Robustness Analysis of Uncertain Discrete-Time Linear Systems based on System Lifting and LMIs

Yoshio Ebihara[†], D. Peaucelle[‡] and D. Arzelier[‡]

[†]Department of Electrical Engineering, Kyoto University,
Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan.

[‡]LAAS-CNRS, Universite de Toulouse,
7 Av. du Colonel Roche, 31077, Toulouse Cedex 4, France.

Abstract: In this paper, we propose novel LMI conditions for the stability and l_2 gain performance analysis of discrete-time linear periodically time-varying (LPTV) systems. These LMIs are convex with respect to all of the coefficient matrices of the LPTV systems and this property is expected to be promising when dealing with several control system analysis and synthesis problems. For example, we can apply those LMIs straightforwadly to robust performance analysis problems of LPTV systems that are affected by polytopic-type uncertainties. Even though our approach for robust performance analysis is conservative in general, we can reduce the conservatism gradually by artificially regarding the original N-periodic system as pN-periodic and increasing p. In addition, thanks to the simple structure of the LMI conditions, we can readily derive a viable test to verify the exactness of the computation results.

Keywords: robustness analysis, discrete-time LPTV systems, real parametric uncertainty, system lifting, LMI.

1. INTRODUCTION

In this paper, we consider robust performance analysis problems of discrete-time linear periodically time-varying (LPTV) systems that are affected by polytopic-type uncertainties. We admit that this kind of analysis problem is classical, well-studied, and effective approaches are already established. Indeed, by means of the periodic Lyapunov Lemma [2, 3], we can cast the analysis problem into a robust LMI problem for which intensive research effort has been made. In particular, it has been shown recently that we can construct a hierarchy of LMI relaxation problems for robust LMI problems so that we can solve them in an asymptotically exact fashion (see, ex., [13, 14] and references there in). These recent approaches strongly rely on the theoretical advances on polynomial optimization by means of sum of squares decomposition of positive polynomials [10, 11]. Therefore the aspect of mathematical programming and numerical optimization is rather overemphasized, and it might be a sound way to incorporate basic techniques continuously polished in the field of linear system theory.

In the present paper, we make another effort to solve the analysis problem by fully exploiting the inherent nature of discrete-time linear systems. The key tool is the discrete-time system lifting [3], which has been indeed a powerful tool to deal with discrete-time LPTV/LTI systems. By applying the discrete-time lifting to an LPTV system, we can obtain a fictitious LPTV system with decreased period, while preserving the input-to-output behavior of the original LPTV system. With this property and known LMI results for LTI systems [4-6], we firstly derive LMI conditions that characterize the stability and the l_2 gain performance of uncertainty-free LPTV systems. Similarly to the LTI case results [4-6], these LMIs are convex with respect to all of the coefficient matrices of the LPTV systems. Therefore we can apply them straightforwardly to robust stability and robust l_2 gain performance of polytopic-type uncertain LPTV systems. Even though this approach is expected to be conservative in general, it is shown that we can reduce the conservatism by simply regarding the original N-periodic system as pN-periodic where $p \geq 2$. It follows that, by increasing the parameter p, we can construct a sequence of LMI problems that enables us to reduce the conservatism gradually (at the expense of increased computational burden). With this achievement in mind, we finally derive a viable test to verify the exactness of the computed results. This test is given in terms of an equality condition with respect to the dual variables of the LMI problem. If this equality condition is satisfied for some p, we can ensure that the obtained result is non-conservative, and hence we have no need to increase p further. These results are fully illustrated via numerical examples.

We note that the lifting-based robustness analysis in the present paper is partly motivated from our independent, and ongoing study in [7, 8, 12]. Our main concern there is periodically time-varying memory state-feedback controller synthesis for uncertain LPTV/LTI systems.

We use the following notations in this paper. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{n,m}$ stand for the identity and zero matrices of the size n and $n \times m$, respectively. We omit the size if it is clear from context. For real matrices A and B where A is square, we define $\mathrm{He}\{A\} := A + A^T$ and $\mathrm{Sq}\{B\} := BB^T$. For a matrix $A \in \mathbf{R}^{n\times m}$ with $\mathrm{rank}(A) = r < n$, $A^\perp \in \mathbf{R}^{(n-r)\times n}$ is a matrix such that $A^\perp A = 0$ and $A^\perp A^{\perp T} \succ 0$. We denote by $\mathbf{S}_n$, $\mathbf{P}_n$ and $\mathbf{Z}^+$ the set of symmetric matrices, positive-definite symmetric matrices and the set of positive integers, respectively. Finally, for a given set $\mathcal{W}_N$ that consists of N matrices, i.e., $\mathcal{W}_N = \{W_0, \cdots, W_{N-1}\}$, we define

$$\mathcal{I}(\mathcal{W}_N) := \text{block-diag}(W_{N-1}, \cdots, W_0). \tag{1}$$

