Robust Performance Analysis of Linear Time-Invariant Uncertain Systems by Taking Higher-Order Time-Derivatives of the States

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Abstract

In this paper, we propose new LMI-based conditions for robust stability/performance analysis of linear time-invariant (LTI) uncertain systems. To get around the conservatism of existing conditions resulting from Lyapunov's stability theory, we first consider to employ Lyapunov functions that can be associated with higher-order derivatives of the state vectors. This motivates us to introduce a redundant system description so that we can take the behavior of the higher-order derivatives of the state into consideration. Indeed, by considering suitable redundant system descriptions, the existence conditions of those Lyapunov functions can be reduced into constrained inequality conditions, to which we can apply Finsler's Lemma. Thus we can readily obtain new LMI-based conditions for (robust) stability/performance analysis of LTI systems in a constructive way. It turns out that the proposed LMI conditions can be regarded as a natural extension of those known as extended or dilated LMIs in the literature.

I. Introduction

Robustness analysis of linear time-invariant (LTI) systems against parametric uncertainties has been studied intensively in the community of control theory [1]. When studying those analysis problems, one of the effective strategies should be to recast those analysis problems into feasibility tests of linear matrix inequalities (LMIs) via Lyapunov's stability theory. In particular, LMI-based approaches have become very promising since they enable us to employ parameter-dependent Lyapunov functions to assess the robust performance so that accurate analysis results can be achieved. Since the basic ideas to employ parameter-dependent Lyapunov functions were proposed [7], [8], notable contributions have been made in this direction [3], [4], [9], [11], [13].

In the literature dealing with robustness analysis problems of LTI systems using Lyapunov's stability theory, the Lyapunov functions employed are almost always restricted to those given by quadratic forms of the state vectors. This is because, when assessing the stability/performance of a system without uncertainties, it is certainly enough to seek for Lyapunov functions of this form. However, from the viewpoints of Lyapunov's stability theory, it is not necessary to restrict our attention to those specific forms. In particular, by employing Lyapunov functions of a suitable form, it is expected that we can derive less conservative conditions for robust stability/performance analysis problems.

From these observations, in this paper, we introduce Lyapunov functions that can be associated with higher-order derivatives of the state and explore the existence conditions of those Lyapunov functions by following similar discussions to [10]. When seeking for Lyapunov functions of these particular forms, we should be very careful on the existence of higher-order derivatives of the state as well as their behavior. This motivates us to employ a redundant description of the considered system so that we can grasp the behavior of the higher-order derivatives of the state exactly. Indeed, by employing suitable redundant system descriptions, we can reduce the existence conditions of those Lyapunov functions into constrained inequality conditions, to which Finsler's Lemma [2], [12] can be applied. It follows that we can readily obtain new LMI-based conditions for (robust) stability analysis. Furthermore, by following similar lines, we can also derive new LMI conditions for (robust) H_∞ performance analysis.

As expected from the fact that we derive new LMI-based conditions by employing Lyapunov functions of particular form and following the methodology shown in [10], the resulting conditions can be regarded as a natural extension of those known as extended or dilated LMI conditions [9], [10], [11], [5]. In particular, we can prove rigorously that the proposed robust stability/performance analysis conditions encompass those [10], [11] as particular cases. Through numerical experiments, it turns out that the proposed conditions are indeed effective to achieve less conservative analysis results than those in [10], [11].

We use the following notations in this paper. For a matrix $A \in \mathbf{R}^{n\times n}$, we define $\mathrm{He}\{A\} := A + A^T$. For a matrix $A \in \mathbf{R}^{m\times n}$ with $\mathrm{rank}(A) = r < n$, $A^\perp \in \mathbf{R}^{n\times(n-r)}$ is a matrix such that $AA^\perp = 0$ and $A^{\perp T}A^\perp > 0$. The symbol $\mathbf{P}_n$ denotes the set of $n \times n$ positive-definite real matrices. Given a positive integer N, let $\mathbf{Z}_N$ denote the set of positive integers up to N, i.e., $\mathbf{Z}_N := \{1, \cdots, N\}$. In this paper, we make an extensive use of the following lemma.

Lemma 1: (Finsler's Lemma) [10], [12] Let matrices $Q \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{m\times n}$ be given such that $\mathrm{rank}(B) < n$. Then, the following conditions are equivalent.

(i) The condition $x^TQx < 0$ holds for all $x \in \mathcal{N}_B$ where $\mathcal{N}_B := \{x \in \mathbf{R}^n : \ x \neq 0, \quad Bx = 0\}$.
(ii) The condition $B^{\perp T}QB^\perp < 0$ holds.
(iii) There exists $F \in \mathbf{R}^{n\times m}$ such that $Q + \mathrm{He}\{FB\} < 0$.

II. Stability Analysis

In this section, we propose new LMI-based conditions for (robust) stability analysis of continuous-time LTI systems by introducing Lyapunov functions that can be associated with higher-order derivatives of the state vectors.

A. New LMI-based Conditions for Stability Analysis

Let us consider the continuous-time LTI system given by

$$\dot{x}(t) = Ax(t), \quad A \in \mathbf{R}^{n\times n}. \tag{1}$$

According to Lyapunov's stability theory, this system is asymptotically stable if there exists a Lyapunov function $V(x(t))$ that satisfies the following conditions.

$$V(x(t)) > 0, \quad \dot{V}(x(t)) < 0, \quad \forall \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \in \mathcal{N}, \quad \mathcal{N} := \left\{ y \in \mathbf{R}^{2n} : \ y \neq 0, \quad [\ A \ \ -I\ ]y = 0 \right\}. \tag{2}$$

In addition, when assessing the asymptotic stability of the LTI system (1), it is well-known that we can restrict the class of the Lyapunov functions into

$$V(x(t)) = x^T(t)Px(t), \quad P \in \mathbf{P}_n. \tag{3}$$

Namely, the system (1) is asymptotically stable iff there exists $P \in \mathbf{P}_n$ that satisfies the requirements in (2) with $V(x(t))$ given by (3). In [10], the existence condition of those $P \in \mathbf{P}_n$ has been reduced into feasibility tests of LMIs in the following constructive way. Indeed, we see that the first condition in (2) can be ensured by $P \in \mathbf{P}_n$ whereas the second condition can be rewritten equivalently as

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0 \quad \forall \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \in \mathcal{N}. \tag{4}$$

By applying Lemma 1 to this constrained inequality condition, we can obtain the following results.

