Robust $\mathcal{H}_\infty$ Performance of Periodic Systems with Memory: New Formulations, Analysis and Design Results

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Abstract— **This paper is devoted to $\mathcal{H}_\infty$ analysis and synthesis conditions of state-feedback periodic memory controllers, in the framework of periodic uncertain discrete-time systems. The proposed conditions are such that the user is allowed to freely add degrees-of-freedom to the control law which effectively reduces the conservatism of the synthesis condition and decrease the guaranteed $\mathcal{H}_\infty$ induced norm of the obtained uncertain closed-loop systems. Numerical examples show that for a particular structure of controllers the efficiency of the design theorem can be significantly enhanced by relaxing the structure of slack-variables.**

I. INTRODUCTION

In recent years, increasing attention has been devoted to the analysis and control of periodic systems mainly due to the variety and the originality of the possible applications. Among them, one can recall the now classic examples of control of vibrations in helicopters studied in [2] and [4] as well as autonomous orbit control discussed in [12], or the attitude control systems of satellites equipped with magnetorquers, [14] and [16].

Despite this interest, results in this study area still leaves some room for improvement, especially concerning robustness aspects. In this framework, state-feedback synthesis conditions, relying on Linear Matrix Inequality (LMI), have been proposed in [1] and [5] for discrete-time periodic models subject to polytopic uncertainties.

In [7], a further attempt was made to significantly outperform these results by introducing a new kind of controllers. The class of periodic static control laws is extended by letting them memorize past states of the plant in order to actively use them to build current inputs. This new study area recently received a lot of attention and several control structures, characterized by their specific use of the state memory, have been proposed (see [6], [7], [8], [13] and [15]).

In line with [15], where the stabilization problem is considered, this paper deals with the most general formulation of state-feedback periodic memory controllers to tackle the $\mathcal{H}_\infty$ analysis and synthesis problems for periodic discrete-time systems affected by polytopic uncertainties. Numerical results confirm that this addition of degrees of freedom to the control law contributes to effectively decrease the $\mathcal{H}_\infty$ induced norm of the obtained closed-loop system. Furthermore, it is shown that certain particular managements of the memory allows to drastically reduce the conservatism of the synthesis conditions.

The core idea of the paper is then to exploit the correspondences existing between time-invariant reformulations such that the obtained analysis condition is first established for a classical LTI system and then extended to a descriptor system where the uncertainties appear linearly. To proceed this way, lifting procedure has been revisited in the context of memory periodic model and a new descriptor-like lifting has been introduced in this paper for the first time. Furthermore, this work has required to propose a dual model of the periodic systems with memory.

We use the following notations in this paper. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{m\times n}$ stand for the identity and zero matrices of dimensions $n \times n$ and $m \times n$, respectively. When it is clear from the context, dimensions are omitted. The set of symmetric matrices and positive-definite symmetric matrices of size l are denoted by $\mathbb{S}^l$ and $\mathbb{S}^l_+$ respectively. For a real square matrix A, we define He$\{A\} = A + A^T$. The operator diag builds block-diagonal matrix from input arguments. The convex hull of the collection of L elements $A^{[1]}, \cdots, A^{[L]}$ is denoted by co$\{A^{[1]}, \cdots, A^{[L]}\}$. The symbol σ stands for the shift operator forward in time. The standard operator of modular arithmetic is referred as mod. The Kronecker product is denoted by $\otimes$.

Throughout this paper, rather than considering each instant time, periods are treated globally via liftings. To this end, every time instant is expressed as $Nq + k + \tau$ with $0 \le k \le N-1$ such that $q \in \mathbb{Z}$ characterizes the considered period which starts at $Nq+\tau$. Nonetheless, as it has already been pointed out by [10], robust analysis results depend on the choice of τ. However, due to space limitation, this paper focuses on the special case where $\tau = 0$. Future works will investigate this problem in details.

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II. PROBLEM STATEMENT

Consider the linear uncertain discrete-time N-periodic system $\Sigma(\theta)$ described by

$$\begin{bmatrix} x_{Nq+k+1} \\ z_{Nq+k} \end{bmatrix} = \underbrace{\begin{bmatrix} A_k(\theta) & B_k(\theta) & E_k(\theta) \\ C_k(\theta) & D_k(\theta) & F_k(\theta) \end{bmatrix}}_{M_k(\theta)} \begin{bmatrix} x_{Nq+k} \\ w_{Nq+k} \\ u_{Nq+k} \end{bmatrix} \tag{1}$$

where $x_{Nq+k} \in \mathbb{R}^n$, $w_{Nq+k} \in \mathbb{R}^m$, $u_{Nq+k} \in \mathbb{R}^{m_u}$ and $z_{Nq+k} \in \mathbb{R}^p$. The model $\Sigma(\theta)$ is subject to polytopic uncertainties gathered in θ such that for all θ in the uncertain domain Θ, each $M_k(\theta)$ are such that

$$\begin{bmatrix} M_0(\theta) \\ \vdots \\ M_{N-1}(\theta) \end{bmatrix} \in \text{co} \left\{ \begin{bmatrix} M_0^{[1]} \\ \vdots \\ M_{N-1}^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} M_0^{[L]} \\ \vdots \\ M_{N-1}^{[L]} \end{bmatrix} \right\} \tag{2}$$

Following the line of [15], a control law defined in (3) is considered since it can be regarded as a unifying way for combining existing results on discrete-time periodic state-feedback controllers.

$$\begin{aligned} u_{Nq+k} = K_{k,0} x_{Nq+k} + K_{k,1} x_{Nq+k-1} \\ + \cdots + K_{k,\alpha_k} x_{Nq+k-\alpha_k} \end{aligned} \tag{3}$$

Thus, the controlled input u_{Nq+k} is allowed to depend not only on the current state x_{Nq+k}, but also on the state history. The sequence $\{\alpha_k\}_{k=0}^{N-1}$ characterizes the control structure.