2. PROBLEM FORMULATION

Let us consider the discrete-time N-periodic system described by

$$\mathcal{P}_N : \left\{ \begin{array}{rcl} x_{k+1} & = & A_k x_k + B_k w_k, \\ z_k & = & C_k x_k + D_k w_k \end{array} \right. \tag{2}$$

where $A_k \in \mathbf{R}^{n\times n}$, $B_k \in \mathbf{R}^{n\times m_k}$, $C_k \in \mathbf{R}^{l_k\times n}$, $D_k \in \mathbf{R}^{l_k\times m_k}$ are all N-periodic, i.e., $A_{k+N} = A_k$, etc. Note that we allow the size of B_k, C_k and D_k to be N-periodically time-varying. For the ease of notation, we define

$$M_N := \sum_{k=0}^{N-1} m_k, \quad L_N := \sum_{k=0}^{N-1} l_k.$$

In this paper, we are interested in the case where the system (2) is subject to the polytopic-type uncertainty of the following form:

$$M_k := \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad M_k^{[l]} = \begin{bmatrix} A_k^{[l]} & B_k^{[l]} \\ C_k^{[l]} & D_k^{[l]} \end{bmatrix},$$

$$M_k \in \left\{ \sum_{l=1}^{L} \alpha_l M_k^{[l]} : \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_L \end{bmatrix} \in \boldsymbol{\alpha} \right\} \tag{3}$$

$$(k = 0, \cdots, N-1),$$

$$\boldsymbol{\alpha} := \left\{ \alpha : \alpha \in \mathbf{R}^L, \sum_{l=1}^{L} \alpha_l = 1, \alpha_l \geq 0 \right\}.$$

Here, $M_k^{[l]}$ $(k = 0, \cdots, N-1,\ l = 1, \cdots, L)$ are given matrices that define the vertices of the polytope. We emphasize that α is a *time-invariant* uncertain parameter vector. For each $\alpha \in \boldsymbol{\alpha}$, we denote by $\nu(\alpha)$ the l_2 gain of the corresponding LPTV system.

For the uncertain LPTV system described by (2) and (3), we consider the robust stability analysis and robust l_2 gain performance analysis problems. These problems can be stated formally as follows:

Problem 1 (Robust Stability Analysis) Determine whether the LPTV system described by (2) and (3) is stable for all $\alpha \in \boldsymbol{\alpha}$.

Problem 2 (Robust l_2 Gain Performance Analysis) Suppose the LPTV system described by (2) and (3) is robustly stable. Then, compute the worst case l_2 gain $\nu_{\rm w}$ defined by $\nu_{\rm w} := \max_{\alpha\in\boldsymbol{\alpha}} \nu(\alpha)$.

3. MAIN RESULTS

To smoothen the subsequent discussion, let us define

$$\mathcal{A}_N := \{A_0, \cdots, A_{N-1}\}, \quad \mathcal{A}_N^{[l]} := \{A_0^{[l]}, \cdots, A_{N-1}^{[l]}\}.$$

The definitions of $\mathcal{B}_N$, $\mathcal{B}_N^{[l]}$, $\mathcal{C}_N$, $\mathcal{C}_N^{[l]}$ and $\mathcal{D}_N$, $\mathcal{D}_N^{[l]}$ follow similarly. Moreover, we define

$$\begin{aligned} \mathcal{I}_A(\mathcal{A}_N) &:= \begin{bmatrix} \mathcal{I}(\mathcal{A}_N) \\ \mathbf{0}_{n,Nn} \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{n,Nn} \\ \mathbf{1}_{Nn} \end{bmatrix} \in \mathbf{R}^{(N+1)n\times Nn}, \\ \mathcal{I}_B(\mathcal{B}_N) &:= \begin{bmatrix} \mathcal{I}(\mathcal{B}_N) \\ \mathbf{0}_{n,M_N} \end{bmatrix} \in \mathbf{R}^{(N+1)n\times M_N}. \end{aligned} \tag{4}$$

Under these definitions, we can state the next results that form an important basis of our study. The proofs are given in the appendix section.

Lemma 1: The uncertainty-free system (2) is stable if and only if there exist $X \in \mathbf{P}_n$ and $\mathcal{F} \in \mathbf{R}^{Nn\times(N+1)n}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + \mathrm{He}\{\mathcal{I}_A(\mathcal{A}_N)\mathcal{F}\} \prec \mathbf{0}. \tag{5}$$

Lemma 2: The uncertainty-free system (2) is stable and its squared l_2 gain is less than $\nu_{\rm sq}$ if and only if there exist $X \in \mathbf{P}_n$ and $\mathcal{F} \in \mathbf{R}^{Nn\times((N+1)n+L_N)}$ such that

$$\begin{aligned} &\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu_{\rm sq}\mathbf{1}_{L_N} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{I}_B(\mathcal{B}_N) \\ \mathcal{I}(\mathcal{D}_N) \end{bmatrix} \right\} \\ &+\mathrm{He}\left\{ \begin{bmatrix} \mathcal{I}_A(\mathcal{A}_N) \\ \mathcal{I}(\mathcal{C}_N) \end{bmatrix} \mathcal{F} \right\} \prec \mathbf{0}. \end{aligned} \tag{6}$$

Since the LMIs in (5) and (6) are convex with respect to all of the coefficient matrices, we can readily apply them to robustness analysis problems. The next results follow immediately from Lemmas 1 and 2, respectively.