Proposition 1: [10], [11] The system (1) is asymptotically stable iff there exist $P \in \mathbf{P}_n$ and $F_j \in \mathbf{R}^{n\times n}$ $(j = 1, 2)$ such that

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} A & -I \end{bmatrix} \right\} < 0. \tag{5}$$

It is well-known that the LMI condition (5) is effective for robust stability analysis of the LTI system (1) where the matrix A is affected by uncertain parameters. Roughly speaking, since the Lyapunov matrix P has no multiplication relation with A, the LMI condition (5) enables us to assess the robust stability via parameter-dependent Lyapunov functions (PDLFs) [7], [8], which are quite promising to achieve less conservative analysis results as shown in [9], [10], [11].

In the above discussions, we see that the dilated LMI condition (5) readily follows from (2) by representing $\dot{V}(x(t))$ in a quadratic form associated with the derivative of the state as in (4). This implies that, if we are to derive new LMI conditions by achieving further dilation on (5), it is promising to employ Lyapunov functions whose time-derivatives can be associated with yet higher-order derivatives of the state.

Motivated by these observations, let us consider to take a (candidate of) Lyapunov function

$$V(x(t)) := x^T(t)P_2x(t), \quad P_2 = \begin{bmatrix} I \\ A \end{bmatrix}^T \Pi \begin{bmatrix} I \\ A \end{bmatrix}, \quad \Pi \in \mathbf{P}_{2n} \tag{6}$$

which can be rewritten, equivalently as

$$V = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \Pi \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}. \tag{7}$$

Note that $P_2 \in \mathbf{P}_n$ since $\Pi \in \mathbf{P}_{2n}$. Namely, even though the function $V(x(t))$ given by (6) is an ordinary quadratic form as in (3), it can be associated with the derivative of the state so that its derivative can be associated with the second-order derivative of the state. This is the key property that motivates us to introduce a Lyapunov function of the form (6). Indeed, by working with the Lyapunov function (6), we can derive novel LMI conditions for the asymptotic stability of the system (1).

To see this, let us examine the asymptotic stability of the system (1) by means of (6). It is apparent that if there exists $\Pi \in \mathbf{P}_{2n}$ such that (2) holds with $V(x(t))$ given by (6), then the system (1) is asymptotically stable. Conversely, suppose that the system (1) is asymptotically stable. Then, there exists a matrix $P \in \mathbf{P}_n$ that satisfies the condition (2) with $V(x(t))$ given by (3). Here, from the Schur complement arguments, we see that there exists a positive scalar $\bar{\varepsilon}(P)$ such that the positivity condition

$$\Pi = \begin{bmatrix} P & -\varepsilon A^T \\ -\varepsilon A & 2\varepsilon I \end{bmatrix} \in \mathbf{P}_{2n} \tag{8}$$

holds for any $0<\varepsilon<\bar{\varepsilon}(P)$. It is obvious that the matrix Π given by (8) satisfies (2) with $V(x(t))$ given by (6) for any $0 < \varepsilon < \bar{\varepsilon}(P)$, since we have

$$\begin{bmatrix} I \\ A \end{bmatrix}^T \begin{bmatrix} P & -\varepsilon A^T \\ -\varepsilon A & 2\varepsilon I \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = P. \tag{9}$$

It follows that the system (1) is asymptotically stable iff there exists $\Pi \in \mathbf{P}_{2n}$ that satisfies (2) with $V(x(t))$ given by (6).

Recalling (7), we see that the time-derivative of $V(x(t))$ can be written in a form associated with the second-order derivative of the state as follows.

$$\dot{V} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}^T W_2^T(\Psi_c \otimes \Pi)W_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}. \tag{10}$$

Here, the matrices W_2 and Ψ_c are given by

$$W_2 := \begin{bmatrix} W_{21} & W_{22} \end{bmatrix}^T, \quad W_{21} = \begin{bmatrix} I_{2n} \\ 0_{n,2n} \end{bmatrix}, \quad W_{22} = \begin{bmatrix} 0_{n,2n} \\ I_{2n} \end{bmatrix}, \quad \Psi_c := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

With this in mind, we now consider to recast the condition (2) with $V(x(t))$ given by (6) into a dilated LMI condition as in (5). To this end, let us introduce

$$\mathcal{N}_2 := \left\{ y \in \mathbf{R}^{3n} : \ y \neq 0, \quad \begin{bmatrix} A & -I & 0 \\ 0 & A & -I \end{bmatrix} y = 0 \right\}. \tag{12}$$

Then, it is easy to see that for each t, the condition $[\, x^T(t)\ \dot{x}^T(t)\,]^T \in \mathcal{N}$ holds iff $\ddot{x}(t)$ exists and $[\, x^T(t)\ \dot{x}^T(t)\ \ddot{x}^T(t)\,]^T \in \mathcal{N}_2$ since we can associate the system (1) with the following redundant description:

$$\begin{bmatrix} A & -I & 0 \\ 0 & A & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = 0. \tag{13}$$

Hence, from (10), we can conclude that the system (1) is asymptotically stable iff there exists $\Pi \in \mathbf{P}_{2n}$ such that

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}^T W_2^T(\Psi_c \otimes \Pi) W_2 \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} < 0 \quad \forall \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} \in \mathcal{N}_2. \tag{14}$$

Noting that the form of (14) is exactly the same as (i) in Lemma 1, we can readily obtain the following results.

Theorem 1: The system (1) is asymptotically stable iff there exist $\Pi \in \mathbf{P}_{2n}$ and $F_{jk} \in \mathbf{R}^{n\times n}$ $(j = 1, 2, 3,\ k = 1, 2)$ such that

$$W_2^T(\Psi_c \otimes \Pi) W_2 + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix} \begin{bmatrix} A & -I & 0 \\ 0 & A & -I \end{bmatrix} \right\} < 0. \tag{15}$$

Here, the matrices W_2 and Ψ_c are given by (11).

Now we have derived a new LMI condition (15) via the Lyapunov function (6) that can be associated with the derivatives of the state as in (7) and the corresponding redundant system description (13). Similarly to (5), we see in the new LMI condition (15) that the matrix variable Π has no multiplication relation with A. This property enables us to employ parameter-dependent Lyapunov functions when dealing with robust stability analysis problems.

