COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART I.2

Lagrangian and SDP duality

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Duality

- Versatile notion
- Theoritical results and numerical methods
- Certificates of infeasibility

Lagrangian duality has many applications and interpretations (price or tax, game, geometry...)

Applications of SDP duality:

- numerical solvers design
- problems reduction
- new theoretical insights into control problems

In the sequel we will recall some basic facts about Lagrangian duality and SDP duality

Lagrangian duality

Let the primal problem

$$
\begin{array}{rcll}
p^\star & = & \min\limits_{x \in \mathbb{R}^n} & f_0(x) \\
 & & \text{s.t.} & f_i(x) \leq 0 \quad i = 1, \cdots, m \\
 & & & h_i(x) = 0 \quad i = 1, \cdots, p
\end{array}
$$

Define Lagrangian $L(.,.,.)\ \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$
L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)
$$

where λ, μ are Lagrange multipliers vectors or dual variables

Let the Lagrange dual function

$$
g(\lambda, \mu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \mu)
$$

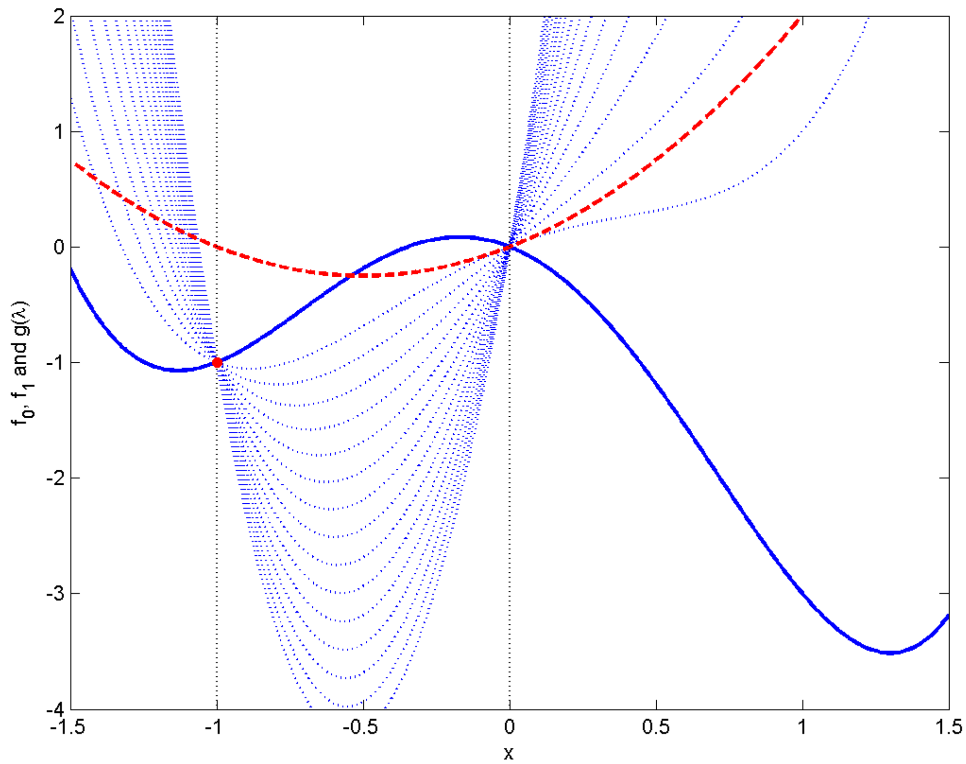
- g is always concave
- $g(\lambda, \mu) = -\infty$ if there is no finite infimum

Lagrangian duality (2)

A pair (λ, μ) s.t. $\lambda \succeq \mathbf{0}$ and $g(\lambda, \mu) > -\infty$ is dual feasible

For any primal feasible x and dual feasible pair (λ, μ)

$$g(\lambda, \mu) \leq p^* \leq f_0(x)$$



$$\min_x \quad x^4 - 3x^2 - x$$
$$\text{under} \quad x(x+1) \leq 0$$

Lagrangian duality (3)

Lagrange dual problem

$$\begin{array}{rcll} d^\star & = & \max\limits_{\lambda,\mu} & g(\lambda,\mu) \\ & & \text{s.t.} & \lambda \succeq \mathbf{0} \end{array}$$

The Lagrange dual problem is a convex optimization problem

Primal | **Dual**

$$\inf_{x\in\mathbb{R}^n} \begin{array}{ll} \sup\limits_{\lambda,\mu} & L(x,\lambda,\mu) \\ \text{s.t.} & \lambda \succeq \mathbf{0} \end{array} \qquad \begin{array}{ll} \sup\limits_{\lambda,\mu} & \inf\limits_{x\in\mathbb{R}^n} \quad L(x,\lambda,\mu) \\ \text{s.t.} & \lambda \succeq \mathbf{0} \end{array}$$

