Using polynomial optimization to solve the fuel-optimal linear impulsive rendezvous problem

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Nomenclature

- a = semi-major axis ;
- e = eccentricity ;
- ν = true anomaly ;
- $\phi(\nu)$ = fundamental matrix of relative motion ;
- $B(\nu)$ = input matrix in the dynamic model of relative motion ;
- $R(\nu) = \phi^{\#t}(\nu)B(\nu) = \phi^{-1}(\nu)B(\nu)$ = primer vector evolution matrix ;
- $u_f = \phi^{-1}(\nu_f)X_f - \phi^{-1}(\nu_1)X_1 \neq 0$ = boundary conditions ;
- N = number of velocity increments ;
- $\nu_i, \forall\, i = 1, \cdots, N$ = impulses application times ;
- Δv_i = impulse modulus at ν_i ;

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- $\beta(\nu_i)$ = impulse direction vector at ν_i ;

- $\Delta V(\nu_i) = \Delta v_i \beta(\nu_i)$ = velocity increment vector at ν_i ;

- $\{b_i\}_{i=1,\cdots,N}$ = sequence of variables b_i, $\forall\, i = 1, \cdots, N$;

- $\leftarrow$ = an affectation symbol inside an algorithm ;

- $\mathbb{R}[x_1, \cdots, x_n]$ stands for the algebra of polynomials in variables $(x_1, \cdots, x_n)$ with coefficients in $\mathbb{R}$;

I. Introduction

The first space missions involving more than one vehicle (Gemini, Apollo, Vostok) have highlighted the fact that the space rendezvous between two spacecraft is a technology raising relevant open control issues. Strictly speaking, the space rendezvous maneuver is an orbital transfer between a passive target and an actuated spacecraft called the chaser, within a fixed or floating time period. Since the rendezvous maneuver can only be performed within a certain period of time while utilizing as little fuel as possible to extend the chaser lifetime, we mainly focus on the so-called time-fixed fuel optimal rendezvous problem [1], [2].

In this paper, far range rendezvous in a linearized gravitational field is viewed as a time-fixed minimum-fuel impulsive orbital transfer between two known elliptical orbits. Because of the constraints of on-board guidance algorithms, numerical solutions based on linear relative motion are particularly appealing. Indirect approaches based on the solution of optimality conditions derived from Pontryagin's Maximum Principle, leading to the development of the so-called primer vector theory presented in [3], have been used in numerous studies [2], [4], [5], [6]. For a fixed number of maneuvers, imposed by operational time constraints for instance, a geometric *adhoc* approach, first proposed in the seminal work [4] and extended in [7] is given for the rendezvous between circular orbits while [8] considers elliptical orbits and four-impulse solutions. Our first contribution is to provide the designer with a clear and systematic numerical method of solution for the rendezvous problem with any number of impulses fixed *a priori* in the case of elliptical orbits. Exploiting the polynomial nature of the necessary and sufficient conditions of optimality of the associated optimal control problem, a convergent hierarchy of convex relaxations expressed in terms of linear matrix inequalities may be built for the genuine nonlinear polynomial optimization problem [9], [10]. Convex optimization problems involving linear matrix inequalities can be solved efficiently using linear matrix inequalities dedicated parsers

[11] and semidefinite programming solvers [12]. One of the key advantages of the procedure is to provide with a certificate of global optimality of the optimal solution when it exists and a certificate of infeasibility when the problem has no solution.

For comparison's sake, the efficiency of the proposed algorithm is illustrated with three different numerical examples. One academic example taken from Carter's reference [2] is first studied. Also reviewed are two realistic scenarios based on PRISMA which is a "technology in-orbit testbed mission" demonstrating formation flight [13] and SIMBOL-X formation flying mission which is a particular example of a high elliptical reference orbit [14].

II. The time-fixed optimal rendezvous problem as a polynomial optimization problem

A. Time-fixed optimal rendezvous problem in a linear setting

This paper focuses on transfers between closed non-circular orbits for the fixed-time minimum-fuel rendezvous of an active (actuated) spacecraft called the chaser with a passive target spacecraft assuming a linear impulsive setting and an unperturbed Keplerian relative motion described by Tschauner-Hempel (TH) equations [15] and Yamanaka-Ankersen's state transition matrix [16]. The impulsive approximation for the thrust means that instantaneous velocity increments are applied to the chaser instead of finite-thrust powered phases of finite duration. The thrust per unit mass vector is therefore defined by:

$$\Gamma(\nu) = \sum_{i=1}^{N} \Delta V(\nu_i)\delta(\nu - \nu_i) = \sum_{i=1}^{N} \Delta v_i \beta(\nu_i)\delta(\nu - \nu_i) \tag{1}$$

where $0 \leq \nu_1 \leq \cdots \leq \nu_N \leq T$, N is the number of impulsive controls and $\Delta V(\nu_i)$ the discontinuity in the velocity vector due to a thrust impulse at ν_i, defined by $\delta(\nu - \nu_i)$. Note that the true anomaly has been chosen as the independent variable. For a minimum-fuel rendezvous, the cost function is defined by the total characteristic velocity:

$$J = \int_0^T \sum_{i=1}^{N} \|\Delta V(\nu_i)\|_2 \delta(\nu - \nu_i) d\nu = \sum_{i=1}^{N} \Delta v_i \tag{2}$$