Among every possible choice, the case $\alpha_k = k+\kappa$, corresponding to the so-called Periodic Full Memory Controller (PFMC) of order κ, plays a particular role. Indeed, according to [15], PFMCs offer the largest number of degrees of freedom for a given knowledge about past states of the plant. As a consequence, every control structure can be worked out as a particular PFMC of sufficient order and hereafter the following expression of (3) coming up with such a controller will be considered:

$$u_{Nq+k} = \sum_{j=0}^{l+k-1} K_{k,j} x_{Nq+k-j} \tag{4}$$

where l is defined by

$$l = \max_{k \in [0,\cdots,N-1]} \alpha_k - k + 1 \geq 1 \tag{5}$$

The periodic autonomous system with memory $\Sigma_{cl}(\theta)$ arising from (1) with (4) is described by

$$\begin{bmatrix} x_{Nq+k+1} \\ z_{Nq+k} \end{bmatrix} = \sum_{j=0}^{l+k-1} \begin{bmatrix} A_{k,j}(\theta) \\ C_{k,j}(\theta) \end{bmatrix} x_{Nq+k-j} + \begin{bmatrix} B_k(\theta) \\ D_k(\theta) \end{bmatrix} w_{Nq+k} \tag{6}$$

where $A_{k,j}$ and $C_{k,j}$ are given by

$$\begin{bmatrix} A_{k,j}(\theta) \\ C_{k,j}(\theta) \end{bmatrix} = \begin{cases} \begin{bmatrix} A_k(\theta) + E_k(\theta) K_{k,0} \\ C_k(\theta) + F_k(\theta) K_{k,0} \end{bmatrix}, \ (j=0) \\ \begin{bmatrix} E_k(\theta) K_{k,j} \\ F_k(\theta) K_{k,j} \end{bmatrix}, \ (1 \leq j \leq l+k-1) \end{cases} \tag{7}$$

Remark 1: In the rest of this paper, notations are sometimes lighten by omitting the dependency of matrices with respect to θ.

This paper is first concerned with robust $\mathcal{H}_\infty$ performance analysis, i.e. find the squared worst-case $\mathcal{H}_\infty$ norm of $\Sigma_{cl}(\theta)$ over the whole uncertain domain:

Problem 1: (Worst-case $\mathcal{H}_\infty$ Analysis) For $\Sigma_{cl}(\theta)$ robustly stable, find ν_{wc} such that

$$\nu_{wc} = \max_{\theta \in \Theta} \|\Sigma_{cl}(\theta)\|_\infty^2 \tag{8}$$

Then, the corresponding synthesis problem is considered:

Problem 2: (Worst-case $\mathcal{H}_\infty$ Synthesis) Find a controller (3) characterized by a given sequence $\{\alpha_k\}_{k=0}^{N-1}$ which robustly stabilizes $\Sigma(\theta)$ along with minimizing ν_{wc}.

Even in the non-periodic case, these problems are considered to be hard and, to our best knowledge, no systematic solutions can be found in the literature. For this reason, relaxations are performed, which leads to upper bounds of ν_{wc} corresponding to guaranteed norms of $\Sigma_{cl}(\theta)$.

III. ROBUST PERFORMANCE ANALYSIS

Time-invariant reformulation offers a suitable way to analyze periodic systems. Indeed, the performance conditions stated for the time-invariant model can serve for the periodic one as this transformation preserves input/output relationship. Among procedure available to achieve this purpose, time-lifting is probably the most classical one, mainly because of its simplicity.

A. Robust Analysis Strategy Via Time Lifting Procedure

Lifting procedure consists in gathering respectively the inputs and the outputs over one period in single vectors, denoted $\hat{w}_q$ and $\hat{z}_q$, such that[1]:

$$\hat{w}_q = \begin{bmatrix} w_{qN+N-1}^T & \cdots & w_{qN}^T \end{bmatrix}^T \in \mathbb{R}^{mN} \tag{9}$$

$$\hat{z}_q = \begin{bmatrix} z_{qN+N-1}^T & \cdots & z_{qN}^T \end{bmatrix}^T \in \mathbb{R}^{pN} \tag{10}$$

In the literature, the relationship between these vectors is modeled by keeping only the first sample of the state vector as it conduces to the minimal realization. As an example, when $N=2$, the memoryless periodic model (1) is transformed as

$$\begin{bmatrix} \eta_{q+1} \\ \hline \hat{z}_q \end{bmatrix} = \left[\begin{array}{c|cc} A_1 A_0 & B_1 & A_1 B_0 \\ \hline C_1 A_0 & D_1 & C_1 B_0 \\ C_0 & \mathbf{0} & D_0 \end{array}\right] \begin{bmatrix} \eta_q \\ \hline \hat{w}_q \end{bmatrix} \tag{11}$$

where η_q corresponds to x_{2q}. However, this example clearly shows that this procedure destroys the linear dependence w.r.t. to the state matrices of (1) which is a strong requirement for robust analysis with polytopic uncertainties.

To overcome this difficulty, this paper exploits the degrees of freedom available when defining the internal representation of the time-invariant reformulation. Thus, the same periodic model can be alternatively reformulated as

$$\left[\begin{array}{ccc|cc|cc} -\mathbf{1} & A_1 & \mathbf{0} & B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & A_0 & \mathbf{0} & B_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & -\sigma\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C_1 & \mathbf{0} & D_1 & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_0 & \mathbf{0} & D_0 & \mathbf{0} & -\mathbf{1} \end{array}\right] \begin{bmatrix} \hat{x}_q \\ \hline \hat{w}_q \\ \hline \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{12}$$

[1]The convention is such that $\hat{w}_q$ stacks w_q's from the bottom to the top when going forward in time along the period although the opposite way is sometimes used in the literature.

where $\hat{x}_q^T$ corresponds to $[x_{2q+2}^T, x_{2q+1}^T, x_{2q}^T]$. In the sequel, this new reformulation is coined as descriptor lifting while the procedure leading to (11) is referred as monodromy lifting as it let the so-called monodromy matrix appears explicitly, equals here to $A_1 A_0$.