On the other hand, from the discussions around (8), it is expected that the matrix variable $\Pi \in \mathbf{P}_{2n}$ in (15) can be constructed from $P \in \mathbf{P}_n$ in (5). Indeed, regarding the matrix variables that satisfy (5) and (15), we can verify that the following connections hold.

Proposition 2: If (5) holds with $P = \mathcal{P} \in \mathbf{P}_n$ and $F_j = \mathcal{F}_j$ $(j = 1, 2)$, then there exists $\bar{\varepsilon} > 0$ such that (15) holds with

$$\Pi = \begin{bmatrix} \mathcal{P} & -\varepsilon A^T \\ -\varepsilon A & 2\varepsilon I \end{bmatrix} \in \mathbf{P}_{2n}, \quad \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1 & 0 \\ \mathcal{F}_2 & 0 \\ 0 & \varepsilon I \end{bmatrix} \tag{16}$$

for any $0 < \varepsilon < \bar{\varepsilon}$.

This result shows that the LMI condition (15) is still necessary and sufficient even when we restrict the variables as $F_{12} = 0$, $F_{22} = 0$ and $F_{31} = 0$. In addition, it turns out in the next subsection that the connection (16) plays an important role to ensure an explicit advantage of (15) over (5) when dealing with robust stability analysis problems.

Before proceeding to robust stability analysis problems, we briefly discuss further extensions in the direction of stability analysis using redundant system descriptions. It should be noted that, by taking Lyapunov functions that can be associated with yet higher-order derivatives of the state and considering the corresponding redundant system descriptions, we can derive new LMI conditions successively. For example, let us take a Lyapunov function of the form

$$V(x(t)) := x^T(t) \left(\begin{bmatrix} I \\ A \\ A^2 \end{bmatrix}^T \Xi \begin{bmatrix} I \\ A \\ A^2 \end{bmatrix} \right) x(t), \quad \Xi \in \mathbf{P}_{3n}$$

and consider the corresponding redundant system description

$$\begin{bmatrix} A & -I & 0 & 0 \\ 0 & A & -I & 0 \\ 0 & 0 & A & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \dddot{x}(t) \end{bmatrix} = 0.$$

Then, by following similar lines to (14), we can show that the system (1) is asymptotically stable iff the following LMI condition holds.

$$W_3^T(\Psi_c \otimes \Xi) W_3 + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \end{bmatrix} \begin{bmatrix} A & -I & 0 & 0 \\ 0 & A & -I & 0 \\ 0 & 0 & A & -I \end{bmatrix} \right\} < 0. \tag{17}$$

Here, $\Xi \in \mathbf{P}_{3n}$ and $F_{jk} \in \mathbf{R}^{n\times n}$ $(j = 1, \cdots, 4,\ k = 1, \cdots, 3)$ are matrix variables to be determined whereas

$$W_3 := \begin{bmatrix} W_{31} & W_{32} \end{bmatrix}^T,\ W_{31} = \begin{bmatrix} I_{3n} \\ 0_{n,3n} \end{bmatrix},\ W_{32} = \begin{bmatrix} 0_{n,3n} \\ I_{3n} \end{bmatrix}.$$

In particular, it can be easily verified that if (15) holds with $\Pi = \widehat{\Pi} \in \mathbf{P}_{2n}$ and $F_{jk} = \mathcal{F}_{jk}$ $(j = 1, 2, 3,\ k = 1, 2)$, then there exists $\bar{\varepsilon} > 0$ such that (17) holds with

$$\Xi = \left[\begin{array}{cc:c} \widehat{\Pi}_{11} & \widehat{\Pi}_{12} & -\varepsilon A^{2T} \\ \widehat{\Pi}_{12}^T & \widehat{\Pi}_{22} & 0 \\ \hdashline -\varepsilon A^2 & 0 & 2\varepsilon I \end{array}\right] \in \mathbf{P}_{3n},\quad \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & 0 \\ \mathcal{F}_{21} & \mathcal{F}_{22} & 0 \\ \mathcal{F}_{31} & \mathcal{F}_{32} & 0 \\ 0 & 0 & \varepsilon I \end{bmatrix} \tag{18}$$

for any $0 < \varepsilon < \bar{\varepsilon}$. In this way, we can obtain new LMI-based conditions successively while retaining the structural properties as in (16) and (18).

B. Robust Stability Analysis of Uncertain Systems

Let us consider the polytopic-type uncertain LTI system described by

$$\dot{x}(t) = A(\alpha)x(t), \quad \alpha \in \boldsymbol{\alpha} \tag{19}$$

where

$$\boldsymbol{\alpha} := \left\{ \alpha \in \mathbf{R}^N :\ \alpha_i \geq 0 \quad \forall i \in \mathbf{Z}_N, \quad \sum_{i=1}^N \alpha_i = 1 \right\}, \tag{20}$$

$$A(\alpha) := \sum_{i=1}^N \alpha_i A_i. \tag{21}$$

Here, $A_i \in \mathbf{R}^{n\times n}$ $(i \in \mathbf{Z}_N)$ are given matrices.

To assess the robust stability of the system (19) in a less conservative fashion, Oliveira and Skelton [10] and Peaucelle et al. [11] proposed the following condition based on (5).

Proposition 3: [10], [11] The system (19) is robustly asymptotically stable if there exists $P_i \in \mathbf{P}_n$ and $F_j \in \mathbf{R}^{n\times n}$ $(j = 1, 2)$ such that

$$\begin{bmatrix} 0 & P_i \\ P_i & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} A_i & -I \end{bmatrix} \right\} < 0 \quad \forall i \in \mathbf{Z}_N. \tag{22}$$

It should be noted that the above LMI-based condition has been obtained by employing fixed (i.e., parameter-independent) matrix variables F_1 and F_2 over the whole uncertainty domain $\boldsymbol{\alpha}$. By applying similar ideas to (15) and considering fixed variables F_{jk} $(j = 1, 2, 3,\ k = 1, 2)$, we can readily obtain the next theorem.

