COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART II.1

LMIs IN SYSTEMS CONTROL
STATE-SPACE METHODS
STABILITY ANALYSIS

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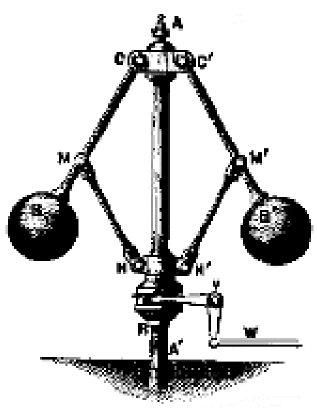

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State-space methods

Developed by Kalman and colleagues in the 1960s as an alternative to frequency-domain techniques (Bode, Nichols...) for optimal control and estimation

RADAR SRC-584

Starting in the 1980s, numerical analysts developed powerful linear algebra routines for matrix equations: numerical stability, low computational complexity, large-scale problems

Matlab launched by Cleve Moler (1977-1984) heavily relies on LINPACK, EISPACK & LAPACK packages

Matlab toolboxes development during the eighties and explosion for the millenium

- Math and analysis (optimization, statistics, spline...)
- Control (robust, predictive, fuzzy...)
- Signal and image processing (wavelet, identification...)
- Finance and economics (financial, GARCH...)

Linear systems and Lyapunov stability

The continuous-time linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ is asymptotically stable, meaning

$$\lim_{t \to \infty} x(t) = 0 \quad \forall\ x_0 \neq 0$$

if and only if

• there exists a quadratic Lyapunov function $V(x) = x'Px$ such that

$$\begin{array}{rcl} V(x(t)) & > & 0 \\ \dot{V}(x(t)) & < & 0 \end{array}$$

along system trajectories
• or matrix A satisfies

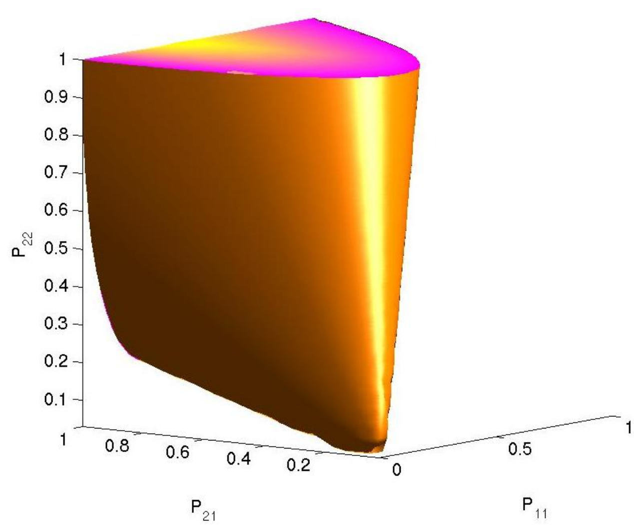
$$\max_{1 \leq i \leq n} \text{real}\, \lambda_i(A) < 0$$

Linear systems and Lyapunov stability (2)

Note that $V(x) = x'Px = x'(P + P')x/2$
so that Lyapunov matrix P can be chosen
symmetric without loss of generality

Since $\dot{V}(x) = \dot{x}'Px + x'P\dot{x} = x'A'Px + x'PAx$ positivity of $V(x)$ and negativity of $\dot{V}(x)$ along system trajectories can be expressed as an LMI

$$\exists\ P \in \mathbb{S}_n : \quad \begin{bmatrix} -\begin{bmatrix} \mathbf{1} & A' \end{bmatrix} \begin{bmatrix} \mathbf{0} & P \\ P & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ A \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \succ \mathbf{0}$$



Matrices P satisfying Lyapunov's LMI with $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Linear systems and Lyapunov stability (3)

The Lyapunov LMI can be written equivalently as the Lyapunov equation

$$A'P + PA + Q = \mathbf{0}$$

where $Q \succ \mathbf{0}$

The following statements are equivalent

- the system $\dot{x} = Ax$ is asymptotically stable
- for some matrix $Q \succ \mathbf{0}$ the matrix P solving the Lyapunov equation satisfies $P \succ \mathbf{0}$
- for all matrices $Q \succ \mathbf{0}$ the matrix P solving the Lyapunov equation satisfies $P \succ \mathbf{0}$

The Lyapunov LMI can be solved numerically by solving the linear system of $n(n+1)/2$ equations in $n(n+1)/2$ unknowns

