Periodically Time-Varying Dynamical Controller Synthesis for Polytopic-Type Uncertain Discrete-Time Linear Systems *

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***Abstract*— This is a continuation of our preceding study dealing with robust stabilizing controller synthesis for uncertain discrete-time linear periodic/time-invariant systems. In this preceding study, we dealt with the case where the underlying systems are affected by polytopic-type uncertainties and revealed a particular periodically time-varying dynamical controller (PTVDC) structure that allows LMI-based robust stabilizing controller synthesis. Based on these preliminary results, in this paper, we provide LMI conditions for robust H_2 and H_∞ PTVDC synthesis. One of the salient features of the proposed method is that we can reduce the conservatism and improve the control performance gradually by increasing the period of the controller to be designed. In addition, we prove rigorously that the proposed design method encompasses the well-known extended-LMI-based design methods as particular cases. Through numerical experiments, we illustrate that our design method is indeed effective to achieve less conservative results under both the periodic and time-invariant settings.**
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I. Introduction

Robust performance analysis and controller synthesis for linear systems affected by parametric uncertainties have been a challenging topic in the community of control theory. As for the robust performance analysis, we have observed drastic theoretical advances in the past few years, and those linear matrix inequality (LMI) approaches [26], [27] based on the idea of sum-of-squares (SOS) decomposition of positive polynomials [21] are surely effective to achieve exact analysis results in an asymptotic fashion. These approaches can be more strengthened in conjunction with the exactness verification tests suggested in [25], [26], [13], which are also closely related to the dual of the SOS approach known with the name of the theory of moments [7], [18], [19].

Unfortunately, however, these powerful LMI-based results do not preserve convexity when we deal with robust controller synthesis problems. Due to this technical reason, the best synthesis result available in the literature dates back to de Oliveira et al. [8] appeared in the late 90's, where the authors investigated robust *static* state-feedback stabilization problems of discrete-time linear time-invariant (LTI) systems subject to polytopic uncertainties. More specifically, the authors provided an "extended" LMI that characterizes Schur stability of a matrix, which enables us to design robust controllers in a less conservative fashion than the quadratic-stability-based approach [3]. The result in [8] was successfully extended to other control problems such as robust performance synthesis [9], robust filtering [16], robust stability and performance analysis [22], [17], [11] and continuous-time robust controller synthesis [1], [28], [10]. Recently, Arzelier et al. [2] and Farges et al. [15] showed an intriguing extensions of [8], [9] to robust controller synthesis of periodic systems subject to polytopic uncertainties. Similarly to the LTI case, less conservative extended-LMI-based synthesis methods of periodically time-varying *static* controllers were suggested.

Even though the approaches in [8], [9], [2] and [15] are promising, they are still conservative and leave some rooms for improvement. Nevertheless, if we persist in *static* controller synthesis, it is really hard to obtain a systematic single-shot LMI-based design method that outperforms these existing results. This is the motivation of our preceding study in [14], where we dealt with robust stabilizing controller synthesis problems for polytopic-type uncertain linear periodic/time-invariant systems and revealed a particular periodically time-varying dynamical controller (PTVDC) structure that allows LMI-based synthesis.

Our goal in this paper is to extend these preceding results to robust H_2 and H_∞ PTVDC synthesis. To this end, we firstly consider a periodic system that has a particular structure. Similarly to [14], the analysis of this particularly structured periodic system brings us some important insights for the desired structure of the PTVDCs that allows LMI-based synthesis. One of the salient features of the proposed design method is that we can reduce the conservatism and improve the control performance gradually by increasing the period of the controller to be designed. In addition, we prove rigorously that the proposed design method encompasses the well-known extended-LMI-based design methods as particular cases. Through numerical experiments, we illustrate that our design method is indeed effective to achieve less conservative results under both the periodic and time-invariant system settings.

We use the following notations in this paper. For given two integers k and N, we denote by $\lceil k \rceil_N$ the remainder of k divided by N. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{n,m}$ stand for the identity and zero matrices of the size n and $n \times m$,

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respectively. We omit the size of these matrices if it is clear from the context. The set of symmetric matrices and positive-definite symmetric matrices of the size n are denoted by $\mathbf{S}_n$ and $\mathbf{P}_n$, respectively. For a matrix $A \in \mathbf{R}^{n\times m}$ with $\mathrm{rank}(A) = r < n$, $A^\perp \in \mathbf{R}^{(n-r)\times n}$ is a matrix such that $A^\perp A = 0$ and $A^\perp A^{\perp T} > 0$. For a real square matrix A, we define $\mathrm{He}\{A\} := A + A^T$. The convex hull of the collection of L elements $A^{[1]}, \cdots, A^{[L]}$ is denoted by $\mathrm{co}\{A^{[1]}, \cdots, A^{[L]}\}$.

In this paper, we make extensive use of the next lemma.

Lemma 1: [14] For given $P \in \mathbf{S}_n$, $Q, S \in \mathbf{S}_m$, $R \in \mathbf{S}_l$, $V \in \mathbf{R}^{n\times m}$ and $W \in \mathbf{R}^{m\times l}$, the following conditions are equivalent.

1) There exists $\mathcal{X} \in \mathbf{S}_m$ such that
$$\begin{bmatrix} P & V \\ V^* & Q+\mathcal{X} \end{bmatrix} \prec \mathbf{0}, \quad \begin{bmatrix} S-\mathcal{X} & W \\ W^* & R \end{bmatrix} \prec \mathbf{0}. \tag{1}$$
2) The following condition holds:
$$\begin{bmatrix} P & V & \mathbf{0} \\ V^* & Q+S & W \\ \mathbf{0} & W^* & R \end{bmatrix} \prec \mathbf{0}. \tag{2}$$

II. Periodic Systems of Particular Structure

Let us consider the discrete-time N-periodic system that has the following particular structure:
$$\begin{cases} x_{k+1} = \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (A_{k,j}x_{k-j} + B_{k,j}w_{k-j}), \\ \;\;z_k \;\;= \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (C_{k,j}x_{k-j} + D_{k,j}w_{k-j}). \end{cases} \tag{3}$$
Here, $x_k \in \mathbf{R}^n$ $w_k \in \mathbf{R}^{m_w}$ and $z_k \in \mathbf{R}^{l_z}$. For all $k \geq 0$, the matrices $A_{k,0}$ and so on satisfy the N-periodic condition, i.e., $A_{k+N,0} = A_{k,0}$, etc. Contrary to the standard state-space description of periodic systems, we note here that those matrices $A_{k,j}$, $B_{k,j}$, $C_{k,j}$, and $D_{k,j}$ $(k = 1, \cdots, N-1, j = 1, \cdots, k)$ are introduced in (3). Similarly to [14], the analysis of this particularly structured system brings us some important insights for the desired structure of the PTVDCs to presented in the next section.

