# Robust Stability of Periodic Systems with Memory: New Formulations, Analysis and Design Results ⋆

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**Abstract:** In this paper, the general formulation of periodically time-varying state-feedback controllers with memory is considered for the first time. New analysis and synthesis conditions for robust stability are proposed. The flexibility of these new results allows the user to freely add degrees-of-freedom to the control law which appears to effectively reduce the conservatism of the synthesis condition and to increase the stability domain of the closed-loop system in the presence of uncertainties. Furthermore, it is shown that for a particular structure of controllers a more efficient version of the design theorem can be derived by enriching the matrix of slack-variables.

## 1. INTRODUCTION

During the last three decades, periodic systems have been attracting significant interest within the control community mainly due to the variety and the originality of the possible applications, see e.g. Bolzern and Colaneri [1988]. One can first recall the now classic examples of control of vibrations in helicopters studied by Bittanti and Lovera [1996] and Bittanti and Cuzzola [2002], as well as autonomous orbit control discussed in Schubert [2001], or the attitude control systems of satellites equipped with magnetorquers, the topic of Trégouët et al. [2011b] and Wisniewski and Blanke [1999].

Surprisingly, few results exist in the robust control framework for this class of models. Among them, a synthesis condition leading to a stabilizing static periodic controller for discrete-time periodic systems is provided by De Souza and Trofino [2000] and then improved in Arzelier et al. [2005]. Both works rely on Linear Matrix Inequality (LMI) techniques, whose relevance has been amply demonstrated when dealing with uncertain systems. They focus on the case of polytopic uncertainties which is probably the most general way of capturing the structured uncertainty that may affect the system parameters.

Nevertheless, as mentioned in Ebihara et al. [2008], sticking to static control law should prevent us to obtain a systematic single-shot LMI-based design method that outperforms these results. As a consequence, this paper introduces a new class of periodic controllers keeping past states of the plant in memory in order to actively use them to construct current inputs. This new study area recently received a lot of attention, see Ebihara et al. [2009], Trégouët et al. [2011a] and references therein. Several control structures, characterized by their specific use of the memory, have been proposed along with their corresponding convex synthesis conditions.

Pursuing this work, this paper aims at proposing a new robust stabilizing synthesis condition for the most general formulation of periodic memory controllers. Arbitrary number of states kept in memory driven by any periodic reset law can be considered. In pursuing a unified framework, we make a constant effort to derive LMI conditions that are of least conservatism, according to the memory structure of the controller to be designed. Furthermore, guidelines are provided for the choice of this structure, which appears to be a trade-off between its complexity and its performance.

Previous works in this area were concerned with specific management of the memory that renders the problem fairly tractable. The generalization of these studies has required to revisit system duality for the class of autonomous periodic systems with memory. Furthermore, in this paper, a particular effort has been made to derive analysis and synthesis theorems by relying on correspondences between different time-invariant reformulations.

Due to space limitation, proofs and nonessential remarks have been removed from this paper. For the full version, see Trégouët et al. [2012].

We use the following notations in this paper. The symbols $\mathbf{1}$ and $\mathbf{0}$ stand for the identity and zero matrices of appropriate dimensions, respectively. The set of symmetric matrices and positive-definite symmetric matrices of size $l$ are denoted by $\mathbb{S}^l$ and $\mathbb{S}^l_+$ , respectively. For a real square matrix $A$, we define $\mathrm{He}\{A\} = A + A^T$. The operator diag

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builds block-diagonal matrix from input arguments. The convex hull of the collection of $L$ elements $A^{[1]}, \cdots, A^{[L]}$ is denoted by $\mathrm{co}\left\{A^{[1]}, \cdots, A^{[L]}\right\}$. The symbol $\sigma$ stands for the shift operator forward in time. The standard operator of modular arithmetic is referred as mod.