$(A' \oplus A')\text{svec}(P) = (A' \otimes \mathbf{1} + \mathbf{1} \otimes A')\text{svec}(P) = -\text{svec}(Q)$

Theorem of alternatives and Lyapunov LMI

Recall the theorem of alternatives for LMI

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i$$

Exactly one statement is true

- there exists x s.t. $F(x) \succ 0$
- there exists a nonzero $Z \succeq 0$ s.t.
  trace $F_0 Z \leq 0$ and trace $F_i Z = 0$, $i = 1, \cdots, n$

Alternative to Lyapunov LMI

$$F(x) = \begin{bmatrix} -A'P - PA & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \succ \mathbf{0}$$

is the existence of a nonzero matrix

$$Z = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix} \succeq \mathbf{0}$$

such that

$$AZ_1 + Z_1 A' - Z_2 = \mathbf{0}$$

Discrete-time Lyapunov LMI

Similarly, the discrete-time LTI system

$$x_{k+1} = Ax_k \quad x(0) = x_0$$

is asymptotically stable iff
• there exists a quadratic Lyapunov function $V(x) = x'Px$ such that

$$\begin{array}{c} V(x_k) > 0 \\ V(x_{k+1}) - V(x_k) < 0 \end{array}$$

along system trajectories
• equivalently, matrix A satisfies

$$\max_{1 \le i \le n} |\lambda_i(A)| < 1$$

This can be expressed as an LMI

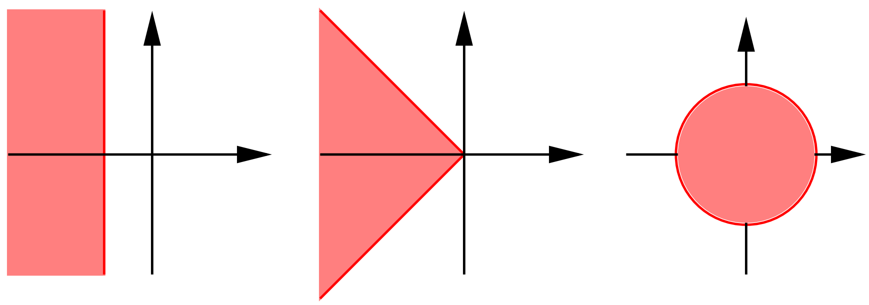
$$\exists\ P \in \mathbb{S}_n : \quad \begin{bmatrix} \begin{bmatrix} \mathbf{1} & A' \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & -P \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ A \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \succ \mathbf{0}$$

$\mathcal{D}$ stability regions

Let $D_i \in \mathbb{C}^{d\times d}$ and

$$\mathcal{D} = \{s \in \mathbb{C} \,:\, D_0 + D_1 s + D_1^\star s^\star + D_2 s^\star s \prec 0\}$$

be a region of the complex plane



Matrix A is said $\mathcal{D}$-stable if $\Lambda(A) \in \mathcal{D}$

Equivalent to generalized Lyapunov LMI

$$\exists\ P \in \mathbb{S}_n :$$
$$\begin{bmatrix} -\begin{bmatrix} \mathbf{1} & \mathbf{1}\otimes A' \end{bmatrix}\begin{bmatrix} D_0 & D_1 \\ D_1^\star & D_2 \end{bmatrix} \otimes P \begin{bmatrix} \mathbf{1} \\ \mathbf{1}\otimes A \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \succ \mathbf{0}$$

Literally replace $s\mathbf{1}$ with $\mathbf{1}\otimes A$ and D with $D \otimes P$!

$\mathcal{D}$ stability regions (2)

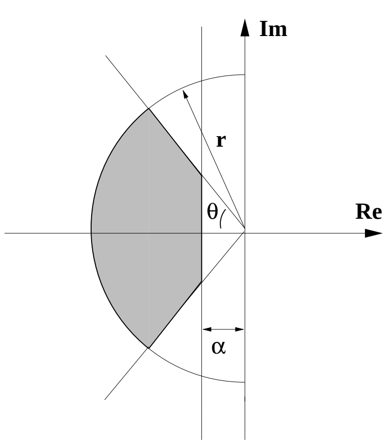
- symmetric with respect to real axis
- convex for $D_2 \succeq 0$ or not
- parabolae, hyperbolae, ellipses...
- intersections of $\mathcal{D}$ regions