To simplify our discussion, let us firstly consider the 2-periodic case. Then, if we denote the "hidden" state in (3) by $\xi_{x,k} \in \mathbf{R}^n$, $\xi_{w,k} \in \mathbf{R}^{m_w}$ and define $\zeta_k = [x_k^T \; \xi_{x,k}^T \; \xi_{w,k}^T]^T$, we can rewrite (3) into the standard 2-periodic state-space form as follows:
$$\begin{cases} \zeta_{k+1} = \overline{A}_k\zeta_k + \overline{B}_k w_k, \\ \;\;z_k \;\;= \overline{C}_k\zeta_k + \overline{D}_k w_k, \end{cases} \tag{4}$$
$$\begin{bmatrix} \overline{A}_0 & \overline{B}_0 \\ \overline{C}_0 & \overline{D}_0 \end{bmatrix} := \left[\begin{array}{ccc:c} A_{0,0} & \mathbf{0} & \mathbf{0} & B_{0,0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hdashline C_{0,0} & \mathbf{0} & \mathbf{0} & D_{0,0} \end{array}\right],$$
$$\begin{bmatrix} \overline{A}_1 & \overline{B}_1 \\ \overline{C}_1 & \overline{D}_1 \end{bmatrix} := \left[\begin{array}{ccc:c} A_{1,0} & A_{1,1} & B_{1,1} & B_{1,0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline C_{1,0} & C_{1,1} & D_{1,1} & D_{1,0} \end{array}\right]. \tag{5}$$

To assess the performance of (4) (or the underlying system (3)), we can readily apply the discrete-time system lifting [6] so that we can obtain an equivalent LTI representation
$$\begin{cases} \hat{\zeta}_{k+1} = \overline{A}_1\overline{A}_0 \;\; \hat{\zeta}_k + \begin{bmatrix} \overline{A}_1\overline{B}_0 & \overline{B}_1 \end{bmatrix} \hat{w}_k, \\ \;\;\hat{z}_k \;\;= \begin{bmatrix} \overline{C}_0 \\ \overline{C}_1\overline{A}_0 \end{bmatrix} \hat{\zeta}_k + \begin{bmatrix} \overline{D}_0 & \mathbf{0} \\ \overline{C}_1\overline{B}_0 & \overline{D}_1 \end{bmatrix} \hat{w}_k. \end{cases} \tag{6}$$
Here, $\hat{\zeta}_k \in \mathbf{R}^{2n+m_w}$, $\hat{w}_k \in \mathbf{R}^{2m_w}$ and $\hat{z}_k \in \mathbf{R}^{2l_z}$. In particular, from the matrix structure in (5), we see that those states corresponding to $\xi_{x,k}$ and $\xi_{w,k}$ in $\hat{\zeta}_k$ are stable uncontrollable and unobservable modes. Thus (6) can be reduced into
$$\begin{cases} \hat{x}_{k+1} = \widehat{A}_2\hat{x}_k + \widehat{B}_2\hat{w}_k, \\ \;\;\hat{z}_k \;\;= \widehat{C}_2\hat{x}_k + \widehat{D}_2\hat{w}_k \end{cases}, \tag{7}$$
$$\begin{bmatrix} \widehat{A}_2 & \widehat{B}_2 \\ \widehat{C}_2 & \widehat{D}_2 \end{bmatrix} := \left[\begin{array}{c|cc} A_{1,0}A_{0,0} + A_{1,1} & A_{1,0}B_{0,0} + B_{1,1} & B_{1,0} \\ \hline C_{0,0} & D_{0,0} & \mathbf{0} \\ C_{1,0}A_{0,0} + C_{1,1} & C_{1,0}B_{0,0} + D_{1,1} & D_{1,0} \end{array}\right].$$
where $\hat{x}_k \in \mathbf{R}^n$. It follows that we can assess the performance of the periodic system (3) by investigating this LTI system. For example, we can conclude from (7) that the system (3) is stable if and only if $\widehat{A}_2 = A_{1,0}A_{0,0} + A_{1,1}$ is Schur stable. Here, it is important to note that, as shown in [14], the Schur stability of $\widehat{A}_2$ can be characterized by an LMI that preserves the matrix structure of (3). Namely, we see that $\widehat{A}_2$ is Schur stable if and only if there exist $X_0 \in \mathbf{P}_n$ and a *slack variable* $\mathcal{F} \in \mathbf{R}^{2n\times 3n}$ such that
$$\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A_{1,0} & A_{1,1} \\ -\mathbf{1} & A_{0,0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \mathcal{F} \right\} \prec \mathbf{0}. \tag{8}$$
The derivation of this result is strongly inspired by [12].

Even though we have restricted our attention to the 2-periodic case, similar results readily follow for the particularly structured N-periodic systems. Namely, by following exactly the same procedure as above, we can derive an equivalent LTI representation of the form
$$\begin{cases} \hat{x}_{k+1} = \widehat{A}_N\hat{x}_k + \widehat{B}_N\hat{w}_k, \\ \;\;\hat{z}_k \;\;= \widehat{C}_N\hat{x}_k + \widehat{D}_N\hat{w}_k \end{cases} \tag{9}$$
where $\hat{x}_k \in \mathbf{R}^n$, $\hat{w}_k \in \mathbf{R}^{Nm_w}$ and $\hat{z}_k \in \mathbf{R}^{Nl_z}$. We denote the transfer matrix of this LTI system by $T_{N,\hat{z}\hat{w}}(z)$. In addition, for compact notation, we define $\mathcal{A}_N \in \mathbf{R}^{(N+1)n\times Nn}$, $\mathcal{B}_N \in \mathbf{R}^{(N+1)n\times Nm_w}$, $\mathcal{C}_N \in \mathbf{R}^{Nl_z\times Nn}$, $\mathcal{D}_N \in \mathbf{R}^{Nl_z\times Nm_w}$, $\overline{\mathcal{AC}}_k \in \mathbf{R}^{((k+1)n+l_z)\times(k+1)n}$ and $\overline{\mathcal{BD}}_k \in \mathbf{R}^{((k+1)n+l_z)\times(k+1)m_w}$ $(k = 0, \cdots, N-1)$ by
$$\mathcal{A}_N := \begin{bmatrix} A_{N-1,0} & A_{N-1,1} & \cdots & \cdots & A_{N-1,N-1} \\ -\mathbf{1} & A_{N-2,0} & A_{N-2,1} & \cdots & A_{N-2,N-2} \\ \mathbf{0} & -\mathbf{1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -\mathbf{1} & A_{0,0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix},$$