Under the previous assumptions, if the relative equations of motion of the chaser are supposed to be

linear and the considered rendezvous problem may be reformulated as the following optimal control problem:

$$\begin{aligned}
\min_{N,\nu_i,\Delta v_i,\beta(\nu_i)} \quad & J = \sum_{i=1}^{N} \Delta v_i \\
\text{s.t.} \quad & \dot{X}(\nu) = A(\nu)X(\nu) + B(\nu)\sum_{i=1}^{N} \beta(\nu_i)\Delta v_i \delta(\nu - \nu_i) \\
& \|\beta(\nu_i)\| = 1 \\
& \Delta v_i \geq 0
\end{aligned} \tag{3}$$

where matrices $A(\nu)$ and $B(\nu)$ define the state-space model of relative dynamics. One of the main difficulties encountered with problem (3) is that the number of impulses N is part of the optimization process. A classical relaxation of (3) consists of considering a fixed-scenario optimal rendezvous for which the number of impulses N is *a priori* set and where the first and last impulses are applied at the beginning and end of the rendezvous respectively [4]. It should be pointed out that the number of impulses N may be chosen equal to the upper-bound N_N on the optimal number of impulses according to Neustadt [17]. Moreover, since a transition matrix may be computed in closed form for the TH equations (see [16]), it may be appropriate to replace the differential constraint on dynamics by the equivalent algebraic constraint involving this transition matrix. Assuming boundedness conditions on relative position and velocity, problem (3) for a fixed number of impulses N may then be reformulated as the following optimization problem [6]:

$$\begin{aligned}
\min_{\nu_i,\Delta v_i,\beta(\nu_i)} \quad & J = \sum_{i=1}^{N} \Delta v_i \\
\text{s.t.} \quad & u_f = \sum_{i=1}^{N} \phi^{-1}(\nu_i)B(\nu_i)\Delta v_i \beta(\nu_i) = \sum_{i=1}^{N} R(\nu_i)\Delta v_i \beta(\nu_i) \\
& \|\beta(\nu_i)\| = 1 \\
& \Delta v_i \geq 0
\end{aligned} \tag{4}$$

where the optimization decision variables are $\{\nu_i\}_{i=1,\cdots,N}$, $\{\Delta v_i\}_{i=1,\cdots,N}$, $\{\beta_i\}_{i=1,\cdots,N}$.

We assume that $\nu_1 = 0$ and $\nu_N = \nu_f$. The algorithm presented here may deal with initial or terminal coasts that may be of interest for reducing the cost but this will not be considered for simplicity and conciseness of exposition.

B. Primer vector theory and Carter's necessary and sufficient conditions for a fixed number of impulses

Applying the Maximum Principle on problem (3) for a fixed number of impulses N as described in Lawden [3] or a Lagrange multiplier rule for the equivalent problem (4) as in [8], one can derive necessary

conditions of optimality in terms of the co-state vector associated with the relative velocity and referred to as the primer vector (see conditions (5) to (9) in theorem II.1 below). Prussing has first shown in [18] that these conditions are also sufficient in the case of linear relative motion with the strengthening semi-infinite constraint (12) that should be fulfilled on the continuum $[\nu_1, \nu_N]$ and is expressed in terms of the transition matrix $R(\nu)$ of the primer vector denoted $\lambda_v(\nu) = R(\nu)\lambda$. These results are summarized using the formalism of Carter in the following theorem.

Theorem II.1 *[6]*

$(\nu_1, ..., \nu_N, \Delta v_1, ..., \Delta v_N, \beta(\nu_1), ..., \beta(\nu_N))$ *is the optimal solution of problem* (4) *if and only if there exists a non-zero vector* $\lambda \in \mathbb{R}^m,\ m = \dim(\phi)$ *that verifies the necessary and sufficient conditions:*

$$\Delta v_i = 0 \text{ or } \beta(\nu_i) = -R'(\nu_i)\lambda,\ \forall\, i = 1, \cdots, N = 2 + r \tag{5}$$

$$\Delta v_i = 0 \text{ or } \lambda' R(\nu_i) R(\nu_i)' \lambda = 1,\ \forall\, i = 1, \cdots, N \tag{6}$$

$$\Delta v_{k_i} = 0 \text{ or } \nu_{k_i} = \nu_1 \text{ or } \nu_{k_i} = \nu_f \text{ or } \lambda' \frac{dR(\nu_{k_i})}{d\nu} R(\nu_{k_i})' \lambda = 0,\ \forall\, i = 2, \cdots, N-1 \tag{7}$$

$$\sum_{i=1}^{N} \left[R(\nu_i) R'(\nu_i)\right] \lambda \Delta v_i = -u_f \tag{8}$$

$$\Delta v_i \geq 0,\ \forall\, i = 1, \cdots, N \tag{9}$$

$$\sum_{i=1}^{N} \Delta v_i = -u_f' \lambda > 0 \tag{10}$$

$$-u_f' \lambda \text{ is the minimum of the set defined as : } \{\lambda \in \mathbb{R}^m \ :\ (5) - (10) \text{ are verified}\} \tag{11}$$

$$\|\lambda_v(\nu)\| \leq 1 \ \ \forall\, \nu \in [\nu_1, \nu_N] \tag{12}$$

A numerical solution of optimality conditions (5) to (12) in the unknowns $\lambda \in \mathbb{R}^m$, $\{\nu_i\}_{i=1,\cdots,N}$, $\{\beta_i\}_{i=1,\cdots,N}$, $\{\Delta v_i\}_{i=1,\cdots,N}$ is still hard to find for a fixed number of impulses N due to the nonconvex and transcendental nature of these polynomial equalities and inequalities. A numerical procedure based on polynomial optimization is now proposed to address this numerical issue.