A Lagrangian relaxation consists in solving the dual problem instead of the primal problem

Weak and strong duality

Weak duality (max-min inequality):

$$p^\star \geq d^\star$$

because

$$g(\lambda,\mu) \leq f_0(x) + \sum_{i=1}^{m} \lambda_i \underbrace{f_i(x)}_{\leq 0} + \sum_{i=1}^{p} \mu_i \underbrace{h_i(x)}_{=0} \leq f_0(x)$$

for any primal feasible x and dual feasible λ, μ

The difference $p^\star - d^\star \geq 0$ is called duality gap

Strong duality (saddle-point property):

$$p^\star = d^\star$$

Sometimes, constraint qualifications ensure that strong duality holds
Example: Slater's condition = strictly feasible convex primal problem

$$f_i(x) < 0,\ i = 1, \cdots, m \quad h_i(x) = 0,\ i = 1, \cdots, p$$

Geometric interpretation of duality (1)

Consider the primal optimization problem

$$
\begin{array}{rcll}
p^\star & = & \min\limits_{x \in \mathbb{R}} & f_0(x) \\
 & & \text{s.t.} & f_1(x) \leq 0
\end{array}
$$

with Lagrangian and dual function

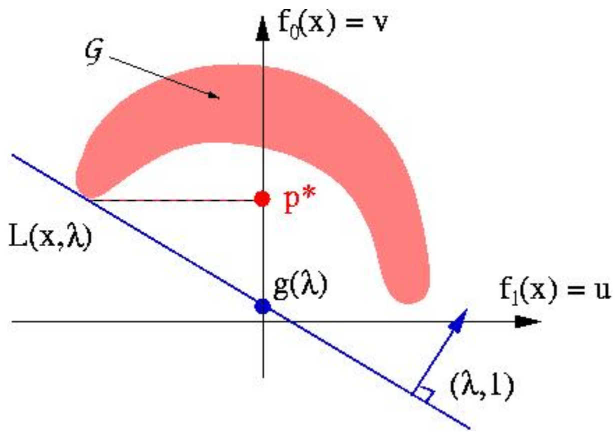
$$L(x, \lambda) = f_0(x) + \lambda f_1(x) \quad g(\lambda) = \inf_x L(x, \lambda)$$

The dual problem is given by:

$$
\begin{array}{rcll}
d^\star & = & \max\limits_{\lambda} & g(\lambda) \\
 & & \text{s.t.} & \lambda \succeq \mathbf{0}
\end{array}
$$

Geometric interpretation of duality (2)

Set of values $\mathcal{G} = (f_1(x),\ f_0(x)),\ \forall\ x \in \mathcal{D}$



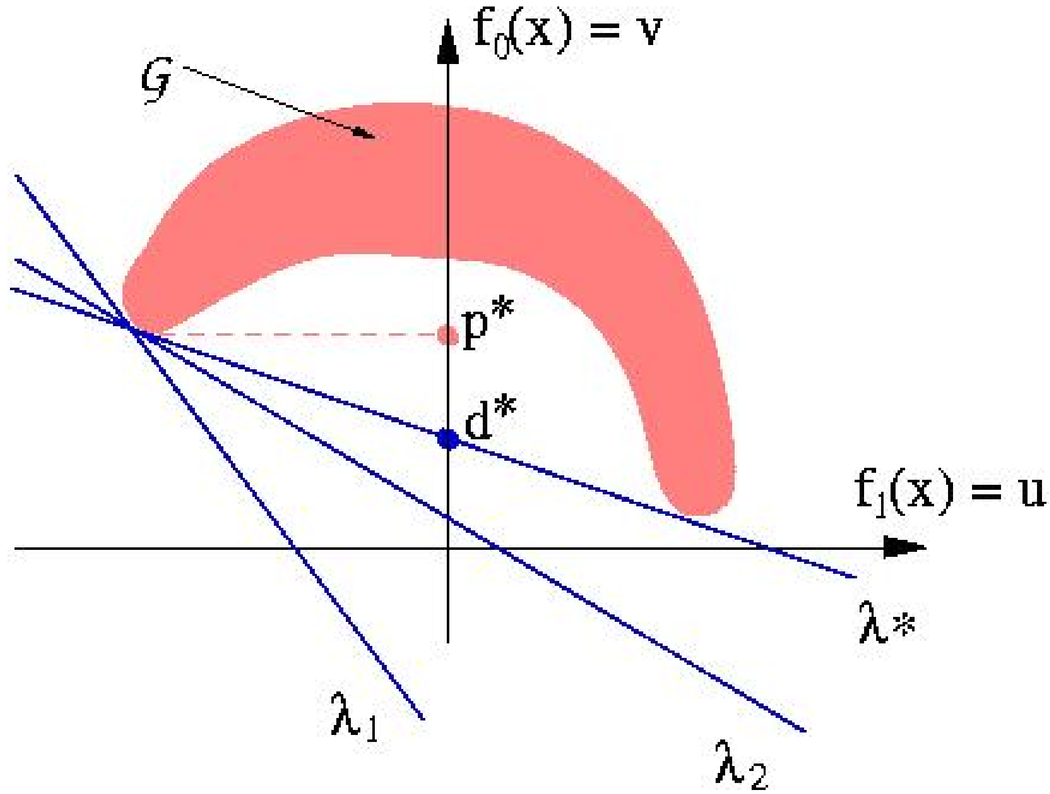
$$L(x,\lambda) = f_0(x) + \lambda f_1(x) = \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_0(x) \end{bmatrix}$$