$$\mathcal{B}_N := \begin{bmatrix} B_{N-1,0} & B_{N-1,1} & \cdots & \cdots & B_{N-1,N-1} \\ \mathbf{0} & B_{N-2,0} & B_{N-2,1} & \cdots & B_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & B_{0,0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{bmatrix},$$

$$\mathcal{C}_N := \begin{bmatrix} C_{N-1,0} & C_{N-1,1} & \cdots & \cdots & C_{N-1,N-1} \\ \mathbf{0} & C_{N-2,0} & C_{N-2,1} & \cdots & C_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & C_{0,0} \end{bmatrix},$$

$$\mathcal{D}_N := \begin{bmatrix} D_{N-1,0} & D_{N-1,1} & \cdots & \cdots & D_{N-1,N-1} \\ \mathbf{0} & D_{N-2,0} & D_{N-2,1} & \cdots & D_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & D_{0,0} \end{bmatrix},$$

$$\overline{\mathcal{AC}}_k := \begin{bmatrix} C_{k,0} & C_{k,1} & \cdots & \cdots & C_{k,k} \\ -\mathbf{1} & A_{k-1,0} & A_{k-1,1} & \cdots & A_{k-1,k-1} \\ \mathbf{0} & -\mathbf{1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -\mathbf{1} & A_{0,0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix},$$

$$\overline{\mathcal{BD}}_k := \begin{bmatrix} D_{k,0} & D_{k,1} & \cdots & \cdots & D_{k,k} \\ \mathbf{0} & B_{k-1,0} & B_{k-1,1} & \cdots & B_{k-1,k-1} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & B_{0,0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{bmatrix}.$$

Under these notations, we can state the next results that provide LMI-based characterizations for the H_2 and H_∞ performance of the N-periodic system (3).

Lemma 2 (Generalized H_2 Performance): Let us denote the generalized H_2 norm of the N-periodic system (3) by γ_N. Then, $\gamma_N < \gamma$ holds if and only if there exist $X_0 \in \mathbf{P}_n$, $\mathcal{F} \in \mathbf{R}^{Nn\times(N+1)n}$, $Z_k \in \mathbf{P}_{l_z}$ and $\mathcal{F}_k \in \mathbf{R}^{(k+1)n\times((k+1)n+l_z)}$ $(k = 0, \cdots, N-1)$ such that

$$\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \mathcal{B}_N\mathcal{B}_N{}^T + \mathrm{He}\{\mathcal{A}_N\mathcal{F}\} \prec \mathbf{0}, \quad (10a)$$

$$\begin{bmatrix} -Z_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{kn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \overline{\mathcal{BD}}_k\overline{\mathcal{BD}}_k^T + \mathrm{He}\{\overline{\mathcal{AC}}_k\mathcal{F}_k\} \prec \mathbf{0} \quad (10b)$$
$$(k = 0, \cdots, N-1),$$

$$\frac{1}{N}\mathrm{trace}\left(\sum_{k=0}^{N-1} Z_k\right) < \gamma^2. \quad (10c)$$

Lemma 3 (H_∞ Performance): Let us denote the H_∞ norm of (3) by ν_N. Then, $\nu_N < \nu$ holds if and only if there exist $X_0 \in \mathbf{P}_n$ and $\mathcal{F}_\infty \in \mathbf{R}^{Nn\times(N(n+l_z)+n)}$ such that

$$\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu^2\mathbf{1}_{Nl_z} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix}\begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix}^T + \mathrm{He}\left\{\begin{bmatrix} \mathcal{A}_N \\ \mathcal{C}_N \end{bmatrix}\mathcal{F}_\infty\right\} \prec \mathbf{0}. \quad (11)$$

From [5], we can define the square of the generalized H_2 norm of the N-periodic system (3) as $||T_{N,\hat{z}\hat{w}}||_2^2/N$, where $||T_{N,\hat{z}\hat{w}}||_2$ stands for the H_2 norm of the LTI system (9). The generalized H_2 norm corresponds to the mean of all the responses corresponding to impulsive inputs applied to each of the m_w input channels at each time k in the N-period. On the other hand, the H_∞ norm of (3) is naturally defined in [5] as $||T_{N,\hat{z}\hat{w}}||_\infty$, which is exactly the same of the H_∞ norm of the LTI system (9). In the time-domain, this norm can be interpreted as an input-to-output l_2 induced norm. Similarly to the LTI case, these criteria are reasonable measure to assess the performance of periodic systems.

Once we have constructed the equivalent LTI system (9), it should be obvious from the above observation that we can characterize the H_2 and H_∞ performances of the system (3) via LMIs by simply writing down $\{\widehat{A}_N, \widehat{B}_N, \widehat{C}_N, \widehat{D}_N\}$ in (9) using $A_{k,j}$ $(k = 0, \cdots, N-1, j = 0, \ldots, k)$, etc, and applying LMI results for the LTI systems. The importance of Lemmas 2 and 3 lies in the fact that those LMIs can be *rewritten* equivalently as in (10) and (11), where these LMIs are in particular *convex with respect to all of the coefficient matrices* $A_{k,j}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$, etc. The proofs of these lemmas are given in the appendix section.

Remark 1: Even though the main interest of the present paper is robust PTVDC synthesis, we believe that the results in Lemmas 2 and 3 have prominence for robust performance analysis as well. In our preceding study [24], we clarified that, by applying the discrete-time system lifting repeatedly to the periodic/time-invariant systems with polytopic uncertainties, we can obtain a sequence of LMIs of the form (10) and (11) that gradually reduces the conservatism (see the related discussion in Subsection III-C). The effectiveness of this repeated application of the system lifting is also reported in [20]. As detailed in [24], the robustness analysis based on (10) and (11) with unstructured slack variables can be interpreted that we examine the robustness of causal uncertain systems with *non-causal systems*, which again exhibit similarities to [20] where the idea of *non-causal scaling* plays an important role.