In this paper, a tractable $\mathcal{H}_\infty$ analysis condition for the periodic models with memory (6) is derived as follows:

1) obtain and analyze the monodromy lifting to establish the first analysis condition;
2) derive the new descriptor lifting and exhibit the linear map existing between η_q and $\hat{x}_q$ which, for the previous example, corresponds to

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \hat{x}_q \tag{13}$$

3) reformulate the first analysis condition to consider matrices of the descriptor lifting such that weaker relaxations allows the derivation of tractable conditions for handling uncertainties.

B. Robust Performance Using Monodromy Lifting

The monodromy lifting of (6) can be obtained by recasting it as a memoryless periodic model with time-varying dimensions that stores history of the states along the period:

Proposition 1: (Memoryless representation) The following model is equivalent to Σ_{cl}:

$$\begin{bmatrix} \bar{x}_{qN+k+1} \\ z_{qN+k} \end{bmatrix} = \begin{bmatrix} \bar{A}_k & \bar{B}_k \\ \bar{C}_k & \bar{D}_k \end{bmatrix} \begin{bmatrix} \bar{x}_{qN+k} \\ w_{qN+k} \end{bmatrix} \tag{14}$$

with $\bar{D}_k = D_k$ and

$$\bar{A}_k = \begin{cases} \begin{bmatrix} \grave{A}_k \\ \mathbf{1}_{n(l+k)} \end{bmatrix}, & (0 \le k \le N-2) \\ \begin{bmatrix} \grave{A}_{N-1} \\ \begin{bmatrix} \mathbf{1}_{n(l-1)} & \mathbf{0} \end{bmatrix} \end{bmatrix}, & (k = N-1) \end{cases} \tag{15}$$

$$\grave{A}_k = \begin{bmatrix} A_{k,0} & \cdots & A_{k,l+k-1} \end{bmatrix} \tag{16}$$

$$\bar{B}_k = \begin{bmatrix} B_k \\ \mathbf{0} \end{bmatrix}, \quad \bar{C}_k = \begin{bmatrix} C_{k,0} & \cdots & C_{k,l+k-1} \end{bmatrix} \tag{17}$$

From this reformulation, the following monodromy lifted model follows readily (see [3]).

Proposition 2: (Monodromy lifting) The periodic model Σ_{cl} can always be rewritten as the following time-invariant model:

$$\begin{bmatrix} \eta_{q+1} \\ \hat{z}_q \end{bmatrix} = \begin{bmatrix} \Psi & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} \eta_q \\ \hat{w}_q \end{bmatrix} \quad \text{with} \quad \eta_q \in \mathbb{R}^{nl} \tag{18}$$

Matrices of (18) are not explicitly written down for space reasons but their expression may be easily derived. It is important to note that they are nonlinear functions of the matrices of Σ_{cl}.

From this reformulation, well-known results established for time-invariant systems can be readily applied to analyze the periodic model Σ_{cl}.

Theorem 1: The worst-case $\mathcal{H}_\infty$ analysis problem is *equivalent* to the following optimization problem leading to ν_{wc}:

$$\nu_{wc} = \max_{\theta \in \Theta} \min_{P(\theta) \in \mathbb{S}_+^{nl}} \nu \tag{19}$$

$$\begin{bmatrix} \Psi^T P \Psi - P + \mathfrak{C}^T \mathfrak{C} & \Psi^T P \mathfrak{B} + \mathfrak{C}^T \mathfrak{D} \\ \mathfrak{B}^T P \Psi + \mathfrak{D}^T \mathfrak{C} & \mathfrak{B}^T P \mathfrak{B} + \mathfrak{D}^T \mathfrak{D} - \nu 1_{mN} \end{bmatrix} \prec \mathbf{0} \tag{20}$$

This problem is known to be non tractable due to its non convexity. Consequently, our goal is to derive tight upper bound of ν_{wc} via convex relaxation. To this end, the problem will first reformulated relying on correspondences between liftings exhibit by the next subsection.

C. New Descriptor Lifting

Based on the previous remark, the new lifting (21) is introduced as an alternative to (18).

Proposition 3: (Descriptor lifting) The periodic model Σ_{cl} can always be rewritten as the following time-invariant model:

$$\begin{bmatrix} \mathcal{N} & \mathcal{B} & \mathbf{0} \\ \begin{bmatrix} \mathbf{1}_{nl} & \mathbf{0} \end{bmatrix} - \sigma \begin{bmatrix} \mathbf{0} & \mathbf{1}_{nl} \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathcal{C} & \mathcal{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{21}$$

where $\mathcal{N} = [\mathcal{E}, \mathcal{A}] \in \mathbb{R}^{nN \times n(N+l)}$, $\mathcal{B} \in \mathbb{R}^{nN \times mN}$, $\mathcal{C} = [\mathcal{C}_1, \mathcal{C}_2] \in \mathbb{R}^{pN \times n(N+l)}$ and $\mathcal{D} \in \mathbb{R}^{pN \times mN}$ are given by

$$\mathcal{E} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1,0} & A_{N-1,1} & \cdots & A_{N-1,N-2} \\ \mathbf{0}_n & -\mathbf{1}_n & A_{N-2,0} & \cdots & A_{N-2,N-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\mathbf{1}_n & A_{1,0} \\ \mathbf{0}_n & \cdots & \cdots & \mathbf{0}_n & -\mathbf{1}_n \end{bmatrix} \tag{22}$$

$$\mathcal{A} = \begin{bmatrix} A_{N-1,N-1} & A_{N-1,N} & \cdots & A_{N-1,l+N-2} \\ A_{N-2,N-2} & A_{N-2,N-1} & \cdots & A_{N-2,l+N-3} \\ \vdots & & & \vdots \\ A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ A_{0,0} & A_{0,1} & \cdots & A_{0,l-1} \end{bmatrix} \tag{23}$$

$$\mathcal{C}_1 = \begin{bmatrix} \mathbf{0}_{p\times n} & C_{N-1,0} & \cdots & \cdots & C_{N-1,N-2} \\ \vdots & \ddots & C_{N-2,0} & \cdots & C_{N-2,N-3} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & C_{1,0} \\ \mathbf{0}_{p\times n} & \cdots & \cdots & \cdots & \mathbf{0}_{p\times n} \end{bmatrix} \tag{24}$$

$$\mathcal{C}_2 = \begin{bmatrix} C_{N-1,N-1} & C_{N-1,N} & \cdots & C_{N-1,l+N-2} \\ C_{N-2,N-2} & C_{N-2,N-1} & \cdots & C_{N-2,l+N-3} \\ \vdots & & & \vdots \\ C_{1,1} & C_{1,2} & \cdots & C_{1,l} \\ C_{0,0} & C_{0,1} & \cdots & C_{0,l-1} \end{bmatrix} \tag{25}$$

$$\mathcal{B} = \operatorname{diag}(B_{N-1}, \cdots, B_0), \quad \mathcal{D} = \operatorname{diag}(D_{N-1}, \cdots, D_0) \tag{26}$$