## 2. PROBLEM STATEMENT

Throughout this paper, rather than considering each instant time, periods are treated globally. To this end, every time instant is expressed as $Nq + k$ with $0 \leq k \leq N - 1$ such that $q \in \mathbb{Z}$ characterizes the period. Referring to this notation, consider the linear uncertain discrete-time $N$-periodic system described by

$$x_{Nq+k+1} = A_k(\theta)x_{Nq+k} + B_k(\theta)u_{Nq+k} \qquad (1)$$

where $x_{Nq+k} \in \mathbb{R}^n$ and $u_{Nq+k} \in \mathbb{R}^m$. The model (1) is subject to polytopic uncertainties gathered in $\theta$ such that for all $\theta$ in the uncertain domain $\Theta$, each $M_k(\theta) = [A_k(\theta) \;\; B_k(\theta)]$ over one period are such that

$$\begin{bmatrix} M_0(\theta) \\ \vdots \\ M_{N-1}(\theta) \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} M_0^{[1]} \\ \vdots \\ M_{N-1}^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} M_0^{[L]} \\ \vdots \\ M_{N-1}^{[L]} \end{bmatrix} \right\} \qquad (2)$$

The periodic control law considered in this paper is given by

$$\begin{aligned} u_{Nq+k} = K_{k,0}x_{Nq+k} + K_{k,1}x_{Nq+k-1} \\ + \cdots + K_{k,\alpha_k}x_{Nq+k-\alpha_k} \end{aligned} \qquad (3)$$

Thus, the controlled input $u_{Nq+k}$ is allowed to depend not only on the current state $x_{Nq+k}$, but also on the state history kept in memory. The sequence $\{\alpha_k\}_{k=0}^{N-1}$ characterizes the control structure as it corresponds to the number of past states required to evaluate $u_{Nq+k}$ for every $0 \leq k \leq N - 1$. Among every possible choice, attention of the reader is drawn to the following cases described by means of $\kappa \in \mathbb{N}$:

- $\alpha_k = 0$: Classical memoryless Periodic State-Feedback Controller (PSFC);
- $\alpha_k = k+\kappa$: Periodic Full Memory Controller (PFMC) of order $\kappa$;
- $\alpha_k = \kappa$: Periodic FIR Controller (PFIRC) of order $\kappa$.

Referring to this new terminology, De Souza and Trofino [2000], Ebihara et al. [2008] and Trégouët et al. [2011a] deal respectively with the following structures: PSFC (or PFIRC of order 0), PFMC of order 0 and PFIRC of order $N - 1$. Thus, the control law (3) can be regarded as a unifying way for combining existing results on discrete-time periodic state-feedback controllers.

**Remark** Changing the starting point of the considered period while keeping the same control structure gives rise to different controllers. As an example, let us consider the 2-periodic control law characterized by $\{\alpha_0, \alpha_1\} = \{0, 2\}$. In a new time axis coming up from a shifting of 1 forward in time, the same choice for $\alpha_0$ and $\alpha_1$ leads to $\{\alpha_0, \alpha_1\} = \{2, 0\}$ in the former time reference, which is a different controller.

An intuitive understanding of the reasons underlying the introduction of PFMCs despite the apparent complexity of their structure is provided now. Consider a 3-periodic controller characterized by the sequence $\{\alpha_0, \alpha_1, \alpha_2\} = \{0, 2, 1\}$. Control inputs can be written down over the period $q$ as:

$$\begin{cases} u_{3q} = & & [K_{0,0} \;\; \mathbf{0}]\,\eta_q \\ u_{3q+1} = & [K_{1,0}]\,\beta_{3q+1} + & [K_{1,1} \;\; K_{1,2}]\,\eta_q \\ u_{3q+2} = & [K_{2,0} \;\; K_{2,1}]\,\beta_{3q+2} + & [\mathbf{0} \;\; \mathbf{0}]\,\eta_q \end{cases} \qquad (4)$$

with

$$\beta_{3q+1} = [x_{3q+1}]\,,\;\; \beta_{3q+2} = \begin{bmatrix} x_{3q+2} \\ x_{3q+1} \end{bmatrix},\;\; \eta_q = \begin{bmatrix} x_{3q} \\ x_{3q-1} \end{bmatrix} \qquad (5)$$

Thus, construction of $u_{3q+k}$ requires both:

- states from current instant time to $3q+1$ which form $\beta_{3q+k}$;
- states prior to the instant time $3q + 1$ which correspond to the first $\alpha_k - k + 1$ entries of $\eta_q$.