A particular case is given by LMI regions

$$\mathcal{D} = \{s \in \mathbb{C} \,:\, D(s) = D_0 + D_1 s + D_1^\star s^\star \prec 0\}$$

such as

$\mathcal{D}$	dynamics
$\text{real}(s) < -\alpha$	dominant behavior
$\lvert s-\alpha\rvert < r$	oscillations
$\text{real}(s)\tan\theta < -\lvert\text{imag}(s)\rvert$	damping cone



Example:

$$D_0 = \text{diag}(\mathbf{0}, \alpha_1 - r^2, -2\alpha_2)$$
$$D_1 = \text{diag}(D_\theta, -\alpha_1, 1)$$
$$D_2 = \text{diag}(\mathbf{0}, 1, 0)$$

Stability as a quadratic optimization problem

$\mathcal{D}$-stability of matrix A can be cast as a quadratic optimization problem $(d = 1)$

$\Lambda(A) \subset \mathcal{D}$ iff $\mu > 0$ where

$$\begin{array}{rcll} \mu & = & \min\limits_{s, q \neq 0} & q^\star (A - s\mathbf{1})^\star (A - s\mathbf{1}) q \\ & & \text{s.t.} & s \in \mathcal{D}^C \end{array}$$

where $\mathcal{D}^C$ complementary of $\mathcal{D}$ in $\mathbb{C}$

Equivalently, $(p = sq)$

$$\begin{array}{rcll} \mu & = & \min\limits_{(q,p) \neq 0} & \left[\begin{array}{cc} q^\star & p^\star \end{array}\right] \left[\begin{array}{c} A' \\ -\mathbf{1} \end{array}\right] \left[\begin{array}{cc} A & -\mathbf{1} \end{array}\right] \left[\begin{array}{c} q \\ p \end{array}\right] \\ & & \text{s.t.} & \left[\begin{array}{cc} q & p \end{array}\right] D \left[\begin{array}{c} q^\star \\ p^\star \end{array}\right] \succeq \mathbf{0} \end{array}$$

Lyapunov matrix as Lagrangian variable

Define $\mathcal{A} = [A \quad -\mathbf{1}]$

If $\exists\ P \succ \mathbf{0}$ such that:

$$\left[\begin{array}{cc} q^\star & p^\star \end{array}\right] \mathcal{A}'\mathcal{A} \left[\begin{array}{c} q \\ p \end{array}\right] > \mathrm{tr}\left[P \left[\begin{array}{cc} q & p \end{array}\right] D \left[\begin{array}{c} q^\star \\ p^\star \end{array}\right]\right]$$

then $\mu^* > 0$ and equivalently

$$D \otimes P - \mathcal{A}'\mathcal{A} \prec \mathbf{0}$$

By projection

$$\left[\begin{array}{c} \mathbf{1} \\ A \end{array}\right]' \left[\begin{array}{cc} d_0 P & d_1 P \\ d_1^\star P & d_2 P \end{array}\right] \left[\begin{array}{c} \mathbf{1} \\ A \end{array}\right] \prec \mathbf{0} \quad P \succ \mathbf{0}$$

we obtain the generalized Lyapunov LMI

Lyapunov matrix P can be interpreted as a Lagrange dual variable or multiplier

Rank-one LMI problem

Define $\mathcal{Q} = [\mathbf{1} \quad \mathbf{0}]$ and $\mathcal{P} = [\mathbf{0} \quad \mathbf{1}]$
Define also dual map

$$F^D(P) = \begin{bmatrix} \mathcal{Q} \\ \mathcal{P} \end{bmatrix}' \begin{bmatrix} d_0 P & d_1 P \\ d_1^\star P & d_2 P \end{bmatrix} \begin{bmatrix} \mathcal{Q} \\ \mathcal{P} \end{bmatrix} = D \otimes P$$

such that trace $F^D(P)X =$ trace $F(X)P$

X is the non-zero rank-one matrix

$$X = xx^\star = \begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}^\star \succeq \mathbf{0}$$

It follows that LMI

$$\begin{aligned} &\mathcal{A}'\mathcal{A} \succ F^D(P) \\ &P \succ \mathbf{0} \end{aligned}$$

is feasible iff $\mu > 0$ in the primal

$$\begin{aligned} \mu \quad = \quad \min_{X \neq \mathbf{0}} \quad & \text{trace}\, \mathcal{A}'\mathcal{A}X \\ \text{s.t.} \quad & F(X) \succeq \mathbf{0} \\ & X \succeq \mathbf{0} \\ & \text{rank}\, X = 1 \end{aligned}$$

