0. \end{array} \right. \end{array} \tag{41}$$

Finally, the optimal Lagrange multipliers and the optimal primer vector are

$$\lambda_1 = -\frac{z_{f_1}}{|z_f|},\ \lambda_2 = \varepsilon_\flat e - \frac{z_{f_2}}{|z_f|},\ p(\theta) = \frac{z_{f_1}\sin(\theta) + (\varepsilon_\flat e|z_f| - z_{f_2})\cos(\theta)}{(1 + e\cos(\theta))|z_f|}. \tag{42}$$

A particular case of proposition 2 is when the primer vector has a unit norm extremum at θ_0 (respectively at θ_f). The results in this case are summarized below.

Corollary 1 *When* $\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2} = 0$ *or* $\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2} = 0$, *the optimal solution comes down to a one impulse boundary solution for which there may exist an infinite number of optimal Lagrange multipliers. Without loss of generality, the primer vector may be chosen as in (42) where* $\varepsilon_\flat = -\mathrm{sgn}(-\sin(\theta_\flat)z_{f_1} + \cos(\theta_\flat)z_{f_2})$ *and* $\flat = 0$ *or* $\flat = f$.

1. *If* $\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2} = 0$, *then the optimal solution is a one initial impulse solution and the associated optimal thrust is given by:*

$$\Delta V(\theta_0) = (-\sin(\theta_0)z_{f_1} + \cos(\theta_0)z_{f_2})(1 + e\cos(\theta_0)).$$

2. *If* $\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2} = 0$, *then the optimal solution is a one final impulse solution and the associated optimal thrust is given by:*

$$\Delta V(\theta_f) = (-\sin(\theta_f)z_{f_1} + \cos(\theta_f)z_{f_2})(1 + e\cos(\theta_f)).$$

An example of the particular case ruled by Corollary 1 may be obtained by choosing the following parameters: $e = 0.8$, $z_f^T = \begin{bmatrix}-0.7297 & 2.0305\end{bmatrix}$, $\theta_0 = 3.4866$, $\theta_f = 5.9998$. In this case, the optimal Lagrange multipliers are given by : $\lambda_1 = 0.3382$, $\lambda_2 = -0.1411$. The optimal primer vector is depicted at Figure 1 alongside samples of non optimal primer vectors built from λ_1 and λ_2 verifying the optimality condition (9) but not the optimality condition (8).

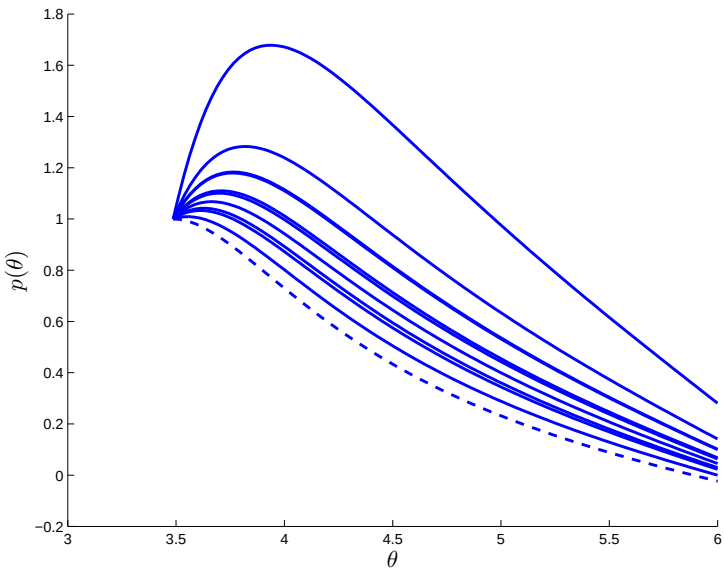


Fig. 1 Optimal (dashed line) and non optimal (solid lines) primer vector in particular case of Corollary 1

5.3 Initial (or final) and one interior impulses

The focus is now on optimal solutions with one interior impulse and at least one boundary (initial or final) impulse. In that case, one extremum of the primer vector has a norm equal to 1 while the other has a norm greater than 1. To give a concise statement of the optimal solution, the following notations will be needed.

$$\theta_{i0}^+ = \theta_0 + \arccos(-1 - 2e\cos(\theta_0)) \tag{43}$$
$$\theta_{if}^- = \theta_f - \arccos(-1 - 2e\cos(\theta_f)) \tag{44}$$
$$\hat{\theta}_{i0}^+ = \theta_0 + 2\pi - \arccos(-1 - 2e\cos(\theta_0)) \tag{45}$$
$$\hat{\theta}_{if}^- = \theta_f - 2\pi + \arccos(-1 - 2e\cos(\theta_f)) \tag{46}$$

$$g^+(\theta_0) = e\sin(\theta_0) + \sqrt{-e\cos(\theta_0)(1 + 2e\cos(\theta_0))} \tag{47}$$
$$g^-(\theta_f) = -e\sin(\theta_f) + \sqrt{-e\cos(\theta_f)(1 + 2e\cos(\theta_f))} \tag{48}$$

θ_{i_0} denotes the extremum anomaly on $[\theta_0, \theta_f]$ and ε the direction of the impulse located at θ_{i_0}:

$$\varepsilon = \frac{\Delta V(\theta_{i_0})}{|\Delta V(\theta_{i_0})|}. \tag{49}$$

Note that if the primer vector norm reaches 1 at any other anomaly on $[\theta_0, \theta_f]$, it can only be at θ_0 or θ_f and the directions of the optimal thrusts at these locations are in the opposite direction to the interior impulse:

$$\frac{\Delta V(\theta_0)}{|\Delta V(\theta_0)|} = \frac{\Delta V(\theta_f)}{|\Delta V(\theta_f)|} = -\varepsilon. \tag{50}$$

Combining the optimality conditions (5), (6) and (7), the Lagrange multipliers are uniquely defined by:

$$\lambda_1 = \varepsilon \sin(\theta_{i_0}),\ \lambda_2 = -\varepsilon(e + \cos(\theta_{i_0})), \tag{51}$$

and the primer vector is then given by:

$$p(\theta) = -\varepsilon \frac{\cos(\theta - \theta_{i_0}) + e\cos(\theta)}{1 + e\cos(\theta)}. \tag{52}$$

Since the study of the case of optimal solutions with final and one interior impulses, is very similar, only the derivation of optimality conditions for optimal solutions with one initial and one interior impulses is detailed here.

Now, it is needed to determine the location of the interior impulse. By the primer vector definition (52), having an impulse at θ_0 leads to:

$$1 + 2e\cos(\theta_0) + \cos(\theta_{i_0} - \theta_0) = 0. \tag{53}$$

This equation has a solution θ_{i_0} on $]\theta_0, \theta_0 + 2\pi]$ if and only if:

$$\cos(\theta_0) \leq 0 \tag{54}$$

and if this condition is satisfied, the solutions are:

$$\{\theta_0 + \arccos(-(1 + 2e\cos(\theta_0)); \theta_0 + 2\pi - \arccos(-(1 + 2e\cos(\theta_0))\}. \tag{55}$$

Note that if $\theta_f - \theta_0 < \pi$, then (53) has a unique solution on $[\theta_0, \theta_0 + \pi]$, and has a solution on $[\theta_0, \theta_f]$ if and only if condition (56) is satisfied:

$$1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) \leq 0. \tag{56}$$

Otherwise, when $\theta_f - \theta_0 \geq \pi$, then equation (53) has a unique solution on $[\theta_0, \theta_f]$ if and only if:

$$1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) < 0, \tag{57}$$

and two solutions if condition (57) does not hold. Next, two cases are discussed: $\theta_{i_0} = \theta_0 + \arccos(-(1 + 2e\cos(\theta_0)))$ and $\theta_{i_0} = \theta_0 + 2\pi - \arccos(-(1 + 2e\cos(\theta_0)))$.

5.3.1 Case I

The results in the case where $\theta_{i_0} = \theta_0 + \arccos(-(1 + 2e\cos(\theta_0)))$, are summarized in Proposition 3. To make the results clearer to the reader, it is important to emphasize that $+$ is associated with $\# = 0$ (initial impulse) while $-$ is associated with $\# = f$ (final impulse) as is indicated by the notations (43)-(46).

Proposition 3 *An optimal solution for the linearized impulsive out-of-plane rendezvous problem is a 2-impulse trajectory defined by the optimal locations $(\theta_\sharp, \theta_{i\sharp}^{\pm})$ and the associated optimal thrusts,*

$$\begin{aligned} \Delta V(\theta_\sharp) &= (1 + e\cos(\theta_\sharp)) \frac{\cos(\theta_{i\sharp}^{\pm}) z_{f_1} + \sin(\theta_{i\sharp}^{\pm}) z_{f_2}}{\sin(\theta_{i\sharp}^{\pm} - \theta_\sharp)}, \\ \Delta V(\theta_{i\sharp}^{\pm}) &= -(1 + e\cos(\theta_{i\sharp}^{\pm})) \frac{\cos(\theta_\sharp) z_{f_1} + \sin(\theta_\sharp) z_{f_2}}{\sin(\theta_{i\sharp}^{\pm} - \theta_\sharp)}. \end{aligned} \tag{58}$$

if the conditions:

$$\sharp = 0 \text{ and } d_\theta < \pi \text{ and } \sin(\theta_0) < \sqrt{1-e^2} \text{ and } 1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) \leq 0$$
and

$$\begin{cases} \varepsilon_0 = \varepsilon_f \\ \text{or} \\ \varepsilon_0 = -\varepsilon_f \text{ and } |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_0) + \varepsilon_0 z_{f_1}\sin(\theta_0) \leq 0 \end{cases} \tag{59}$$

or

$$\sharp = f \text{ and } d_\theta < \pi \text{ and } \sin(\theta_f) > -\sqrt{1-e^2} \text{ and } 1 + 2e\cos(\theta_f) + \cos(\theta_f - \theta_0) \leq 0$$
and

$$\begin{cases} \varepsilon_0 = \varepsilon_f \\ \text{or} \\ \varepsilon_0 = -\varepsilon_f \text{ and } |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_f) + \varepsilon_0 z_{f_1}\sin(\theta_f) \leq 0 \end{cases} \tag{60}$$

or

$$\sharp = 0 \text{ and } \pi \leq d_\theta < 2\pi \text{ and } \sin(\theta_0) < \sqrt{1-e^2} \text{ and } \cos(\theta_0) \leq 0$$
and