$$g(\lambda) = \inf_{x \in \mathcal{D}} L(\lambda, x) = \inf_{x \in \mathcal{D}} \left\{ \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ (u,v) \in \mathcal{G} \right\}$$

Supporting hyperplane with slope $-\lambda$

$$\begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq g(\lambda)\ (u,v) \in \mathcal{G}$$

Geometric interpretation of duality (3)



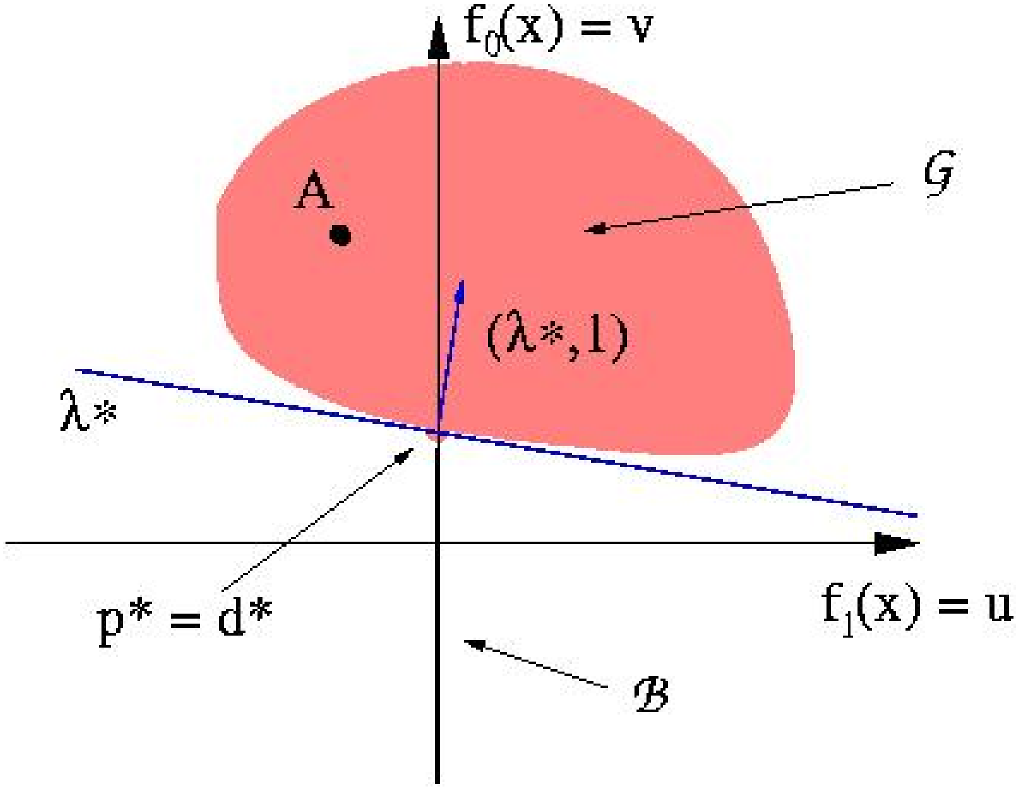
Three supporting hyperplanes, including the optimum $\lambda^\star$ yielding $d^\star < p^\star$
No strong duality here

$$p^* - d^* > 0$$

Duality gap $\neq 0$

Geometric interpretation of duality (4)

$$\mathcal{B} = \{(0, s) \in \mathbb{R} \times \mathbb{R} \ : \ s < p^*\}$$



- Separating hyperplane theorem for $\mathcal{G}$ and $\mathcal{B}$
- The separating hyperplane is a supporting hyperplane to $\mathcal{G}$ in $(0, p^*)$
- Slater's condition ensures the hyperplane is non vertical

Optimality conditions

Suppose that strong duality holds, let $x^\star$ be primal optimal and $(\lambda^\star, \mu^\star)$ be dual optimal,

$$
\begin{aligned}
f_0(x^\star) &= g(\lambda^\star, \mu^\star) \\
&= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^\star f_i(x) + \sum_{i=1}^{p} \mu_i^\star h_i(x) \right) \\
&\leq f_0(x^\star) + \sum_{i=1}^{m} \lambda_i^\star f_i(x^\star) + \sum_{i=1}^{p} \mu_i^\star h_i(x^\star) \\
&< f_0(x^\star)
\end{aligned}
$$

$$
\lambda_i^\star f_i(x^\star) = 0 \quad i = 1, \cdots, m
$$

This is complementary slackness condition

$$
\lambda_i^\star > 0 \;\Rightarrow\; f_i(x^\star) = 0 \quad \text{or} \quad f_i(x^\star) < 0 \;\Rightarrow\; \lambda_i^\star = 0
$$