Theorem 2: The system (19) is robustly asymptotically stable if there exist $\Pi_i \in \mathbf{P}_{2n}$ and $F_{jk} \in \mathbf{R}^{n\times n}$ $(j = 1, 2, 3\ k = 1, 2)$ such that

$$W_2^T(\Psi_c \otimes \Pi_i)W_2 + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix} \begin{bmatrix} A_i & -I & 0 \\ 0 & A_i & -I \end{bmatrix} \right\} < 0 \quad \forall i \in \mathbf{Z}_N. \tag{23}$$

It is meaningful to examine the new robust stability analysis condition (23) in comparison with (22). As is well-known, the condition (22) can be interpreted as a sufficient condition for the existence of PDLFs of the form

$$V(x(t), \alpha) = x^T(t)P(\alpha)x(t), \quad P(\alpha) := \sum_{i=1}^N \alpha_i P_i \in \mathbf{P}_n$$

that ensures the asymptotic stability of the system (19). On the other hand, the new condition (23) corresponds to a sufficient condition for the existence of PDLFs of the form

$$V(x(t), \alpha) = x^T(t) \left(\begin{bmatrix} I \\ A(\alpha) \end{bmatrix}^T \Pi(\alpha) \begin{bmatrix} I \\ A(\alpha) \end{bmatrix} \right) x(t), \quad \Pi(\alpha) := \sum_{i=1}^N \alpha_i \Pi_i \in \mathbf{P}_{2n}. \tag{24}$$

It follows that the new condition (23) enables us to assess the robust stability with Lyapunov functions that depend cubically on the uncertain parameter α as in (24).

From these observations, it is expected that the new condition (23) is promising to achieve less conservative analysis results than (22). In particular, we can ensure the advantage of (23) over (22) more rigorously as follows.

Theorem 3: If the robust stability analysis condition (22) holds with $P_i = \mathcal{P}_i \in \mathbf{P}_n$ $(i \in \mathbf{Z}_N)$ and $F_j = \mathcal{F}_j$ $(j = 1, 2)$, then there exists $\varepsilon > 0$ such that the condition (23) holds with

$$\Pi_i = \begin{bmatrix} \mathcal{P}_i & -\varepsilon A_i^T \\ -\varepsilon A_i & 2\varepsilon I \end{bmatrix} \in \mathbf{P}_{2n} \ (i \in \mathbf{Z}_N), \quad \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1 & 0 \\ \mathcal{F}_2 & 0 \\ 0 & \varepsilon I \end{bmatrix}.$$

Namely, the robust stability analysis condition (23) provides no more conservative results than (22).

Proof: Since there are only finitely many vertices in (19), the assertion readily follows from (16). ■

As is easily seen from (17), we can derive new LMI-based robust stability analysis conditions successively by taking Lyapunov functions that can be associated with yet higher-order derivatives of the state vectors. In addition, since the structural properties as in (18) holds, we can also ensure the non-conservativeness of the resulting LMI conditions as in Theorem 3. However, it is not necessarily apparent whether those LMI conditions indeed work effectively to reduce the conservatism and achieve more accurate analysis results. In the next subsection, we examine these points through numerical experiments.

C. Numerical Experiments

For given matrices $\bar{A}_0 \in \mathbf{R}^{n\times n}$ and $\bar{A}_i \in \mathbf{R}^{n\times n}$ $(i \in \mathbf{Z}_N)$, let us consider the polytopic-type uncertain system (19) where the vertex matrices A_i are given by

$$A_i(\eta) = \bar{A}_0 + \eta \bar{A}_i \quad i \in \mathbf{Z}_N. \tag{25}$$

We assume that the matrix $\bar{A}_0$ is Hurwitz stable. In this subsection, we follow the discussions in [4] and consider the problem to compute the stability margin ρ defined by

$$\rho := \sup \{\bar{\eta} \in \mathbf{R} : \text{ The system (19) is asymptotically stable for all } (\alpha, \eta) \in \boldsymbol{\alpha} \times [0 \ \bar{\eta}]\}.$$

1) Example 1: We first consider the case where $N = 2$. The matrices in (25) are given by

$$\bar{A}_0 = \begin{bmatrix} -2.0 & 1.0 & -1.0 \\ 2.5 & -3.0 & 0.5 \\ -1.0 & 1.0 & -3.5 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -0.7 & -0.5 & -2.0 \\ -0.8 & 0.0 & 0.0 \\ 1.5 & 2.0 & 2.4 \end{bmatrix}$$

and $\bar{A}_2 = -\bar{A}_1$, respectively. Based on the proposed robust stability analysis conditions, we carry out bisection search and obtain the estimates of the stability margin $\hat{\rho}$ as shown in Table I. In this example, we can compute the exact value ρ by using the notion of guardian map [1], which turns out to be $\rho = 3.551$. From Table I, we see that the proposed LMI conditions yield better analysis results than the existing condition (22). In particular, the proposed conditions successfully achieve the exact stability margin. Note however that the proposed conditions are computationally demanding as indicated by the number of scalar variables N_s in Table I.

TABLE I

STABILITY MARGIN FOR A POLYTOPIC-TYPE UNCERTAIN SYSTEM WITH TWO VERTICES.

Robust stability condition (22) [10], [11].	$\hat{\rho}_1 = 3.207$ $(N_s = 30)$
Robust stability condition (23) (Theorem 2).	$\hat{\rho}_2 = 3.551$ $(N_s = 96)$
Robust stability condition based on (17).	$\hat{\rho}_3 = 3.551$ $(N_s = 198)$

(N_s: The number of scalar variables in each LMI condition.)

2) Example 2: We next consider the case where $N = 3$. The matrices in (25) are

$$\bar{A}_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5.0 & -1.6 \\ -0.6 & 2.9 & -2.0 & -0.6 \end{bmatrix}, \ \bar{A}_1 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2.0 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2.0 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix}, \ \bar{A}_3 = \begin{bmatrix} -1.0 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}.$$

For this problem, we apply the proposed robust stability analysis conditions and obtain the estimates of the stability margin $\hat{\rho}$ as shown in Table II. The proposed two conditions yield the same result $\hat{\rho} = 1.930$, even though the condition (17) takes a higher-order derivative of the state vectors.

TABLE II

STABILITY MARGIN FOR A POLYTOPIC-TYPE UNCERTAIN SYSTEM WITH THREE VERTICES.