$$\begin{cases} 1 - \cos(\theta_f - \theta_0) + 2\sin(\theta_f - \theta_0)g^-(\theta_0) \geq 0 \\ \text{and} \\ |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_0) + \varepsilon_0 z_{f_1}\sin(\theta_0) \leq 0 \\ \text{and} \\ \sin(\theta_f - \theta_0) + e(\sin(\theta_f) - \sin(\theta_0)) + 2(\cos(\theta_f - \theta_0) + e\cos(\theta_0))g^-(\theta_0) \leq 0 \end{cases} \tag{61}$$

or

$$\sharp = f \text{ and } \pi \leq d_\theta < 2\pi \text{ and } \sin(\theta_f) > -\sqrt{1-e^2} \text{ and } \cos(\theta_f) \leq 0$$
and
$$1 - \cos(\theta_f - \theta_0) + 2\sin(\theta_f - \theta_0)g^+(\theta_f) \geq 0$$
and

$$\begin{cases} |z_f| + (2e|z_f| + \varepsilon_f z_{f_2})\cos(\theta_f) - \varepsilon_f z_{f_1}\sin(\theta_f) \leq 0 \\ \text{and} \\ \sin(\theta_f - \theta_0) + e(\sin(\theta_f) - \sin(\theta_0)) + 2(\cos(\theta_f - \theta_0) + e\cos(\theta_f))g^+(\theta_f) \geq 0 \end{cases} \tag{62}$$

are verified. The optimal Lagrange multipliers are given by:

$$\begin{aligned} \lambda_1 &= \varepsilon_\sharp \left(\pm\sin(\theta_\sharp)(1 + 2e\cos(\theta_\sharp) - 2\cos(\theta_\sharp)\sqrt{-e\cos(\theta_\sharp)(1 + e\cos(\theta_\sharp))} \right), \\ \lambda_2 &= \varepsilon_\sharp \left(\pm e \mp \cos(\theta_\sharp)(1 + e\cos(\theta_\sharp) - 2\sin(\theta_\sharp)\sqrt{-e\cos(\theta_\sharp)(1 + 2e\cos(\theta_\sharp))} \right), \end{aligned} \tag{63}$$

while the optimal primer vector is defined by:

$$p(\theta) = p(\theta_{i\sharp}^{\pm})\frac{\cos(\theta - \theta_{i\sharp}^{\pm}) + e\cos(\theta)}{1 + e\cos(\theta)}. \tag{64}$$

5.3.2 Case II

The results in the case where $\theta_{i_0} = \theta_0 + 2\pi - \arccos(-(1 + 2e\cos(\theta_0)))$, are summarized in Proposition 4. To make the results clearer to the reader, it is important to emphasize again that $+$ is associated with $\# = 0$ (initial impulse) while $-$ is associated with $\# = f$ (final impulse) as is indicated by the notations (43)-(46).

Proposition 4 *An optimal solution for the linearized impulsive out-of-plane rendezvous problem is a 2-impulse trajectory defined by the optimal locations* $(\theta_\sharp, \hat{\theta}_{i\sharp}^{\pm})$ *and the associated optimal thrusts given by (58), if the conditions:*

$$\sharp = 0 \text{ and } \pi \leq d_\theta < 2\pi \text{ and} \sin(\theta_0) < -\sqrt{1-e^2} \text{ and } \cos(\theta_0) \leq 0$$
and

$$\begin{cases} 1-\cos(\theta_f-\theta_0)-2\sin(\theta_f-\theta_0)g^+(\theta_0)\geq 0 \\ \text{and} \\ 1+2e\cos(\theta_0)+\cos(\theta_f-\theta_0)\geq 0 \\ \text{and} \\ |z_f|+(2e|z_f|+\varepsilon_0 z_{f_2})\cos(\theta_0)-\varepsilon_0 z_{f_1}\sin(\theta_0)\leq 0 \\ \text{and} \\ -\sin(\theta_f-\theta_0)-e(\sin(\theta_f)-\sin(\theta_0))+2(\cos(\theta_f-\theta_0)+e\cos(\theta_0))g^+(\theta_0)\geq 0 \end{cases} \tag{65}$$

or

$$\begin{aligned} &\sharp = f \text{ and } \pi\leq d_\theta<2\pi \text{ and } \sin(\theta_f)>\sqrt{1-e^2} \text{ and } \cos(\theta_f)\leq 0 \\ &\text{and} \end{aligned}$$

$$\begin{cases} 1-\cos(\theta_f-\theta_0)-2\sin(\theta_f-\theta_0)g^-(\theta_f)\geq 0 \\ \text{and} \\ 1+2e\cos(\theta_f)+\cos(\theta_f-\theta_0)\geq 0 \\ \text{and} \\ |z_f|+(2e|z_f|-\varepsilon_f z_{f_2})\cos(\theta_f)+\varepsilon_f z_{f_1}\sin(\theta_f)\leq 0 \\ \text{and} \\ -\sin(\theta_f-\theta_0)-e(\sin(\theta_f)-\sin(\theta_0))+2(\cos(\theta_f-\theta_0)+e\cos(\theta_f))g^-(\theta_f)\geq 0 \end{cases} \tag{66}$$

are verified. The optimal Lagrange multipliers are given by:

$$\begin{aligned} \lambda_1 &= \varepsilon_\sharp\left(\mp\sin(\theta_\sharp)(1+2e\cos(\theta_\sharp))-2\cos(\theta_\sharp)\sqrt{-e\cos(\theta_\sharp)(1+e\cos(\theta_\sharp))}\right), \\ \lambda_2 &= \varepsilon_\sharp\left(\mp e\pm\cos(\theta_\sharp)(1+e\cos(\theta_\sharp))-2\sin(\theta_\sharp)\sqrt{-e\cos(\theta_\sharp)(1+2e\cos(\theta_\sharp))}\right), \end{aligned} \tag{67}$$

while the optimal primer vector is defined by:

$$p(\theta)=p(\hat{\theta}^{\pm}_{i\sharp})\frac{\cos(\theta-\hat{\theta}^{\pm}_{i\sharp})+e\cos(\theta)}{1+e\cos(\theta)} \tag{68}$$

5.4 Boundary solutions

In this section, the case of optimal solutions with no interior impulse is discussed, which can occur only if $\theta_f-\theta_0<2\pi$. In that case, the norm of the primer vector can reach 1 only at θ_0 and θ_f. The objective equation given by the optimality condition (8) leads to make the difference between two cases: $\theta_f-\theta_0\neq\pi$ and $\theta_f-\theta_0=\pi$.

5.4.1 Case I: $\theta_f-\theta_0\neq\pi$

Proposition 5 *A optimal solution for the linearized impulsive out-of-plane rendezvous problem is a 2-impulse trajectory defined by the optimal locations (θ_0,θ_f) and the associated optimal thrusts,*

$$\begin{aligned} \Delta V(\theta_0) &= (1+e\cos(\theta_0))\frac{\cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2}}{\sin(\theta_f-\theta_0)}, \\ \Delta V(\theta_f) &= -(1+e\cos(\theta_f))\frac{\cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2}}{\sin(\theta_f-\theta_0)}, \end{aligned} \tag{69}$$

if the conditions

$$\begin{aligned} &d_\theta<\pi \text{ and } ((\sin(\theta_0)<\sqrt{1-e^2}) \text{ or } (\sin(\theta_f)>-\sqrt{1-e^2})) \\ &\text{and} \end{aligned}$$

$$\begin{cases} \varepsilon_0=\varepsilon_f \\ \text{and} \\ 1+2e\cos(\theta_f)+\cos(\theta_f-\theta_0)\geq 0 \\ \text{and} \\ 1+2e\cos(\theta_0)+\cos(\theta_f-\theta_0)\geq 0 \\ \text{and} \\ \cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2}\neq 0 \\ \text{and} \\ \cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2}\neq 0 \end{cases} \tag{70}$$

are verified and for which the optimal Lagrange multipliers are given by:

$$\begin{aligned}\lambda_1 &= -\varepsilon_0 \frac{\cos(\theta_f)+\cos(\theta_0)+2e\cos(\theta_0)\cos(\theta_f)}{\sin(\theta_f-\theta_0)},\\ \lambda_2 &= -\varepsilon_0 \frac{\sin(\theta_f)+\sin(\theta_0)+e\sin(\theta_f+\theta_0)}{\sin(\theta_f-\theta_0)},\end{aligned} \tag{71}$$

while the optimal primer vector is defined by:

$$p(\theta) = \varepsilon_0 \frac{(1+e\cos(\theta_f))\sin(\theta-\theta_0)}{\sin(\theta_f-\theta_0)(1+e\cos(\theta))} - \varepsilon_0 \frac{\sin(\theta_f-\theta)(1+e\cos(\theta_0)}{\sin(\theta_f-\theta_0)(1+e\cos(\theta))}, \tag{72}$$

or

$$\begin{array}{l}\pi < d_\theta < 2\pi \text{ and } \varepsilon_0 = -\varepsilon_f\\ \text{and}\end{array}$$

$$\begin{cases}\cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2} \neq 0 \text{ and } \cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2} \neq 0\\ \text{and}\\ \dfrac{|\lambda_2|(1+Q^2)}{\sqrt{1+Q^2(1-e^2)}+e} < 1\end{cases} \tag{73}$$

are verified and for which the optimal Lagrange multipliers are given by:

$$\lambda_1 = \varepsilon_f \frac{\cos(\theta_f)-\cos(\theta_0)}{\sin(\theta_f-\theta_0)},\ \lambda_2 = \varepsilon_f \left(\frac{\sin(\theta_f)-\sin(\theta_0)}{\sin(\theta_f-\theta_0)}+e\right), \tag{74}$$

while the optimal primer vector is defined by:

$$p(\theta) = -\varepsilon_f \frac{\sin(\theta-\theta_0)+\sin(\theta_f-\theta)+e\cos(\theta_0)}{\sin(\theta_f-\theta_0)(1+e\cos(\theta))}, \tag{75}$$

or

$$\begin{array}{l}\pi < d_\theta < 2\pi \text{ and } \varepsilon_0 = \varepsilon_f\\ \text{and}\end{array}$$

$$\begin{cases}1+2e\cos(\theta_f)+\cos(\theta_f-\theta_0) \leq 0\\ \text{and}\\ 1+2e\cos(\theta_0)+\cos(\theta_f-\theta_0) \leq 0\\ \text{and}\\ \cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2} \neq 0 \textit{ and } \cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2} \neq 0\\ \text{and}\\ \begin{cases}|\sin(\theta_0)| < \sqrt{1-e^2} \text{ and } (e+\cos(\theta_0))(e+\cos(\theta_f)) < 0\\ \text{or}\\ \sin(\theta_0) < -\sqrt{1-e^2} \text{ and } \sin(\theta_f) > -\sqrt{1-e^2}\end{cases}\end{cases} \tag{76}$$

are verified and for which the optimal Lagrange multipliers are given by:

$$\begin{aligned}\lambda_1 &= \varepsilon_0 \frac{\cos(\theta_f)+\cos(\theta_0)+2e\cos(\theta_0)\cos(\theta_f)}{\sin(\theta_f-\theta_0)},\\ \lambda_2 &= \varepsilon_0 \frac{\sin(\theta_f)+\sin(\theta_0)+e\sin(\theta_f+\theta_0)}{\sin(\theta_f-\theta_0)},\end{aligned} \tag{77}$$

while the optimal primer vector is defined by:

$$p(\theta) = -\varepsilon_0 \frac{(1+e\cos(\theta_f))\sin(\theta-\theta_0)}{\sin(\theta_f-\theta_0)(1+e\cos(\theta))} + \varepsilon_0 \frac{\sin(\theta_f-\theta)(1+e\cos(\theta_0)}{\sin(\theta_f-\theta_0)(1+e\cos(\theta))}. \tag{78}$$