In words, the ith optimal Lagrange multiplier is zero unless the ith constraint is active at the optimum

LP duality (1)

Primal LP (standard form):

$$
\begin{aligned}
p^\star \;=\; & \min_{x\in\mathbb{R}^n} && c'x \\
& \text{s.t.} && Ax = b \quad b \in \mathbb{R}^p \\
& && x \succeq \mathbf{0}
\end{aligned}
$$

Lagrange dual function:

$$
\begin{aligned}
g(\lambda,\mu) &= \inf_{x\in\mathcal{D}} \left(c'x + \mu'(b - Ax) - \lambda'x\right) \\
&= \begin{cases} b'\mu & \text{if } c - A'\mu - \lambda = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Lagrange dual problem:

$$
\begin{aligned}
& \max_{\lambda\in\mathbb{R}^n} && g(\lambda,\mu) = \begin{cases} b'\mu & \text{if } c - A'\mu - \lambda = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases} \\
& \text{s.t.} && \lambda \succeq \mathbf{0}
\end{aligned}
$$

LP duality (2)

Dual LP:

$$d^\star = \max_{\mu \in \mathbb{R}^p} \; b'\mu \quad \text{s.t.} \quad \lambda = c - A'\mu \succeq \mathbf{0}$$

Complementary slackness:

$$(x^\star)'\lambda^\star = 0$$

Weaker form of Slater's condition:

If primal (dual) is feasible then strong duality holds

Strong duality fails for LPs when both dual and primal are infeasible

LP duality (3): Example

$$d^* = \min_x \quad x$$

s.t.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} x \preceq \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$p^* = \max_\mu \quad \begin{bmatrix} 1 - 1 \end{bmatrix} \mu$$

s.t.

$$\begin{bmatrix} 0 & -1 \end{bmatrix} \mu = 1$$

$$\mu \succeq 0$$

then

$$p^* = -\infty \quad d^* = \infty$$

KKT optimality conditions

f_i, h_i are differentiable and strong duality holds

$$
\begin{gathered}
h_i(x^\star) = 0,\ i = 1, \cdots, p, \quad \text{(primal feasible)} \\
f_i(x^\star) \le 0,\ i = 1, \cdots, m, \ \text{(primal feasible)} \\
\lambda_i^\star \succeq \mathbf{0},\ i = 1, \cdots, m, \ \text{(dual feasible)} \\
\lambda_i^\star f_i(x^\star) = 0,\ i = 1, \cdots, m, \ \text{(complementary)} \\
\nabla f_0(x^\star) + \sum_{i=1}^{p} \lambda_i^\star \nabla f_i(x^\star) + \sum_{i=1}^{p} \mu_i^\star \nabla h_i(x^\star) = 0
\end{gathered}
$$

Necessary Karush-Kuhn-Tucker conditions satisfied by any primal and dual optimal pair $x^\star$ and $(\lambda^\star, \mu^\star)$

For convex problems, KKT conditions are also sufficient

History of KKT conditions

"Nonlinear programming" paper written jointly by Albert W. Tucker and Harold W. Kuhn (Princeton Univ) launched the theory of NLP in 1950

Later on, it turned out that this theorem had been proved already:

• First in 1939 in a MSc thesis by William Karush supervised by Lawrence M. Graves (Univ Chicago)

• Second in 1948 by Fritz John in a paper rejected by the Duke Math J, later on published in a collection of essays for Richard Courant's 60th birthday

Feasibility of inequalities (1)

$$\exists\ x \in \mathbb{R}^n \ \ : \ \ \begin{cases} f_i(x) \leq 0 & i = 1, \cdots, m \\ h_i(x) = 0 & i = 1, \cdots, p \end{cases}$$

Dual function: $g(.,.) \ : \ \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

The dual feasibility problem is

$$\exists\ (\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p \ \ : \ \ \begin{cases} g(\lambda, \mu) > 0 \\ \lambda \succeq \mathbf{0} \end{cases}$$

Theorem of weak alternatives

At most, one of the two (primal and dual) is feasible

If the dual problem is feasible then the primal problem is infeasible

Feasibility of inequalities (2)