Robust stability condition (22) [10], [11].	$\hat{\rho}_1 = 1.497$ ($N_s = 62$)
Robust stability condition (23) (Theorem 2).	$\hat{\rho}_2 = 1.930$ ($N_s = 204$)
Robust stability condition based on (17).	$\hat{\rho}_3 = 1.930$ ($N_s = 426$)

(N_s: The number of scalar variables in each LMI condition.)

This problem was dealt with by Chesi et al. [4] and they obtained an estimate of the stability margin $\hat{\rho} = 2.224$ by using polynomial-type PDLFs. Thus, in this example, the proposed conditions fail to achieve an exact stability margin.

In addition to the above two examples, we carried out numerous numerical experiments. It turns that, in the case where $N = 2$, the proposed analysis condition always achieves an exact stability margin irrespective of the system dimension n, by taking sufficiently many higher-order derivatives of the state vectors. However, rigorous proofs for this assertion are still open.

Remark 1: In [6], the authors dealt with the stability analysis problems of the system (1) by reformulating them into nonsingularity analysis problems of the matrix polynomial $sI - A$ over $s \in \bar{\mathbf{C}}_+$, where $\bar{\mathbf{C}}_+ \subset \mathbf{C}$ denotes the closed right half plane. In particular, by employing suitable polynomial-type multipliers $\mathcal{F}(s)$, exactly the same conditions as in (15) and (17) have also been derived. In this sense, the present paper has contributed to giving a new interpretation on those conditions from the viewpoints of Lyapunov's stability theory in the time-domain. More importantly, the idea to employ Lyapunov functions that can be associated with higher-order derivatives of the state and introduce redundant system descriptions can readily be applied to *performance* analysis problems. Indeed, in the next section, we derive a new robust H_∞ performance analysis condition based on a particular redundant system description, which should be far from attainable if we pursue the direction of [6].

III. H_∞ PERFORMANCE ANALYSIS

In this section, we derive a new LMI-based condition for the H_∞ performance analysis of continuous-time LTI systems based on a particular redundant system description.

A. New LMI-based Condition for H_∞ Performance Analysis

Let us consider the LTI system described by

$$\dot{x}(t) = Ax(t) + Bw(t), \quad z(t) = Cx(t) + Dw(t), \quad x(0) = 0 \tag{26}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$ and $D \in \mathbf{R}^{l\times m}$. The $\mathcal{L}_2$ gain of the system (26) is defined by

$$\gamma_\infty := \sup_{w(t)\in\mathcal{L}_2, w(t)\neq 0} \frac{||z(t)||_2}{||w(t)||_2} \tag{27}$$

where

$$\mathcal{L}_2 := \{\xi(t) : ||\xi(t)||_2 < \infty\}, \ ||\xi(t)||_2 := \left(\int_0^\infty \xi^T(t)\xi(t)dt\right)^{\frac{1}{2}}.$$

Note that if the system (26) is asymptotically stable (i.e., the matrix A is Hurwitz stable), the $\mathcal{L}_2$ gain of the system (26) coincides with the H_∞ norm of the associated transfer function

$$G(s) := C(sI - A)^{-1}B + D. \tag{28}$$

In view of this fact, our goal here is to derive new LMI-based conditions for the analysis of the H_∞ norm $||G(s)||_\infty$, by investigating the $\mathcal{L}_2$ gain of the system (26) defined in the time-domain as in (27).

For a prescribed value $0 < \gamma < \infty$, it is well-known [10] that the system (26) is asymptotically stable and $\gamma_\infty < \gamma$ holds iff there exists a Lyapunov function

$$V(x(t)) = x^T(t)Px(t), \quad P \in \mathbf{P}_n \tag{29}$$

that satisfies

$$\begin{aligned} &\dot{V}(x(t)) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \quad \forall \begin{bmatrix} x(t) \\ \dot{x}(t) \\ z(t) \\ w(t) \end{bmatrix} \in \mathcal{M}, \\ &\mathcal{M} := \left\{ y \in \mathbf{R}^{2n+l+m} : \ y \neq 0, \ \begin{bmatrix} A & -I & 0 & B \\ C & 0 & -I & D \end{bmatrix} y = 0 \right\}. \end{aligned} \tag{30}$$

By applying Lemma 1 to the constrained inequality condition (30), the following results have been obtained [10].

Proposition 4: [10] Let us consider the system described by (28). Then, the matrix A is Hurwitz stable and $||G(s)||_\infty < \gamma$ holds iff there exist $P \in \mathbf{P}_n$ and F_{jk} $(j = 1, \cdots, 4, \ k = 1, 2)$ such that

$$\begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \\ F_{41} & F_{42} \end{bmatrix} \begin{bmatrix} A & -I & 0 & B \\ C & 0 & -I & D \end{bmatrix} \right\} < 0. \tag{31}$$

We now consider to derive new LMI-based conditions for the analysis of the H_∞ norm of the system (28), by investigating the $\mathcal{L}_2$ gain condition of the system (26). Since we have successfully obtained new LMIs for (robust) stability analysis by employing redundant system descriptions, we pursue this direction for the system (26) and explore to take higher-order derivatives of the state vector $x(t)$. Unfortunately, however, this cannot be done straightforwardly since in the analysis of the $\mathcal{L}_2$ gain, the input signal $w(t)$ is not necessarily differentiable.

In order to get around this difficulty, in the sequel, we assume that the input matrix $B \in \mathbf{R}^{n\times m}$ satisfies $\mathrm{rank}(B) = r < n$ and define $\bar{B} := B^{T\perp T} \in \mathbf{R}^{(n-r)\times n}$. Under this assumption, we can successfully obtain the following redundant description of the system (26).

$$\bar{B}\ddot{x}(t) = \bar{B}A\dot{x}(t), \quad \dot{x}(t) = Ax(t) + Bw(t), \quad z(t) = Cx(t) + Dw(t). \tag{32}$$

This can be rewritten, equivalently as

$$\Omega \begin{bmatrix} x(t) \\ \dot{x}(t) \\ z(t) \\ w(t) \\ \bar{B}\ddot{x}(t) \end{bmatrix} = 0, \quad \Omega := \begin{bmatrix} A & -I & 0 & B & 0 \\ C & 0 & -I & D & 0 \\ 0 & \bar{B}A & 0 & 0 & -I \end{bmatrix}. \tag{33}$$

It should be noted that $\bar{B}\ddot{x}(t)$ is well-defined even though $\ddot{x}(t)$ may not be.