As demonstrated in [15], a linear map exists from the state vector of (21) to the one of (18) such that:

$$\forall l, \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = T \hat{x}_q \text{ with } T = \begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nl} & \mathbf{0}_{nl\times nN} \\ \mathbf{0}_{nl\times nN} & \mathbf{1}_{nl} \end{bmatrix} \end{bmatrix} \tag{27}$$

As a result, equalities (28) and (29) fully capture the input/output behavior of the monodromy lifting (18):

$$\begin{bmatrix} -\mathbf{1} & \Psi & \mathfrak{B} & \mathbf{0} \\ \mathbf{0} & \mathfrak{C} & \mathfrak{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \quad (28)$$

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \begin{bmatrix} T & & \\ & \mathbf{1} & \\ & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix} \mathcal{N} & \mathcal{B} & \mathbf{0} \\ \mathcal{C} & \mathcal{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \quad (29)$$

D. A New Robust $\mathcal{H}_\infty$ Performance Analysis Condition

1) Robust analysis using descriptor lifting: From these results, a new version of the Th. 1 can be stated.

Theorem 2: The worst-case $\mathcal{H}_\infty$ analysis problem is *equivalent* to the following optimization problem leading to ν_{wc}:

$$\nu_{wc} = \min_{P(\theta)\in\mathbb{S}_+^{nl},\ \mathcal{F}(\theta)\in\mathbb{R}^{(n(N+l)+mN)\times nN}} \nu \quad (30)$$

$$\begin{bmatrix} -\mathcal{P}(P) & \\ & -\nu\mathbf{1}_{mN} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} \right\} + \mathrm{He}\left\{ \mathcal{F}\begin{bmatrix} \mathcal{N} & \mathcal{B} \end{bmatrix} \right\} \prec 0 \quad (31)$$

where

$$\mathcal{P}(P) = \begin{bmatrix} -P & \\ & \mathbf{0}_{nN} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{nN} & \\ & P \end{bmatrix} \quad (32)$$

Proof: It is well known that the real bounded lemma (20) can be equivalently written as:

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \end{bmatrix}^T \begin{bmatrix} P & & \\ & -P & \\ & & -\nu\mathbf{1} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \end{bmatrix} + \hat{z}_q^T \hat{z}_q < 0 \quad \text{s.t.} \quad (28) \text{ holds} \quad (33)$$

Referring to the relationship between (28) and (29), this condition can be rewritten in an equivalent way as

$$\begin{bmatrix} \hat{x}_q \\ \hat{w}_q \end{bmatrix}^T \begin{bmatrix} T^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} T & \\ & -\nu\mathbf{1} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \end{bmatrix} + \hat{z}_q^T \hat{z}_q < 0 \quad \text{s.t.} \quad \begin{bmatrix} \mathcal{N} & \mathcal{B} & \mathbf{0} \\ \mathcal{C} & \mathcal{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \quad (34)$$

Then introducing $\mathcal{P}(P) = T^T \mathrm{diag}\{-P, P\} T$, Finsler's Lemma is invoked to obtain (31). ■

Unlike (20), the linear dependency of (31) upon the $A_{i,j}$'s is preserved which makes this condition suitable for convex relaxation.

2) Convex conditions for robust analysis: The previous remark suggests that (31) may be evaluated only on every vertex of the polytope Θ. In addition to that, it can be noticed that the Lyapunov matrix P and the matrices of (6) are decoupled which allows a more complex dependence of P upon the uncertain parameter θ. Here, P is chosen to be polytopic w.r.t. θ.

Theorem 3: (Primal worst-case $\mathcal{H}_\infty$ analysis) The solution ν_g of the following optimization problem:

$$\nu_g = \min_{P^{[i]}\in\mathbb{S}_+^{nl},\ \mathcal{F}\in\mathbb{R}^{(n(N+l)+mN)\times nN}} \nu \quad (35)$$

such that, for $i = \{1, \cdots, L\}$,

$$\begin{bmatrix} -\mathcal{P}(P^{[i]}) & \\ & -\nu\mathbf{1}_{mN} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} (\mathcal{C}^{[i]})^T \\ (\mathcal{D}^{[i]})^T \end{bmatrix} \right\} + \mathrm{He}\left\{ \mathcal{F}\begin{bmatrix} \mathcal{N}^{[i]} & \mathcal{B}^{[i]} \end{bmatrix} \right\} \prec 0 \quad (36)$$

is a guaranteed solution of the worst-case $\mathcal{H}_\infty$ analysis problem, i.e. $\nu_{wc} \leq \nu_g$.

Proof: A Schur complement applied to (36) makes it linear with respect to $\mathcal{C}^{[i]}$ and $\mathcal{D}^{[i]}$. The convex combination of the obtained conditions over all vertices followed by a new Schur complement leads to (31) with $\mathcal{F}(\theta) = \mathcal{F}$ and $\{P, \mathcal{N}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ polytopic functions of θ. Finally, the inequality $\nu_{wc} \leq \nu_g$ comes from the suboptimality of the problem which stems from the enforcement for $\mathcal{F}$ to be independent of θ. ■

IV. ROBUST PERFORMANCE SYNTHESIS

Following the line of [15], construction of a dual model of Σ_{cl} is first performed as a preliminary step toward the derivation of synthesis condition. Indeed, analysis condition for the dual model exhibits direct multiplication between controller gains and LMI variables. It is well-known that this allows to implement the classical linearizing change of variables to render the synthesis condition convex.