C. Numerical insights for the fixed-scenario optimal rendezvous

The difficulty related to transcendental equations may be overcome through the use of a gridding technique providing a numerical approximation of the optimal impulse times. The major drawback of gridding techniques is the definition of the resolution value (density) of grid points in order to adequately handle hard equality constraints as defined in (7). Therefore, we propose a dynamic gridding strategy to overcome the first drawback while equality constraints (7) are relaxed as:

$$-\epsilon_1 < \lambda' \frac{dR(\nu_{k_i})}{d\nu} R(\nu_{k_i})' \lambda < \epsilon_1, \ \forall\, i = 2, \cdots, N-1 \tag{13}$$

where $\epsilon_1 > 0$ is a small parameter representing the tolerance over equality constraints.

Furthermore, verifying (12) requires imposing the positivity of the quadratic polynomial $1 - \lambda' R(\nu) R(\nu) \lambda$ on the continuum of the whole rendezvous interval $[\nu_1, \nu_f]$. This inequality constraint is not as hard as the previous ones and may be roughly discretized over a static grid of M_d equidistant points considering that a few number of points (typically 50) will be necessary to satisfy this constraint due to the usual shape of the function $\|\lambda_v\|$. In every case, the primer vector trajectory will be propagated and checked *a posteriori* on a finer grid of points.

Another issue is related to the poor scaled representation due to large differences between initial time and positions and final ones. One can easily see that (6) is already in a Cholesky factorized form and other conditions may be factorized using λ and Δv_i's decision variables. Therefore, the problem could be scaled using Schur decomposition. However, as the scaling matrix entries differ by several orders of magnitude in this case, a different scaling strategy has been preferred. First, diagonal temporary scaling matrices (S_{T_i}) are calculated for each impulse ($i = 1, \ldots, N$) in order to make diagonal entries of $Q_{S_i} = S_{T_i} R(\nu_i) R'(\nu_i) S_{T_i}$ matrix less than or equal to 1. Second, entries of diagonal scaling matrix (S) are defined using the corresponding entries of minimum value of S_{T_i} matrices.

For a given grid of impulse times, the optimization problem is then given by:

$$
\begin{aligned}
&\min_{\lambda,\Delta v_i} \; -u_f'\lambda \\
&\text{s.t.} \\
&-u_f = \sum_{i=1}^{N} R(\nu_i)R'(\nu_i)S\lambda\Delta v_i \\
&\lambda' SR(\nu_i)R(\nu_i)'S\lambda = 1, \; \forall\, i = 1, \cdots N \\
&\left|\lambda' S\frac{dR(\nu_)}{d\nu}R(\nu_i)'S\lambda\right| < \epsilon, \; \forall\, i = 2, \cdots, N-1 \\
&\Delta v_i \geq 0, \; \forall\, i = 1, \cdots N \\
&\lambda' SR(\nu_j)R(\nu_j)'S\lambda \leq 1, \; \forall\, j = 0, \cdots, M_d
\end{aligned}
\tag{14}
$$

The problem (14) belongs to the class of polynomial nonconvex optimization problems with respect to the variables $\lambda \in \mathbb{R}^m$ and Δv_i, $i = 1, ..., N$ for which a hierarchy of convex relaxations has been proposed [9], [10]. Given below in the next section are some basic facts about convex relaxations of nonconvex polynomial optimization problems.

D. Convex relaxations for polynomial optimization

The problem (14) can be written under the general form:

$$
\begin{aligned}
\mathbb{P}: \; g^\star \; &= \; \min \;\; g_0(x) \\
&\qquad \text{s.t.} \quad g_i(x) \geq 0, \; i = 1, \ldots, m
\end{aligned}
\tag{15}
$$

where $g_i(x) \in \mathbb{R}[x_1, \cdots, x_n] \; : \; \mathbb{R}^n \to \mathbb{R}$. This class of optimization problems is known as NP-hard. The feasible set of (15) is denoted:

$$
\mathbb{K} = \{x \in \mathbb{R}^n \mid g_i \geq 0, \; i = 1, \ldots, m\} \tag{16}
$$

Computing the global optimum of $\mathbb{P}$ is effected by finding g^*, where $g_0(x) - g^\star \geq 0$ is a globally positive polynomial on the set $\mathbb{K}$ i.e. $g_0(x) - g^\star \in \mathcal{P}_n^d$, the convex cone of semi definite positive polynomials (SDP) in $\mathbb{R}^D$ having degree $\leq d$ defined by:

$$
\mathcal{P}_n^d = \{p \in \mathbb{R}[x_1, \cdots, x_n] \mid p(x) \geq 0 \; \forall\, x \in \mathbb{R}^n\} \qquad D = \begin{pmatrix} n+d \\ d \end{pmatrix} \tag{17}
$$

If $\mathcal{S}_n^d$ is the convex cone of sum of squares polynomials (SOS) in $\mathbb{R}^D$ given by:

$$
\mathcal{S}_n^d = \left\{ p \in \mathbb{R}[x_1, \cdots, x_n] \mid p(x) = \sum_{i=1}^{r} q_i(x)^2 \right\} \tag{18}
$$

then each element of $\mathcal{S}_n^d$ can be characterized by a linear matrix inequality (LMI) formulation [9], [10], [19]:

$$p(x) = \sum_{\alpha} p_\alpha x^\alpha \in \mathcal{S}_n^d \;\Leftrightarrow\; \exists\, X \;:\; p(x) = z' X z \quad X \succeq 0 \tag{19}$$

where z is the monomials of degrees $\leq d$ array. For a feasible matrix X, Cholesky factorization gives:

$$X = Q'Q \;\; Q' = \Big[\; q_1, \;\cdots\; , q_r \;\Big] \tag{20}$$

and

$$p(x) \;=\; z'Q'Qz = \|Qz\|_2 = \sum_{i=1}^{r} (q_i' z)^2 = \sum_{i=1}^{r} q_i^2(x) \tag{21}$$