Proof of the theorem of alternatives

Suppose $\overline{x} \in \mathcal{D}$ is a feasible point for the primal problem

$$\begin{aligned} g(\lambda, \mu) &= \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x) \\ &\leq \sum_{i=1}^{m} \lambda_i \underbrace{f_i(\overline{x})}_{\leq 0} + \sum_{i=1}^{p} \mu_i \underbrace{h_i(\overline{x})}_{=0} \\ &\quad \forall\ (\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p \end{aligned}$$

and so $g(\lambda, \mu) \leq 0$ for all $\lambda \succeq 0$

If f_i are convex functions, h_i are affine functions and some type of constraint qualification holds:

Theorem of strong alternatives

Exactly one of the two alternative holds

A dual feasible pair (λ, μ) gives a certificate (proof) of infeasibility of the primal

Feasibility of inequalities (3)

Example of Farkas'lemma

$$\exists \; x \in \mathbb{R}^n \quad : \quad \begin{cases} Ax \preceq b \\ c'x < 0 \end{cases}$$

$$\exists \; \lambda \in \mathbb{R}^m \quad : \quad \begin{cases} A'\lambda + c = 0 \\ b'\lambda < 0 \\ \lambda \succeq \mathbf{0} \end{cases}$$

Given the infeasible set of linear inequalities

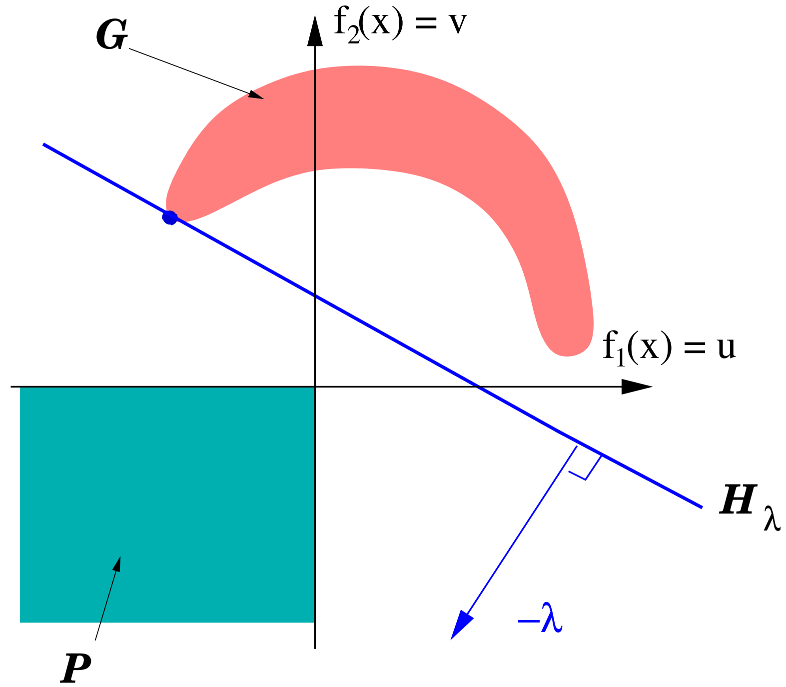
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} x \preceq \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x < 0$$

A certificate of infeasibility is given by
$\lambda = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}'$ solution of the alternative

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \lambda + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \quad \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \lambda < 0$$

$$\lambda \succeq 0$$

Feasibility of inequalities (4)
Geometric interpretation



$$P = \left\{ (u,v) \in \mathbb{R}^2 \;:\; \begin{bmatrix} u \\ v \end{bmatrix} \preceq 0 \right\}$$

$$H_\lambda = \left\{ (u,v) \in \mathbb{R}^2 \;:\; \lambda' \begin{bmatrix} u \\ v \end{bmatrix} = g(\lambda) \right\}$$

If $g(\lambda) > 0$ and $\lambda \succeq 0$ then H_λ is a separating hyperplane for P from

$$G = \left\{ \begin{bmatrix} f_1(x) & f_2(x) \end{bmatrix} \;:\; x \in \mathbb{R}^n \right\}$$

Conic duality (1)

Let the primal:

$$p^\star = \min_{x\in\mathbb{R}^n} \; f_0(x) \qquad \text{s.t.} \; f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots m$$

Lagrange dual function: $g(.) \; : \; \mathbb{R}^m \to \mathbb{R}$

$$g(\lambda) = \inf_{x\in\mathcal{D}} f_0(x) + \sum_{i=1}^{m} \lambda_i' f_i(x)$$

Lagrange dual problem:

$$d^\star = \max_{\lambda\in\mathbb{R}^m} \; g(\lambda) \qquad \text{s.t.} \; \lambda_i \succeq_{\mathcal{K}_i^*} \mathbf{0}, \quad i = 1, \cdots, m$$

Conic duality (2)

- Weak duality
- Strong duality:

- if primal is s.f. with finite $p^\star$ then $d^\star$ is reached by dual
- if dual is s.f. with finite $d^\star$ then $p^\star$ is reached by primal
- if primal and dual are s.f. then $p^\star = d^\star$