Theorem 1: The uncertain system (2) with (3) is robustly stable if there exist $X^{[l]} \in \mathbf{P}_n$ $(l = 1, \cdots, L)$ and $\mathcal{F} \in \mathbf{R}^{Nn\times(N+1)n}$ such that

$$\begin{bmatrix} -X^{[l]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X^{[l]} \end{bmatrix} + \mathrm{He}\{\mathcal{I}_A(\mathcal{A}_N^{[l]})\mathcal{F}\} \prec \mathbf{0} \tag{7}$$

$$(l = 1, \cdots, L).$$

Theorem 2: The uncertain system (2) with (3) is robustly stable and $\nu_{\rm w}^2 < \nu_{\rm sq}$ holds if there exist $X^{[l]} \in \mathbf{P}_n$ $(l = 1, \cdots, L)$ and $\mathcal{F} \in \mathbf{R}^{Nn\times((N+1)n+L_N)}$ such that

$$\begin{aligned} &\begin{bmatrix} -X^{[l]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X^{[l]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu_{\rm sq}\mathbf{1}_{L_N} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{I}_B(\mathcal{B}_N^{[l]}) \\ \mathcal{I}(\mathcal{D}_N^{[l]}) \end{bmatrix} \right\} \\ &+\mathrm{He}\left\{ \begin{bmatrix} \mathcal{I}_A(\mathcal{A}_N^{[l]}) \\ \mathcal{I}(\mathcal{C}_N^{[l]}) \end{bmatrix} \mathcal{F} \right\} \prec \mathbf{0} \quad (l = 1, \cdots, L). \end{aligned} \tag{8}$$

Thus we have obtained very concise LMI-based conditions for robust stability and robust l_2 gain performance analysis of polytopic-type uncertain LPTV systems.

Before proceeding, we briefly discuss the impact of the novel LMI conditions (5) and (6). One may think that we can obtain similar results to Theorems 1 and 2 via standard lifting-based treatment of periodic systems. However, this is not necessarily true.

To be more precise, let us revisit the standard lifting-based treatment of periodic systems. As is well-known, the behavior of the periodic system (2) can be fully captured via an LTI system that is obtained by the discrete-time system lifting and described as follows:

$$\widehat{\mathcal{P}}_N : \left\{ \begin{array}{rcl} \xi_{j+1} & = & \widehat{A}\xi_j + \widehat{B}\widehat{w}_j, \\ \widehat{z}_j & = & \widehat{C}\xi_j + \widehat{D}\widehat{w}_j \end{array} \right. . \tag{9}$$

Here,

$$\xi_j = x_{jN}, \quad \widehat{w}_j = \begin{bmatrix} w_{(j+1)N-1} \\ \vdots \\ w_{jN} \end{bmatrix}, \quad \widehat{z}_j = \begin{bmatrix} z_{(j+1)N-1} \\ \vdots \\ z_{jN} \end{bmatrix}$$

and

$$\left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & \widehat{D} \end{array}\right] = \left[\begin{array}{c|cccc} \prod_{k=0}^{N-1} A_k & B_{N-1} & \left(\prod_{k=N-1}^{N-1} A_k\right) B_{N-2} & \cdots & \left(\prod_{k=1}^{N-1} A_k\right) B_0 \\ \hline C_{N-1}\prod_{k=0}^{N-2} A_k & D_{N-1} & C_{N-1}B_{N-2} & \cdots & C_{N-1}\left(\prod_{k=1}^{N-2} A_k\right) B_0 \\ C_{N-2}\prod_{k=0}^{N-3} A_k & \mathbf{0} & D_{N-2} & & C_{N-2}\left(\prod_{k=1}^{N-3} A_k\right) B_0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ C_0 & \mathbf{0} & \cdots & \mathbf{0} & D_0 \end{array}\right],$$

$$\prod_{k=p}^{q} A_k := A_q \cdots A_{p+1} A_p.$$

Namely, it is known that the original periodic system $\mathcal{P}_N$ is stable if and only if the fictitious LTI system $\widehat{\mathcal{P}}_N$ is stable, and the l_2 gain of the system $\mathcal{P}_N$ coincides with the H_∞ norm of the system $\widehat{\mathcal{P}}_N$. It follows that we can readily obtain LMI-based conditions for the stability and the l_2 gain performance analysis of $\mathcal{P}_N$ by simply applying the LTI-case results to $\widehat{\mathcal{P}}_N$. However, such LMI conditions are not suitable for robust stability and robust l_2 gain performance analysis, since the associated coefficient matrices $\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}\}$ become very complicated functions with respect to the original coefficient matrices $\mathcal{A}_N, \mathcal{B}_N, \mathcal{C}_N$ and $\mathcal{D}_N$ as shown above. In the case of robustness analysis, this prevents us from obtaining "vertex conditions" as in Theorems 1 and 2. Therefore we explored another direction and arrived at the LMI conditions in Lemmas 1 and 2, where these LMIs are convex with respect to all of the coefficient matrices. This convex property is very important not only for robustness analysis but also for robust controller synthesis. The latter research topic is pursued independently in our ongoing study. See [7, 8].