where the number of squared terms is r = $\mathrm{rang}(X)$. By identifying the coefficients of $p(x) = z'Xz = \sum_\alpha p_\alpha x^\alpha \geq 0$, we obtain the following LMIs:

$$\begin{aligned} &\mathrm{trace}\, H_\alpha X = p_\alpha \quad \forall\, \alpha \\ &X \succeq 0 \end{aligned} \tag{22}$$

where H_α is a Hankel matrix. Determining whether a polynomial belongs to $\mathcal{S}_n^d$ is therefore equivalent to solving an LMI problem where powerful solvers may be used [12]. Furthermore, $\mathcal{S}_n^d \subset \mathcal{P}_n^d$: A lower bound to the polynomial optimization problem (15) may be easily computed. Finding the multipliers $q_i(x) \in \mathcal{S}_n^d$ such that:

$$p(x) \;=\; g_0(x) - g^* = q_0(x) + \sum_{i=1}^{m} g_i(x) q_i(x) \Rightarrow g_0(x) - g^* \geq 0 \;\; \forall\, x \in \mathbb{K} \tag{23}$$

for a fixed $\deg(q_i(x))$, is a semi-definite programming problem. For $\deg(p(x)) = 2k$, the kth order convex LMI relation stating that $p(x) = g_0(x) - g^* \in \mathcal{S}_n^d,\; \forall\, x \in \mathbb{K}$ is an LMI problem whose optimal solution $p_k^\star$ gives a lower bound to the global optimum g^*. Under some assumptions, it has been shown in [9] that a hierarchy of monotone convex relaxations can be constructed which asymptotically converges to the global optimum of the problem (15). On the contrary, if the set $\mathbb{K}$ is empty, a certificate of infeasibility may be computed.

III. Using a polynomial optimization algorithm to solve the rendezvous problem for a fixed number of impulses

In this section, a rendezvous algorithm with a fixed number N of impulses, relying on a dynamic gridding strategy and relaxations of the genuine polynomial optimization problem is presented. Algorithm input

arguments are the initial state (position/velocity) X_1, the final state vector X_f, the desired resolution (res_d) on the impulse times, dimension of Θ_i sets (d), first impulse date (ν_1), final impulse time (ν_f), number of impulses (N), initial precision value ϵ_1 for (13) and precision value ϵ_2 on the norm of the primer vector. The output arguments are the optimum impulse times ($\nu_2^\star, \ldots, \nu_{N-1}^\star$), optimal primer vector λ^*, optimal impulses (optimal amplitude Δv_i^* and optimal direction β_i^*) and the optimal fuel consumption J^*.

Let $\Theta_2, \ldots, \Theta_{N-1}$ be nonempty ordered sets given by $\Theta_i := \{\nu_{i_1}, \ldots, \nu_{i_d}\}$ where $\nu_{i_1}, \ldots, \nu_{i_d}$ represent the possible candidates for i^{th} impulse date. The distances between sequential candidates are equal and represented by the current resolution value (res). In addition to these $N-2$ sets, it is necessary to define the two singletons $\Theta_1 = \{\nu_1\}$ and $\Theta_N = \{\nu_f\}$.

A. A Polynomial Rendezvous Delta-V (PRDV) algorithm

1 Let $\nu_{i_1} := \nu_1$, $\nu_{i_d} := \nu_f$ for all $i = 2, \cdots, N-1$.

2 Generate all Θ_i sets for all $i = 2, \ldots, N-1$.

3 Generate the set $\mathcal{S}$ consisting of all the grid vectors of impulses times:

$$\mathcal{S} := \left\{ \begin{bmatrix} \nu_1 & x_2 & \cdots & x_{N-1} & \nu_f \end{bmatrix}^t \in \mathrm{R}^N : x_i \in \Theta_i, \;\; x_i < x_{i+1}, \forall i = 2, \ldots, N-1 \right\} \tag{24}$$

4 Compute the global optimal solution α_k, $k = 1, \cdots, \mathrm{card}(\mathcal{S})$, if it exists, of the polynomial optimization problem (14) for every grid vector in $\mathcal{S}$.

5 If there exists no global solution α_i, let $\epsilon_1 \leftarrow \epsilon_1 + \frac{\epsilon_1}{2}$ and go to previous step.

6 Find the best solution $\alpha^* = \min_i \alpha_i$ and its argument (optimal grid vector $[\nu_1, x_2^\star, \ldots, x_{N-1}^\star, \nu_f]^t$ in $\mathcal{S}$ set, optimal amplitudes Δv_i^*, $i = 1, \cdots, N$, optimal costate vector λ^*) w.r.t. fuel consumption.

7 Define $l_i := \{x_i \in \Theta_i : x_i < x_i^\star\}$ and $u_i := \{x_i \in \Theta_i : x_i > x_i^\star\}$ sets for all $i = 2, \ldots, N-1$.

8 Calculate the resolution value ($res = \nu_{2_2} - \nu_{2_1}$).

9 If the l_i set is nonempty then assign $\nu_{i_1} \leftarrow \max(l_i)$ else $\nu_{i_1} \leftarrow x_i^\star$.

10 If the u_i set is nonempty then assign $\nu_{i_d} \leftarrow \min(u_i)$ else $\nu_{i_d} \leftarrow x_i^\star$.

11 If the resolution value is greater than res_d, let $\epsilon_1 \leftarrow \frac{\epsilon_1}{2}$ and go to Step 2 else go to step 12.

12 Check the primer vector with $(\lambda^*, (\Delta v_i^*)_{i=1,\cdots,N}, (\nu_i^*)_{i=1,\cdots,N})$

$$\|R'(\nu)\lambda\|_2 - 1 \leq \epsilon_2 \tag{25}$$

13 If the previous test is positive, compute the direction and amplitude of the optimal impulses:

$$\beta^*(\nu_i^*) = -R(\nu_i^*)'\lambda^* \quad \Delta V^*(\nu_i^*) = \beta^*(\nu_i^*)\Delta v_i^* \tag{26}$$

Some additional comments are now given for a greater clarity of each step.