A. System Duality for Memory Periodic Models

As system duality is in general well-known in the time-invariant framework, this subsection first deals with a dual version of (21) as an intermediate step. To this end, $l \geq 1$, given by (5), is decomposed as $l = bN + r$ such that $1 \leq r \leq N$ and $b \in \mathbb{N}$.

Proposition 4: (Dual descriptor lifting) A dual version of the descriptor lifting (21) is:

$$\left[\begin{array}{cc|c} \breve{\mathcal{N}}^T & \breve{\mathcal{C}}^T & \mathbf{0} \\ \sigma^{-1}\begin{bmatrix} \mathbf{1}_{nl} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{1}_{nl} \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^{-1}\begin{bmatrix} \mathbf{1}_{p(l-1)} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{1}_{p(l-1)} \end{bmatrix} & \mathbf{0} \\ \hline \begin{bmatrix} \mathbf{0} & \breve{\mathcal{B}}^T \end{bmatrix} & \begin{bmatrix} \mathbf{0} & \breve{\mathcal{D}}^T \end{bmatrix} & \mathbf{1} \end{array}\right] \cdot \left[\begin{array}{c} \begin{bmatrix} \hat{x}_q^d \\ \breve{w}_q^d \end{bmatrix} \\ \hline \breve{z}_q^d \end{array}\right] = \mathbf{0} \text{ with } \hat{x}_q^d \in \mathbb{R}^{n(N+l)+p(N+l-1)} \quad (37)$$

with

$$\breve{\mathcal{N}}^T = \begin{bmatrix} \mathscr{A}_{b+1}^T & \cdots & \mathscr{A}_1^T & \begin{bmatrix} \mathbf{0} \\ \mathscr{A}_{0,0}^T \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{nN\times n(N+l)} \quad (38)$$

$$\breve{\mathcal{C}}^T = \begin{bmatrix} \mathscr{C}_{b+1}^T & \cdots & \mathscr{C}_1^T & \begin{bmatrix} \mathbf{0} \\ \mathscr{C}_{0,0}^T \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{nN\times p(N+l-1)} \quad (39)$$

$$\breve{\mathcal{B}}^T = \begin{cases} \begin{bmatrix} \mathcal{B}^T & \mathbf{0}_{mN\times n} \end{bmatrix} & (r=1) \\ \begin{bmatrix} \mathcal{W}(B^T) & \mathbf{0}_{mN\times n} \end{bmatrix} & (r>1) \end{cases} \in \mathbb{R}^{mN\times n(N+1)} \quad (40)$$

$$\breve{\mathcal{D}}^T = \begin{cases} \mathcal{D}^T & (r=1) \\ \mathcal{W}(D^T) & (r>1) \end{cases} \in \mathbb{R}^{mN\times pN} \quad (41)$$

referring to the operator $\mathcal{W}$ defined by

$$\mathcal{W}(X^T) = \mathrm{diag}\{X_{N-r}^T, \cdots, X_0^T, X_{N-1}^T, \cdots, X_{N-r+1}^T\} \tag{42}$$

Matrices $\mathscr{A}_{0,0} \in \mathbb{R}^{nr\times nr}$, $\mathscr{C}_{0,0} \in \mathbb{R}^{p(r-1)\times n(r-1)}$ and, for $j = \{1, \cdots, b+1\}$, $\mathscr{A}_j \in \mathbb{R}^{nN\times nN}$ and $\mathscr{C}_j \in \mathbb{R}^{pN\times nN}$ are described by

$$\mathscr{A}_{0,0} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1,0} & \cdots & A_{N-1,r-2} \\ \mathbf{0}_n & -\mathbf{1}_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{N-r+1,0} \\ \mathbf{0}_n & \cdots & \mathbf{0}_n & -\mathbf{1}_n \end{bmatrix} \tag{43}$$

$$\mathscr{A}_1 = \begin{bmatrix} A_{N-1,r-1} & \cdots & A_{N-1,N-2} & A_{N-1,N-1} & \cdots & A_{N-1,N+r-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{N-r,0} & & \vdots & \vdots & & \vdots \\ -\mathbf{1}_n & \ddots & \vdots & \vdots & & \vdots \\ & \ddots & A_{1,0} & \vdots & & \vdots \\ \mathbf{0} & & -\mathbf{1}_n & A_{0,0} & \cdots & A_{0,r-1} \end{bmatrix} \tag{44}$$

$$\mathscr{A}_j = \begin{bmatrix} A_{N-1,(j-1)N+r-1} & \cdots & A_{N-1,jN+r-2} \\ \vdots & & \vdots \\ A_{0,(j-2)N+r} & \cdots & A_{0,(j-1)N+r-1} \end{bmatrix}, (2 \leq j \leq b+1) \tag{45}$$

$$\mathscr{C}_{0,0} = \begin{bmatrix} C_{N-1,0} & \cdots & \cdots & C_{N-1,r-2} \\ \mathbf{0}_{p\times n} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{p\times n} & \cdots & \mathbf{0}_{p\times n} & C_{N-r+1,0} \end{bmatrix} \tag{46}$$

$$\mathscr{C}_1 = \begin{bmatrix} C_{N-1,r-1} & \cdots & C_{N-1,N-2} & C_{N-1,N-1} & \cdots & C_{N-1,N+r-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{N-r,0} & & \vdots & \vdots & & \vdots \\ \mathbf{0}_{p\times n} & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & C_{1,0} & \vdots & & \vdots \\ \mathbf{0}_{p\times n} & \cdots & \mathbf{0}_{p\times n} & C_{0,0} & \cdots & C_{0,r-1} \end{bmatrix} \tag{47}$$

$$\mathscr{C}_j = \begin{bmatrix} C_{N-1,(j-1)N+r-1} & \cdots & C_{N-1,jN+r-2} \\ \vdots & & \vdots \\ C_{0,(j-2)N+r} & \cdots & C_{0,(j-1)N+r-1} \end{bmatrix}, (2 \leq j \leq b+1) \tag{48}$$