From the implementation point of view, vector $\beta_{Nq+k}$ can be easily constructed by storing states along the period while $\eta_q$ represents the minimal information about the history of the system required to compute control inputs of the period $q$. Without modifying $\eta_q$, it is possible to enrich (4) by redefining $\{\alpha_0, \alpha_1, \alpha_2\}$ . At the end, this leads to $\{\alpha_0, \alpha_1, \alpha_2\} = \{1, 2, 3\}$ which coincides with a PFMC of order 1. Generalization of this observation reveals that PFMCs offer the largest number of degrees of freedom for a given knowledge about past states of the plant. As a consequence, every control structure can be worked out as a particular PFMC of sufficient order and hereafter the following expression of (3) coming up with such a controller will be considered:

$$u_{Nq+k} = \sum_{j=0}^{l+k-1} K_{k,j}x_{Nq+k-j} \qquad (6)$$

where $l$ is defined by

$$l = \max_{k\in[0,\cdots,N-1]} \alpha_k - k + 1 \geq 1 \qquad (7)$$

The periodic autonomous system with memory arising from the closed-loop of (1) with (6) is described by

$$x_{Nq+k+1} = \sum_{j=0}^{l+k-1} A_{k,j}(\theta)x_{Nq+k-j} \qquad (8)$$

with $A_{k,j}$ given by

$$A_{k,j}(\theta) = \begin{cases} A_k(\theta) + B_k(\theta)K_{k,0}, \;\; (j = 0) \\ B_k(\theta)K_{k,j}, \;\; (1 \leq j \leq l + k - 1) \end{cases} \qquad (9)$$

For this reason, this paper is first concerned with robust stability analysis of periodic systems described by (8). Then, robust stabilizing synthesis conditions for (1) is proposed and leads to a controller (3) characterized by a given sequence $\{\alpha_k\}_{k=0}^{N-1}$.

**Remark** In the rest of this paper, notations are sometimes lighten by omitting the dependency of matrices with respect to $\theta$.

**Remark** In the sequel, time-invariant reformulations relying on partial or full lifting of states of the periodic model are intensively used. This requires to specify the starting point of the considered period. Nevertheless, as it has already been pointed out by Hosoe and Hagiwara [2011], this definition may have a large influence on the conservatism of subsequent analysis results. However, due to space limitation, this paper does not consider this problem in details and leaves it for further investigations.

## 3. ROBUST STABILITY ANALYSIS

Time-invariant reformulation offers a suitable way to analyze periodic systems. Indeed, the stability condition stated for the time-invariant model can serve for the periodic one as this transformation preserves stability. Among procedures provided by Bittanti and Colaneri [2008] to perform this reformulation, the classical time lifted representation stands as a nice approach by restricting the dynamics of the periodic system to its essential feature by retaining only one sample of the states for each period. The transition matrix corresponds to the well-known time-invariant monodromy matrix, denoted by $\Psi$. In the sequel, this reformulation is referred as the monodromy representation.

### 3.1 Robust analysis using monodromy reformulation

The monodromy representation of (8) can be obtained by recasting it as a memoryless periodic model with time-varying dimensions that stores history of the states along the period.

*Proposition 1.* (Monodromy representation) The following time-invariant model can always be derived from (8):

$$\eta_{q+1} = \Psi(\theta)\eta_q \quad \text{with} \quad \eta_q \in \mathbb{R}^{nl} \tag{10}$$

where

$$\Psi(\theta) = \prod_{k=N-1}^{0} \bar{A}_k \tag{11}$$

$$\bar{A}_k = \begin{cases} \begin{bmatrix} \grave{A}_k \\ \mathbf{1}_{n(l+k)} \end{bmatrix}, & (0 \le k \le N-2) \\ \begin{bmatrix} \grave{A}_k \\ [\mathbf{1}_{n(l-1)} \ \mathbf{0}] \end{bmatrix}, & (k = N-1) \end{cases} \tag{12}$$

$$\grave{A}_k = [A_{k,0} \ \cdots \ A_{k,l+k-1}] \tag{13}$$

The matrix $\Psi$ is the generalization of the monodromy matrix for (8). According to Bittanti and Colaneri [2008], the Schur stability of $\Psi$ is a necessary and sufficient condition to prove the stability of (8).

*Theorem 2.* The periodic model (8) is stable *if and only if* the following condition holds:

$$\exists P_l(\theta) \in \mathbb{S}_+^{nl} \ : \ \Psi^T(\theta)P_l(\theta)\Psi(\theta) - P_l(\theta) \prec 0. \tag{14}$$

Unfortunately, although this theorem is suitable for nominal analysis (i.e., when there is no uncertainties in the system matrices), the robust case cannot be handled directly. As $\Psi$ is a nonlinear function in $\theta$, (14) is non convex with respect to this variable.