Once we obtain the redundant system description (33), it is straightforward to derive a new condition for the $\mathcal{L}_2$ gain analysis. To this end, let us consider the Lyapunov function of the form

$$V(x(t)) = x^T(t)P_a x(t), \quad P_a = \begin{bmatrix} I \\ \bar{B}A \end{bmatrix}^T \Pi \begin{bmatrix} I \\ \bar{B}A \end{bmatrix}, \quad \Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \in \mathbf{P}_{2n-r}, \quad \Pi_{11} \in \mathbf{P}_n. \tag{34}$$

We see that $P_a \in \mathbf{P}_n$ since $\Pi \in \mathbf{P}_{2n-r}$ and hence the above $V(x(t))$ is nothing but an ordinary quadratic form as in (29). Note however that it can be rewritten equivalently in a from associated with the derivative of the state as follows.

$$V = \begin{bmatrix} x(t) \\ \bar{B}\dot{x}(t) \end{bmatrix}^T \Pi \begin{bmatrix} x(t) \\ \bar{B}\dot{x}(t) \end{bmatrix}. \tag{35}$$

With this in mind, we derive a new condition for the $\mathcal{L}_2$ gain analysis of the system (26). The following lemma plays an important role for this purpose.

Lemma 2: The system (26) is asymptotically stable and the $\mathcal{L}_2$ gain condition $\gamma_\infty < \gamma$ holds iff there exists a Lyapunov function $V(x(t))$ of the form (34) that satisfies (30).

Proof: As we have stated, the system (26) is asymptotically stable and $\gamma_\infty < \gamma$ holds iff there exists $P \in \mathbf{P}_n$ satisfying (30) with $V(x(t))$ given by (29). Hence, noting that $P_a \in \mathbf{P}_n$ in (34), the sufficiency of the assertion is apparent. On the other hand, to prove the necessity, let us take a matrix $P \in \mathbf{P}_n$ that satisfies (30) with (29). Then, similarly to what we have shown around (8) and (9), we can verify that there exists sufficiently small $\bar{\varepsilon}(P) > 0$ such that the function $V(x(t))$ given by (34) satisfies (30) with

$$\Pi := \begin{bmatrix} P & -\varepsilon A^T \bar{B}^T \\ -\varepsilon \bar{B}A & 2\varepsilon I_{n-r} \end{bmatrix} \in \mathbf{P}_{2n-r} \tag{36}$$

for any $0 < \varepsilon < \bar{\varepsilon}(P)$. This completes the proof. ■

To assess the $\mathcal{L}_2$ gain of the system (26) by means of Lemma 2, let us take the time-derivative of $V(x(t))$ given by (34). Here, we see from (35) that the time-derivative of $V(x(t))$ can be written in a form associated with the second-order derivative of the state as follows.

$$\dot{V} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \bar{B}\ddot{x}(t) \end{bmatrix}^T \begin{bmatrix} 0 & \Pi_{11} & \Pi_{12} \\ \Pi_{11} & \Pi_{12}\bar{B} + \bar{B}^T\Pi_{12}^T & \bar{B}^T\Pi_{22} \\ \Pi_{12}^T & \Pi_{22}\bar{B} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \bar{B}\ddot{x}(t) \end{bmatrix}. \tag{37}$$

On the other hand, from the discussions around (32) and (33), it can be readily seen that we can rewrite the condition (30) in the following form so that the term $\bar{B}\ddot{x}$ appears explicitly in the constraints.

$$\dot{V}(x(t)) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \quad \forall \begin{bmatrix} x(t) \\ \dot{x}(t) \\ z(t) \\ w(t) \\ \bar{B}\ddot{x}(t) \end{bmatrix} \in \mathcal{M}_a, \tag{38}$$

$$\mathcal{M}_a := \left\{ y \in \mathbf{R}^{3n-r+l+m} : \ y \neq 0, \quad \Omega y = 0 \right\}.$$

By substituting (37) into (38) and applying Lemma 1 to the resulting constraint inequality condition, we can readily obtain the next theorem.

Theorem 4: Let us consider the system described by (28). Then, the matrix A is Hurwitz stable and $||G(s)||_\infty < \gamma$ holds iff there exist $\Pi \in \mathbf{P}_{2n-r}$, F_{jk} $(j = 1, \cdots, 5,\ k = 1, 2, 3)$ such that

$$\begin{bmatrix} 0 & \Pi_{11} & 0 & 0 & \Pi_{12} \\ \Pi_{11} & \Pi_{12}\bar{B} + \bar{B}^T\Pi_{12}^T & 0 & 0 & \bar{B}^T\Pi_{22} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I & 0 \\ \Pi_{12}^T & \Pi_{22}\bar{B} & 0 & 0 & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \\ F_{51} & F_{52} & F_{53} \end{bmatrix} \begin{bmatrix} A & -I & 0 & B & 0 \\ C & 0 & -I & D & 0 \\ 0 & \bar{B}A & 0 & 0 & -I \end{bmatrix} \right\} < 0 \tag{39}$$

Now we have derived a new LMI condition (39) for the H_∞ performance analysis based on the redundant system description (32) and the Lyapunov function (34), which can be associated with the derivatives of the state as in (35). It should be noted here that, when introducing the redundant system description (32), we rely only on the relation $\bar{B}B = 0$. Hence, we see that the results in Theorem 4 are still valid even if we replace the matrix $\bar{B}$ with arbitrary $\hat{B}$ satisfying $\hat{B}B = 0$.

B. Connections with the Existing Results

In this subsection, we examine the new H_∞ performance analysis condition (39) in comparison with the existing results and clarify explicit connections among them.

1) Connection with the KYP Lemma: Regarding the new LMI condition (39), we see

$$\begin{bmatrix} A & -I & 0 & B & 0 \\ C & 0 & -I & D & 0 \\ 0 & \bar{B}A & 0 & 0 & -I \end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 \\ A & B \\ C & D \\ 0 & I \\ \bar{B}A^2 & \bar{B}AB \end{bmatrix}. \tag{40}$$

Hence, from Lemma 1, the new condition (39) reduces to

$$\begin{bmatrix} \mathrm{He}\{P(\Pi, A, \bar{B})A\} & P(\Pi, A, \bar{B})B + C^T D \\ B^T P^T(\Pi, A, \bar{B}) + D^T C & D^T D - \gamma^2 I \end{bmatrix} < 0$$

where

$$P(\Pi, A, \bar{B}) := \begin{bmatrix} I \\ \bar{B}A \end{bmatrix}^T \Pi \begin{bmatrix} I \\ \bar{B}A \end{bmatrix}. \tag{41}$$

This implies that, in the case where we deal with LTI systems without uncertainties, the new condition (39) is essentially equivalent to the well-known KYP type LMI [2], [12].