- Polynomial optimization is achieved under MATLAB©, using the free academic Gloptipoly software developed at LAAS [19]. This software package requires the use of SeDuMi [12] and YALMIP [11].
- Each optimal solution obtained at 4th step for each grid point is certified to be global if it exists.

IV. Applications and numerical examples

Only coplanar elliptical rendezvous problems are considered for numerical illustration of the results proposed. Under Keplerian assumptions and for an elliptical rendezvous, the complete rendezvous problem may be decoupled between the out-of-plane rendezvous problem for which an analytical solution may be found [5] and the coplanar problem. For the latter problem, the bound of Neustadt [17] on the optimal number of impulses is 4. Finally, all numerical examples are processed under Matlab 2008b© running on a Pentium D 3.4GHz system with 1GB ram.

A. Case study 1

Consider the first example presented by Carter in [2]. It consists of a coplanar four-impulse circle-to-circle rendezvous. The rendezvous maneuver must be completed in one orbital period (see Table 1).

Eccentricity	$e = 0$
ν_0	0 rad
$X_0^t = [R_0^t \; v_0^t]$	$[\,1 \;\; 0 \;\; 0 \;\; 0\,]$
ν_f	2π rad
$X_f^t = [R_f^t \; v_f^t]$	$[\,0 \;\; 0 \;\; 0 \;\; 0.427\,]$
N_{max}	4

Table 1: Data for Carter's first example

For the sake of comparison, the results are shown in Table 2 (absolute precision less than 0.01 for the times of application of velocity increments) alongside those of reference [2].

	Carter [2]	PRDV Algorithm
λ^*	$\begin{bmatrix} -9.8201 \cdot 10^{-2} \\ -5.2929 \cdot 10^{-2} \\ 1.8228 \cdot 10^{-1} \\ -4.3629 \cdot 10^{-1} \end{bmatrix}$	$\begin{bmatrix} -6.5876 \cdot 10^{-2} \\ -6.4727 \cdot 10^{-1} \\ 2.53 \cdot 10^{-4} \\ -6.2193 \cdot 10^{-1} \end{bmatrix}$
ν_{int_1} (rad)	$\frac{\pi}{2} \simeq 1.57$	1.7077
ν_{int_2} (rad)	$\frac{3\pi}{2} \simeq 4.7124$	4.5859
$\Delta V(\nu_0)^t$	$[\,-0.0273 \;\; 0.0344\,]$	$[\,-0.02655 \;\; 0.0329\,]$
$\Delta V(\nu_1)^t$	$[\,0.0897 \;\; 0.0119\,]$	$[\,0.0917 \;\; 0.004614\,]$
$\Delta V(\nu_2)^t$	$[\,-0.0897 \;\; 0.0119\,]$	$[\,-0.09011 \;\; 0.00434\,]$
$\Delta V(\nu_f)^t$	$[\,0.0273 \;\; 0.0344\,]$	$[\,0.0259 \;\; 0.0322\,]$
Fuel-cost	0.2688	0.2667

Table 2: Result comparison for Carter's first example

Interestingly, the results are not so different due to Carter's particularly smart *a priori* choice of the impulse times. Nevertheless, Carter's results are clearly not optimal. More details about Carter's solution and PRDV solution are given in the plot of the in-plane trajectories of both solutions shown in Fig. 1 where red stars depict the positions of the application of the velocity increments. If these solutions lead to almost similar trajectories and consumption, only the one obtained with the PRDV algorithm is guaranteed to be

optimal for the fixed number of impulses $N = 4$.

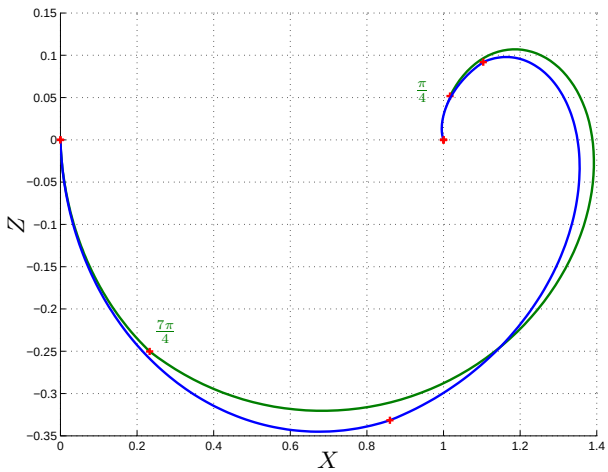

Figure 1: Detailed trajectories for solutions: PRDV algorithm (blue), Carter's solution (green)

Primer vector trajectory and impulse vectors are shown in Fig. 2 in the $(\lambda_{vx}, \lambda_{vy})$ plane.

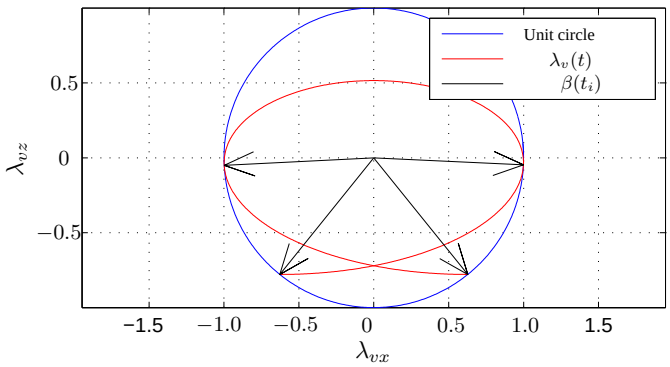

Figure 2: Primer vector in-plane trajectory for case study 1

B. Case study 2

Following the first academic numerical example, a more realistic illustration based on PRISMA [13] is now presented. The orbital elements of the target orbit as well as initial and final rendezvous conditions are listed in Table 3.