### 3.2 Robust analysis using descriptor reformulation

Another time-invariant reformulation can be obtained for which the matrices $A_{k,j}$ appear linearly. As the next subsection will make it clear, the stability condition stated for this new model can be easily extended to the robust case by invoking convexity argument.

*Proposition 3.* (Descriptor representation) The periodic memory model (8) can always be rewritten as the following time-invariant model:

$$\begin{bmatrix} \mathcal{N}(\theta) \\ [\mathbf{1}_{nl} \ \mathbf{0}] - \sigma [\mathbf{0} \ \mathbf{1}_{nl}] \end{bmatrix} \hat{x}_q = \mathbf{0} \quad \text{with} \quad \hat{x}_q \in \mathbb{R}^{n(N+l)} \tag{15}$$

with $\mathcal{N} = [\mathcal{E} \ \mathcal{A}]$ where $\mathcal{E} \in \mathbb{R}^{nN\times nN}$ and $\mathcal{A} \in \mathbb{R}^{nN\times nl}$ are described by

$$\mathcal{E} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1,0} & A_{N-1,1} & \cdots & A_{N-1,N-2} \\ \mathbf{0}_n & -\mathbf{1}_n & A_{N-2,0} & \cdots & A_{N-2,N-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\mathbf{1}_n & A_{1,0} \\ \mathbf{0}_n & \cdots & \cdots & \mathbf{0}_n & -\mathbf{1}_n \end{bmatrix} \tag{16}$$

$$\mathcal{A} = \begin{bmatrix} A_{N-1,N-1} & A_{N-1,N} & \cdots & A_{N-1,l+N-2} \\ A_{N-2,N-2} & A_{N-2,N-1} & \cdots & A_{N-2,l+N-3} \\ \vdots & & & \vdots \\ A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ A_{0,0} & A_{0,1} & \cdots & A_{0,l-1} \end{bmatrix} \tag{17}$$

By construction, there exists a linear map from the state vector of (15) to the one of (10) such that for every value of $l$:

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = T\hat{x}_q \quad \text{with } T = \begin{bmatrix} [\mathbf{1}_{nl} \ \mathbf{0}_{nl\times nN}] \\ [\mathbf{0}_{nl\times nN} \ \mathbf{1}_{nl}] \end{bmatrix} \tag{18}$$

As a result, equalities (19) and (20) fully capture the dynamics of the monodromy representation (10):

$$[-1 \ \Psi] \begin{bmatrix} \eta_{k+1} \\ \eta_k \end{bmatrix} = \mathbf{0} \tag{19}$$

$$\mathcal{N}\hat{x}_q = \mathbf{0} \quad \text{with} \quad \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = T\hat{x}_q \tag{20}$$

**Remark** The size of $T \in \mathbb{R}^{2nl\times n(l+N)}$, given by (18), reveals that the monodromy representation (10) derives from (15) by contraction when $l < N$ and by expansion when $l > N$. As a matter of fact, subvector $\zeta_q$ of $\hat{x}_q$ has been dropped in the first case whereas in the second case $n(l-N)$ lines have been repeated. When $l = N$, $T = \mathbf{1}_{2nN}$ and both state vectors are equal.

Once this relationship between time-invariant models has been stated, use of Finsler's Lemma conduces to the following stability condition as an alternative of Th.2.

*Theorem 4.* The periodic model (8) is stable *if and only if* the following condition holds:

$$\begin{array}{c} \exists P_l(\theta) \in \mathbb{S}_+^{nl}, \exists \mathcal{F}(\theta) \in \mathbb{R}^{n(l+N)\times nN} \ : \\ -\mathcal{P}(P_l(\theta)) + \mathrm{He}\{\mathcal{F}(\theta)\mathcal{N}(\theta)\} \prec 0 \end{array} \tag{21}$$

where

$$\mathcal{P}(P_l) = \begin{bmatrix} -P_l & \\ & \mathbf{0}_{nN} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{nN} & \\ & P_l \end{bmatrix} \tag{22}$$

and $\mathcal{N} = [\mathcal{E} \ \mathcal{A}]$ is given by (16) and (17).