III. State-Feedback PTVDC Synthesis

A. PTVDC Synthesis for Periodic Systems

Let us consider the "standard" N-periodic system described by

$$\begin{cases} x_{k+1} = A_k x_k + B_k w_k + E_k u_k, \\ z_k = C_k x_k + D_k w_k + F_k u_k. \end{cases} \quad (12)$$

For this system, the controller discussed in [2], [15] is the N-periodic static state-feedback controller of the form

$$u_k = K_k x_k, \quad K_{k+N} = K_k \ (\forall k \geq 0). \quad (13)$$

Contrary to this conventional controller structure, here we are interested in designing N-periodic dynamical controllers. In particular, motivated by the analysis results in Section II, we are interested in the N-PTVDC of the form

$$u_k = \sum_{j=0}^{\lceil k \rceil_N} K_{k,j} x_{k-j} \quad K_{k+N,j} = K_{k,j} \ (\forall k \geq 0). \tag{14}$$

This controller is obviously causal, and surely dynamical with (hidden) states of dimension $(N-1)n$. It is also obvious that (14) reduces to (13) if we let $K_{k,0} = K_k$ and $K_{k,j} = 0$ $(j \neq 0)$. From (12) and (14), the closed-loop system is described by

$$\begin{cases} x_{k+1} \ = \ \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (A_{k,j}^{\mathrm{cl}} x_{k-j} \ + \ B_{k,j}^{\mathrm{cl}} w_{k-j}), \\ z_k \ = \ \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (C_{k,j}^{\mathrm{cl}} x_{k-j} \ + \ D_{k,j}^{\mathrm{cl}} w_{k-j}). \end{cases} \tag{15}$$

where

$$\begin{aligned} &A_{k,0}^{\mathrm{cl}} := A_k + E_k K_{k,0}, \ A_{k,j}^{\mathrm{cl}} := E_k K_{k,j} \ (j \neq 0) \\ &B_{k,0}^{\mathrm{cl}} := B_k, \ B_{k,j}^{\mathrm{cl}} := \mathbf{0} \ (j \neq 0), \\ &C_{k,0}^{\mathrm{cl}} := C_k + F_k K_{k,0}, \ C_{k,j}^{\mathrm{cl}} := F_k K_{k,j} \ (j \neq 0) \\ &D_{k,0}^{\mathrm{cl}} := D_k, \ D_{k,j}^{\mathrm{cl}} := \mathbf{0} \ (j \neq 0). \end{aligned}$$

We note that this closed-loop system has exactly the same structure as (3). Thus, we can apply Lemmas 2 and 3 to assess its performance. To this end, let us denote by $\mathcal{A}_N^{\mathrm{cl}}$ the matrix resulting from $\mathcal{A}_N$ with $A_{k,j}$ replaced by $A_{k,j}^{\mathrm{cl}}$. We also introduce $\mathcal{B}_N^{\mathrm{cl}}$, $\mathcal{C}_N^{\mathrm{cl}}$, $\mathcal{D}_N^{\mathrm{cl}}$, $\overline{\mathcal{AC}}_N^{\mathrm{cl}}$ and $\overline{\mathcal{BD}}_N^{\mathrm{cl}}$ in a obvious fashion. Then, we can obtain matrix inequality conditions to assess the H_2 and H_∞ performance of the closed-loop system (15) by simply replacing $\mathcal{A}_N$ by $\mathcal{A}_N^{\mathrm{cl}}$ and so on in (10) and (11).

Unfortunately, those inequalities resulting from (10) and (11) are not suitable for controller synthesis due to multiple bilinear terms among the variables $K_{k,j}$ $(k = 0, \cdots, N-1, \ j = 0, \cdots, k)$ and $\mathcal{F}$, $\mathcal{F}_i$ $(i = 1, \cdots, N)$, $\mathcal{F}_\infty$. To get around this difficulty, by following [14], we consider to restrict the structure of the variables $\mathcal{F}$, $\mathcal{F}_i$ $(i = 1, \cdots, N)$ and $\mathcal{F}_\infty$. Then, we can obtain the next results that provide LMIs for PTVDC synthesis.

Theorem 1 (H_2 PTVDC Synthesis): Let us denote by γ_N the generalized H_2 norm of the closed-loop system (15) constructed from (12) and (14). Then, $\gamma_N < \gamma$ holds if there exist $X_0 \in \mathbf{P}_n$, $Z_k \in \mathbf{P}_{l_z}$ and $G_k \in \mathbf{R}^{n \times n}$ $(k = 0, \cdots, N-1)$ such that

$$\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \mathcal{B}_N^{\mathrm{cl}} \mathcal{B}_N^{\mathrm{cl}T} + \mathrm{He}\{\mathcal{A}_N^{\mathrm{cl}} \mathcal{G}\} \prec \mathbf{0}, \tag{16a}$$

$$\begin{bmatrix} -Z_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{kn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \overline{\mathcal{BD}}_k^{\mathrm{cl}} \overline{\mathcal{BD}}_k^{\mathrm{cl}T} + \mathrm{He}\{\overline{\mathcal{AC}}_k^{\mathrm{cl}} \mathcal{G}_k\} \prec \mathbf{0} \quad (k = 0, \cdots, N-1), \tag{16b}$$

$$\frac{1}{N} \mathrm{trace} \left(\sum_{k=0}^{N-1} Z_k \right) < \gamma^2 \tag{16c}$$

where

$$\begin{aligned} \mathcal{G} \ &:= \ \begin{bmatrix} \mathbf{0}_{Nn,n} & \text{block-diag}(G_{N-1}, \cdots, G_0) \end{bmatrix}, \\ \mathcal{G}_k \ &:= \ \begin{bmatrix} \mathbf{0}_{(k+1)n,l_z} & \text{block-diag}(G_k, \cdots, G_0) \end{bmatrix} \\ &\qquad\qquad (k = 0, \cdots, N-1). \end{aligned}$$

The matrix inequalities in (16) can be reduced into LMIs via change of variables $Y_{k,j} = K_{k,j} G_{k-j}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$.

Theorem 2 (H_∞ PTVDC Synthesis): Let us denote by ν_N the H_∞ norm of the closed-loop system (15) constructed from (12) and (14). Then, $\nu_N < \nu$ holds if there exist $X_0 \in \mathbf{P}_n$ and $G_i \in \mathbf{R}^{n \times n}$ $(i = 1, \cdots, N)$ such that

$$\begin{aligned} &\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu^2 \mathbf{1}_{N l_z} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}} \\ \mathcal{D}_N^{\mathrm{cl}} \end{bmatrix} \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}} \\ \mathcal{D}_N^{\mathrm{cl}} \end{bmatrix}^T \\ &+\mathrm{He} \left\{ \begin{bmatrix} \mathcal{A}_N^{\mathrm{cl}} \\ \mathcal{C}_N^{\mathrm{cl}} \end{bmatrix} \mathcal{G}_\infty \right\} \prec \mathbf{0}. \end{aligned} \tag{17}$$

where

$$\mathcal{G}_\infty := \begin{bmatrix} \mathbf{0}_{Nn,n} & \text{block-diag}(G_{N-1}, \cdots, G_0) & \mathbf{0}_{Nn, N l_z} \end{bmatrix}.$$

The matrix inequalities in (17) can be reduced into LMIs via change of variables $Y_{k,j} = K_{k,j} G_{k-j}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$.