5.4.2 Case II: $\theta_f - \theta_0 = \pi$

Corollary 2 *When $d_\theta = \pi$ and $\theta_0 = -\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, the optimal solution may be concentrated on one boundary impulse or scattered in two boundary impulses. In that case, there may exist an infinite number of optimal Lagrange multipliers $(-\varepsilon_1, \lambda_2)$, $|\lambda_2| \leq e$, and the optimal directions and amplitudes of thrust are characterized by:*

$$\frac{\Delta V(\theta_0)}{|\Delta V(\theta_0)|} = \varepsilon_1 \text{ and } \frac{\Delta V(\theta_f)}{|\Delta V(\theta_f)|} = -\varepsilon_1, \tag{79}$$

The optimal consumption is given by: $|\Delta V(\theta_0)| + |\Delta V(\theta_f)| = |z_f|$.

5.5 Summary of solutions and conditions

The solutions derived in the previous subsections are quite complex due to the different cases and conditions that have to be considered. Two tables are now given to help the reader to grasp the essentials of the paper and to design the associated algorithm providing one optimal solution in every case. Table 1 summarizes the conditions to be met for each type of solution depending on the duration of the rendezvous. For a particular duration d_θ identified in the first line of Table 1, the designer will have to check which conditions involving the parameters θ_0, θ_f, e and z_f are satisfied in the corresponding column, leading to the identification of the specific case to be chosen on the corresponding row. Once the case to be dealt with is known, Table 2 gives the corresponding fuel-optimal solutions in terms of their velocity increments and the location of the optimal impulses. In the Table 1, n.a. stands for the abbreviation of not applicable.

Case \ duration	$d_\theta < \pi$	$\pi < d_\theta < 2\pi$	$d_\theta \geq 2\pi$
A 2 int. imp.	$e\|z_f\| > \|z_{f_2}\|$ and $\sin(\theta_0) \geq \sqrt{1-e^2}$ and $\sin(\theta_f) \leq -\sqrt{1-e^2}$	$e\|z_f\| > \|z_{f_2}\|$ and (33)	$e\|z_f\| > \|z_{f_2}\|$
B_1 1 int. imp.	$e\|z_f\| \leq \varepsilon_0 z_{f_2}$ and $\varepsilon_f = -\varepsilon_0$ or $e\|z_f\| > \varepsilon_0 z_{f_2}$ and $\varepsilon_f = -\varepsilon_0$ and (38)	$e\|z_f\| \leq \varepsilon_0 z_{f_2}$ and $\varepsilon_f = -\varepsilon_0$ or $e\|z_f\| > \varepsilon_0 z_{f_2}$ and $\varepsilon_f = -\varepsilon_0$ and (38)	n.a.
B_2 1 int. imp.	n.a.	$e\|z_f\| \leq \varepsilon_0 z_{f_2}$ and $\varepsilon_f = \varepsilon_0$ or $e\|z_f\| > \varepsilon_0 z_{f_2}$ and $\varepsilon_f = \varepsilon_0$ and (41)	$e\|z_f\| < \|z_{f_2}\|$
C_1 1 init. 1 int. imp.	$\sin(\theta_0) < \sqrt{1-e^2}$ and $1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) \leq 0$ and (59)	$\sin(\theta_0) < \sqrt{1-e^2}$ and $\cos(\theta_0) \leq 0$ and (61)	n.a.
C_2 1 int. imp. 1 final imp.	$\sin(\theta_f) > -\sqrt{1-e^2}$ and $1 + 2e\cos(\theta_f) + \cos(\theta_f - \theta_0) \leq 0$ and (60)	$\sin(\theta_f) > -\sqrt{1-e^2}$ and $\cos(\theta_f) \leq 0$ and (62)	n.a.
C_3 1 init. imp. 1 int. imp.	n.a.	$\sin(\theta_0) < -\sqrt{1-e^2}$ and $\cos(\theta_0) \leq 0$ and (65)	n.a.
C_4 1 int. imp. 1 final imp.	n.a.	$\sin(\theta_f) > \sqrt{1-e^2}$ and $\cos(\theta_f) \leq 0$ and (66)	n.a.
D 1 init. imp. 1 final imp.	$\sin(\theta_0) < \sqrt{1-e^2}$ and (70) or $\sin(\theta_f) > -\sqrt{1-e^2}$ and (70)	$\varepsilon_0 = -\varepsilon_f$ and (73) or $\varepsilon_0 = \varepsilon_f$ and (76)	n.a.

Table 1 Summary of the conditions/solutions depending on the duration of the rendezvous

Solutions	Velocity increments	Impulse Locations
A 2 int. imp.	$\Delta V(\theta_+) = \frac{\sqrt{1-e^2}}{2e}(-ez_{f_1} - \sqrt{1-e^2}z_{f_2})$ $\Delta V(\theta_-) = \frac{\sqrt{1-e^2}}{2e}(+ez_{f_1} - \sqrt{1-e^2}z_{f_2})$	$\theta_+ = \min\{\theta \geq \theta_0 \ / \ \begin{matrix}\cos(\theta) = -e \\ \sin(\theta) = \sqrt{1-e^2}\end{matrix}\}$ $\theta_- = \min\{\theta \geq \theta_0 \ / \ \begin{matrix}\cos(\theta) = -e \\ \sin(\theta) = -\sqrt{1-e^2}\end{matrix}\}$
B_1 1 int. imp.	$\Delta V(\theta_{i0}) = -\varepsilon_0\lvert z_f\rvert + ez_{f_2}$	$\cos(\theta_{i_0}) = -\varepsilon_0\frac{z_{f_2}}{\lvert z_f\rvert},\ \sin(\theta_{i_0}) = \varepsilon_0\frac{z_{f_1}}{\lvert z_f\rvert}$
B_2 1 int. imp.	$\Delta V(\theta_{i2}) = -\varepsilon_2\lvert z_f\rvert + ez_{f_2}$	$\cos(\theta_{i_2}) = -\varepsilon_2\frac{z_{f_2}}{\lvert z_f\rvert},\ \sin(\theta_{i_2}) = \varepsilon_2\frac{z_{f_1}}{\lvert z_f\rvert}$
C_1 1 init. imp. 1 int. imp.	$\Delta V(\theta_0) = (1+e\cos(\theta_0))\frac{\cos(\theta_{i0}^+)z_{f_1} + \sin(\theta_{i0}^+)z_{f_2}}{\sin(\theta_{i0}^+ - \theta_0)}$ $\Delta V(\theta_{i0}^+) = -(1+e\cos(\theta_{i0}^+))\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_{i0}^+ - \theta_0)}$	θ_0 $\theta_{i0}^+ = \theta_0 + \arccos(-1-2e\cos(\theta_0))$
C_2 1 int. Imp. 1 final imp.	$\Delta V(\theta_f) = (1+e\cos(\theta_f))\frac{\cos(\theta_{if}^-)z_{f_1} + \sin(\theta_{if}^-)z_{f_2}}{\sin(\theta_{if}^- - \theta_f)}$ $\Delta V(\theta_{if}^-) = -(1+e\cos(\theta_{if}^-))\frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\theta_{if}^- - \theta_f)}$	$\theta_{if}^- = \theta_f - \arccos(-1-2e\cos(\theta_f))$ θ_f
C_3 1 init. imp. 1 int. imp.	$\Delta V(\theta_0) = (1+e\cos(\theta_0))\frac{\cos(\hat{\theta}_{i0}^+)z_{f_1} + \sin(\hat{\theta}_{i0}^+)z_{f_2}}{\sin(\hat{\theta}_{i0}^+ - \theta_0)}$ $\Delta V(\hat{\theta}_{i0}^+) = -(1+e\cos(\hat{\theta}_{i0}^+))\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\hat{\theta}_{i0}^+ - \theta_\sharp)}$	θ_0 $\hat{\theta}_{i0}^+ = \theta_0 + 2\pi - \arccos(-1-2e\cos(\theta_0))$
C_4 1 int. imp. 1 final imp.	$\Delta V(\theta_f) = (1+e\cos(\theta_f))\frac{\cos(\hat{\theta}_{if}^-)z_{f_1} + \sin(\hat{\theta}_{if}^-)z_{f_2}}{\sin(\hat{\theta}_{if}^- - \theta_f)}$ $\Delta V(\hat{\theta}_{if}^-) = -(1+e\cos(\hat{\theta}_{if}^-))\frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\hat{\theta}_{if}^- - \theta_f)}$	$\hat{\theta}_{if}^- = \theta_f - 2\pi + \arccos(-1-2e\cos(\theta_f))$ θ_f
D 1 init. imp. 1 final imp.	$\Delta V(\theta_0) = (1+e\cos(\theta_0))\frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\theta_f - \theta_0)},$ $\Delta V(\theta_f) = -(1+e\cos(\theta_f))\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_f - \theta_0)}$	θ_0 and θ_f

Table 2 Summary of fuel-optimal solutions for each case

6 Numerical examples

6.1 Example 1

The first example is based on the PROBA-3 mission whose main goals are to demonstrate the technologies required for Formation Flying of two spacecraft in highly elliptical orbit [15]. This mission is made of two independent mini-satellites in HEO (Highly-elliptical Earth Orbit) in Precise Formation Flying. These two satellites are close to one another with the capacity to accurately control their attitude and separation. Among the different demonstrations scheduled for the PROBA-3 mission, rendezvous experiments will be one of the key technologies tested for on-board autonomy [3]. The necessary orbital elements and conditions for the out-of-plane rendezvous definition are given in Table 3. Two different initial state vectors of rendezvous $X_{0_1}^T$, $X_{0_2}^T$ and two different final time of rendezvous θ_{f_1}, θ_{f_2} are considered while the other parameters defining the conditions of the rendezvous remain unchanged.

a km.	e	θ_0 rad.	$X_{0_1}^T$ km m/s	$X_{0_2}^T$ km m/s	θ_{f_1} rad.	θ_{f_2} rad.	X_f^T m m/s
37039.887	0.80621	2.042	$[-5\ 0.5]$	$[-5\ 0]$	3π	4π	$[20\ 0.2]$

Table 3 Rendezvous parameters for PROBA-3 example: case 1 and 2

6.1.1 Case 1: $X_0 = X_{0_1}$ and $\theta_f = \theta_{f_1}$

The duration of the rendezvous d_θ is greater than 2π and therefore the conditions to be checked in order to identify which case will lead to the optimal solution are given in the last column of Table 1. Here,

$e|z_f| > |z_{f_2}|$ and the optimal solution corresponds to the case A. The first line of the Table 2 gives the optimal velocity increments and the optimal impulse locations.