4. REDUCTION OF CONSERVATISM

Unfortunately, the LMI conditions in Theorems 1 and 2 are expected to be conservative due to the fact that we have enforced a common $\mathcal{F}$ for all vertices of the polytope. However, we can reduce the conservatism by simply regarding the original N-periodic system as pN-periodic ($p \geq 2$) and apply the same results in Theorems 1 and 2. The rest of this section is devoted to the technical details to validate this assertion.

To this end, we firstly make the following definition. Namely, for given $\mathcal{W}_N = \{W_0, \cdots, W_{N-1}\}$ and $p \in \mathbf{Z}^+$, we define

$$\mathcal{T}^p(\mathcal{W}_N) := \underbrace{\{\mathcal{W}_N, \cdots, \mathcal{W}_N\}}_{p \text{ times}}. \tag{10}$$

For example, if $p = 2$, we have

$$\mathcal{T}^2(\mathcal{W}_N) = \{W_0, \cdots, W_{N-1}, W_0, \cdots, W_{N-1}\}.$$

Then, it is obvious that we can regard the original N-periodic system $\mathcal{P}_N$ with the coefficient matrices $\mathcal{A}_N$, $\mathcal{B}_N$, $\mathcal{C}_N$, $\mathcal{D}_N$ as pN-periodic where the corresponding coefficient matrices are $\mathcal{T}^p(\mathcal{A}_N)$, $\mathcal{T}^p(\mathcal{B}_N)$, $\mathcal{T}^p(\mathcal{C}_N)$ and $\mathcal{T}^p(\mathcal{D}_N)$. In view of this fact and Theorems 1 and 2, the next two results obviously hold.

Proposition 1: The uncertain system (2) with (3) is robustly stable if there exist $p \in \mathbf{Z}^+$, $X^{[l]} \in \mathbf{P}_n$ ($l = 1, \cdots, L$) and $\mathcal{F} \in \mathbf{R}^{pNn \times (pN+1)n}$ such that

$$\begin{bmatrix} -X^{[l]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(pN-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X^{[l]} \end{bmatrix} + \mathrm{He}\{\mathcal{I}_A(\mathcal{T}^p(\mathcal{A}_N^{[l]}))\mathcal{F}\} \prec \mathbf{0} \tag{11}$$
$$(l = 1, \cdots, L).$$

Proposition 2: The uncertain system (2) with (3) is robustly stable and $\nu_{\mathrm{w}}^2 < \nu_{\mathrm{sq}}$ holds if there exist $p \in \mathbf{Z}^+$, $X^{[l]} \in \mathbf{P}_n$ ($l = 1, \cdots, L$) and $\mathcal{F} \in \mathbf{R}^{pNn \times ((pN+1)n + pL_N)}$ such that

$$\begin{bmatrix} -X^{[l]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(pN-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X^{[l]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu_{\mathrm{sq}}\mathbf{1}_{pL_N} \end{bmatrix} + \mathrm{Sq}\left\{\begin{bmatrix} \mathcal{I}_B(\mathcal{T}^p(\mathcal{B}_N^{[l]})) \\ \mathcal{I}(\mathcal{T}^p(\mathcal{D}_N^{[l]})) \end{bmatrix}\right\} \tag{12}$$
$$+\mathrm{He}\left\{\begin{bmatrix} \mathcal{I}_A(\mathcal{T}^p(\mathcal{A}_N^{[l]})) \\ \mathcal{I}(\mathcal{T}^p(\mathcal{C}_N^{[l]})) \end{bmatrix}\mathcal{F}\right\} \prec \mathbf{0} \quad (l = 1, \cdots, L).$$

By means of these propositions, we can construct a sequence of LMI problems by simply increasing the parameter $p \in \mathbf{Z}^+$. What is important here is that, by increasing $p \in \mathbf{Z}^+$, it is theoretically guaranteed that we can obtain better (no worse) results as formally stated in the next theorem.

Theorem 3: For given $p_1, p_2 \in \mathbf{Z}^+$, suppose p_2 is a multiple of p_1. Then, the following assertions hold:

(i) Suppose the LMI (11) holds for $p = p_1$. Then it holds for $p = p_2$.

(ii) For given $\nu_{\mathrm{sq},0}$, suppose the LMI (12) holds for $p = p_1$ and $\nu_{\mathrm{sq}} = \nu_{\mathrm{sq},0}$. Then it holds for $p = p_2$ and $\nu_{\mathrm{sq}} = \nu_{\mathrm{sq},0}$.

The proof of this theorem is given in the appendix section. Roughly speaking, this theorem implies that we can obtain no more conservative results by simply increasing the period from N to pN. This strategy is indeed promising, and we show via numerical experiments in Section 6 that we can indeed reduce conservatism by increasing p at the expense of increased computational burden.

Remark 1: We should note that the validity of the assertion in (i) in Theorem 3 is ensured only in the case where p_2 is a multiple of p_1. Indeed, it is often observed in numerical experiments that, even in the case where the LMI (11) is feasible for $p = p_1$, it becomes infeasible if we increase the period from p_1 to p_2 unless p_2 is a multiple of p_1. Similar comment applies also to (ii). For illustration, see the computation results in Section 6.