Proof: The first idea is to append forward to $\hat{x}_q$ the vector $\xi_q = \sigma \begin{bmatrix} \mathbf{0} & \mathbf{1}_{n(N-r)} & \mathbf{0}_{n(N-r)\times nl} \end{bmatrix} \hat{x}_q$ without changing the model. The concatenation of ξ_q and $\hat{x}_q$ can then be divided in $b+2$ terms of equal size, linked with each others by the shift operator σ, such that (21) can be written as:

$$\begin{bmatrix} \sum_{j=0}^{b+1} \mathscr{A}_j \sigma^{-j} & \mathcal{B} & \mathbf{0} \\ \sum_{j=0}^{b+1} \mathscr{C}_j \sigma^{-j} & \mathcal{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \tilde{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{49}$$

where

$$\begin{bmatrix} \mathscr{A}_0 \cdots \mathscr{A}_{b+1} \\ \mathscr{C}_0 \cdots \mathscr{C}_{b+1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}\, \mathcal{N} \\ \mathbf{0}\, \mathcal{C} \end{bmatrix}, \begin{bmatrix} \mathscr{A}_0 \\ \hline \mathscr{C}_0 \end{bmatrix} = \left[\begin{array}{cc} \mathbf{0} & \mathscr{A}_{0,0} \\ \mathbf{0}_{n(N-r)} & \mathbf{0} \\ \hline \mathbf{0} & \mathscr{C}_{0,0} \\ \mathbf{0}_{p(N-r+1)\times n(N-r+1)} & \mathbf{0} \end{array}\right] \tag{50}$$

This new formulation of (21) is suitable for applying the well-known theory of duality for discrete-time time-invariant models proposed in [11]. Thus, the dual version of (49) follows as:

$$\begin{bmatrix} \sum_{j=0}^{b+1} \mathscr{A}_j^T \sigma^j & \sum_{j=0}^{b+1} \mathscr{C}_j^T \sigma^j & \mathbf{0} \\ \mathcal{B}^T & \mathcal{D}^T & \mathbf{1} \end{bmatrix} \begin{bmatrix} \tilde{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{51}$$

Then, as (21) has been artificially enlarged to get (49) by incorporating ξ_q, the size of (51) is now reduced. To this end, let consider first the following change of variables which can be interpreted as a time-shifting:

$$\tilde{x}_q^d = \begin{bmatrix} \begin{bmatrix} \mathbf{0}\ \mathbf{1}_{nr} \end{bmatrix} \breve{x}_q^d \\ \begin{bmatrix} \mathbf{1}_{n(N-r)}\ \mathbf{0} \end{bmatrix} \breve{x}_{q-1}^d \end{bmatrix}, \ \hat{w}_q^d = \begin{bmatrix} \begin{bmatrix} \mathbf{0}\ \mathbf{1}_{p(r-1)} \end{bmatrix} \breve{w}_q^d \\ \begin{bmatrix} \mathbf{1}_{p(N-r+1)}\ \mathbf{0} \end{bmatrix} \breve{w}_{q-1}^d \end{bmatrix} \tag{52}$$

where $\breve{x}_q^d \in \mathbb{R}^{nN}$ and $\breve{w}_q^d \in \mathbb{R}^{pN}$. Following the same rule, $\breve{z}_q^d$ is defined. It comes that $\breve{x}_{q-1}^d$ and $\breve{w}_{q-1}^d$ are not involved in the state equation because of the structure of $\mathscr{A}_0^T$ and $\mathscr{C}_0^T$. Consequently, they can be dropped since $\breve{x}_{q-1}^d$ goes to zero if $\tilde{x}_q^d$ does (the time goes backward).

Then, the expression (37) follows by letting $\hat{x}_q^d$ stacks the useful part of states and inputs from $\breve{x}_{q+b+1}^d$ to $\breve{x}_q^d$ and from $\breve{w}_{q+b+1}^d$ to $\breve{w}_{q+1}^d$. ■

It is worth noticing that the structures of $\mathscr{A}_{0,0}^T$ and $\mathscr{A}_1^T$ match such that the rightest square block of size nN of $\breve{\mathcal{N}}^T$ is a lower triangular matrix with N blocks -1_n on the diagonal. This remark allows to derive a periodic system, which can be considered as a dual of Σ_{cl}, in the same way as (21) has been obtained from Σ_{cl}.

Theorem 4: A dual model of Σ_{cl} is given by

$$\begin{cases} x_{Nq-k-r-1}^d = \sum_{j=0}^{l+k-1} \Big(A_{\vartheta(j,k+r),j}^T x_{Nq-k-r+j}^d + \\ \qquad\qquad C_{\vartheta(j,k+r),j}^T w_{Nq-k-r+j}^d \Big) \\ z_{Nq-k-r}^d = -B_{\vartheta(0,k+r)}^T x_{Nq-k-r}^d - D_{\vartheta(0,k+r)}^T w_{Nq-k-r}^d \end{cases} \tag{53}$$

with $\vartheta(j,k) = j - k \pmod N$.

B. Analysis of the Dual Model

Following the same line as section III, robust analysis theorem for the dual model (53) can be obtained. As duality preserves worst-case $\mathcal{H}_\infty$ performance, the obtained condition can be used for the primal model as well.