2) Connection with the Dilated LMI (31): In the proof of Lemma 2, we have clarified in part the connections among the matrix variables $P \in \mathbf{P}_n$ in (29) and $\Pi \in \mathbf{P}_{2n-r}$ in (34) that satisfy (30). As a natural result of (36), we can verify that the following connections hold regarding the matrix variables in the LMIs (31) and (39).

Proposition 5: For a prescribed value $\hat{\gamma} > 0$, suppose (31) holds for $\gamma = \hat{\gamma}$ with $P = \mathcal{P} \in \mathbf{P}_n$ and $F_{jk} = \mathcal{F}_{jk}$ $(j = 1, \cdots, 4,\ k = 1, 2)$. Then, there exists $\bar{\varepsilon} > 0$ such that (39) holds for $\gamma = \hat{\gamma}$ with

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{P} & -\varepsilon A^T \bar{B}^T \\ -\varepsilon \bar{B}A & 2\varepsilon I_{n-r} \end{bmatrix} \in \mathbf{P}_{2n-r}, \tag{42}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \\ F_{51} & F_{52} & F_{53} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & 0 \\ \mathcal{F}_{21} & \mathcal{F}_{22} & 0 \\ \mathcal{F}_{31} & \mathcal{F}_{32} & 0 \\ \mathcal{F}_{41} & \mathcal{F}_{42} & 0 \\ 0 & 0 & \varepsilon I \end{bmatrix}, \tag{43}$$

where ε is an arbitrary scalar that satisfies $0 < \varepsilon < \bar{\varepsilon}$.

This result clearly corresponds to Proposition 2 and plays a key role to ensure the advantage of the new condition (39) over the existing one (31) when dealing with robust H_∞ performance analysis problems.

C. Robust H_∞ Performance Analysis of Uncertain Systems

Let us consider the polytopic-type uncertain LTI system described by

$$\dot{x} = A(\alpha)x + B(\alpha)w, \quad z = C(\alpha)x + D(\alpha)w, \quad x(0) = 0, \quad \alpha \in \boldsymbol{\alpha} \tag{44}$$

where $\boldsymbol{\alpha}$ is given by (20) and

$$\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} := \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}. \tag{45}$$

Here, A_i, B_i, C_i and D_i $(i \in \mathbf{Z}_N)$ are given matrices. The transfer function from w to z is given by

$$G(s,\alpha) = C(\alpha)(sI - A(\alpha))^{-1}B(\alpha) + D(\alpha). \tag{46}$$

Our goal here is to compute the worst case H_∞ norm of the above system over $\alpha \in \boldsymbol{\alpha}$.

To deal with this robust H_∞ performance analysis problem, we can obtain the following results based on (31).

Proposition 6: [10] Let us consider the system described by (46). Then, the matrix $A(\alpha)$ is Hurwitz stable and $||G(s,\alpha)||_\infty < \gamma$ holds for all $\alpha \in \boldsymbol{\alpha}$ if there exist $P_i \in \mathbf{P}_n$ and F_{jk} $(j = 1, \cdots, 4,\ k = 1, 2)$ such that

$$\begin{bmatrix} 0 & P_i & 0 & 0 \\ P_i & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \\ F_{41} & F_{42} \end{bmatrix} \begin{bmatrix} A_i & -I & 0 & B_i \\ C_i & 0 & -I & D_i \end{bmatrix} \right\} < 0 \quad \forall i \in \mathbf{Z}_N. \tag{47}$$

On the other hand, since we have already established those results as Theorem 4 and Proposition 5, it is straightforward to see that the following two theorems hold.

Theorem 5: Let us consider the system described by (46). Then, the matrix $A(\alpha)$ is Hurwitz stable and $||G(s,\alpha)||_\infty < \gamma$ holds for all $\alpha \in \boldsymbol{\alpha}$ if there exist $\Pi_i \in \mathbf{P}_{2n-r}$ and F_{jk} $(j = 1, \cdots, 5,\ k = 1, 2, 3)$ such that

$$\begin{aligned} &\begin{bmatrix} 0 & \Pi_{11,i} & 0 & 0 & \Pi_{12,i} \\ \Pi_{11,i} & \Pi_{12,i}\bar{B} + \bar{B}^T\Pi_{12,i}^T & 0 & 0 & \bar{B}^T\Pi_{22,i} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I & 0 \\ \Pi_{12,i}^T & \Pi_{22,i}\bar{B} & 0 & 0 & 0 \end{bmatrix} \\ &+\mathrm{He}\left\{ \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \\ F_{51} & F_{52} & F_{53} \end{bmatrix} \begin{bmatrix} A_i & -I & 0 & B_i & 0 \\ C_i & 0 & -I & D_i & 0 \\ 0 & \bar{B}A_i & 0 & 0 & -I \end{bmatrix} \right\} < 0 \quad \forall i \in \mathbf{Z}_N. \end{aligned} \tag{48}$$

Here, $\bar{B} \in \mathbf{R}^{(n-r)\times n}$ is an arbitrary matrix that satisfies $\bar{B}B_i = 0$ $(\forall i \in \mathbf{Z}_N)$.

Theorem 6: For a prescribed value $\hat{\gamma} > 0$, suppose (47) holds for $\gamma = \hat{\gamma}$ with $P_i = \mathcal{P}_i \in \mathbf{P}_n$ $(i \in \mathbf{Z}_N)$ and $F_{jk} = \mathcal{F}_{jk}$ $(j = 1, \cdots, 4,\ k = 1, 2)$. Then, there exists $\varepsilon > 0$ such that (48) holds for $\gamma = \hat{\gamma}$ with

$$\begin{bmatrix} \Pi_{11,i} & \Pi_{12,i} \\ \Pi_{12,i}^T & \Pi_{22,i} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_i & -\varepsilon A_i^T\bar{B}^T \\ -\varepsilon\bar{B}A_i & 2\varepsilon I_{n-r} \end{bmatrix} > 0 \ (i \in \mathbf{Z}_N), \quad \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \\ F_{51} & F_{52} & F_{53} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & 0 \\ \mathcal{F}_{21} & \mathcal{F}_{22} & 0 \\ \mathcal{F}_{31} & \mathcal{F}_{32} & 0 \\ \mathcal{F}_{41} & \mathcal{F}_{42} & 0 \\ 0 & 0 & \varepsilon I \end{bmatrix}.$$

Hence, the robust H_∞ performance analysis condition (48) provides no more conservative results than (47).