In the above theorems, those LMIs (16) and (17) are derived by restricting the slack variables introduced in (10) and (11). We emphasize that these restrictions have been done in such a sound way that the resulting LMIs (16) and (17) for PTVDC synthesis encompass the corresponding extended-LMI-based static controller synthesis as particular cases. To see this more concretely, let us consider the H_∞ controller synthesis problem for the N-periodic system (12). We see from [4], [5] and the extended-LMI-based approach in [8], [9] that the static controller (13) that satisfies $\nu_N < \nu$ exists if and only if there exist $X_k, G_k, Y_{k,0}$ $(k = 0, \cdots, N-1)$ such that the following LMIs hold:

$$\begin{aligned} &\begin{bmatrix} -X_{k+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1}_{l_z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_k \end{bmatrix} + \begin{bmatrix} B_k \\ D_k \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} B_k \\ D_k \\ \mathbf{0} \end{bmatrix}^T \\ &\quad +\mathrm{He} \left\{ \begin{bmatrix} A_k G_k + E_k Y_{k,0} \\ C_k G_k + F_k Y_{k,0} \\ -G_k \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1}_n \end{bmatrix} \right\} \prec \mathbf{0}. \end{aligned} \tag{18}$$

Here, $k = 0, \cdots, N-1$ and $X_N = X_0$. If these LMIs are satisfied, the desired feedback gains can be obtained by $K_k = Y_{k,0} G_k^{-1}$ $(k = 0, \cdots, N-1)$.

To reveal a connection between (17) and (18), let us consider the case where $N = 2$ and apply Lemma 1 to the two inequalities in (18). Then, we see that (18) holds if and only if there exist X_0, G_k, $Y_{k,0}$ $(k = 0, 1)$ such that

$$\begin{aligned} &\begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1}_{l_z} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu^2 \mathbf{1}_{l_z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & X_0 \end{bmatrix} + \begin{bmatrix} B_1 & \mathbf{0} \\ D_1 & \mathbf{0} \\ \mathbf{0} & B_0 \\ \mathbf{0} & D_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} B_1 & \mathbf{0} \\ D_1 & \mathbf{0} \\ \mathbf{0} & B_0 \\ \mathbf{0} & D_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \\ &+\mathrm{He} \left\{ \begin{bmatrix} A_1 G_1 + E_1 Y_{1,0} & \mathbf{0} \\ C_1 G_1 + F_1 Y_{1,0} & \mathbf{0} \\ -G_1 & A_0 G_0 + E_0 Y_{0,0} \\ \mathbf{0} & C_0 G_0 + F_0 Y_{0,0} \\ \mathbf{0} & -G_0 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_n \end{bmatrix} \right\} \prec \mathbf{0}. \end{aligned} \tag{19}$$

Applying a congruence transformation with

$$T = \begin{bmatrix} \mathbf{1}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_n \\ \mathbf{0} & \mathbf{1}_{l_z} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{l_z} & \mathbf{0} \end{bmatrix},$$

we see that (19) reduces to (17) with exactly the same X_0, G_k, $Y_{k,0}$ $(k = 0, 1)$ and $Y_{1,1} = \mathbf{0}$.

We can confirm that similar results do follow in the general N-periodic cases. Namely, if (18) holds with $X_k, G_k, Y_{k,0}$ $(k = 0, \cdots, N-1)$, then (17) holds with the identical $X_0, G_k, Y_{k,0}$ $(k = 0, \cdots, N-1)$ and $Y_{k,j} = \mathbf{0}$ $(j \neq 0)$. It follows that, when we deal with the uncertainty-free system (12), the restriction on the slack variable in Theorem 2 does not introduce any conservatism and the LMI (17) corresponds to a *necessary and sufficient* condition for the existence of the desired static controller of the form (13). Similar comments also apply to the LMI (16) in Theorem 1. This point is crucial to ensure explicit advantages of the PTVDCs when dealing with robust controller synthesis problems for polytopic-type uncertain systems. As we see in the next subsection, the extra freedom introduced by those gains $K_{k,j}$ $(j \neq 0)$ can be used to obtain more sharpened results in comparison with the extended-LMI-based approaches [8], [9], [2], [15].

B. Robust PTVDC Synthesis

Now we are ready to state an explicit advantage of the PTVDC (14) over the conventional form (13). To this end, let us consider the case where the system (12) is subject to a polytopic uncertainty given in the following:

$$\begin{aligned} &\mathcal{M}_{\mathrm{s},k} := \begin{bmatrix} A_k & B_k & E_k \\ C_k & D_k & F_k \end{bmatrix}, \\ &\mathcal{M}_{\mathrm{s},k}^{[l]} = \begin{bmatrix} A_k^{[l]} & B_k^{[l]} & E_k^{[l]} \\ C_k^{[l]} & D_k^{[l]} & F_k^{[l]} \end{bmatrix} \ (l = 1, \cdots, L), \\ &\begin{bmatrix} \mathcal{M}_{\mathrm{s},0} \\ \vdots \\ \mathcal{M}_{\mathrm{s},N-1} \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} \mathcal{M}_{\mathrm{s},0}^{[1]} \\ \vdots \\ \mathcal{M}_{\mathrm{s},N-1}^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} \mathcal{M}_{\mathrm{s},0}^{[L]} \\ \vdots \\ \mathcal{M}_{\mathrm{s},N-1}^{[L]} \end{bmatrix} \right\}. \end{aligned} \tag{20}$$

Here, $\mathcal{M}_{\mathrm{s},k}^{[l]}$ $(k = 0, \cdots, N-1,\ l = 1, \cdots, L)$ are given matrices that define the vertices of the polytope.

1) Advantage of PTVDCs: For concrete illustration, let us consider the robust H_∞ state-feedback controller synthesis problem. If we seek for the robust static controller of the form (13), the following LMIs readily follow from the extended LMI (18):

$$\begin{aligned} &\begin{bmatrix} -X_{k+1}^{[l]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1}_{l_z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_k^{[l]} \end{bmatrix} + \begin{bmatrix} B_k^{[l]} \\ D_k^{[l]} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} B_k^{[l]} \\ D_k^{[l]} \\ \mathbf{0} \end{bmatrix}^T \\ &\quad + \mathrm{He}\left\{ \begin{bmatrix} A_k^{[l]} G_k + E_k^{[l]} Y_{k,0} \\ C_k^{[l]} G_k + F_k^{[l]} Y_{k,0} \\ -G_k \end{bmatrix} \begin{bmatrix} \mathbf{0} \ \mathbf{0} \ \mathbf{1}_n \end{bmatrix} \right\} \prec \mathbf{0}. \end{aligned} \tag{21}$$

Here, $k = 0, \cdots, N-1$, $l = 1, \cdots, L$ and $X_N^{[l]} = X_0^{[l]}$ $(l = 1, \cdots, L)$. If these LMIs hold, the desired feedback gains are obtained by $K_k = Y_{k,0} G_k^{-1}$ $(k = 0, \cdots, N-1)$.