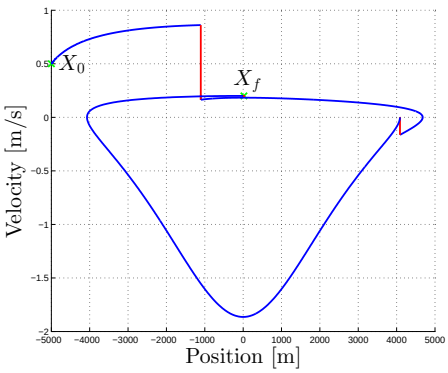
$$\begin{aligned}
\Delta V(\theta_+) &= \tfrac{\sqrt{1-e^2}}{2e}(-ez_{f_1} - \sqrt{1-e^2}z_{f_2}),\\
\Delta V(\theta_-) &= \tfrac{\sqrt{1-e^2}}{2e}(+ez_{f_1} - \sqrt{1-e^2}z_{f_2}),\\
\theta_+ &= \min\{\theta \geq \theta_0 \ / \ \cos(\theta) = -e, \ \sin(\theta) = \sqrt{1-e^2}\},\\
\theta_- &= \min\{\theta \geq \theta_0 \ / \ \cos(\theta) = -e, \ \sin(\theta) = -\sqrt{1-e^2}\}.
\end{aligned} \tag{80}$$

Applying formulae (80) leads to the numerical optimal solution in Table 4.

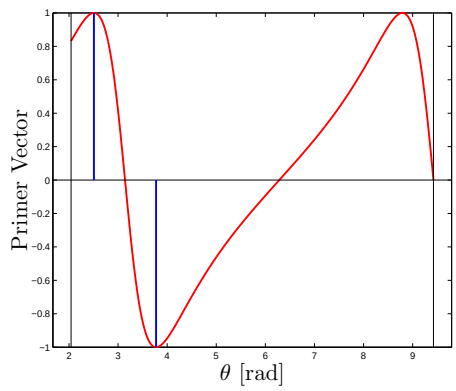
θ_1 (rad)	$\Delta V(\theta_1)$ m/s	θ_2 (rad)	$\Delta V(\theta_2)$ m/s	Fuel Cost m/s
2.5085	−0.6975	3.7747	0.1639	0.8614

Table 4 Optimal solution with a minimum number of impulses for PROBA-3 example

Figures 2(a) and 2(b) respectively depict the optimal out-of-plane trajectory in the phase plane and the optimal primer vector attached to this particular solution.



(a) Optimal trajectory in phase plane: PROBA-3.



(b) Optimal primer vector: PROBA-3.

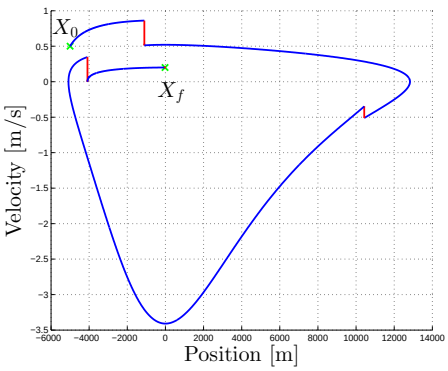
Fig. 2 Numerical Example: PROBA-3

Note that, since $d_\theta > 2\pi$, it is always possible to choose an optimal solution scattered over the maximum number of impulsive maneuvers while preserving the optimal consumption as indicated by Remark 1. This 3-impulse optimal solution is given in Table 5. Note also that $\theta_3 = \theta_1 + 2\pi$.

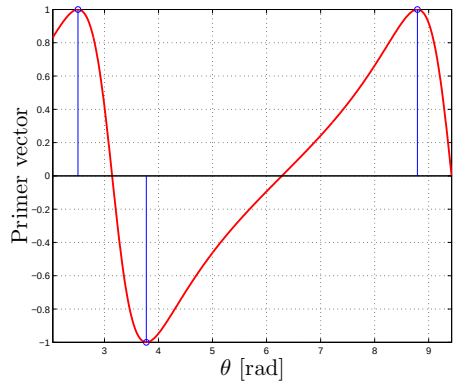
θ_1 (rad)	$\Delta V(\theta_1)$ m/s	θ_2 (rad)	$\Delta V(\theta_2)$ m/s	θ_3 (rad)	$\Delta V(\theta_3)$ m/s	Fuel Cost m/s
2.5085	−0.34875	3.7747	0.1639	8.7917	−0.34875	0.8614

Table 5 Optimal solution for PROBA-3 example

Figures 3(a) and 3(b) respectively depict the optimal out-of-plane trajectory in the phase plane and the optimal primer vector.



(a) Optimal trajectory in phase plane for 3 impulses: PROBA-3.



(b) Optimal primer vector for 3 impulses: PROBA-3.

Fig. 3 Numerical Example for 3 impulses: PROBA-3

The choice between two equivalent minimum-fuel solution with a different number of thrusts (2-impulse solution and 3-impulse solution) may be justified by different operational constraints. For instance, considering that a maximum velocity increment for this type of mission is set to 0.5 m/s, it is easily seen that the 2-impulse optimal solution would not be eligible when the 3-impulse optimal solution would be. On the contrary, if some operational constraint has to be met in the second part of the rendezvous, preventing to fire again the engine, the 2-impulse mission could be preferable if it respects the maximum Delta-V allowed by the designer.

6.1.2 Case 2: $X_0 = X_{0_2}$ and $\theta_f = \theta_{f_2}$

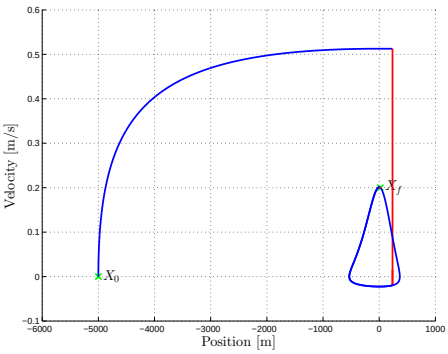
When $X_0 = X_{0_2}$ and $\theta_f = \theta_{f_2}$, the duration of the rendezvous is still such that $d_\theta > 2\pi$ but we have $e|z_f| < |z_{f_2}|$, corresponding to the case B_2 in Table 1. Table 2 indicate that the optimal solution is a 1-impulse solution defined by its velocity increment and its optimal lacation:

$$\begin{aligned} &\Delta V(\theta_{i2}) = -\varepsilon_2|z_f| + ez_{f_2}, \\ &\cos(\theta_{i_2}) = -\varepsilon_2\tfrac{z_{f_2}}{|z_f|}, \sin(\theta_{i_2}) = \varepsilon_2\tfrac{z_{f_1}}{|z_f|}. \end{aligned} \tag{81}$$

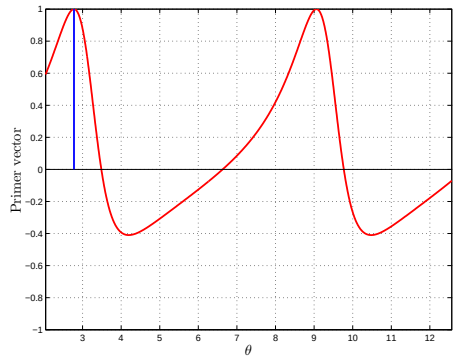
The corresponding numerical solution is given in Table 6 while the optimal out-of-plane trajectory in the phase plane and the optimal primer vector history are depicted in Figures 4(a) and 4(b) respectively.

θ (rad)	$\Delta V(\theta)$ m/s	Fuel Cost m/s
2.777	-0.5322	0.5322

Table 6 Optimal solution with a minimum number of impulses for PROBA-3 example with $X_0 = X_{0_2}$



(a) Optimal trajectory in phase plane for 1 impulse: PROBA-3.



(b) Optimal primer vector for 1 impulse: PROBA-3.

Fig. 4 Numerical Example for 1 impulse: PROBA-3

6.2 Example 2

Let consider the numerical example borrowed from the reference [22], for which the target spacecraft is in the geostationary orbit transfer (GTO). It is a highly elliptical Earth orbit with apogee of 42,164 km. The rendezvous characteristics are summarized in the Table 7.

Semi-major axis	Eccentricity	θ_0	X_0^T	θ_{f_1}	θ_{f_2}	X_f^T
$a = 24616$ km.	$e = 0.73074$	0.1π rad.	$[10\ -3]$ km m/s	5.2 rad.	3 rad.	$[0\ 0]$ m m/s

Table 7 Rendezvous parameters: [22]

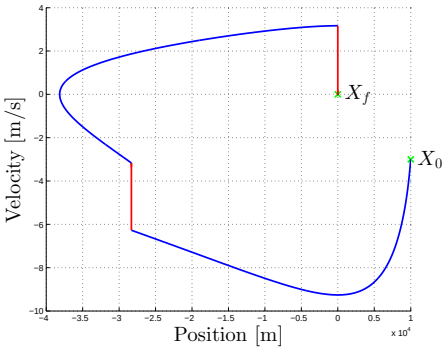
Two different durations of rendezvous are considered. The first case is directly the one presented in [22] with $\pi < d_{\theta_1} < 2\pi$ while the duration of the second rendezvous has been shortened, $d_{\theta_2} < \pi$, when the initial and final conditions remain unchanged.