Semi-major axis	$a = 7011$ km
Inclination	$i = 98$ deg.
Argument of Perigee	$\omega = 0$ deg.
Right Ascension of the Ascending Node	$\Omega = 190$ deg.
Eccentricity	$e = 0.004$
True Anomaly	$\nu_0 = 0$ rad.
t_0	0 s
$X_0^t = [R_0^t \; v_0^t]$	$[\,-10 \;\; 0 \;\; 0 \;\; 0\,]$ km -km/s
t_f	64620 s
$X_f^t = [R_f^t \; v_f^t]$	$[\,-100 \;\; 0 \;\; 0 \;\; 0\,]$ m -m/s
N_{max}	4

Table 3: PRISMA rendezvous characteristics

Duration of the rendezvous is approximately 12 hours for an expected average cost of 20 cm/s [13].

Running the PDRV algorithm with $N = 3$ leads to a three-impulse solution whose global optimality is certified.

	PRDV Algorithm
t_{int} (s)	3171.95
ν_{int} (rad)	3.4071
$\Delta V(\nu_0)^t$	$[\,-0.04911 \;\; 0.001994\,]$
$\Delta V(\nu_1)^t$	$[\,-0.002037 \;\; 0.000007\,]$
$\Delta V(\nu_f)^t$	$[\,0.051313 \;\; 0.001466\,]$
Fuel-cost m/s	0.102525

Table 4: Results of the PRDV algorithm for the PRISMA case study

Figure 3 shows primer vector magnitude during transfer.

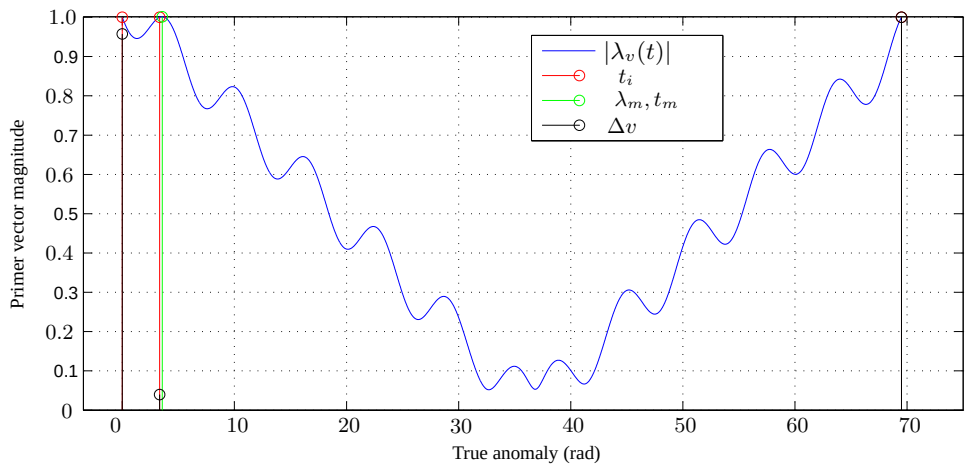

Figure 3: Primer vector magnitude during PRISMA mission

Note the low magnitude of the second impulse (0.002 m/s) with respect to the initial and final velocity increments (0.0492 m/s and 0.0513 m/s) but these velocity increments play a significant role in the optimality of the result. In particular, they provide the right chaser orientation for the long drift (61400 s) between the second impulse and the final one. The long drifting period of 61400 s of the optimal solution is clearly illustrated in Fig. 4 where the in-plane trajectory and impulse positions are represented.

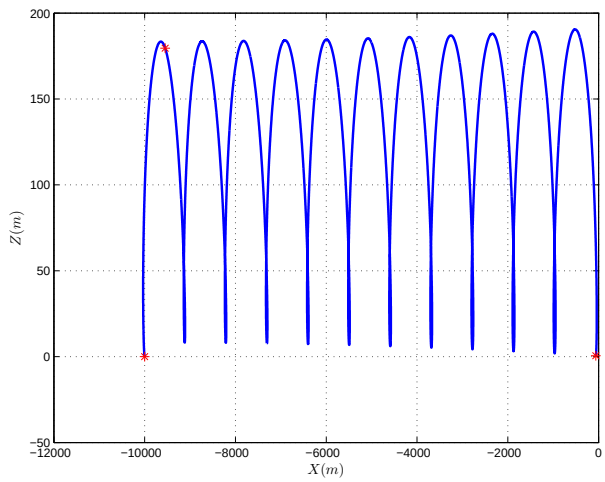

Figure 4: In-plane trajectory and impulse positions * for PRISMA mission

Finally, note that the optimal cost is half the expected average cost of 20 cm/s [13].

C. Case study 3

In order to validate PRDV algorithm for a highly elliptical reference case, the SIMBOL-X mission is now studied. For a comprehensive description of this mission, the interested reader may consult the reference [14]. The characteristics of this rendezvous are described in Table 5 where the coordinates of the initial and final relative positions and velocities are converted into the LVLH frame.