**Proof.** Referring to the monodromy representation, it is well-known that the periodic model (8) is stable iff there exists $P_l \in \mathbb{S}_+^{nl}$ such that

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix}^T \begin{bmatrix} P_l & \mathbf{0} \\ \mathbf{0} & -P_l \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} < 0 \quad \text{s.t.} \quad [-\mathbf{1} \ \Psi] \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = \mathbf{0} \tag{23}$$

Using the relationship between (19) and (20), this condition can be rewritten equivalently as

$$\hat{x}_q^T T^T \begin{bmatrix} P_l & \mathbf{0} \\ \mathbf{0} & -P_l \end{bmatrix} T\hat{x}_q < 0 \quad \text{s.t.} \quad \mathcal{N}\hat{x}_q = \mathbf{0} \tag{24}$$

where the descriptor representation is involved. Then $\mathcal{P}(P_l) = T^T \mathrm{diag}\{-P_l, P_l\} T$ is introduced and Finsler's Lemma is invoked to obtain (21).

### 3.3 Robust analysis condition

The previous subsection has demonstrated that the new LTI reformulation (15) in which the linear dependency upon the $A_{i,j}$'s is preserved may be used for stability analysis instead of the more classical monodromic representation (10). As a consequence, the obtained stability condition is convex with respect to $\theta$ which allows to consider the robust case by evaluating this LMI on every vertex of the polytope. In addition to that, it can be noticed that the Lyapunov matrix $P_l$ and the matrices of (8), forming $\mathcal{N}$, are decoupled which permits to consider $P_l$ as a polytopic matrix with respect to $\theta$.

*Theorem 5.* (Primal robust stability analysis) The periodic model (8) is robustly stable *if* there exist $\mathcal{F} \in \mathbb{R}^{n(N+l)\times nN}$ and $L$ matrices $P_l^{[i]} \in \mathbb{S}_+^{nl}$ such that, for $i = \{1, \cdots, L\}$,

$$-\mathcal{P}(P_l^{[i]}) + \mathrm{He}\left\{\mathcal{F}\mathcal{N}^{[i]}\right\} \prec 0 \tag{25}$$

where $\mathcal{P}(P_l^{[i]})$ and $\mathcal{N}^{[i]}$ may be easily derived from (22), (16) and (17).

Obviously, the lack of necessity for Th. 5 stems from the enforcement for $\mathcal{F}$ to be independent on $\theta$.

## 4. ROBUST STABILIZING SYNTHESIS

Derivation of synthesis conditions from the analysis ones is usually carried out by means of the classical linearizing change of variables. This is unfortunately not possible when considering (25) because of the lack of direct multiplication between controller gains and LMI variables. Nevertheless, as this section makes it clear, system duality offers a way to tackle this issue.

### 4.1 System duality for memory periodic models

As system duality is in general well-known in the time-invariant framework, this subsection first deals with a dual version of (15) as an intermediate step. To this end, $l \geq 1$, given by (7), is decomposed as $l = bN + r$ such that $1 \leq r \leq N$ and $b \in \mathbb{N}$.

*Proposition 6.* (Dual descriptor representation) A dual version of the descriptor representation (15) is:

$$\begin{bmatrix} \breve{\mathcal{N}}^T \\ [\mathbf{1}_{nl}\ \ \mathbf{0}] - \sigma\,[\mathbf{0}\ \ \mathbf{1}_{nl}] \end{bmatrix} \hat{x}_q^d = \mathbf{0} \quad \text{with} \quad \hat{x}_q^d \in \mathbb{R}^{n(N+l)} \tag{26}$$

where

$$\breve{\mathcal{N}}^T = \begin{bmatrix} \mathscr{A}_{b+1}^T & \cdots & \mathscr{A}_1^T & \begin{bmatrix} \mathbf{0} \\ \mathscr{A}_{0,0}^T \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{nN\times n(N+l)} \tag{27}$$

with $\mathscr{A}_{0,0} \in \mathbb{R}^{nr\times nr}$ and $\mathscr{A}_j \in \mathbb{R}^{nN\times nN}$ ($j = \{1, \cdots, b+1\}$) described by

$$\mathscr{A}_{0,0} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1,0} & \cdots & A_{N-1,r-2} \\ \mathbf{0}_n & -\mathbf{1}_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{N-r+1,0} \\ \mathbf{0}_n & \cdots & \mathbf{0}_n & -\mathbf{1}_n \end{bmatrix} \tag{28}$$

$$\mathscr{A}_1 = \begin{bmatrix} A_{N-1,r-1} & \cdots & A_{N-1,N-2} & A_{N-1,N-1} & \cdots & A_{N-1,N+r-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{N-r,0} & & \vdots & \vdots & & \vdots \\ -\mathbf{1}_n & \ddots & \vdots & \vdots & & \vdots \\ & \ddots & A_{1,0} & \vdots & & \vdots \\ \mathbf{0} & & -\mathbf{1}_n & A_{0,0} & \cdots & A_{0,r-1} \end{bmatrix} \tag{29}$$

$$\mathscr{A}_j = \begin{bmatrix} A_{N-1,(j-1)N+r-1} & \cdots & A_{N-1,jN+r-2} \\ \vdots & & \vdots \\ A_{0,(j-2)N+r} & \cdots & A_{0,(j-1)N+r-1} \end{bmatrix}, (2 \leq j \leq b+1) \tag{30}$$