6.2.1 Case 1: $\theta_f = 5.2$ rad.

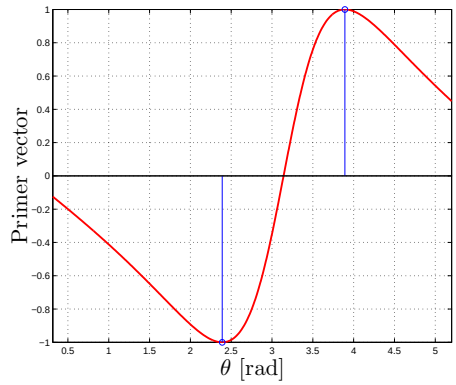
In this case, the optimal out-of-plane solution is a 2-impulse solution with initial and final coastings defined agian as the case A in Table 1. Here, the final coasting may be considered as a degenerate one since the chaser has reached the final conditions after the second maneuver.

θ_1 (rad)	$\Delta V(\theta_1)$ m/s	θ_2 (rad)	$\Delta V(\theta_2)$ m/s	Fuel Cost m/s
2.3902	3.1060	3.8930	-3.1668	6.2728

Table 8 Optimal solution for Example 2 and θ_{f_1}



(a) Optimal trajectory in phase plane for θ_{f_1}.



(b) Optimal primer vector for θ_{f_1}.

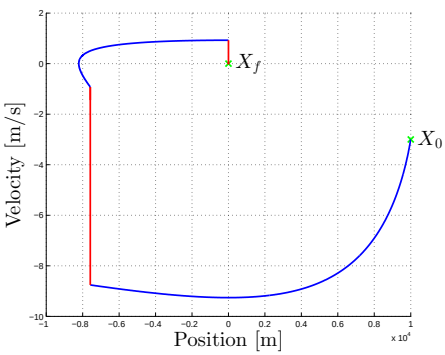
Fig. 5 Numerical Example 2 for θ_{f_1}

6.2.2 Case 2: $\theta_f = 3$ rad.

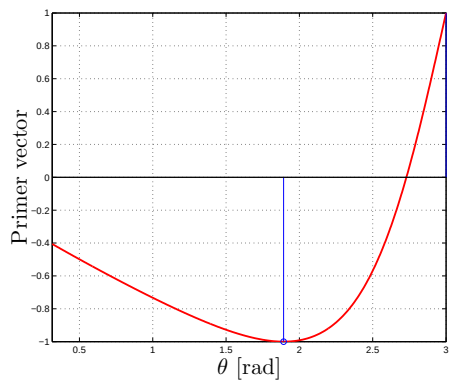
The particular nature of the previous optimal solution is further illustrated by the following case where the duration of the rendezvous has been shortened resulting in a 2-impulse optimal solution with an initial coasting and a final impulse and detected as case C_2 in Table 1. For this case, using formulae given in Table 2 in the corresponding row, we get the numerical results given in Table 9.

θ_1 (rad)	$\Delta V(\theta_1)$ m/s	θ_2 (rad)	$\Delta V(\theta_2)$ m/s	Fuel Cost m/s
1.8924	7.8311	3	-0.9261	8.7572

Table 9 Optimal solution for θ_{f_2}



(a) Optimal trajectory in phase plane for θ_{f_2}.



(b) Optimal primer vector for θ_{f_2}.

Fig. 6 Numerical Example 2 for θ_{f_2}

The reduction of the duration of the rendezvous obviously results in increasing the consumption of almost 40 %.

7 Conclusions

The problem of time-fixed fuel-optimal out-of-plane elliptic rendezvous between spacecraft in a linear setting was still an open problem. This paper presents a new complete analytical closed-form solution to address this problem. It is mainly based on the so-called primer vector theory which is used to capture all possible optimal solutions for any duration and any boundary conditions of the rendezvous. The

derivations of these results are based on the particular properties of the primer vector and on the thorough analysis of the possible primer vector candidates that have to meet the optimality conditions. Depending upon the duration of the rendezvous, conditions are derived to identify if the optimal solutions consist in one interior impulse, two interior impulses, one interior and one boundary impulses or one single boundary solution. In addition, it is shown that when the rendezvous may last more than one revolution, the designer has extra degrees of freedom allowing to split the optimal consumption over a maximum number of impulses that may be greater than the usual upper-bound of Neustadt. Despite its apparent complexity (high number of different cases and conditions), this analytical solution paves the way for onboard implementation in order to develop operational autonomy of future missions. Indeed, the optimal solutions are directly expressed in terms of the data of the problem (eccentricity, initial and final anomaly, boundary conditions, duration of the rendezvous) and do not require time-consuming computations to be obtained.

In addition, it is worth noticing that similar analytical results are likely to be provided for non elliptic Keplerian reference orbits (parabolic $e = 1$ and hyperbolic $e > 1$ using identical derivations and might be an interesting topic in the future.

A Proof of Proposition 1

The case where the rendezvous lasts more than 2π is first considered. Consistent with the notation for the true anomalies θ_+ and θ_-, let:

$$N_\pm = \max\left\{i \in \mathbb{N}_{>0} \ : \ \theta_\pm + 2(i-1)\pi \leq \theta_f\right\}, \ N^* = N_+ + N_-. \tag{82}$$

Due to optimality conditions (6) and (7), the optimal impulses could only occur at the extrema locations in $[\theta_0, \theta_f]$ of the primer vector. From the extremum ratio (18), it is observed that θ_+ and θ_- are necessarily the locations in $[\theta_0, \theta_0 + 2\pi)$ of the extrema of the primer vector:

$$|p(\theta_\pm)| = 1 \ \frac{\mathrm{d}p}{\mathrm{d}\theta}(\theta_\pm) = 0. \tag{83}$$

Any optimal solution with two interior impulses per period, is composed of at most N^* optimal impulses, located at $\theta_i \in \left\{\theta_i^+ \ : \ i = 1, \cdots, N_+\right\} \cup \left\{\theta_i^- \ : \ i = 1, \cdots, N_-\right\}$ where:

$$\theta_i^\pm = \theta_\pm + 2(i-1)\pi, \ i = 1, \cdots, N_\pm. \tag{84}$$

By the way, from (12) and (83), the second optimal Lagrange multiplier is obtained as:

$$\lambda_2^* = 0. \tag{85}$$

The optimality condition (5) leads to the optimal directions of thrust:

$$\forall i = 1, \cdots, N_+, \ \frac{\Delta V(\theta_i^+)}{\left|\Delta V(\theta_i^+)\right|} = \frac{\Delta V(\theta_+)}{|\Delta V(\theta_+)|} = -\frac{\lambda_1}{\sqrt{1-e^2}} = \varepsilon. \tag{86}$$

$$\forall i = 1, \cdots, N_-, \ \frac{\Delta V(\theta_i^-)}{\left|\Delta V(\theta_i^-)\right|} = \frac{\Delta V(\theta_-)}{|\Delta V(\theta_-)|} = \frac{\lambda_1}{\sqrt{1-e^2}} = -\varepsilon. \tag{87}$$

(86) and (87) indicate that the optimal directions are alternating between θ_i^+ and θ_i^-. Condition (8) leads to the following system of equations:

$$\begin{bmatrix} \sqrt{1-e^2} \\ e \end{bmatrix} \sum_{i=1}^{N_+} |\Delta V(\theta_i^+)| + \begin{bmatrix} \sqrt{1-e^2} \\ -e \end{bmatrix} \sum_{i=1}^{N_-} |\Delta V(\theta_i^-)| = \varepsilon z_f (1-e^2), \tag{88}$$

from which may be easily computed the optimal consumption:

$$\sum_{i=1}^{N_+} |\Delta V(\theta_i^+)| = \varepsilon \frac{\sqrt{1-e^2}}{2e} \left(e z_{f_1} + \sqrt{1-e^2} z_{f_2}\right), \tag{89}$$

$$\sum_{i=1}^{N_-} |\Delta V(\theta_i^-)| = \varepsilon \frac{\sqrt{1-e^2}}{2e} \left(e z_{f_1} - \sqrt{1-e^2} z_{f_2}\right), \tag{90}$$

$$\sum_{i=1}^{N_+} |\Delta V(\theta_i^+)| + \sum_{i=1}^{N_-} |\Delta V(\theta_i^-)| = \varepsilon z_{f_1} \sqrt{1-e^2}. \tag{91}$$

From the positive sign of (89) and (90), it is deduced that:

$$\varepsilon = \mathrm{sgn}(z_{f_1}) = \varepsilon_1. \tag{92}$$

$(ez_{f_1} + \sqrt{1-e^2}z_{f_2})$ and $(ez_{f_1} - \sqrt{1-e^2}z_{f_2})$ are necessarily not equal to 0 and have the same sign which is equivalent to:

$$e|z_f| > |z_{f_2}|. \tag{93}$$

Indeed, if $e|z_f| = |z_{f_2}|$, the optimal solution comes down to one interior impulse solution as described in the next section, with at most one positive or negative impulse per period. The first optimal Lagrange multiplier is then obtained from (5) as:

$$\lambda_1^* = -\varepsilon_1\sqrt{1-e^2}. \tag{94}$$

In conclusion, when $d_\theta \geq 2\pi$, the optimal solution of the planning may be chosen to be concentrated over two impulses, as is presented in Proposition 1 or spread over $N^* = N_- + N_+$ impulses verifying (89) and (90), depending on the duration of the rendezvous and operational constraints. Finally, the optimal consumption is defined by $|z_{f_1}|\sqrt{1-e^2}$.

Let now consider the case where the rendezvous lasts less than 2π i.e. $d_\theta < 2\pi$. The discussion is very similar to the case where $\theta_f - \theta_0 \geq 2\pi$ and leads to the conditions w.r.t. $(z_f, \theta_0, \theta_f)$:

$$e|z_f| > |z_{f_2}| \text{ and } (\theta_+, \theta_-) \in [\theta_0, \theta_f]^2. \tag{95}$$

- **Case** $d_\theta < \pi$. There exists $\theta_+,\ \theta_- \in [\theta_0, \theta_f]$ such that

$$\cos(\theta_\pm) = -e \text{ and } \begin{cases} \sin(\theta_+) = \sqrt{1-e^2} \\ \sin(\theta_-) = -\sqrt{1-e^2} \end{cases} \tag{96}$$

  if and only if

$$\sin(\theta_0) \geq \sqrt{1-e^2} \text{ and } \sin(\theta_f) \leq -\sqrt{1-e^2}$$

- **Case** $\pi < d_\theta < 2\pi$. There exists $\theta_+,\ \theta_- \in [\theta_0, \theta_f]$ such that (96) if and only if

$$\begin{cases} \sin(\theta_0) \geq \sqrt{1-e^2} \\ \text{or} \\ \sin(\theta_0) \leq \sqrt{1-e^2} \text{ and } \sin(\theta_f) \leq -\sqrt{1-e^2} \\ \text{or} \\ |\sin(\theta_0)| < \sqrt{1-e^2} \text{ and } (e+\cos(\theta_0))(e+\cos(\theta_f)) > 0 \end{cases}$$

Under these conditions, the optimal solution is a 2-impulse solution defined by the optimal locations θ_+ and θ_- and:

$$\Delta V(\theta_\pm) = \frac{\sqrt{1-e^2}}{2e}(\mp ez_{f_1} - \sqrt{1-e^2}z_{f_2}),\ \lambda_1^* = -\varepsilon_1\sqrt{1-e^2},\ \lambda_2^* = 0. \tag{97}$$

In both cases the optimal primer vector is given by:

$$p^*(\theta) = \varepsilon_1 \frac{\sqrt{1-e^2\sin(\theta)}}{1+e\cos(\theta)}. \tag{98}$$

B Proof of Proposition 2 and Corollary 1

B.1 Proof of Proposition 2

B.1.1 Proof of Proposition 2 when $d_\theta \geq 2\pi$

Optimal solutions with one impulse, at most, per period are looked for: One extremum of the primer vector has a unit norm and the other one has a norm strictly less than 1.