5. EXACTNESS VERIFICATION

From Proposition 2, we can compute upper bounds ν_p ($p \in \mathbf{Z}^+$) for the worst-case l_2 gain ν_{w} by solving the following SDP:

$$\nu_p^2 := \inf\ \nu_{\mathrm{sq}} \quad \text{subject to} \quad (12). \tag{13}$$

Moreover, if p_2 is a multiple of p_1, we see from Theorem 3 that

$$\nu_{\mathrm{w}} \leq \nu_{p_2} \leq \nu_{p_1}.$$

Since this nice property holds, it would be desirable if we could establish a viable test to examine the exactness of the analysis results. This research issue is often called "exactness verification" and intensively studied recently [9, 13, 14]. In particular, by adapting the result in [13] to the SDP (13), we can obtain the next result.

Theorem 4: Suppose the SDP (13) is feasible and let us denote by $\mathcal{H}_{p,l}$ the resulting dual variable with respect to the l-th LMI in (12). For these $\mathcal{H}_{p,l}$ $(l = 1, \cdots, L)$, suppose there exists $\alpha_{\mathrm{w}} = [\alpha_{\mathrm{w},1}, \cdots, \alpha_{\mathrm{w},L}]^T \in \boldsymbol{\alpha}$ such that

$$\mathcal{H}_{p,l} = \alpha_{\mathrm{w},l}\overline{\mathcal{H}}_p \ (l = 1, \cdots, L), \ \overline{\mathcal{H}}_p := \sum_{l=1}^{L} \mathcal{H}_{p,l}. \quad (14)$$

Then, we can conclude that $\nu_p = \nu_{\mathrm{w}}$. Namely, the computed value ν_p is exact, and α_{w} corresponds to the worst-case uncertain parameter.

Remark 2: The dual variable $\mathcal{H}_{p,l}$ $(l = 1, \cdots, L)$ can be readily obtained if we solve the SDP (13) via primal-dual interior point methods. By following convex duality theory [1, 14], it is also possible to derive explicitly the dual SDP of (13) with respect to the dual variables $\mathcal{H}_{p,l}$ $(l = 1, \cdots, L)$.

One of the salient feature of Theorem 4 is that, even in the case where (14) does not hold exactly, we can estimate the worst-case uncertain parameter by solving the following convex optimization problem:

$$\alpha_{\mathrm{w},p}^{\mathrm{est}} := \arg\min_{\alpha \in \boldsymbol{\alpha}} \sum_{l=1}^{L} \sigma_{\max}(\mathcal{H}_{p,l} - \alpha_l \overline{\mathcal{H}}_p). \quad (15)$$

Once we have obtained $\alpha_{\mathrm{w},p}^{\mathrm{est}}$, we can compute $\nu(\alpha_{\mathrm{w},p}^{\mathrm{est}})$, which corresponds to a lower bound of ν_{w}. In this way, we can examine the quality of the upper bound ν_p in conjunction with the efficiently-computable lower bound $\nu(\alpha_{\mathrm{w},p}^{\mathrm{est}})$.

6. NUMERICAL EXAMPLES

In this section, we apply the results in the preceding sections to the robust H_∞ performance analysis problem of uncertain LTI systems. Since the l_2 gain coincides with the H_∞ norm for LTI systems, the analysis can be accomplished by simply letting $N = 1$ in (2). We consider the case where $L = 3$ in (3) and the corresponding vertex matrices are given by

$$A^{[1]} = \begin{bmatrix} -0.22 & 0.44 & -0.44 \\ 0.33 & -0.71 & -0.71 \\ 0.11 & -0.11 & 0.11 \end{bmatrix}, \quad A^{[2]} = \begin{bmatrix} 0.44 & 0.33 & 0.44 \\ 0.71 & 0.11 & -0.22 \\ 0.11 & -0.11 & 0.44 \end{bmatrix},$$
$$A^{[3]} = \begin{bmatrix} 0.82 & 0.11 & 0.00 \\ 0.82 & -0.11 & 0.33 \\ 0.71 & -1.02 & 0.93 \end{bmatrix}, \ B^{[1]} = B^{[2]} = B^{[3]} = \begin{bmatrix} 0.1 \\ -0.5 \\ 0.2 \end{bmatrix},$$
$$C^{[1]} = \begin{bmatrix} 0.2 & 0.0 & 0.3 \end{bmatrix}, \quad C^{[2]} = \begin{bmatrix} 0.0 & -0.8 & -0.1 \end{bmatrix},$$
$$C^{[3]} = \begin{bmatrix} -0.1 & -0.6 & -0.2 \end{bmatrix}, \ D^{[1]} = D^{[2]} = D^{[3]} = -0.2.$$

In Tables 1 and 2, we show the computation results. From Table 1, we see that we can indeed obtain sharpened upper bounds by increasing p. However, the improvement is not monotonic, i.e., we see that p_3 is worse than p_2 (as noted in Remark 1). In the case where $p = 4$, the exactness verification test (14) is numerically verified, yielding $\alpha_{\mathrm{w}} = [0.0000 \ \ 0.1753 \ \ 0.8247]$. Thus we can conclude that $\nu_{\mathrm{w}} = 1.2505$.