Noticing that the structure of $\mathscr{A}_{0,0}^T$ and $\mathscr{A}_1^T$ match such that the rightest square block of size $N$ of $\breve{\mathcal{N}}^T$ is a lower triangular matrix with $N$ blocks $-1_n$ on the diagonal, a periodic system, which can be considered as a dual of (8), can be derived from (26) in the same way as (15) has been obtained from (8).

*Theorem 7.* A dual model of (8) is given by

$$x_{Nq-k-1}^d = \sum_{j=0}^{l+k-1} A_{\vartheta(j,k),j}^T(\theta) x_{Nq-k+j}^d \tag{31}$$

with $\vartheta(j,k) = j - k - r \pmod N$.

### 4.2 Robust synthesis condition

Having derived the dual model of periodic autonomous systems with memory (8), an alternative sufficient condition of robust stability of this model may be easily obtained.

*Theorem 8.* (Dual robust stability analysis) Both periodic models (31) and (8) are stable *if* there exist $\mathcal{F} \in \mathbb{R}^{n(N+l)\times nN}$ and $L$ matrices $P_l^{[i]} \in \mathbb{S}_+^{nl}$ such that, for $i = \{1, \cdots, L\}$,

$$\mathcal{P}(P_l^{[i]}) + \mathrm{He}\left\{\breve{\mathcal{N}}^{[i]}\mathcal{F}\right\} \prec 0 \tag{32}$$

where $\mathcal{P}(P_l^{[i]})$ and $\breve{\mathcal{N}}^{[i]}$ may be easily derived from (22), (27), (28), (29) and (30).

We are now in position to move on to the problem of memory controller synthesis which is first tackled by the following theorem expressed for any memory controller.

*Theorem 9.* (Robust stabillizing synthesis - General case) The periodic model (1) can be robustly stabilized by the memory controller (3) *if* there exist $\mathcal{G} \in \mathbb{R}^{nN\times nN}$, $\mathcal{Y} \in \mathbb{R}^{m(l+N-1)\times nN}$ and $L$ matrices $P_l^{[i]} \in \mathbb{S}_+^{nl}$ such that, for $i = \{1, \cdots, L\}$,

$$\mathcal{P}(P_l^{[i]}) + \mathrm{He}\left\{\left(\breve{\mathcal{A}}^{[i]}\mathcal{G} + \breve{\mathcal{B}}^{[i]}\mathcal{Y}\right)[\mathbf{0}_{nN\times nl}\ \ \mathbf{1}_{nN}]\right\} \prec 0 \tag{33}$$

where the matrices $\breve{\mathcal{A}}^{[i]}$ and $\breve{\mathcal{B}}^{[i]}$ may be easily derived from

$$\breve{\mathcal{A}} = \begin{bmatrix} \breve{\mathcal{A}}_{b+1} \\ \vdots \\ \breve{\mathcal{A}}_1 \\ \mathbf{0} \;\; \breve{\mathcal{A}}_{0,0} \end{bmatrix}, \; \breve{\mathcal{A}}_{0,0} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & \ddots & A_{N-r+1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\mathbf{1}_n \end{bmatrix} \quad (34)$$

$$\breve{\mathcal{A}}_1 = \begin{bmatrix} \multicolumn{8}{c}{\mathbf{0}_{n(l-1)\times nN}} \\ \hline A_{N-r} & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ -\mathbf{1}_n & \ddots & \ddots & & & & & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{1}_n & A_0 & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \quad (35)$$

$$\breve{\mathcal{A}}_j = \mathbf{0}_{nN\times nN} \;, \; (2 \leq j \leq b+1) \quad (36)$$

$$\breve{\mathcal{B}} = \mathrm{diag}\{\breve{\mathcal{B}}_{b+1}, \cdots, \breve{\mathcal{B}}_1, \breve{\mathcal{B}}_{0,0}\} \quad (37)$$

$$\breve{\mathcal{B}}_{0,0} = \mathrm{diag}\{B_{N-1}, \cdots, B_{N-r+2}, \begin{bmatrix} B_{N-r+1} \\ \mathbf{0}_{n\times m} \end{bmatrix}\} \quad (38)$$

$$\breve{\mathcal{B}}_j = \mathrm{diag}\{B_{N-1}, \cdots, B_0\} \;, \; (1 \leq j \leq b+1) \quad (39)$$