The assertions in Theorem 6 clearly correspond to those in Theorem 3. Note however that, when dealing with robust H_∞ performance analysis problems, the analysis condition (48) depends on the choice of the matrix $\bar{B}$. In particular, if we let $\bar{B} = 0$, we can confirm that the condition (48) reduces essentially to (47). In other words, the new condition (48) is effective only when we can take $\bar{B} \neq 0$ that satisfies $\bar{B}B_i = 0$ $(\forall i \in \mathbf{Z}_N)$.

If this is the case, the new condition (48) should be promising in comparison with (47). This is because, in addition to the advantage shown in Theorem 6, the new condition (48) enables us to assess the robust H_∞ performance via PDLFs of the form

$$V(x(t),\alpha) = x^T(t)\left(\begin{bmatrix} I \\ \bar{B}A(\alpha) \end{bmatrix}^T \Pi(\alpha) \begin{bmatrix} I \\ \bar{B}A(\alpha) \end{bmatrix} \right) x(t), \quad \Pi(\alpha) = \sum_{i=1}^{N} \alpha_i \Pi_i.$$

Namely, we can assess the robust H_∞ performance via Lyapunov functions that depend cubically on the uncertain parameter α.

D. Numerical Experiments

1) The case where the matrix B has no uncertainties: Let us consider the polytopic-type uncertain system described by (44) and (45) with $N = 3$. The vertex matrices are given by

$$A_1 = \begin{bmatrix} -0.5008 & -0.4073 & 0.0336 \\ 0.2194 & 0.3801 & 0.5698 \\ -0.4972 & -0.6033 & -0.2826 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1603 & -0.1433 & -0.3133 \\ -0.1536 & -1.2454 & 0.3023 \\ 0.5079 & 0.2159 & -0.1128 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.2436 & 0.1515 & -0.1856 \\ -0.5742 & 0.0975 & 0.6841 \\ -0.3823 & -0.4388 & -0.9544 \end{bmatrix}, \quad B_i = B, \quad C_i = C, \quad D_i = -0.4786 \ (i = 1, 2, 3)$$

where

$$B = \begin{bmatrix} 0.2190 \\ -0.6587 \\ 0.7503 \end{bmatrix}, \quad C = \begin{bmatrix} -0.0890 \\ 0.3838 \\ -0.0374 \end{bmatrix}^T.$$

By applying the analysis conditions (47) and (48), we obtain the upper bounds of the worst case H_∞ norm as shown in Table III. Here we apply those analysis conditions also to the dual system of (44). When applying the proposed analysis condition (48) to the primal system, we take $\bar{B} = B^{T\perp T}$ whereas in the case of the dual system, we take $\bar{B} = C^{\perp T}$. In both cases, we see that the proposed condition (48) yields better analysis results than (47).

TABLE III
COMPUTATION RESULTS FOR ROBUST H_∞ PERFORMANCE ANALYSIS

	Primal system	Dual system
LMI condition (47) [10]	3.0896 $(N_s = 51)$	2.3681 $(N_s = 51)$
LMI condition (48)	1.9345 $(N_s = 106)$	1.8716 $(N_s = 106)$

(N_s: The number of scalar variables in each LMI condition.)

2) The case where the matrix B is subject to uncertainties: We next consider the case where $N = 2$ in (44). The vertex matrices are given by

$$A_1 = \begin{bmatrix} -0.4820 & 0.2475 & 0.5523 \\ 0.6564 & -0.0541 & 0.6678 \\ 0.4631 & -0.8679 & -0.7465 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3280 & -0.4585 & -0.0199 \\ 0.2639 & 0.1258 & -0.0179 \\ -0.0156 & 0.1079 & -0.5112 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.7089 \\ 0.0738 \\ 0.5886 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0118 \\ -0.5969 \\ 0.0295 \end{bmatrix}, \quad C_i = C, \quad D_i = 0.2713 \ (i = 1, 2)$$

where

$$C = \begin{bmatrix} 0.3126 & 0.3343 & -0.3119 \end{bmatrix}.$$

Similarly to the preceding experiments, we apply the analysis conditions (47) and (48) to (44) as well as its dual system and obtain the results in Table IV. When applying (48) to the primal system, we take $\bar{B} = [B_1 \ B_2]^{T\perp T}$ whereas in the case of the dual system, we take $\bar{B} = C^{\perp T}$. In both cases, we can again confirm that the proposed condition is effective to achieve less conservative analysis results.

TABLE IV
COMPUTATION RESULTS FOR ROBUST H_∞ PERFORMANCE ANALYSIS

	Primal system	Dual system
LMI condition (47) [10]	1.4704 $(N_s = 45)$	1.4056 $(N_s = 45)$
LMI condition (48)	1.0413 $(N_s = 66)$	1.0258 $(N_s = 91)$

(N_s: The number of scalar variables in each LMI condition.)

IV. Conclusion

In this paper, we proposed new LMI conditions for robust stability/performance analysis of uncertain LTI systems. By introducing Lyapunov functions that can be associated with higher-order derivatives of the state and considering corresponding redundant system descriptions, we have shown a constructive way to derive those analysis conditions. It turned out that the proposed LMI conditions can be regarded as a natural extension of those in [9], [10], [11]. Through numerical experiments, we illustrated that the proposed conditions are indeed effective to reduce the conservatism of the existing results and achieve more accurate analysis results.

We restricted our attention to the analysis of continuous-time systems in this paper but it is straightforward to extend the present results to discrete-time system analysis. An outstanding issue is the extension of the proposed analysis conditions to controller synthesis. Indeed, it is our current topic to find out a reasonable way to design (robust) state-feedback controllers by means of the proposed analysis conditions.

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