Table 1 Upper bounds computation.

p	ν_p	CPU time [s]
1	1.4220	0.24
2	1.2596	0.34
3	1.3034	0.49
4	1.2505	0.98

Table 2 Lower bounds computation and estimation of the worst case parameter.

p	$\nu(\alpha_{\mathrm{w},p}^{\mathrm{est}})$	$\alpha_{\mathrm{w},p}^{\mathrm{est}}$
1	0.6103	$[0.4465 \ \ 0.1434 \ \ 0.4101]$
2	0.5986	$[0.2608 \ \ 0.1318 \ \ 0.6074]$
3	0.6068	$[0.3173 \ \ 0.1594 \ \ 0.5233]$
4	1.2505	$[0.0000 \ \ 0.1753 \ \ 0.8247]$

7. CONCLUSION

In this paper, we provided novel LMI conditions for the stability and the l_2 gain performance analysis of discrete-time linear periodically time-varying (LPTV) systems. In particular, we applied them to robust performance analysis problems and showed that we can construct a sequence of LMI problems that enables us to reduce the conservatism gradually. We also derived a viable test to verify the exactness of the computation results.

APPENDIX

A PROOF OF LEMMAS 1 AND 2

For the proof, we firstly introduce the following lemma. From the inherent nature of the discrete-time lifting and the l_2 gain, we can easily confirm that the next result holds.

Lemma 3: For the N-periodic system $\mathcal{P}_N$ described by (2) where $N \geq 2$, let us consider a fictitious $N-1$ periodic system $\widehat{\mathcal{P}}_{N-1}$ that is defined by the following coefficient matrices:

$$\widehat{\mathcal{A}}_{N-1} = \{A_0, \cdots, A_{N-3}, \widehat{A}_{N-2}\},$$
$$\widehat{\mathcal{B}}_{N-1} = \{B_0, \cdots, B_{N-3}, \widehat{B}_{N-2}\},$$
$$\widehat{\mathcal{C}}_{N-1} = \{C_0, \cdots, C_{N-3}, \widehat{C}_{N-2}\},$$
$$\widehat{\mathcal{D}}_{N-1} = \{D_0, \cdots, D_{N-3}, \widehat{D}_{N-2}\}.$$

Here,

$$\widehat{A}_{N-2} = A_{N-1}A_{N-2},$$
$$\widehat{B}_{N-2} = \begin{bmatrix} B_{N-1} & A_{N-1}B_{N-2} \end{bmatrix},$$
$$\widehat{C}_{N-2} = \begin{bmatrix} C_{N-1}A_{N-2} \\ C_{N-2} \end{bmatrix},$$
$$\widehat{D}_{N-2} = \begin{bmatrix} D_{N-1} & C_{N-1}B_{N-2} \\ \mathbf{0} & D_{N-2} \end{bmatrix}.$$

Then, the system $\mathcal{P}_N$ is stable if and only if $\widehat{\mathcal{P}}_{N-1}$ is stable. Moreover, if these systems are stable, they share equivalent l_2 gain.

We now state the proofs of Lemmas 1 and 2 by means of Lemma 3.

Proof of Lemma 1: We complete the proof via induction with respect to the period N.
(i) $N = 1$: If we let $N = 1$, then the LMI (5) reduces to the well-known extended LMI that characterizes the stability of LTI systems [4-6]. Thus the result is surely valid for $N = 1$.
(ii) $N = q + 1$: Suppose the assertion holds for $N = q$ $(q \geq 1)$. Then, we see from Lemma 3 that the system (2) of period $N = q+1$ is stable if and only if there exist $X \in \mathbf{P}_n$ and $\widehat{\mathcal{F}}_2 \in \mathbf{R}^{qn\times(q+1)n}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(q-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + \mathrm{He}\{\mathcal{I}_A(\widehat{\mathcal{A}}_q)\widehat{\mathcal{F}}_2\} \prec \mathbf{0}.$$

Then, from the reciprocal application of the Elimination Lemma [15], we see that the above LMI holds if and only if there exist $X \in \mathbf{P}_n$, $\mathcal{F}_1 \in \mathbf{R}^{n\times(q+2)n}$ and $\mathcal{F}_2 \in \mathbf{R}^{qn\times(q+2)n}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{qn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} A_q \\ -\mathbf{1}_n \\ \mathbf{0}_{qn,n} \end{bmatrix}\mathcal{F}_1\right\} + \mathrm{He}\left\{\begin{bmatrix} \mathbf{0}_{n,qn} \\ \mathcal{I}_A(\mathcal{A}_q) \end{bmatrix}\mathcal{F}_2\right\} \prec \mathbf{0} \tag{16}$$

This can be readily seen if we note

$$\begin{bmatrix} A_q \\ -\mathbf{1} \\ \mathbf{0}_{qn,n} \end{bmatrix}^{\perp} = \begin{bmatrix} \mathbf{1}_n & A_q & \mathbf{0}_{n,qn} \\ \mathbf{0}_{qn,n} & \mathbf{0}_{qn,n} & \mathbf{1}_{qn} \end{bmatrix}$$

and therefore the following relation holds:

$$\begin{bmatrix} A_q \\ -\mathbf{1} \\ \mathbf{0}_{qn,n} \end{bmatrix}^{\perp} \begin{bmatrix} \mathbf{0}_{n,qn} \\ \mathcal{I}_A(\mathcal{A}_q) \end{bmatrix} = \mathcal{I}_A(\widehat{\mathcal{A}}_q).$$

The LMI (16) is equivalent to (5) for $N = q + 1$ where

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{bmatrix}.$$

This completes the proof. Q.E.D.