The matrix $\mathcal{G}$ is equal to $\mathrm{diag}\{G_{N-1} \cdots, G_0\}$ with $G_k \in \mathbb{R}^{n\times n}$ $(k = \{0, \cdots N-1\})$ and $\mathcal{Y}$ depends on the desired sequence $\{\alpha_k\}_{k=0}^{N-1}$. Indeed, the controller gains can then be recovered by solving

$$\breve{\mathcal{K}} = \mathcal{Y}\mathcal{G}^{-1} \quad (40)$$

which means that $\mathcal{Y} = \breve{\mathcal{K}}\mathcal{G}$ inherites its structure from $\breve{\mathcal{K}}$, subsequently defined for the PFMC case as any other controller can be obtained by constraining $K_{k,j}$'s to be equal to $\mathbf{0}_{m\times n}$.

$$\breve{\mathcal{K}} = \begin{bmatrix} \breve{\mathcal{K}}_{b+1} \\ \vdots \\ \breve{\mathcal{K}}_1 \\ \mathbf{0} \;\; \breve{\mathcal{K}}_{0,0} \end{bmatrix}, \; \breve{\mathcal{K}}_{0,0} = \begin{bmatrix} \mathbf{0}_{m\times n} & K_{N-1,0} & \cdots & K_{N-1,r-2} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{m\times n} & \cdots & \mathbf{0}_{m\times n} & K_{N-r+1,0} \end{bmatrix} \quad (41)$$

$$\breve{\mathcal{K}}_1 = \begin{bmatrix} K_{N-1,r-1} & \cdots & \cdots & \cdots & \cdots & K_{N-1,N+r-2} \\ \vdots & & & & & \vdots \\ K_{N-r,0} & \ddots & & & & \vdots \\ \mathbf{0}_{m\times n} & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ \mathbf{0}_{m\times n} & \cdots & \mathbf{0}_{m\times n} & K_{0,0} & \cdots & K_{0,r-1} \end{bmatrix} \quad (42)$$

$$\breve{\mathcal{K}}_j = \begin{bmatrix} K_{N-1,(j-1)N+r-1} & \cdots & K_{N-1,jN+r-2} \\ \vdots & & \vdots \\ K_{0,(j-2)N+r} & \cdots & K_{0,(j-1)N+r-1} \end{bmatrix} \quad (43)$$
$$,(2 \leq j \leq b+1)$$

Note that $\mathcal{F}$ has been restricted to $[\,\mathbf{0}\;\mathcal{G}\,]$ when deriving (33) from the dual analysis condition (32) in order to ensure the recovery of $\breve{\mathcal{K}}$ from the knowledge of $\breve{\mathcal{Y}}$ and $\mathcal{G}$. As a result, it cannot be ensured that every controllers (3), which has been proved to be stabilizing using Th 8, can be parameterized by (33). However, it can be demonstrated at least that (33) is always feasible if the system (1) is stabilizable by means of the PFMC of order 0 computed with $\mathcal{G}$ block-diagonal, see Trégouët et al. [2012].

Another motivation for the introduction of PFMCs can now be stated: For this class of controllers, the structure of $\mathcal{G}$ can be chosen to be upper triangular while preserving its invertibility and therefore the possibility of recovering $\breve{\mathcal{K}}$ from the knowledge of $\mathcal{G}$ and $\mathcal{Y}$. At our knowledge, it is not possible to choose $\mathcal{G}$ less sparse than upper triangular without obtaining a non-causal controller using (40).

*Theorem 10.* (Robust stabillizing synthesis - PFMC case) In the PFMC case, Th. 9 can be used with $\mathcal{G}$ given by

$$\mathcal{G} = \begin{bmatrix} G_{N-1,0} & \cdots & \cdots & G_{N-1,N-1} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & G_{0,0} \end{bmatrix} \in \mathbb{R}^{nN\times nN} \quad (44)$$

where $G_{k,j} \in \mathbb{R}^{n\times n}$ $(k = \{0, \cdots, N-1\}, j = \{0, \cdots, k\})$. Furthermore, if (33) holds then $\mathcal{G}$ is invertible.