- Complementary slackness:

$$\lambda_i^{\star\prime} f_i(x^\star) = 0$$

$$\lambda_i^\star \succ_{\mathcal{K}_i^\star} \mathbf{0} \Rightarrow f_i(x^\star) = 0$$

$$f_i(x^\star) \prec_{\mathcal{K}_i} \mathbf{0} \Rightarrow \lambda_i^\star = \mathbf{0}$$

- KKT conditions:

$$f_i(x^\star) \preceq_{\mathcal{K}_i} \mathbf{0}$$

$$\lambda_i^\star \succeq_{\mathcal{K}_i^\star} \mathbf{0}$$

$$\nabla f_0(x^\star) + \sum_{i=1}^{m} \nabla f_i(x^\star)' \lambda_i^\star = \mathbf{0}$$

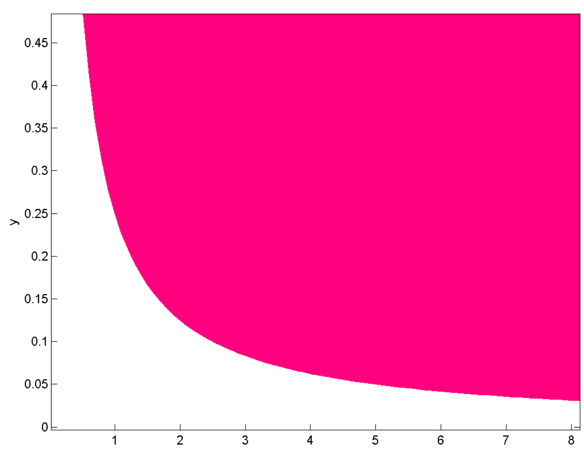
Example of conic duality

Consider the primal conic program

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \succeq_{\mathbb{L}^3} \mathbf{0} \Leftrightarrow \begin{array}{l} x_1 + x_2 > 0 \\ 4x_1x_2 \geq 1 \end{array} \end{array}$$

with dual

$$\begin{array}{ll} \max & -\lambda_2 \\ \text{s.t.} & \left\{ \begin{array}{l} \lambda_1 + \lambda_3 = 1 \\ -\lambda_1 + \lambda_3 = 0 \\ \lambda \in \mathbb{L}^3 \end{array} \right. \Leftrightarrow \begin{array}{l} \lambda_1 = \lambda_3 = 1/2 \\ 1/2 \geq \sqrt{1/4 + \lambda_2{}^2} \end{array} \end{array}$$



The primal is strictly feasible and bounded below with $p^\star = 0$ which is not reached since dual problem is infeasible $d^\star = -\infty$

SDP duality (1)

Primal SDP:

$$p^\star = \min_{x\in\mathbb{R}^n} \; c'x \quad \text{s.t.} \quad F_0 + \sum_{i=1}^{n} x_i F_i \preceq \mathbf{0}$$

Lagrange dual function:

$$\begin{aligned} g(Z) &= \inf_{x\in\mathcal{D}} \left(c'x + \operatorname{tr} ZF(x)\right) \\ &= \begin{cases} \operatorname{tr} F_0 Z & \text{if } \operatorname{tr} F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Dual SDP:

$$\begin{aligned} d^\star = \max_{Z\in\mathbb{S}^m} \; & \operatorname{tr} F_0 Z \\ \text{s.t.} \; & \operatorname{tr} F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ & Z \succeq \mathbf{0} \end{aligned}$$

Complementary slackness:

$$\operatorname{tr} F(x^\star)Z^\star = 0 \iff F(x^\star)Z^\star = Z^\star F(x^\star) = \mathbf{0}$$

SDP duality (2)
KKT optimality conditions

$$
\begin{aligned}
F_0 + \sum_{i=1}^{n} x_i F_i + Y &= \mathbf{0} \quad Y \succeq \mathbf{0} \\
\forall\ i \ \text{trace}\, F_i Z + c_i &= \mathbf{0} \quad Z \succeq \mathbf{0} \\
Z^\star F(x^\star) = Z^\star Y^\star &= \mathbf{0}
\end{aligned}
$$

Nota:

Since $Y^\star \succeq \mathbf{0}$ and $Z^* \succeq \mathbf{0}$ then

$$\text{trace}\, F(x^\star) Z^\star = \mathbf{0} \iff F(x^\star) Z^\star = Z^\star F(x^\star) = \mathbf{0}$$

Theorem:

Under the assumption of strict feasibility for the primal and the dual, the above conditions form a system of necessary and sufficient optimality conditions for the primal and the dual

Example of SDP duality gap

Consider the primal semidefinite program

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{bmatrix} 0 & x_1 & 0 \\ x_1 & -x_2 & 0 \\ 0 & 0 & -1-x_1 \end{bmatrix} \preceq \mathbf{0} \end{array}$$

with dual

$$\begin{array}{ll} \max & -z_6 \\ \text{s.t.} & \begin{bmatrix} z_1 & (1-z_6)/2 & z_4 \\ (1-z_6)/2 & 0 & z_5 \\ z_4 & z_5 & z_6 \end{bmatrix} \succeq 0 \end{array}$$