On the other hand, it is obvious from (17) that we can design robust H_∞ PTVDC of the form (14) by solving the LMIs given in the following:

$$\begin{aligned} &\begin{bmatrix} -X_0^{[l]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0^{[l]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu^2 \mathbf{1}_{N l_z} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}[l]} \\ \mathcal{D}_N^{\mathrm{cl}[l]} \end{bmatrix} \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}[l]} \\ \mathcal{D}_N^{\mathrm{cl}[l]} \end{bmatrix}^T \\ &\quad + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_N^{\mathrm{cl}[l]} \\ \mathcal{C}_N^{\mathrm{cl}[l]} \end{bmatrix} \mathcal{G}_\infty \right\} \prec \mathbf{0}. \end{aligned} \tag{22}$$

Here, again $k = 0, \cdots, N-1$ and $l = 1, \cdots, L$. Those matrices $\mathcal{A}_N^{\mathrm{cl}[l]}$, $\mathcal{B}_N^{\mathrm{cl}[l]}$, $\mathcal{C}_N^{\mathrm{cl}[l]}$ and $\mathcal{D}_N^{\mathrm{cl}[l]}$ are readily defined from $\mathcal{A}_N^{\mathrm{cl}}$, $\mathcal{B}_N^{\mathrm{cl}}$, $\mathcal{C}_N^{\mathrm{cl}}$ and $\mathcal{D}_N^{\mathrm{cl}}$, respectively, by simply replacing A_k by $A_k^{[l]}$, etc. If the LMIs in (22) hold, the desired feedback gains in (14) can be obtained by $K_{k,j} = Y_{k,j} G_{k-j}^{-1}$ $(k = 0, \ldots, N-1, j = 0, \cdots, k)$.

By comparing (21) and (22), it is obvious from the discussion in the preceding subsection that if (21) holds, then (22) holds with exactly the same $X_0^{[l]}$ $(l = 1, \cdots, L)$, G_k, $Y_{k,0}$ $(k = 0, \cdots, N-1)$ and $Y_{k,j} = 0$ $(j \neq 0)$. Hence, in the context of robust H_∞ controller synthesis for polytopic-type uncertain systems, we can obtain no more conservative results by (22). In fact, the PTVDC synthesis based on (22) and its counterpart for the robust H_2 synthesis is surely effective as wee see in the next examples.

2) Numerical Examples: To illustrate the effectiveness of the suggested PTVDCs, we solved the robust H_2 controller synthesis problem discussed in [15]. Before proceeding to numerical computation, we briefly outline the advantage of the PTVDC synthesis over the static controller design suggested in [15].

For the uncertainty-free system (12), the next extend-LMIs are suggested in [15] to design a periodically time-varying static state-feedback H_2 controller of the form (13).

$$\begin{bmatrix} B_k B_k^T - X_{k+1} & A_k G_k + E_k Y_{k,0} \\ * & X_k - G_k - G_k^T \end{bmatrix} \prec \mathbf{0}, \tag{23a}$$

$$\begin{bmatrix} D_k D_k^T - Z_k & C_k G_k + F_k Y_{k,0} \\ * & X_k - G_k - G_k^T \end{bmatrix} \prec \mathbf{0}, \tag{23b}$$

$$\frac{1}{N} \mathrm{trace}\left(\sum_{i=1}^{N} Z_i \right) < \gamma^2. \tag{23c}$$

Here, $k = 0, \cdots, N-1$ and $X_N = X_0$. If these LMIs hold, the feedback gains in (13) are obtained by $K_k = Y_{k,0} G_k^{-1}$ $(k = 0, \cdots, N-1)$. These LMIs may seem completely different from (16) for PTVDC synthesis. However, by applying Lemma 1 repeatedly, we can prove that if (23) holds, then (16) holds with exactly the same X_0, Z_k, G_k, $Y_{k,0}$ $(k = 0, \cdots, N-1)$ and $Y_{k,j} = 0$ $(j \neq 0)$. For example, in the case of period two, the two LMIs in (23a) in conjunction with Lemma 1 leads to (16a) (with

$Y_{k,j} = 0$ $(j \neq 0)$). The first LMI in (23b) is nothing but the first one in (16b). Finally, it is apparent that the second LMI in (23b) and the first one in (23a) ensures the second LMI in (16b) (with $Y_{k,j} = 0$ $(j \neq 0)$. Similar observations are also valid in the general N-periodic case. This clearly indicates that, under the robust H_2 synthesis setting for polytopic systems, we can obtain no more conservative results by (the robust version of) (16).

To illustrate this point practically, we solved the robust H_2 state-feedback synthesis problem for uncertain 3-periodic system discussed in Section 5 of [15]. This system has two uncertain parameters α and β and thus modeled as a polytopic-type uncertain system with four vertices. By letting the margin of the variation of α as $|\alpha| \leq \bar{\alpha}$ and β as $0 \leq \beta \leq 1$, we minimized γ subject to (16) evaluated on all four vertices of the polytope. The resulting value for $\bar{\alpha} = 0.1$ was $\gamma^2 = 2.3795$. If we enforce $K_{k,j} = 0$ $(j \neq 0)$ and seek for a static controller of the form (13), we obtained $\gamma_s^2 = 2.7513$. On the other hand, if $\bar{\alpha} = 0.3$, we obtained $\gamma^2 = 3.6591$ whereas $\gamma_s^2 = 5.2173$. Finally, if we let $\bar{\alpha} = 0.5$, we obtained $\gamma^2 = 10.5923$ whereas (16) was identified to be infeasible if we let $K_{k,j} = 0$ $(j \neq 0)$. These results clearly illustrate the effectiveness of designing PTVDCs.

C. Application to LTI System Synthesis

The goal of this subsection is to clarify that the suggested PTVDC structure and the associated LMI-based synthesis method are promising for LTI system synthesis as well. It is of course meaningless to consider the complicated controller structure (14) for nominal system synthesis. However, when we consider such "difficult" problems as robust controller synthesis for polytopic uncertain systems [8], [9] to which definite solution is not currently available, the PTVDCs bring some improvements over the existing methods (at the expense of complicated controller structure).

1) PTVDC Synthesis via LTI System Lifting: Let us consider the polytopic-type uncertain LTI system described by

$$\begin{cases} x_{k+1} = Ax_k + Bw_k + Eu_k, \\ \;\; z_k \;\; = Cx_k + Dw_k + Fu_k \end{cases} \tag{24}$$

where

$$\begin{bmatrix} A\,B\,E \\ C\,D\,F \end{bmatrix} \in \mathrm{co} \left\{ \begin{bmatrix} A^{[1]}\,B^{[1]}\,E^{[1]} \\ C^{[1]}\,D^{[1]}\,F^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} A^{[L]}\,B^{[L]}\,E^{[L]} \\ C^{[L]}\,D^{[L]}\,F^{[L]} \end{bmatrix} \right\}.$$

To design a robust LTI controller of the form $u_k = Kx_k$, we can readily apply the extended-LMI-based method shown in [8], [9]. On the other hand, by artificially regarding this LTI system as N-periodic (i.e, $A_k = A$ $(k = 0, \cdots, N-1)$ and so on in (12)), we can apply Theorem 1 and Theorem 2 to design robust PTVDC of the form (14). The advantage of the PTVDC synthesis over the extended-LMI-based LTI controller synthesis is obvious and can be stated exactly in the same fashion as in the preceding periodic system case.