Semi-major axis	$a = 106246.9753$ km
Inclination	$i = 5.2$ deg.
Argument of Perigee	$\omega = 180$ deg.
Right Ascension of the Ascending Node	$\Omega = 90$ deg.
Eccentricity	$e = 0.798788$
True Anomaly	$\nu_0 = 135$ rad.
t_0	7 s
$X_0^t = [R_0^t \; v_0^t]$	$[\; -18309.5 \;\; 23764.7 \;\; 0.0542 \;\; 0.0418 \;]$ m -m/s
t_f	50002 s
$X_f^t = [R_f^t \; v_f^t]$	$[\; -335.12 \;\; 371.1 \;\; -0.00155 \;\; -0.00140 \;]$ m -m/s
N_{max}	4

Table 5: SIMBOL-X rendezvous characteristics

Rendezvous duration is 49995 s, that is, much shorter than the orbital period. The final solution is a two-impulse transfer as described in Table 6 and certified by PRDV algorithm. Note that a certificate of unfeasibility may be obtained for the 3-impulse and 4-impulse scenarii. As shown in Fig. 5, cost cannot be improved by adding interior impulses since the primer vector norm does not exceed 1 on $[\nu_1, \nu_f]$.

$\begin{bmatrix} t_1 & t_f \end{bmatrix}$ (s)	$\begin{bmatrix} 7 & 50002 \end{bmatrix}$
$\begin{bmatrix} \nu_1 & \nu_f \end{bmatrix}$ (rad)	$\begin{bmatrix} 2.3562 & 2.7859 \end{bmatrix}$
$\Delta_V(\nu_1)^t$	$\begin{bmatrix} 0.6193 & -0.5061 \end{bmatrix}$
$\Delta_V(\nu_f)^t$	$\begin{bmatrix} -0.1748 & 0.4912 \end{bmatrix}$
Cost m/s	1.3212

Table 6: SIMBOL-X results

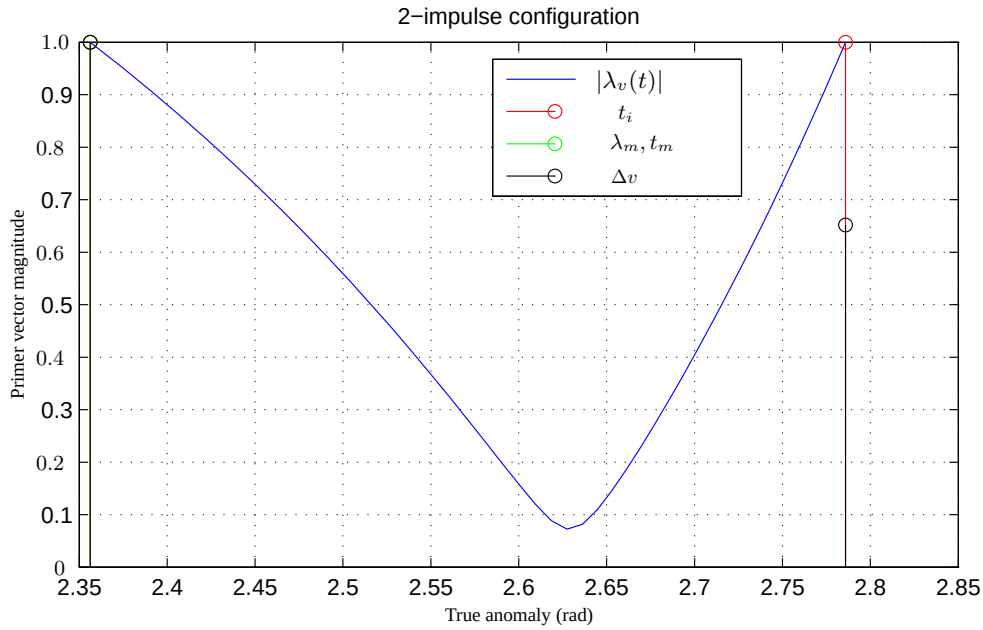

Figure 5: Primer vector magnitude for SIMBOL-X mission

Figure 6 shows the in-plane trajectory for the first rendezvous of the SIMBOL-X mission resulting in a direct transfer between chaser and target.

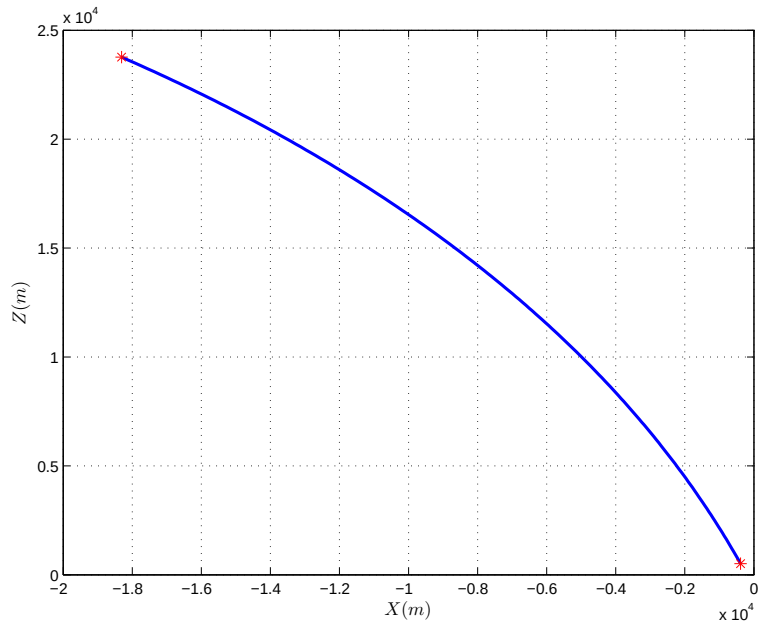

Figure 6: In-plane trajectory for the SIMBOL-X mission

V. Conclusion

A numerical algorithm based on polynomial optimization and tools from algebraic geometry has been proposed to address the issue of time-fixed optimal rendezvous in a linear setting. This algorithm relying on polynomial optimization provides a guarantee of global optimality for its solution for a fixed number of impulses. A numerically reliable procedure to determine the global solution of the impulsive linear rendezvous problem is available for the designer. The numerical results given by this procedure are consistent and may be obtained for different types of rendezvous ranging from circular to highly elliptical orbits and from the classical two-impulse solutions to numerically involved four-impulse solutions.