It is clear that by enriching $\mathcal{G}$, the conservatism of (33) can only decrease. Numerical results of the next section show how this procedure increase the stability domain of the resulting closed-loop.

At our knowledge, these two last theorems encompass every synthesis conditions leading to memory stabilizing controllers considered elsewhere in the literature.

### *4.3 Some hints about the design strategy*

This subsection aims at giving some guidelines for the choice of $\{\alpha_k\}_{k=0}^{N-1}$ which appears to be a trade-off between the complexity of the control law and the reduction of the conservatism of the corresponding synthesis condition.

From the implementation point of view, the PFIRC represents an interesting choice because of the simplicity of its structure. From the corresponding sequence of $\alpha_k$'s and as soon as $l$ does not change, any addition of $K_{i,j}$ terms in the control law induces an enrichment of $\mathcal{Y}$ which decreases the conservatism of (33). Obvioulsy, the resulting controller is not a PFIRC anymore. At the end, the fully enriched controller corresponds to the PFMC of order $l-1$. In that case, $\mathcal{G}$ can be modified in a less sparse triangular structure for free which means that using Th. 10 instead of Th. 9 decreases the conservatism of (33). To summarize, among every control laws leading to the same value of $l$, the different design conditions may be ordered as follows with respect to their decreasing conservatism.

| PFIRC of order $\kappa$ | $\leq$ | PFMC of order $\kappa$ from Th. 9 | $\leq$ | PFMC of order $\kappa$ from Th. 10 |
|---|---|---|---|---|

## 5. NUMERICAL RESULTS

To illustrate the effectiveness of the suggested design methods, the 3-periodic problem of Example 1 provided by De Souza and Trofino [2000] is used as a benchmark. The goal is to maximize a properly defined stability margin denoted by $\grave{\alpha}$, i.e. the allowable maximal absolute value of an uncertain parameter.

Periodic controllers, corresponding to different choices of $\alpha_k$, are designed relying on Th 9 and, when it is possible, on Th 10. Robustness of every resulting closed-loop systems is evaluated by means of Th 8. Results are gathered in Table 1 where numbers appearing in the first column identify the controller.

| Controller | | Synthesis | $\grave{\alpha}$ | |
|---|---|---|---|---|
| id. | $\{\alpha_0, \alpha_1, \alpha_2\}$ | Theorem | Synthesis | Analysis (Th. 8) |
| 1 | $\{2, 2, 2\}$ | Th. 9 | 0.7031 | 0.8672 |
| 2 | $\{2, 3, 4\}$ | Th. 9 | 0.7188 | 0.8398 |
| 3 | $\{2, 3, 4\}$ | Th. 10 | 1.0625 | 1.1563 |

Table 1. Synthesis and analysis stability margins for different choices of $\alpha_k$

The synthesis condition has been derived from (32) of Th. 8 by enforcing the structure of $\mathcal{F}$. That is the reason why the value of $\grave{\alpha}$ always increases from the synthesis to the analysis step.

These experiments illustrate the inclusion relationships stated in subsection 4.3. Indeed, referring to the value of $\grave{\alpha}$ achieved by synthesis theorems, the controllers can be ordered as $1 < 2 < 3$. However, it is worth noticing that this hierarchy may not be preserved at the analysis step: Here the stability domain is larger with controller 1 than 2.

From the controller 2 to 3, the achievement is increased by 48 percent and 37 percent respectively at the synthesis and the analysis steps although the control structure remains the same. This brings to light how crucial is the relaxation of $\mathcal{G}$.

## 6. CONCLUSIONS AND FUTURE WORKS

In the memory periodic controllers framework, the flexibility of the proposed approach allows the user to freely add degrees-of-freedom to the control law which appears to effectively reduce the conservatism of the synthesis condition. Furthermore, it has been shown that for a particular structure of controllers a more efficient version of this condition can be derived by enriching the matrix of slack-variables.

As discussed in section 2, the control structure depends on the sequence of $\alpha_k$'s and on the definition of the starting point of the considered period. Both are closely related as any enrichment of control laws that increases the value of $l$ can be interpreted as a redefinition of the starting point. Providing deeper guidelines for these choices remains a challenging subject.

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