Let define:

$$\theta_{i_0} = \min\left\{\theta \in [\theta_0, \theta_f] :\ |p(\theta)| = 1 \text{ and } \frac{\mathrm{d}p}{\mathrm{d}\theta}(\theta) = 0\right\}. \tag{99}$$

as the first true anomaly in $[\theta_0, \theta_f]$ for which $|p(\theta_{i_0})| = 1$. From the extremum ratio (18), $\cos(\theta_{i_0}) < -e$. The optimal solution will exhibit, at most, N^* impulses, located at

$$\theta_i = \theta_{i_0} + 2(i-1)\pi,\ i = 1, \cdots, N^*, \tag{100}$$

where N^* is defined by:

$$N^* = \max\left\{i \in \mathbb{N}^* \ : \ \theta_{i_0} + 2(i-1)\pi \leq \theta_f\right\}. \tag{101}$$

Applying (5), the direction of the optimal thrusts are identical and may be deduced as:

$$\frac{\Delta V(\theta_i)}{|\Delta V(\theta_i)|} = -p(\theta_i) = -p(\theta_{i_0}) = \frac{\Delta V(\theta_{i_0})}{|\Delta V(\theta_{i_0})|} = \varepsilon. \tag{102}$$

Using (102), the optimality condition (8) leads to the following equation:

$$\begin{bmatrix} -\sin(\theta_{i_0}) \\ \cos(\theta_{i_0}) \end{bmatrix} \sum_{i=1}^{N_*} |\Delta V(\theta_i)| = \varepsilon z_f r(\theta_{i_0}). \tag{103}$$

Multiplying the left of (103) by the invertible matrix $\begin{bmatrix} -\sin(\theta_{i_0}) & \cos(\theta_{i_0}) \\ \cos(\theta_{i_0}) & \sin(\theta_{i_0}) \end{bmatrix}$, the following system of equations is obtained:

$$\begin{cases} \displaystyle\sum_{i=1}^{N^*} |\Delta V(\theta_i)| = \varepsilon\left(\cos(\theta_{i_0}) z_{f_2} - \sin(\theta_{i_0}) z_{f_1}\right) r(\theta_{i_0}), \\ \cos(\theta_{i_0}) z_{f_1} + \sin(\theta_{i_0}) z_{f_2} = 0. \end{cases} \tag{104}$$

Equivalently, the following system is obtained:

$$\begin{cases} \displaystyle\sum_{i=1}^{N^*} |\Delta V(\theta_i)| = |\cos(\theta_{i_0}) z_{f_2} - \sin(\theta_{i_0}) z_{f_1}|\, r(\theta_{i_0}), \\ \varepsilon = \operatorname{sgn}\left(-\sin(\theta_{i_0}) z_{f_1} + \cos(\theta_{i_0}) z_{f_2}\right) \\ \cos(\theta_{i_0}) z_{f_1} + \sin(\theta_{i_0}) z_{f_2} = 0. \end{cases} \tag{105}$$

Working out (105) and keeping in mind that $\cos\theta_{i_0} < -e$, the optimal direction of impulses and the optimal consumption are given by:

$$\varepsilon = -\operatorname{sgn}(z_{f_2}) = -\varepsilon_2, \quad \sum_{i=1}^{N^*} |\Delta V(\theta_i)| = |z_f| - e|z_{f_2}|. \tag{106}$$

Additionally, θ_{i_0} is characterized via the second equation of (104) as:

$$\begin{cases} \cos(\theta_{i_0}) = -\dfrac{|z_{f_2}|}{|z_f|}, \ \sin(\theta_{i_0}) = \varepsilon_2 \dfrac{z_{f_1}}{|z_f|}, \\ \theta_0 \leq \theta_{i_0} < \theta_0 + 2\pi. \end{cases} \tag{107}$$

Note that the last equation of (106) means that all N^* optimal impulses may be chosen arbitrarily as long as their absolute value verifies this equation. Depending on the operational constraints, the optimal solution may be reduced to a single impulse at θ_{i_0}, $\Delta V(\theta_{i_0}) = -\varepsilon_2(|z_f| - e|z_{f_2}|)$ or spread over N^* impulses located at θ_i given by (100) with the constraint (106).

In addition, it is worthwhile to remark that $\cos(\theta_{i_0}) < -e$ may be expressed in terms of z_f as:

$$|z_{f_2}| > e|z_f|. \tag{108}$$

Finally, the optimal Lagrange multipliers are defined by:

$$\lambda_1^* = -\frac{z_{f_1}}{|z_f|}, \ \lambda_2^* = \varepsilon_2\left(e - \frac{|z_{f_2}|}{|z_f|}\right). \tag{109}$$

B.1.2 Proof of Proposition 2 when $d_\theta < 2\pi$

Let θ_{i_0} denote the extremum anomaly in $[\theta_0, \theta_f]$. Assume that there is no other unit norm value. In this case, the optimality conditions (5)-(7) give:

$$\lambda_1 = \varepsilon \sin(\theta_{i_0}),\ \lambda_2 = -\varepsilon(e + \cos(\theta_{i_0})) \tag{110}$$

where $\varepsilon = \frac{\Delta V(\theta_{i_0})}{|\Delta V(\theta_{i_0})|}$ denotes the optimal direction of thrust at θ_{i_0}. The optimality condition (8) is equivalent to:

$$\Delta V(\theta_{i_0}) = (-\sin(\theta_{i_0}) z_{f_1} + \cos(\theta_{i_0}) z_{f_2}) r(\theta_{i_0}), \tag{111}$$

$$\cos(\theta_{i_0}) z_{f_1} + \sin(\theta_{i_0}) z_{f_2} = 0. \tag{112}$$

It follows:

$$\varepsilon = \mathrm{sgn}(-\sin(\theta_{i_0}) z_{f_1} + \cos(\theta_{i_0}) z_{f_2}). \tag{113}$$

Observe that the objective equation (112) has a unique solution θ_{i_0} on $[\theta_0, \theta_f]$ if:

$$(\cos(\theta_0) z_{f_1} + \sin(\theta_0) z_{f_1})(\cos(\theta_f) z_{f_1} + \sin(\theta_f) z_{f_2}) < 0. \tag{114}$$

Otherwise, it has no solution on $[\theta_0, \theta_f]$ if $\theta_f - \theta_0 < \pi$ and two solutions on $[\theta_0, \theta_f]$ if $\pi \leq \theta_f - \theta_0 < 2\pi$.

First case. Let assume that condition (114) holds i.e. $\varepsilon_0 = -\varepsilon_f$. Then the extremum anomaly θ_{i_0} is defined by:

$$\cos(\theta_{i_0}) = -\varepsilon_0 \frac{z_{f_2}}{|z_f|},\ \sin(\theta_{i_0}) = \varepsilon_0 \frac{z_{f_1}}{|z_f|}. \tag{115}$$

The optimal thrust is then defined by:

$$\Delta V(\theta_{i_0}) = -\varepsilon_0 |z_f| + e z_{f_2} \tag{116}$$

whose direction is given by: $\varepsilon = -\varepsilon_0$. The Lagrange multipliers are uniquely determined:

$$\lambda_1 = -\frac{z_{f_1}}{|z_f|},\ \lambda_2 = \varepsilon_0 e - \frac{z_{f_2}}{|z_f|}. \tag{117}$$

It remains to guarantee that the primer vector norm satisfies: $|p(\theta)| \leq 1,\ \forall \theta \in [\theta_0, \theta_f]$. First, note that if $\cos(\theta_{i_0}) + e \leq 0$, or equivalently if:

$$e|z_f| \leq \varepsilon_0 z_{f_2}, \tag{118}$$

then by (18), the second extremum norm of the primer vector is less than 1 and it is automatically satisfied. So, condition (37) is demonstrated. Reversely, suppose now that $e|z_f| > \varepsilon_0 z_{f_2}$. The second extremum norm is greater than 1 and additional conditions are needed to ensure: $|p(\theta)| \leq 1,\ \forall \theta \in [\theta_0, \theta_f]$, namely:

$$\begin{cases} |p(\theta_0)| < 1,\ \mathrm{sgn}(p'(\theta_0)) = \mathrm{sgn}(p(\theta_{i_0})) = -\varepsilon, \\ |p(\theta_f)| < 1,\ \mathrm{sgn}(p'(\theta_f)) = -\mathrm{sgn}(p(\theta_{i_0})) = \varepsilon. \end{cases} \tag{119}$$

Equivalently, the following systems of inequalities are obtained:

$$\begin{cases} \cos(\theta_{i_0} - \theta_0) + 2e\cos(\theta_0) + 1 > 0 \\ \sin(\theta_{i_0} - \theta_0) + e\left(\sin(\theta_{i_0}) - \sin(\theta_0)\right) > 0 \end{cases} \tag{120}$$

$$\begin{cases} \cos(\theta_{i_0} - \theta_f) + 2e\cos(\theta_f) + 1 > 0 \\ \sin(\theta_{i_0} - \theta_f) + e\left(\sin(\theta_{i_0}) - \sin(\theta_f)\right) < 0, \end{cases} \tag{121}$$

which, by (115), are also equivalent to:

$$\begin{cases} |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_0) + \varepsilon_0 z_{f_1}\sin(\theta_0) > 0 \\ \varepsilon_0(e + \cos(\theta_0)) z_{f_1} + (\varepsilon_0 z_{f_2} - e|z_f|)\sin(\theta_0) > 0 \end{cases} \tag{122}$$

$$\begin{cases} |z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_f) + \varepsilon_0 z_{f_1}\sin(\theta_f) > 0 \\ \varepsilon_0(e + \cos(\theta_f)) z_{f_1} + (\varepsilon_0 z_{f_2} - e|z_f|)\sin(\theta_f) < 0. \end{cases} \tag{123}$$

Here, conditions (38) are retrieved.