Theorem 5: (Dual worst-case $\mathcal{H}_\infty$ analysis) The solution ν_g^d of the following optimization problem:

$$\nu_g^d = \min_{P_l^{[i]} \in \mathbb{S}_+^{nl+p(l-1)},\ \mathcal{F} \in \mathbb{R}^{nN\times(n(N+l)+p(N+l-1))}} \nu \tag{54}$$

such that, for $i = \{1, \cdots, L\}$,

$$\mathcal{P}^d(P^{[i]}, -\nu \mathbf{1}_{pN}) + \mathrm{Sq}\left\{ \begin{bmatrix} \mathbf{0}_{n(l-1)\times mN} \\ \breve{\mathcal{B}}^{[i]} \\ \mathbf{0}_{p(l-1)\times mN} \\ \breve{\mathcal{D}}^{[i]} \end{bmatrix} \right\} + \mathrm{He}\left\{ \begin{bmatrix} \breve{\mathcal{N}}^{[i]} \\ \breve{\mathcal{C}}^{[i]} \end{bmatrix} \mathcal{F} \right\} \prec 0 \tag{55}$$

is a guaranteed solution of the worst-case $\mathcal{H}_\infty$ analysis problem, i.e. $\nu_{wc} \leq \nu_g^d$. For $P \in \mathbb{S}^{nl+p(l-1)}$ and $Z \in \mathbb{S}^{pN}$, the operator $\mathcal{P}^d(P,Z)$ is defined by

$$\mathcal{P}^d(P,Z) = \begin{bmatrix} -P_1 & & -P_2 & \\ & \mathbf{0}_{nN} & & \\ -P_2^T & & -P_3 & \\ & & & Z \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{nN} & & & \\ & P_1 & & P_2 \\ & & \mathbf{0}_{pN} & \\ & P_2^T & & P_3 \end{bmatrix} \tag{56}$$

with $P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$, $P_1 \in \mathbb{S}^{nl}$ and $P_3 \in \mathbb{S}^{p(l-1)}$.

Remark 2: In this theorem, the Lyapunov matrix P belongs to $\mathbb{S}^{nl+p(l-1)}$ which is in stark contrast with the primal analysis condition where $P \in \mathbb{S}^{nl}$. This comes from the fact that $\hat{x}_q^d$ of (37) stacks not only the states of the dual model but also the history of its inputs.

Note that, in general, $\nu_g^d \neq \nu_g$ and that there is no particular relationship between these two upper bounds of ν_{wc}.

C. Robust Synthesis Condition

To move on to the problem of memory controller synthesis, open-loop matrices of Σ are reintroduced in (55) using the relationships (7). To this end, $\breve{\mathcal{N}}$ is decomposed in $\breve{\mathcal{N}}_{op} + \breve{\mathcal{E}}\breve{\mathcal{K}}$ where $\breve{\mathcal{N}}_{op}$ and $\breve{\mathcal{E}}$ gather respectively the state and the input matrices of Σ while $\breve{\mathcal{K}}$ is composed of the $K_{k,j}$'s terms. Similarly, $\breve{\mathcal{C}}$ is rewritten as $\breve{\mathcal{C}}_{op} + \breve{\mathcal{F}}\breve{\mathcal{K}}$.

Theorem 6: (Worst-case $\mathcal{H}_\infty$ synthesis - General case) Consider the following problem:

$$\nu_g^s = \min_{\substack{P_l^{[i]} \in \mathbb{S}_+^{nl+p(l-1)},\ \mathcal{G} \in \mathbb{R}^{nN \times nN} \\ \mathcal{Y} \in \mathbb{R}^{m_u(N+l-1) \times nN}}} \nu \tag{57}$$

such that, for $i = \{1, \cdots, L\}$,

$$\begin{aligned} &\mathcal{P}^d(P^{[i]}, -\nu \mathbf{1}_{pN}) \\ &+ \mathrm{Sq}\left\{ \begin{bmatrix} \mathbf{0}_{n(l-1)\times mN} \\ \breve{\mathcal{B}}^{[i]} \\ \mathbf{0}_{p(l-1)\times mN} \\ \breve{\mathcal{D}}^{[i]} \end{bmatrix} \right\} + \mathrm{He}\left\{ \left(\begin{bmatrix} \breve{\mathcal{N}}_{op}^{[i]} \\ \breve{\mathcal{C}}_{op}^{[i]} \end{bmatrix} \mathcal{G} + \begin{bmatrix} \breve{\mathcal{E}}^{[i]} \\ \breve{\mathcal{F}}^{[i]} \end{bmatrix} \mathcal{Y} \right) \right. \\ &\left. \begin{bmatrix} \mathbf{0}_{nN\times nl} & \mathbf{1}_{nN} & \mathbf{0}_{nN\times p(N+l-1)} \end{bmatrix} \right\} \prec 0 \end{aligned} \tag{58}$$

and the controller which gains are recovered by solving

$$\breve{\mathcal{K}} = \mathcal{Y}\mathcal{G}^{-1} \tag{59}$$

Then, the closed-loop system of (1) with this control law is stable and ν_g^s is a guaranteed cost such that $\nu_{wc} \leq \nu_g^d \leq \nu_g^s$.

The structure $\breve{\mathcal{K}}$ follows the one of $\breve{\mathcal{C}}$ by replacing $C_{k,j}$'s by $K_{k,j}$'s according to the given sequence $\{\alpha_k\}_{k=0}^{N-1}$. Matrix $\breve{\mathcal{N}}_{op}$ (respectively $\breve{\mathcal{C}}_{op}$) results from $\breve{\mathcal{N}}$ ($\breve{\mathcal{C}}$) with $A_{k,0}$ ($C_{k,0}$) replaced by A_k (C_k) while $A_{k,j} = \mathbf{0}$ ($C_{k,j} = \mathbf{0}$) for $j \neq 0$. Matrices $\breve{\mathcal{E}}$ and $\breve{\mathcal{F}}$ are defined by

$$\breve{\mathcal{E}} = \begin{bmatrix} \mathcal{V}(E) \\ \mathbf{0}_{n\times m_u(l+N-1)} \end{bmatrix}, \quad \breve{\mathcal{F}} = \mathcal{V}(F) \tag{60}$$

referring to the operator $\mathcal{V}$ given by

$$\begin{aligned} \mathcal{V}(X) = \mathrm{diag}\{ &\mathbf{1}_{b+1} \otimes \mathrm{diag}\{X_{N-1}, \cdots, X_0\}, \\ &X_{N-1}, \cdots, X_{N-r+1}\} \end{aligned} \tag{61}$$

Controller		Synthesis	Synthesis	Analysis (Th. 5)
id.	$\{\alpha_0, \alpha_1, \alpha_2\}$	Theorem	ν_g^s	ν_g^d
1	$\{1,2,3\}$	Th. 7	12.6196	4.6645
2	$\{3,3,3\}$	Th. 6	23.6111	5.6831
3	$\{3,4,5\}$	Th. 6	20.0606	6.7069
4	$\{3,4,5\}$	Th. 7	4.7464	3.9315

TABLE I

SYNTHESIS AND ANALYSIS RESULTS FOR DIFFERENT CHOICES OF α_k

Decision variables $\mathcal{G}$ and $\mathcal{Y}$ are such that $\mathcal{G} = \mathrm{diag}\{G_{N-1} \cdots, G_0\}$ with $G_k \in \mathbb{R}^{n\times n}$ ($k = \{0, \cdots N-1\}$) and $\mathcal{Y} = \breve{\mathcal{K}}\mathcal{G}$ inherits its structure from $\breve{\mathcal{K}}$.