Proof of Lemma 2: Similarly to the proof of Lemma 1 we complete the proof via induction.
(i) $N = 1$: If we let $N = 1$, then the LMI (6) reduces to the extended LMI that characterizes the H_∞ performance of LTI systems [6]. Thus the result is surely valid for $N = 1$.
(ii) $N = q + 1$: Suppose the assertion holds for $N = q$ $(q \geq 1)$. Then, we see from Lemma 3 that the system (2) of period $N = q + 1$ is stable its squared l_2 gain is less than ν_{sq} if and only if there exist $X \in \mathbf{P}_n$ and $\widehat{\mathcal{F}}_2 \in \mathbf{R}^{qn\times((q+1)n+L_{q+1})}$ such

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(q-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu_{\mathrm{sq}}\mathbf{1}_{L_{q+1}} \end{bmatrix} + \mathrm{Sq}\left\{\begin{bmatrix} \mathcal{I}_B(\widehat{\mathcal{B}}_q) \\ \mathcal{I}(\widehat{\mathcal{D}}_q) \end{bmatrix}\right\} + \mathrm{He}\left\{\begin{bmatrix} \mathcal{I}_A(\widehat{\mathcal{A}}_q) \\ \mathcal{I}(\widehat{\mathcal{C}}_q) \end{bmatrix}\widehat{\mathcal{F}}_2\right\} \prec \mathbf{0}. \tag{17}$$

Then, again, we see from the reciprocal application of the Elimination Lemma that the above LMI holds if and only if there exist $X \in \mathbf{P}_n$, $\mathcal{F}_1 \in \mathbf{R}^{n\times((q+2)n+L_{q+1})}$ and $\mathcal{F}_2 \in \mathbf{R}^{qn\times((q+2)n+L_{q+1})}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{qn} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu_{\mathrm{sq}}\mathbf{1}_{L_{q+1}} \end{bmatrix} + \mathrm{Sq}\left\{\begin{bmatrix} \mathcal{I}_B(\mathcal{B}_{q+1}) \\ \mathcal{I}(\mathcal{D}_{q+1}) \end{bmatrix}\right\} + \mathrm{He}\left\{\begin{bmatrix} A_q \\ -\mathbf{1}_n \\ \mathbf{0}_{qn,n} \\ C_q \\ \mathbf{0}_{L_q,n} \end{bmatrix}\mathcal{F}_1\right\} + \mathrm{He}\left\{\begin{bmatrix} \mathbf{0}_{n,qn} \\ \mathcal{I}_A(\mathcal{A}_q) \\ \mathbf{0}_{l_q,n} \\ \mathcal{I}(\mathcal{C}_q) \end{bmatrix}\mathcal{F}_2\right\} \prec \mathbf{0}. \tag{18}$$

This can be readily seen if we note

$$\begin{bmatrix} A_q \\ -\mathbf{1}_n \\ \mathbf{0}_{qn,n} \\ C_q \\ \mathbf{0}_{L_q,n} \end{bmatrix}^{\perp} = \begin{bmatrix} \mathbf{1}_n & A_q & \mathbf{0}_{n,qn} & \mathbf{0}_{n,l_q} & \mathbf{0}_{n,L_q} \\ \mathbf{0}_{qn,n} & \mathbf{0}_{qn,n} & \mathbf{1}_{qn} & \mathbf{0}_{qn,l_q} & \mathbf{0}_{qn,L_q} \\ \mathbf{0}_{l_q,n} & C_q & \mathbf{0}_{l_q,qn} & \mathbf{1}_{l_q} & \mathbf{0}_{l_q,L_q} \\ \mathbf{0}_{L_q,n} & \mathbf{0}_{L_q,n} & \mathbf{0}_{L_q,qn} & \mathbf{0}_{L_q,l_q} & \mathbf{1}_{L_q} \end{bmatrix}$$

and hence the following equalities hold:

$$\begin{bmatrix} A_q \\ -\mathbf{1}_n \\ \mathbf{0}_{pn,n} \\ C_q \\ \mathbf{0}_{L_q,n} \end{bmatrix}^{\perp} \begin{bmatrix} \mathbf{0}_{n,qn} \\ \mathcal{I}_A(\mathcal{A}_q) \\ \mathbf{0}_{l_q,n} \\ \mathcal{I}(\mathcal{C}_q) \end{bmatrix} = \begin{bmatrix} \mathcal{I}_A(\widehat{\mathcal{A}}_q) \\ \mathcal{I}(\widehat{\mathcal{C}}_q) \end{bmatrix},$$

$$\begin{bmatrix} A_q \\ -\mathbf{1}_n \\ \mathbf{0}_{qn,n} \\ C_q \\ \mathbf{0}_{L_q,n} \end{bmatrix}^{\perp} \begin{bmatrix} \mathcal{I}_B(\mathcal{B}_{q+1}) \\ \mathcal{I}(\mathcal{D}_{q+1}) \end{bmatrix} = \begin{bmatrix} \mathcal{I}_B(\widehat{\mathcal{B}}_q) \\ \mathcal{I}(\widehat{\mathcal{D}}_q) \end{bmatrix}.$$

The LMI (18) is equivalent to (6) for $N = q + 1$ where

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{bmatrix}.$$

This completes the proof. Q.E.D.