Second case: Suppose now that condition (114) does not hold i.e.: $\varepsilon_0 = \varepsilon_f$ and $d_\theta = \theta_f - \theta_0 \geq \pi$. Then, the objective equation (112) is equivalent to:

$$\cos(\theta_{i_0}) = \pm\frac{z_{f_2}}{|z_f|},\ \sin(\theta_{i_0}) = \mp\frac{z_{f_1}}{|z_f|} \tag{124}$$

and: $\Delta V(\theta_{i_0}) = \pm|z_f| + ez_{f_2}$. The minimum consumption solution is given by:

$$\Delta V(\theta_{i_0}) = -\varepsilon_2|z_f| + ez_{f_2} \tag{125}$$

whose direction is: $\varepsilon = -\varepsilon_2$. The optimal impulse is located at θ_{i_0} defined by:

$$\cos(\theta_{i_0}) = -\varepsilon_2\frac{z_{f_2}}{|z_f|},\ \sin(\theta_{i_0}) = \varepsilon_2\frac{z_{f_1}}{|z_f|}. \tag{126}$$

The Lagrange multipliers are uniquely determined:

$$\lambda_1 = -\frac{z_{f_1}}{|z_f|},\ \lambda_2 = \varepsilon_2 e - \frac{z_{f_2}}{|z_f|}. \tag{127}$$

It remains to guarantee that the norm of the primer vector satisfies: $|p(\theta)| \leq 1,\ \forall\theta \in [\theta_0, \theta_f]$. As has been done in the first case, note that if $\cos(\theta_{i_0}) + e \leq 0$, or equivalently if:

$$e|z_f| \leq |z_{f_2}| \tag{128}$$

then it is automatically satisfied. The condition (40) is recovered. Reversely, if $e|z_f| > |z_{f_2}|$ then additional conditions are needed to ensure: $|p(\theta)| \leq 1\ \forall\theta \in [\theta_0, \theta_f]$, namely:

$$\begin{cases} |p(\theta_0)| < 1,\ \mathrm{sgn}(p'(\theta_0)) = \mathrm{sgn}(p(\theta_{i_0})) = -\varepsilon, \\ |p(\theta_f)| < 1,\ \mathrm{sgn}(p'(\theta_f)) = -\mathrm{sgn}(p(\theta_{i_0})) = \varepsilon, \end{cases} \tag{129}$$

which, by (124), are also equivalent to:

$$\begin{cases} |z_f| + (2e|z_f| - \varepsilon_2 z_{f_2})\cos(\theta_0) + \varepsilon_2 z_{f_1}\sin(\theta_0) > 0 \\ \varepsilon_2 z_{f_1}(e + \cos(\theta_0)) - (e|z_f| - \varepsilon_2 z_{f_2})\sin(\theta_0) > 0 \end{cases} \tag{130}$$

$$\begin{cases} |z_f| + (2e|z_f| - \varepsilon_2 z_{f_2})\cos(\theta_f) + \varepsilon_2 z_{f_1}\sin(\theta_f) > 0 \\ -\varepsilon_2 z_{f_1}(e + \cos(\theta_f)) + (e|z_f| - \varepsilon_2 z_{f_2})\sin(\theta_f) > 0. \end{cases} \tag{131}$$

In that case, conditions (41) are obtained.

B.2 Proof of Corollary 1

The proof is elementary and follows the same lines as the proof of Proposition 2 where it is only necessary to notice that the particular choice of λ_1 and λ_2 as in (42) verifies the optimality conditions:

$$\lambda_1\sin(\theta_0) - \lambda_2\cos(\theta_0) = \mathrm{sgn}(-\sin(\theta_0)z_{f_1} + \cos(\theta_0)z_{f_2})(1 + e\cos(\theta_0)) \tag{132}$$

$$|p(\theta)| < 1,\ \text{for all } \theta \in (\theta_0, \theta_f] \tag{133}$$

C Proof of Proposition 3

Assume that:

$$\theta_{i_0} = \theta_0 + \arccos(-(1 + 2e\cos(\theta_0))) \in [\theta_0, \theta_0 + \pi] \tag{134}$$

which is possible if and only if:

$$\cos(\theta_0) \leq 0 \text{ and } \begin{cases} \theta_f - \theta_0 < \pi \text{ and } 1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) \leq 0 \\ \text{or} \\ \theta_f - \theta_0 \geq \pi. \end{cases} \tag{135}$$

In that case, the objective equation given by (8) is equivalent to:

$$\Delta V(\theta_0) = \frac{\cos(\theta_{i_0})z_{f_1} + \sin(\theta_{i_0})z_{f_2}}{\sin(\theta_{i_0} - \theta_0)}(1 + e\cos(\theta_0)) \tag{136}$$

$$\Delta V(\theta_{i_0}) = -\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_{i_0} - \theta_0)}(1 + e\cos(\theta_{i_0})). \tag{137}$$

By definition: $\sin(\theta_{i_0} - \theta_0) > 0$, hence:

$$\varepsilon = -\text{sgn}(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}) = -\varepsilon_0. \tag{138}$$

Combining (134) and (51), the Lagrange multipliers are uniquely determined by:

$$\lambda_1 = \varepsilon_0 \left(\sin(\theta_0)(1 + 2e\cos(\theta_0)) - 2\cos(\theta_0)\sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}\right) \tag{139}$$

$$\lambda_2 = \varepsilon_0 \left(e - \cos(\theta_0)(1 + 2e\cos(\theta_0)) - 2\sin(\theta_0)\sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}\right), \tag{140}$$

The sign condition (50) on impulses leads to:

$$\text{sgn}(\cos(\theta_{i_0})z_{f_1} + \sin(\theta_{i_0})z_{f_2}) = \text{sgn}(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}) \tag{141}$$

Observe that if $\theta_f - \theta_0 < \pi$ and $\varepsilon_0 = \varepsilon_f$, then the sign of the function: $\theta \mapsto \cos(\theta)z_{f_1} + \sin(\theta)z_{f_2}$ does not change on $[\theta_0, \theta_f]$ and the condition (141) is automatically satisfied. Otherwise, the previous statement is equivalent to the fact that the smallest anomaly $\bar{\theta}$ on $[\theta_0, \theta_f]$ where the function $\theta \mapsto \cos(\theta)z_{f_1} + \sin(\theta)z_{f_2}$ changes sign, is defined by:

$$\cos(\bar{\theta}) = -\varepsilon_0\frac{z_{f_2}}{|z_f|}, \ \sin(\bar{\theta}) = \varepsilon_0\frac{z_{f_1}}{|z_f|} \tag{142}$$

and is greater than θ_{i_0}. Thus, this is equivalent to state that the angle $\bar{\theta}$ makes the function $\theta \mapsto 1 + 2e\cos(\theta_0) + \cos(\theta - \theta_0)$ negative i.e.:

$$|z_f| + (2e|z_f| - \varepsilon_0 z_{f_2})\cos(\theta_0) + \varepsilon_0 z_{f_1}\sin(\theta_0) \leq 0. \tag{143}$$

Lastly, it remains to ensure that $|p(\theta)| \leq 1, \ \forall\theta \in [\theta_0, \theta_f]$. Since the primer vector norm reaches 1 at θ_0, the other extremum norm is strictly greater than 1 and additional conditions are needed at θ_0 and θ_f, namely:

$$\text{sgn}(p(\theta_0)) = -\text{sgn}(p'(\theta_0)) = \text{sgn}(p'(\theta_f)) \text{ and } |p(\theta_f)| \leq 1 \tag{144}$$

Using (52), these conditions are respectively equivalent to:

$$e(\sin(\theta_{i_0}) - \sin(\theta_0)) + \sin(\theta_{i_0} - \theta_0) > 0, \tag{145}$$

$$e(\sin(\theta_{i_0}) - \sin(\theta_f)) + \sin(\theta_{i_0} - \theta_f) < 0. \tag{146}$$

and

$$\cos(\theta_f - \theta_{i_0}) + 2e\cos(\theta_f) + 1 \geq 0. \tag{147}$$

By (134), the condition (145) becomes: $\sin(\theta_{i_0} - \theta_0) - 2e\sin(\theta_0) > 0$. Keeping in mind the fact that $\cos(\theta_0) \leq 0$, it is equivalent to:

$$\sin(\theta_0) < \sqrt{1 - e^2}. \tag{148}$$

Using a similar approach, the conditions (146) and (147) can be expressed as:

$$1 - \cos(\theta_f - \theta_0) - 2\sin(\theta_f - \theta_0)(e\sin(\theta_0) - \sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}) \geq 0, \tag{149}$$

$$(1+e\cos(\theta_0))(\sin(\theta_f-\theta_0)-e\sin(\theta_0))+(\cos(\theta_f-\theta_0)+e\cos(\theta_0))(\sqrt{-e\cos(\theta_0)(1+e\cos(\theta_0))}-e\sin(\theta_0)) \leq 0 \tag{150}$$

and are automatically satisfied when $\theta_f - \theta_0 < \pi$.