2) Numerical Examples: Let us consider the polytopic-type uncertain LTI system (24) with two vertices where

$$A^{[1]} = \begin{bmatrix} -0.2 & -0.4 & 0.5 \\ -0.6 & 0.1 & 0.7 \\ 0.4 & 0.2 & -0.5 \end{bmatrix},\ A^{[2]} = \begin{bmatrix} -0.2 & 0.0 & -0.4 \\ 0.9 & 0.5 & 0.2 \\ -0.2 & -0.3 & -0.8 \end{bmatrix},$$

$$B^{[1]} = B^{[2]} = \begin{bmatrix} -0.4 \\ -0.2 \\ 0.6 \end{bmatrix},\ E^{[1]} = E^{[2]} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.2 \end{bmatrix},$$

$$C^{[1]} = C^{[2]} = \begin{bmatrix} 100 \\ 010 \\ 000 \end{bmatrix},\ D^{[1]} = D^{[2]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ F^{[1]} = F^{[2]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For this system, we first designed a robust static state-feedback controller of the form $u_k = Kx_k$ by following the extended-LMI-based method in [9]. Then, we obtained an upper bound of the H_2 norm $\gamma_1 = 60.1640$ and the gain

$$K = \begin{bmatrix} 1.2649 & -0.1503 & -1.1286 \end{bmatrix}.$$

The CPU time was 0.24 [sec]. Next, we artificially constructed an equivalent N-periodic system and designed a robust PTVDC based on (16). Then, for $N = 2, \cdots, 6$, we obtained upper bounds $\gamma_2 = 30.6074$, $\gamma_3 = 24.4013$, $\gamma_4 = 23.3218$, $\gamma_5 = 22.7163$ and $\gamma_6 = 22.3195$. The CPU time were 0.32, 0.42, 0.59, 0.81 and 1.07 [sec], respectively. The PTVDC gains for the case $N = 3$ are as follows:

$$\begin{aligned} K_{0,0} &= \begin{bmatrix} 1.2652 & 0.2190 & -1.3953 \end{bmatrix}, \\ K_{1,0} &= \begin{bmatrix} 1.0524 & 0.4969 & -0.8226 \end{bmatrix}, \\ K_{1,1} &= \begin{bmatrix} -1.0203 & -0.5147 & 0.2790 \end{bmatrix}, \\ K_{2,0} &= \begin{bmatrix} 1.0311 & 0.4869 & -0.9831 \end{bmatrix}, \\ K_{2,1} &= \begin{bmatrix} -0.9679 & -0.5641 & 0.2707 \end{bmatrix}, \\ K_{2,2} &= \begin{bmatrix} 0.2924 & 0.1208 & 0.0902 \end{bmatrix}. \end{aligned}$$

We see that $K_{0,0}$, $K_{1,0}$ and $K_{2,0}$ are close to each other. In this numerical example, we successfully gained drastic improvement by designing PTVDCs.

We next solved the robust H_2 synthesis problem for 3-periodic system discussed in [15] and also dealt with in the preceding subsection. In contrast with the preceding subsection, here we design PTVDCs by regarding the original 3 periodic system as 6 and 9 periodic systems. Then, in the case $\bar{\alpha} = 0.1$, we obtained upper bounds $\gamma_3^2 = 2.3795$ (as in the preceding subsection), $\gamma_6^2 = 2.2103$ and $\gamma_9^2 = 2.1651$. The CPU time were 0.53, 1.24 and 2.69 [sec], respectively. If $\bar{\alpha} = 0.3$, we obtained $\gamma_3^2 = 3.6591$, $\gamma_6^2 = 3.2275$ and $\gamma_9^2 = 3.1187$. Finally, if we let $\bar{\alpha} = 0.5$, we obtained $\gamma_3^2 = 10.5923$, $\gamma_6^2 = 7.4130$ and $\gamma_9^2 = 6.8275$.

To summarize, we can obtain favorable results by designing PTVDCs. We believe that, since the publication of [8], [9], it has been an outstanding issue to derive systematic single-shot LMI-based methods for robust state-feedback synthesis that outperform these extended-LMI-based results. We surely achieved this by rendering controllers to be periodically time-varying and dynamic.

Before closing this section, we stress that the suggested PTVDC synthesis method is also effective in other control problems to which definitive solutions are not yet established to this date. For example, decentralized state-feedback control and multiobjective state-feedback H_2/H_∞ control problems discussed in [9] are within the scope. The former problem can readily be dealt with in the present approach by imposing decentralized (i.e., block-diagonal) structures on G_k and $Y_{k,j}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$ in (16) and (17). It is also straightforward to deal with the latter problem by simply taking common G_k $(k = 1, \cdots, N-1)$ in (16) and (17). Due to these restrictions, it is still inevitable to include some conservatism into the design. However, as in the robust state-feedback case, we note that these novel approach is theoretically guaranteed to be no more conservative than [9] and surely promising to achieve more accurate results. Indeed, in some tested numerical examples, we confirmed that the PTVDC synthesis brings drastic improvement over [9] under both of the problem settings.

IV. Conclusion

In this paper, we proposed an LMI-based method to design periodically time-varying dynamical state-feedback controllers for discrete-time uncertain linear periodic/time-invariant systems. Through numerical experiments, we confirmed that the suggested design method is indeed effective to obtain less conservative results than the existing approaches. We also showed that, by applying discrete-time system lifting repeatedly, we can gradually reduce the conservatism (at the expense of the increased computational burden and the complexity of the controllers). This is a striking feature of the present approach, and we stress that such successful reduction of the conservatism has been done without resorting to cumbersome iterative computations.

These synthesis results were derived from the LMI-based analysis results for particularly structured periodic systems. Those LMIs are also of prime importance when dealing with robustness analysis problems and should deserve for independent research. In our ongoing study [24], this topic is fully investigated in conjunction with the idea of descriptor-like system representation [23].

Finally, we believe that the present paper and [20], [24] posed an interesting topic around lifting-based discrete-time system analysis and synthesis. The extension of this idea to output-feedback synthesis, time-delay system synthesis and filtering problems are important subjects of our future research.