In the primal $x_1 = 0$ (x_1 appears in a row with zero diagonal entry) so the primal optimum is $x_1^\star = 0$

Similarly, in the dual necessarily $(1 - z_6)/2 = 0$ so the dual optimum is $z_6^\star = 1$

There is a nonzero duality gap here $(p^\star = 0) > (d^\star = -1)$

Conic theorem of alternatives

$$f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \qquad \mathcal{K}_i \subseteq \mathbb{R}^{k_i}$$

Lagrange dual function

$$g(\lambda) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i' f_i(x) \quad \lambda_i \in \mathbb{R}^{k_i}$$

Weak alternatives:

$$1 - \ \ f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2 - \ \ \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) > 0$$

Strong alternatives:

f_i $\mathcal{K}_i$-convex and $\exists\ x \in \text{relint}\mathcal{D}$

$$1 - \ \ f_i(x) \prec_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2 - \ \ \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) \geq 0$$

Theorem of alternatives for LMIs

For the LMI feasible set

$$F(x) = F_0 + \sum_i x_i F_i \prec \mathbf{0}$$

Exactly one statement is true
1- $\exists\ x$ s.t. $F(x) \prec 0$
2- $\exists\ \mathbf{0} \neq Z \succeq 0$ s.t.
trace $F_0 Z \geq \mathbf{0}$ and trace $F_i Z = \mathbf{0}$ for $i = 1, \cdots, n$

Useful for giving certificate of infeasibility of LMIs

Rich literature on theorems of alternatives for generalized inequalities, e.g. nonpolyhedral convex cones

Elegant proofs of standard results (Lyapunov, ARE) in linear systems control, see later...

S-procedure (1)

S-procedure: also frequently useful in robust and nonlinear control, also an outcome of the theorem of alternatives

1- if $x'A_1x \geq 0, \cdots, x'A_mx \geq 0$
then $x'A_0x \geq 0 \ \forall \ x \in \mathbb{R}^n$

2- $\exists \ \tau_j \geq 0$ s.t. $x'A_0x - \sum_{j=1}^{m} \tau_j x'A_jx \geq 0$

The S-procedure consists in replacing 1 by 2

The converse also holds (no duality gap)

- when $m = 1$ for real quadratic forms and $\exists \ x \mid x'A_1x > 0$ (from the theorem of alternatives)
- when $m = 2$ for complex quadratic forms

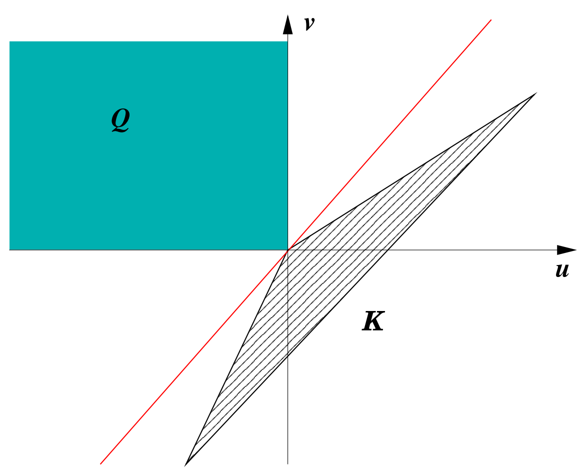
S-procedure (2)
Sketch of the proof for $m = 1$

Dines theorem:

For $(A_0, A_1) \in \mathbb{S}_n$ then

$$\mathcal{K} = \left\{(u,v) = (x'A_0x, x'A_1x) \ : \ x \in \mathbb{R}^n\right\}$$

is a closed convex cone of $\mathbb{R}^2$



Suppose

$$\text{if } x'A_0x \geq 0 \text{ then } x'A_1x \geq 0 \qquad (1)$$

It means that $\mathcal{K} \cap \mathcal{Q} = \emptyset$ where $\mathcal{Q} = \{v \geq 0, \ u < 0\}$

$\tau_1 u - \tau_2 v < 0$	$(u,v) \in \mathcal{Q}$	$\tau_2 \geq 0$	$\tau_1 > 0$
$\forall \ (u,v) \in \mathcal{K}$	$\exists \ \tau = \tau_2/\tau_1 \geq 0$	$u - \tau v \geq 0$	

Finsler's (Debreu) lemma (1)

The following statements are equivalent

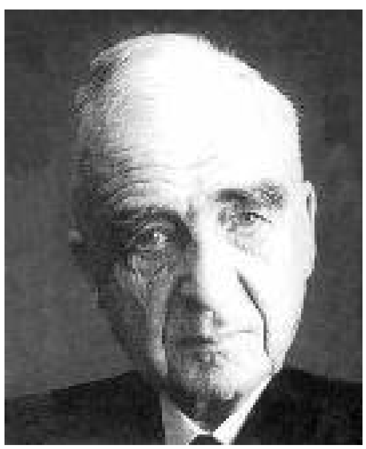
1 − $x'A_0x > 0 \ \forall \ x \neq 0 \in \mathbb{R}^n, \ n \geq 3, $ s.t. $x'A_1x = 0$