D Proof of Proposition 4

Assume that:

$$\theta_{i_0} = \theta_0 + 2\pi - \arccos(-(1 + 2e\cos(\theta_0))) \tag{151}$$

which is only possible when:

$$\theta_f - \theta_0 \geq \pi \quad \text{and} \quad 1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0) \geq 0. \tag{152}$$

The objective equation given by (8) is equivalent to:

$$\Delta V(\theta_0) = \frac{\cos(\theta_{i_0})z_{f_1} + \sin(\theta_{i_0})z_{f_2}}{\sin(\theta_{i_0} - \theta_0)}(1 + e\cos(\theta_0)) \tag{153}$$

$$\Delta V(\theta_{i_0}) = -\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_{i_0} - \theta_0)}(1 + e\cos(\theta_{i_0})). \tag{154}$$

By definition, $\sin(\theta_{i_0} - \theta_0) < 0$, hence:

$$\varepsilon = \text{sgn}(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}) = \varepsilon_0. \tag{155}$$

Combining (151) and (51), the Lagrange multipliers are uniquely determined by:

$$\lambda_1 = -\varepsilon_0 \left(\sin(\theta_0)(1 + 2e\cos(\theta_0)) + 2\cos(\theta_0)\sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}\right) \tag{156}$$

$$\lambda_2 = -\varepsilon_0 \left(e - \cos(\theta_0)(1 + 2e\cos(\theta_0)) + 2\sin(\theta_0)\sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}\right). \tag{157}$$

The sign condition on the impulses leads to:

$$\text{sgn}(\cos(\theta_{i_0})z_{f_1} + \sin(\theta_{i_0})z_{f_2}) = \text{sgn}(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}). \tag{158}$$

As in case I, the previous statement is equivalent to the fact that the biggest anomaly on $[\theta_0, \theta_0 + 2\pi[$ where the sign of the function $\theta \mapsto \cos(\theta)z_{f_1} + \sin(\theta)z_{f_2}$ changes, is smaller than θ_{i_0} i.e.:

$$|z_f| + (2e|z_f| + \varepsilon_0 z_{f_2})\cos(\theta_0) - \varepsilon_0 z_{f_1}\sin(\theta_0) \leq 0. \tag{159}$$

Lastly, the fact that $|p(\theta)| \leq 1 \; \forall\theta \in [\theta_0, \theta_f]$, is equivalent to:

$$\text{sgn}(p(\theta_0)) = -\text{sgn}(p'(\theta_0)) = \text{sgn}(p'(\theta_f)) \text{ and } |p(\theta_f)| \leq 1. \tag{160}$$

Keeping in mind the fact that $\cos(\theta_0) \leq 0$, it is equivalent to:

$$\sin(\theta_0) \leq -\sqrt{1 - e^2} \tag{161}$$

$$(1+e\cos(\theta_0))(\sin(\theta_f-\theta_0)-e\sin(\theta_0))+(\cos(\theta_f-\theta_0)+e\cos(\theta_0))(-\sqrt{-e\cos(\theta_0)(1+e\cos(\theta_0))}-e\sin(\theta_0)) \leq 0 \tag{162}$$

$$1 - \cos(\theta_f - \theta_0) - 2\sin(\theta_f - \theta_0)(e\sin(\theta_0) + \sqrt{-e\cos(\theta_0)(1 + e\cos(\theta_0))}) \geq 0. \tag{163}$$

E Proof of Proposition 5 and Corollary 2

E.1 Proof of Proposition 5

Let first consider the objective equation given by the optimality condition (8). Provided that $d_\theta = \theta_f - \theta_0 \neq \pi$, this condition is equivalent to:

$$\begin{aligned} \Delta V(\theta_0) &= \frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\theta_f - \theta_0)}(1 + e\cos(\theta_0)) \\ \Delta V(\theta_f) &= -\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_f - \theta_0)}(1 + e\cos(\theta_f)). \end{aligned} \tag{164}$$

First case: Assume that:

$$\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2} \neq 0 \text{ and} \cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}) \neq 0, \tag{165}$$

The two impulses are non-zero and the primer vector norm reaches 1 both at θ_0 and θ_f. By (164), the optimal directions of the impulses are given by:

$$\frac{\Delta V(\theta_0)}{|\Delta V(\theta_0)|} = \text{sgn}\left(\frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\theta_f - \theta_0)}\right) \tag{166}$$

$$\frac{\Delta V(\theta_f)}{|\Delta V(\theta_f)|} = -\text{sgn}\left(\frac{\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}}{\sin(\theta_f - \theta_0)}\right) \tag{167}$$

First sub-case Suppose that the two impulses have the same direction:

$$\text{sgn}(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2}) = -\text{sgn}(\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}) \tag{168}$$

i.e.: $\varepsilon_0 = -\varepsilon_f$. Then by (5) and (6), the Lagrange multipliers are uniquely determined:

$$\lambda_1 = \varepsilon_f \frac{\cos(\theta_f) - \cos(\theta_0)}{\sin(\theta_f - \theta_0)},\ \lambda_2 = \varepsilon_f(\frac{\sin(\theta_f) - \sin(\theta_0)}{\sin(\theta_f - \theta_0)} + e) \tag{169}$$

Then, the condition $|p(\theta)| \leq 1,\ \forall\theta \in [\theta_0, \theta_f]$ has to be fulfilled. Since the two boundary impulses have the same direction, the primer vector necessarily decreases and then increases on $[\theta_0, \theta_f]$ so that it reaches a global extremum with norm less than 1. The fact that $|p(\theta)| \leq 1,\ \forall\theta \in [\theta_0, \theta_f]$, is then equivalent to:

$$\text{sgn}(p'(\theta_0)) = -\text{sgn}(p(\theta_0)) = -\varepsilon_f \tag{170}$$

$$\text{sgn}(p'(\theta_f)) = \text{sgn}(p(\theta_f)) = \varepsilon_f \tag{171}$$

$$\min_{\theta\in\mathbb{R}} |p(\theta)| < 1. \tag{172}$$

The first two conditions could be expressed in terms of Lagrange multipliers and by (169) are finally equivalent to $\sin(\theta_f - \theta_0) < 0$. Hence:

$$d_\theta = \theta_f - \theta_0 > \pi. \tag{173}$$

By (25), the third condition (172) is equivalent to:

$$\frac{|\lambda_2|(1 + Q^2)}{\sqrt{1 + Q^2(1 - e^2)} + e} < 1. \tag{174}$$

The conditions (73) are retrieved.

Second sub-case Suppose that the sign condition (168) does not hold i.e.: $\varepsilon_0 = \varepsilon_f$. Then, the two impulses have an opposite direction. By (5) and (6), the Lagrange multipliers are uniquely determined:

$$\lambda_1 = -\varepsilon\frac{\cos(\theta_f) + \cos(\theta_0) + 2e\cos(\theta_0)\cos(\theta_f)}{\sin(\theta_f - \theta_0)},\ \lambda_2 = -\varepsilon\frac{\sin(\theta_f) + \sin(\theta_0) + e\sin(\theta_0 + \theta_f)}{\sin(\theta_f - \theta_0)} \tag{175}$$

where

$$\varepsilon = \text{sign}\left(\frac{\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}}{\sin(\theta_f - \theta_0)}\right) = \begin{cases} \varepsilon_f = \ \ \varepsilon_0 & \text{si } \theta_f - \theta_0 < \pi \\ -\varepsilon_f = -\varepsilon_0 & \text{si } \theta_f - \theta_0 > \pi \end{cases} \tag{176}$$

Because $\theta_f - \theta_0 < 2\pi$ and $|p(\theta)| \leq 1\ \forall\theta \in [\theta_0, \theta_f]$, the primer vector can only be monotonous on $[\theta_0, \theta_f]$ which could be expressed as follows:

$$\text{sgn}(p(\theta_0)) = -\text{sgn}(p'(\theta_0)) = -\text{sgn}(p'(\theta_f)) = -\text{sgn}(p(\theta_f)) = \varepsilon \tag{177}$$

After some computations and using (175), it is equivalent to:

$$\begin{cases} \dfrac{1 + 2e\cos(\theta_f) + \cos(\theta_f - \theta_0)}{\sin(\theta_f - \theta_0)} \geq 0 \\ \dfrac{1 + 2e\cos(\theta_0) + \cos(\theta_f - \theta_0)}{\sin(\theta_f - \theta_0)} \geq 0. \end{cases} \tag{178}$$

Lastly, the primer vector norm cannot reach its largest value on $[\theta_0, \theta_f]$, otherwise it would be greater than 1. Thus:

$$(\theta_+, \theta_-) \notin [\theta_0, \theta_f]^2. \tag{179}$$

Conditions (70) and (76) from Proposition 5 are now proved.

Second case: Suppose that the condition (165) does not hold i.e.:

$$(\cos(\theta_0)z_{f_1} + \sin(\theta_0)z_{f_2})(\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2}) = 0. \tag{180}$$

Then, one of the two impulses is zero. There is infinitely many Lagrange multipliers satisfying the optimality conditions, but a solution of same consumption could be found with a primer vector reaching one unit norm extremum. More precisely, suppose that: $\cos(\theta_0)z_{f_1}+\sin(\theta_0)z_{f_2} = 0$ and $\cos(\theta_f)z_{f_1}+\sin(\theta_f)z_{f_2} \neq 0$. Then the only allowed impulse is located at θ_0. The optimality condition (6) leads to:

$$\sin(\theta_0)\lambda_1 - \cos(\theta_0)\lambda_2 = \varepsilon_f(1 + e\cos(\theta_0)). \tag{181}$$

Hence, infinitely many λ vectors satisfy the optimality condition. As shown previously for the case of one initial impulse, it is always possible to choose:

$$\lambda_1 = -\frac{z_{f_1}}{|z_f|},\ \lambda_2 = -\varepsilon_f e - \frac{z_{f_2}}{|z_f|}. \tag{182}$$

The resulting primer vector reaches an extremum of unit norm at θ_0. This case comes down to a particular case of the one interior impulse solution (located at θ_0) described in Corollary 1. The case where $\cos(\theta_f)z_{f_1} + \sin(\theta_f)z_{f_2} = 0$ is completely similar, and is not detailed here.

E.2 Proof of Corollary 2

When $\theta_f - \theta_0 = \pi$, the optimality conditions (5), (6), (8) and (9) could be rewritten as:

$$\theta_0 = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \tag{183}$$

$$\lambda_1 = \varepsilon\sin(\theta_0),\ \lambda_2 \in \mathbb{R} \tag{184}$$

$$\frac{\Delta V(\theta_0)}{|\Delta V(\theta_0)|} = -\frac{\Delta V(\theta_f)}{|\Delta V(\theta_f)|} = \lambda_1\sin(\theta_0) = \varepsilon. \tag{185}$$

The objective equation given by (8) is equivalent to:

$$\begin{aligned} &|\Delta V(\theta_0)| + |\Delta V(\theta_f)| = -\varepsilon\sin(\theta_0)z_{f_1} \\ &z_{f_2} = 0. \end{aligned} \tag{186}$$

It is deduced that: $\varepsilon = -\mathrm{sgn}(z_{f_1}\sin(\theta_0))$ and:

$$|\Delta V(\theta_0)| + |\Delta V(\theta_f)| = |z_{f_1}| = |z_f|. \tag{187}$$

Now, $|p(\theta)| \leq 1,\ \theta \in [\theta_0, \theta_f]$ has to be ensured. As in the case where $\theta_f - \theta_0 \neq \pi$, the primer vector could only be monotonous on $[\theta_0, \theta_f]$ and therefore, it has to satisfy (177). After some computations:

$$\sin(\theta_0) = -1,\ |\lambda_2| \leq e \text{ and } \varepsilon = \mathrm{sgn}(z_{f_1}) = \varepsilon_1. \tag{188}$$

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