Despite the good numerical results presented, some improvement can still be expected if more sophisticated transition matrices including orbital perturbation effects are used. Another avenue of research deals with the extension of the proposed algorithm for optimal trajectory planning with collision avoidance constraints.

Aknowledgements

This work was supported by CNES Grant R-S07/VF-0001-065 and the work of fourth author was partly funded by TUBITAK. The authors would like to thanks Professor J. E. Prussing for helpful discussions.

References

[1] Prussing, J. E., and Chiu, J. H. "Optimal multiple-impulse time-fixed rendezvous between circular orbits,". *Journal of Guidance, Control and Dynamics*, Vol. 9, No. 1, Jan-Feb 1986, pp. 17-22; also AIAA paper 84-2036, Aug. 1984.
doi: 10.2514/3.20060

[2] Carter, T. E., "Optimal impulsive space trajectories based on linear equations," *Journal on Optimization Theory and Applications*, Vol. 70, No. 2, Aug. 1991, pp. 277-297.
doi: 10.1007/BF00940627

[3] Lawden, D. F., "General theory of optimal rocket trajectories," *Optimal trajectories for space navigation*, Butterworth, London, England, 1963, pp. 54-78.
doi: 10.1016/0032-0633(64)90149-7

[4] Prussing, J. E., "Optimal multiple-impulse orbital rendezvous," Ph. D. Dissertation, Dept. of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, 1967.

[5] Prussing, J. E., "Illustration of the primer vector in time-fixed orbit transfer," *AIAA Journal*, Vol. 7, No. 6, 1969, pp. 1167-1168.
doi: 10.2514/3.5297

[6] Carter, T. E., "Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion," *Dynamics and control*, Vol. 10, No. 3, 2000, pp. 219-227.
doi: 10.1023/A:1008376427023

[7] Carter, T. E., Alvarez, S. A., "Quadratic-based computation of four-impulse optimal rendezvous near circular orbit," *Journal of Guidance, Control and Dynamics*, Vol. 23, No. 1, Jan-Feb 2000, pp. 109-117.
doi: 10.2514/2.4493

[8] Carter, T. E., and Brient, J., "Linearized impulsive rendezvous problem," *Journal of Optimization Theory and Applications*, Vol. 86, No. 3, Sept. 1995, pp. 553-584.
doi: 10.1007/BF02192159

[9] Lasserre, J., "Global optimization with polynomials and the problem of moments," *Society for Industrial and Applied Mathematics (SIAM) Journal of Optimization*, Vol. 11, No. 3, 2001, pp. 796-817.
doi: 10.1.1.1.6947

[10] Parilo, P. A., "Semidefinite Programming Relaxations for Semialgebraic Problems," *Mathematical Programming, Series B*, Vol. 96, No. 2, 2003, pp. 293-320.
doi: 10.1007/s10107-003-0387-5

[11] Löfberg, J., "YALMIP: A Toolbox for Modeling and Optimization in MATLAB," *Proceedings of the Computer Aided Control Systems Design Conference*, Institute of Electrical and Electronic Engineers (IEEE), Taipei, Taiwan, 4 Sep. 2004, pp. 284-289.

[12] Sturm, J. F., "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, Vol. 11-12, 1999, pp. 625-653.
doi: 10.1.1.49.6954

[13] Berges, J. C., Cayeux, P., Gaudel-Vacaresse, A., and Messygnac, B., "CNES approaching guidance experiment within FFIORD," *Proceedings of the* 20th *International Symposium on Space Flight Dynamics*, NASA Conference Publication, Annapolis, Maryland, USA, 24-28 Sept. 2007.

[14] Gaudel, A., Berges, J. C., Trapier, T., Gamet, P., and Djalal, S., "Autonomous rendezvous guidance function of the SIMBOL-X formation flying mission a high elliptical orbit: Preliminary design and performance analysis," *Proceedings of the* 21st *International Symposium on Space Flight Dynamics*, CNES, Toulouse, France, 2-5 Aug. 2010.

[15] Tschauner, J., and Hempel, P., "Rendevous zu einem in elliptischer Bahn umlaufenden Ziel," *Astronautica Acta*,

Vol. II, No. 2, 1965, pp. 104-109.

[16] Yamanaka, K., and Ankersen, F., "New state transition matrix for relative motion on an arbitrary elliptical orbit," *Journal of Guidance, Control and Dynamics*, Vol. 25, No. 1, Jan-Feb 2002, pp. 60-66.
doi: 10.2514/2.4875

[17] Neustadt, L. W., "A general theory of minimum-fuel space trajectories," *Society for Industrial and Applied Mathematics (SIAM) Journal of Control*, Vol. 3, No. 2, 1965, pp. 317-356.

[18] Prussing, J. E., "Optimal impulsive linear systems: Sufficient conditions and maximum number of impulses," *Journal of the Astronautical Sciences*, Vol. 43, No. 2, Apr-Jun 1995 pp. 195-206.

[19] Henrion, D., and Lasserre, J. B., "Solving nonconvex optimization problems - How GloptiPoly is applied to problems in robust and nonlinear control", *Institute of Electrical and Electronic Engineers (IEEE) Control Systems Magazine*, Vol. 24, No. 3, 2003, pp. 72-83.
doi: 10.1.1.101.2803