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Appendices

V. Proof of Lemma 2

Here we give a detailed proof for 2-periodic case only. Once we have established an explicit proof for the 2-periodic case, the proof for the general N-periodic case can readily be deduced.

Proof of Lemma 2: From the definition of the generalized H_2 norm in terms of the corresponding LTI system (7), we see that $\gamma_2 < \gamma$ holds if and only if there exists $X_0 \in \mathbf{P}_n$ and $Z \in \mathbf{P}_{2l_z}$ such that

$$\widehat{A}_2 X_0 \widehat{A}_2^T - X + \widehat{B}_2 \widehat{B}_2^T \prec 0, \tag{25a}$$

$$\widehat{C}_2 X_0 \widehat{C}_2^T + \widehat{D}_2 \widehat{D}_2^T \prec Z, \tag{25b}$$

$$\frac{1}{2}\mathrm{trace}(Z) \prec \gamma^2. \tag{25c}$$

Thus, it suffices to prove that (25) holds if and only if (10) holds.

In view of the fact that

$$\mathcal{A}_2^{\perp} = \begin{bmatrix} \mathbf{1} & A_{1,0} & \widehat{A}_2 \end{bmatrix}$$

and the Elimination Lemma [29], it should be straightforward to verify that (25a) holds if and only if (10a). On the other hand, if we partition Z in (25b) as

$$Z = \begin{bmatrix} Z_0 & Z_{01} \\ Z_{01}^T & Z_1 \end{bmatrix},\ Z_0, Z_1 \in \mathbf{P}_{l_z},$$

we see that (25b) and (25c) hold if and only if there exist $Z_1, Z_2 \in \mathbf{P}_{l_z}$ such that

$$C_{0,0} X_0 C_{0,0}^T + D_{0,0} D_{0,0}^T \prec Z_0, \tag{26}$$

$$\begin{array}{l}(C_{1,0}A_{0,0} + C_{1,1}) X_0 (C_{1,0}A_{0,0} + C_{1,1})^T \\ +[C_{1,0}B_{0,0} + D_{1,1}\ \ D_{1,0}][C_{1,0}B_{0,0} + D_{1,1}\ \ D_{1,0}]^T \prec Z_1,\end{array} \tag{27}$$

$$\frac{1}{2}\mathrm{trace}(Z_0 + Z_1) < \gamma^2. \tag{28}$$

It can be seen that (26) holds if and only if (10b) for $k = 0$ holds if we note

$$\overline{\mathcal{AC}}_0^{\perp} = \begin{bmatrix} \mathbf{1} & C_{0,0} \end{bmatrix}.$$

It is also true that (27) holds if and only if (10b) for $k = 1$ holds since

$$\overline{\mathcal{AC}}_1^{\perp} = \begin{bmatrix} \mathbf{1} & C_{1,0} & C_{1,0}A_{0,0} + C_{1,1} \end{bmatrix}.$$

This completes the proof. ■

VI. Proof of Lemma 3

Here again we give a detailed proof for 2-periodic case only. The proof for the general N-periodic case follows similarly.

Proof of Lemma 3: Since the H_∞ norm of (3) is defined exactly the same as that of (7), we see from the well-known LMI result for LTI systems that $\nu_2 < \nu$ holds if and only if there exists $X_0 \in \mathbf{P}_n$ such that

$$\begin{bmatrix} -X_0 & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1} \end{bmatrix} + \begin{bmatrix} \widehat{A}_2 \\ \widehat{C}_2 \end{bmatrix} X_0 \begin{bmatrix} \widehat{A}_2 \\ \widehat{C}_2 \end{bmatrix}^T + \begin{bmatrix} \widehat{B}_2 \\ \widehat{D}_2 \end{bmatrix} \begin{bmatrix} \widehat{B}_2 \\ \widehat{D}_2 \end{bmatrix}^T \prec 0. \tag{29}$$

To prove that (29) holds if and only if (11) holds, we note that

$$\begin{bmatrix} \mathcal{A}_2 \\ \mathcal{C}_2 \end{bmatrix}^{\perp} = \begin{bmatrix} \mathbf{1}_n & A_{1,0} & \widehat{A}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{1,0} & C_{1,0}A_{0,0} + C_{1,1} & \mathbf{1}_{l_z} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{0,0} & \mathbf{0} & \mathbf{1}_{l_z} \end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix} \mathcal{A}_2 \\ \mathcal{C}_2 \end{bmatrix}^{\perp} \left\{ \begin{bmatrix} -X_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu^2 \mathbf{1}_{2l_z} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_2 \\ \mathcal{D}_2 \end{bmatrix} \begin{bmatrix} \mathcal{B}_2 \\ \mathcal{D}_2 \end{bmatrix}^T \right\} \begin{bmatrix} \mathcal{A}_2 \\ \mathcal{C}_2 \end{bmatrix}^{\perp T}$$

$$= \begin{bmatrix} -X_0 & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1}_{2l_z} \end{bmatrix}$$

$$+ \begin{bmatrix} \widehat{A}_2 \\ C_{1,0}A_{0,0} + C_{1,1} \\ C_{0,0} \end{bmatrix} X_0 \begin{bmatrix} \widehat{A}_2 \\ C_{1,0}A_{0,0} + C_{1,1} \\ C_{0,0} \end{bmatrix}^T$$

$$+ \begin{bmatrix} B_{1,0} & A_{1,0}B_{0,0} + B_{1,1} \\ D_{1,0} & C_{1,0}B_{0,0} + D_{1,1} \\ \mathbf{0} & D_{0,0} \end{bmatrix} \begin{bmatrix} B_{1,0} & A_{1,0}B_{0,0} + B_{1,1} \\ D_{1,0} & C_{1,0}B_{0,0} + D_{1,1} \\ \mathbf{0} & D_{0,0} \end{bmatrix}^T$$

$$= T_2 \left\{ \begin{bmatrix} -X_0 & \mathbf{0} \\ \mathbf{0} & -\nu^2 \mathbf{1} \end{bmatrix} + \begin{bmatrix} \widehat{A}_2 \\ \widehat{C}_2 \end{bmatrix} X_0 \begin{bmatrix} \widehat{A}_2 \\ \widehat{C}_2 \end{bmatrix}^T + \begin{bmatrix} \widehat{B}_2 \\ \widehat{D}_2 \end{bmatrix} \begin{bmatrix} \widehat{B}_2 \\ \widehat{D}_2 \end{bmatrix}^T \right\} T_2^T$$

where

$$T_2 := \left[\begin{array}{c|cc} \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1}_{l_z} \\ \mathbf{0} & \mathbf{1}_{l_z} & \mathbf{0} \end{array} \right].$$

Thus, the equivalence of (29) and (11) readily follows from Elimination Lemma [29]. ■

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