Proof: If $\begin{bmatrix} \breve{\mathcal{N}}_{op} \\ \breve{\mathcal{C}}_{op} \end{bmatrix} \mathcal{G} + \begin{bmatrix} \breve{\mathcal{E}} \\ \breve{\mathcal{F}} \end{bmatrix} \mathcal{Y}$ is rewritten as $(\begin{bmatrix} \breve{\mathcal{N}}_{op} \\ \breve{\mathcal{C}}_{op} \end{bmatrix} + \begin{bmatrix} \breve{\mathcal{E}} \\ \breve{\mathcal{F}} \end{bmatrix} \breve{\mathcal{K}})\mathcal{G}$, then (58) is equivalent to (55) with $\mathcal{F} = \begin{bmatrix} \mathbf{0} & \mathcal{G} & \mathbf{0} \end{bmatrix}$. This proves that the considered optimization problem leads to a suboptimal solution of the worst-case $\mathcal{H}_\infty$ analysis problem for the obtained closed-loop system. In addition to that, the restriction of $\mathcal{F}$ proves that $\nu_g^d \leq \nu_g^s$. Finally, it can easily be verified that if (58) holds then $\mathcal{G}$ is invertible. ■

Note that $\mathcal{F}$ has been restricted to $\begin{bmatrix} \mathbf{0} & \mathcal{G} & \mathbf{0} \end{bmatrix}$ when deriving (58) from the dual analysis condition (55) in order to ensure the recovery of $\breve{\mathcal{K}}$ from the knowledge of $\mathcal{Y}$ and $\mathcal{G}$. As a result, not only that $\nu_g^d \leq \nu_g^s$, but also it cannot be ensured that every controllers (3), conducing to a guaranteed closed-loop $\mathcal{H}_\infty$ cost by means of Th 5, can be parameterized by (58).

Another important feature of PFMCs can now be exposed: For this class of controllers, the matrix $\mathcal{G}$ can be relaxed to be upper triangular while preserving the recovery of $\breve{\mathcal{K}}$.

Theorem 7: (Worst-case $\mathcal{H}_\infty$ synthesis - PFMC case) In the PFMC case, Th. 6 is valid if $\mathcal{G}$ is redefined as

$$\mathcal{G} = \begin{bmatrix} G_{N-1,0} & \cdots & \cdots & G_{N-1,N-1} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & G_{0,0} \end{bmatrix} \in \mathbb{R}^{nN\times nN} \tag{62}$$

where $G_{k,j} \in \mathbb{R}^{n\times n}$ ($k = \{0, \cdots, N-1\}, j = \{0, \cdots, k\}$).

At our knowledge, these theorems encompass every synthesis conditions leading to memory $\mathcal{H}_\infty$ controllers considered elsewhere in the literature.

V. NUMERICAL RESULTS

The efficiency of the proposed theorems is evaluated by using the 3-periodic model provided by [9] as a benchmark. Periodic controllers, corresponding to different choices of α_k, are designed relying on Th 6 and, when it is possible, on Th. 7. Robust $\mathcal{H}_\infty$ performance of every resulting closed-loop systems is measured by means of Th 5. Results are gathered in Table I where numbers appearing in the first column identify the controller.

The controller 1, corresponding to the PFMC of order 1, has been designed in [8]. It can be verified that the results for this control law are recovered here as a special case.

These experiments illustrate that for a given synthesis theorem, the addition of degrees of freedom to the control law may be effective for reducing the conservatism of the synthesis condition. Indeed, referring to their capacity of lowering ν_g^s, the controllers can be ordered as $1 < 4$ and $2 < 3$. Nevertheless, this remark does not hold anymore when different synthesis theorems are employed. The value ν_g^s obtained with controller 2 is larger than the one with 1 although its control structure is richer. From the controller 3 to 4, ν_g^s is decreased by 76 percent although the control structure remains the same. These two observations bring to light how crucial is the relaxation of $\mathcal{G}$.

As far as analysis is concerned, Table I confirms that $\nu_g^d \leq \nu_g^s$. Indeed, the synthesis condition has been derived from Th. 5 by enforcing the structure of $\mathcal{F}$ which implies that $\nu_g^d \leq \nu_g^s$. However, it is worth noticing that the hierarchy between controller may not be preserved from the synthesis step to the analysis one: From the controller 2 to 3, ν_g^s decreases while ν_g^d increases.

Remark 3: For this particular example, using the primal worst-case analysis theorem would have lead to the same results, i.e. $\nu_g = \nu_g^d$ but, as already said, this is not the case in general. Other results experiments, which are not presented in this paper, confirm this assertion.

VI. CONCLUSIONS AND FUTURE WORKS

The flexibility of the proposed approach allows the user to freely add degrees-of-freedom to the control law which appears to effectively decrease the guaranteed cost of the $\mathcal{H}_\infty$ problem. Furthermore, numerical examples have shown that for a particular structure of controllers the efficiency of the design theorem can be significantly enhanced by relaxing the matrix of slack-variables.

As noticed in [13], providing deeper guidelines for the choice of α_k's and for the definition of the starting point of the considered period remain challenging subjects and will be the topic of future works. In addition to that, dealing with other performance measures, like the $\mathcal{H}_2$ induced norm, is under current investigation.

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