B PROOF OF THEOREM 3

For the proof, the next lemma is useful. This lemma is a variant of the one derived in [7].

Lemma 4: For given $P \in \mathbf{S}_{n_1}$, $Q \in \mathbf{S}_{n_2}$, $R \in \mathbf{S}_{n_3}$ and matrices E_1, E_2, F_1, F_2, G_1, G_2, H_1, H_2 with appropriate dimensions, suppose the following two inequality holds:

$$\begin{aligned} &\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\begin{bmatrix} F_1 & F_2 \end{bmatrix}\right\} \prec \mathbf{0}, \\ &\begin{bmatrix} -Q & \mathbf{0} \\ \mathbf{0} & R \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\begin{bmatrix} H_1 & H_2 \end{bmatrix}\right\} \prec \mathbf{0}. \end{aligned} \tag{19}$$

Then, the following inequality holds:

$$\begin{bmatrix} P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R \end{bmatrix} \mathrm{He}\left\{\begin{bmatrix} E_1 & \mathbf{0} \\ E_2 & G_1 \\ \mathbf{0} & G_2 \end{bmatrix}\begin{bmatrix} F_1 & F_2 & \mathbf{0} \\ \mathbf{0} & H_1 & H_2 \end{bmatrix}\right\} \prec \mathbf{0}. \tag{20}$$

Namely, the matrix Q can be eliminated.

Proof of Lemma 4: Let us denote the left-hand side of the inequalities in (19) by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then, it is apparent that

$$\text{block-diag}(\mathcal{M}_1, \mathcal{M}_2) \prec \mathbf{0}$$

and hence

$$\mathcal{V}\left(\text{block-diag}(\mathcal{M}_1, \mathcal{M}_2)\right)\mathcal{V}^T \prec \mathbf{0} \tag{21}$$

holds where

$$\mathcal{V} := \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \mathbf{1}_{n_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n_3} \end{bmatrix}.$$

We can easily confirm that (21) is equivalent to (20). This completes the proof. Q.E.D.

We are now ready to state the proof of Theorem 3.

Proof of Theorem 3: We give a proof for (i) only. The proof for (ii) follows similarly.

From the underlying assumption, we firstly note that there exist $X_{p_1}^{[l]} \in \mathbf{P}_n$ $(l = 1, \cdots, L)$ and $\mathcal{F}_{p_1} \in \mathbf{R}^{p_1 Nn \times (p_1 N+1)n}$ such that

$$\begin{bmatrix} -X_{p_1}^{[l]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(p_1 N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_{p_1}^{[l]} \end{bmatrix} + \text{He}\{\mathcal{I}_A(\mathcal{T}^{p_1}(\mathcal{A}_N^{[l]}))\mathcal{F}_{p_1}\} \prec \mathbf{0} \ (l = 1, \cdots, L). \tag{22}$$

Let us partition the matrix $\mathcal{F}_{p_1}$ as follows:

$$\mathcal{F}_{p_1} =: \begin{bmatrix} \mathcal{F}_{p_1,1} & \mathcal{F}_{p_1,2} \end{bmatrix},$$
$$\mathcal{F}_{p_1,1} \in \mathbf{R}^{p_1 Nn \times p_1 Nn} \mathcal{F}_{p_1,2} \in \mathbf{R}^{p_1 Nn \times n}.$$

Then, applying Lemma 4 to the identical two inequalities given in (22), we see that

$$\begin{bmatrix} -X_{p_1}^{[l]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(2p_1 N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_{p_1}^{[l]} \end{bmatrix} + \text{He}\{\mathcal{I}_A(\mathcal{T}^{2p_1}(\mathcal{A}_N^{[l]}))\mathcal{F}\} \prec \mathbf{0} \ (l = 1, \cdots, L). \tag{23}$$

holds where

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_{p_1,1} & \mathcal{F}_{p_1,2} & \mathbf{0} \\ \mathbf{0} & \mathcal{F}_{p_1,1} & \mathcal{F}_{p_1,2} \end{bmatrix} \in \mathbf{R}^{2p_1 Nn \times (2p_1 N+1)n}.$$

Repeating this procedure recursively by $r-1$ times where $r := p_2/p_1 \in \mathbf{Z}^+$, we can conclude that (11) holds for $p = rp_1 = p_2$ with

$$X^{[l]} = X_{p_1}^{[l]} \ (l = 1, \cdots, L)$$

and

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_{p_1,1} & \mathcal{F}_{p_1,2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathcal{F}_{p_1,1} & \mathcal{F}_{p_1,2} \end{bmatrix} \in \mathbf{R}^{p_2 Nn \times (p_2 N+1)n}.$$

To summarize, we can ensure that if the LMI (11) holds for $p = p_1$, then it holds for $p = p_2$. This complete the proof. Q.E.D.

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