2 − $A_0 + \tau A_1 \succ 0$ for some $\tau \in \mathbb{R}$

Theorem of alternatives

1 − $\exists \ \tau \in \mathbb{R} \mid \tau A_1 + A_0 \succ \mathbf{0}$

2 − $\exists \ Z \in \mathbb{S}^n_+ \ : \ \mathrm{tr}(AZ_1) = 0$ and $\mathrm{tr}(A_0 Z) \leq 0$



Paul Finsler
(1894 Heilbronn - 1970 Zurich)

Finsler's (Debreu) lemma (2)
Counter-examples

Counter-example 1:

$$f_0(x) = x_1^2 - 2x_2^2 - x_3^2 \quad f_1(x) = x_1 - x_2$$

$$f_0(x) \leq 0 \;\text{ if }\; f_1(x) = 0$$

But, no τ exists s.t. $f_0(x) + \tau f_1(x) \leq 0$

$$x' \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \tau \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x \leq 0$$

Counter-example 2:

$$f_0(x) = 2x_1x_2 \;\; f_1(x) = x_1^2 - x_2^2$$

$f_0(x) > 0$ for $x \mid f_1(x) = 0$ but no $\tau \in \mathbb{R}$ exists s.t.

$$f_0(x) + \tau f_1(x) = x' \begin{bmatrix} \tau & 1 \\ 1 & -\tau \end{bmatrix} x > 0$$

Elimination lemma

The following statements are equivalent

$1-$ $H^{\perp}AH^{\perp *} \succ 0$ or $HH^{*} \succ 0$

$2-$ $\exists\ X \mid A + XH + H^{\star}X^{\star} \succ 0$

Theorem of alternatives

$1-$ $\exists\ X \in \mathbb{C}^{m\times n} \mid HX + (XH)^{*} + A \succ \mathbf{0}$

$2-$ $\exists\ Z \in \mathbb{S}^{n}_{+}$: $ZH = 0$ and $\mathrm{tr}(AZ) \geq 0$

Nota: For $H \in \mathbb{C}^{n\times m}$ with rank r, $H^{\perp} \in \mathbb{C}^{(n-r)\times n}$ s.t.

$$H^{\perp}H = 0 \quad H^{\perp}H^{\perp *} \succ 0$$

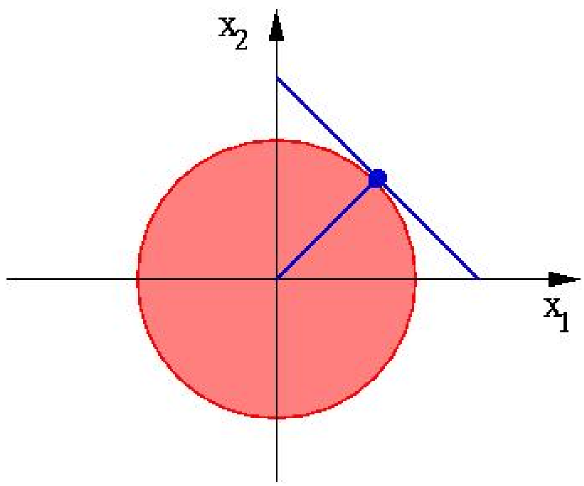
Reformulations

Linear LMI constraint = projection in subspace

Using explicit subspace basis, more efficient formulations (less decision variables) can be obtained

Example: original problem

$$\begin{array}{ll} \max & 2x_1 + 2x_2 \\ \text{s.t.} & \begin{bmatrix} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{bmatrix} \succeq 0 \end{array}$$



with dual

$$\begin{array}{ll} \min & \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z \\ \text{s.t.} & \text{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} Z = 2 \\ & \text{trace} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} Z = 2 \\ & Z \succeq 0 \end{array}$$

Reformulations (2)

Denoting

$$Z = \begin{bmatrix} z_{11} & z_{21} \\ z_{21} & z_{22} \end{bmatrix}$$

the linear trace constraints on Z can be written

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Particular solution and explicit null-space basis

$$\begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bar{z}$$

so we obtain the equivalent dual problem
with less variables

$$\begin{array}{ll} \text{min} & 2\bar{z} \\ \text{s.t.} & \begin{bmatrix} \bar{z}-1 & -1 \\ -1 & \bar{z}+1 \end{bmatrix} \succeq 0 \end{array}$$

and primal

$$\begin{array}{ll} \text{min} & \text{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{X} \\ \text{s.t.} & \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{X} = 2 \\ & \bar{X} \succeq 0 \end{array}$$