Alternatives for Lyapunov

Define the adjoint map

$$G(Z_1, Z_2) = Z_2 - [\mathbf{1} \quad A] \begin{bmatrix} d_0 Z_1 & d_1 Z_1 \\ d_1^\star Z_1 & d_2 Z_1 \end{bmatrix} [\mathbf{1} \quad A]'$$

then from SDP duality $\mu > 0$ iff dual LMI

$$G(Z_1, Z_2) = \mathbf{0}$$
$$Z_1 \succeq \mathbf{0} \quad \text{and} \quad Z_2 \succeq \mathbf{0}$$
$$\text{rank}\, Z_1 = 1$$

is infeasible

• This is the alternative LMI obtained before so we can remove the rank constraint !
• Adequate alternative proves the necessity for the dual

Uncertain systems and robustness

When modeling systems we face several sources of uncertainty, including

- non-parametric (unstructured) uncertainty
  - unmodeled dynamics
  - truncated high frequency modes
  - non-linearities
  - effects of linearization, time-variation..

- parametric (structured) uncertainty
  - physical parameters vary within given bounds
  - interval uncertainty (l_∞)
  - ellipsoidal uncertainty (l_2)
  - diamond uncertainty (l_1)

- How can we overcome uncertainty ?
  - model predictive control
  - adaptive control
  - robust control

A control law is robust if it is valid over the whole range of admissible uncertainty (can be designed off-line, usually cheap)

Uncertainty modeling

Consider the continuous-time LTI system

$$\dot{x}(t) = Ax(t) \quad A \in \mathcal{A}$$

where matrix A belong to an uncertainty set $\mathcal{A}$

For unstructured uncertainties we consider norm-bounded matrices

$$\mathcal{A} = \{A + B\Delta C \ : \ \|\Delta\|_2 \leq \rho\}$$

For structured uncertainties we consider polytopic matrices

$$\mathcal{A} = \text{co}\,\{A_1, \ldots, A_N\}$$

There are other more sophisticated uncertainty models not covered here

Uncertainty modeling is an important and difficult step in control system design !

Robust stability

The continuous-time LTI system

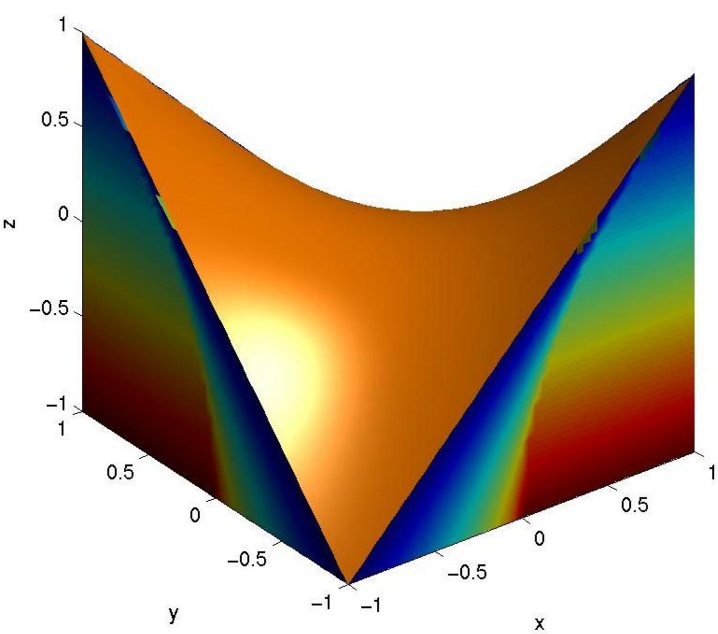
$$\dot{x}(t) = Ax(t) \quad A \in \mathcal{A}$$

is robustly stable when it is asymptotically stable for all $A \in \mathcal{A}$

If $\mathcal{S}$ denotes the set of stable matrices, then robust stability is ensured as soon as

$$\mathcal{A} \subset \mathcal{S}$$

Unfortunately $\mathcal{S}$ is a non-convex cone !



Non-convex set of continuous-time stable matrices

$$\begin{bmatrix} -1 & x \\ y & z \end{bmatrix}$$

Robust and quadratic stability

Because of non-convexity of the cone of stable matrices, robust stability is sometimes difficult to check numerically, meaning that

> computational cost is an exponential function of the number of system parameters

Remedy:
The continuous-time LTI system $\dot{x}(t) = Ax(t)$ is quadratically stable if its robust stability can be guaranteed with the same quadratic Lyapunov function for all $A \in \mathcal{A}$

Obviously, quadratic stability is more conservative than robust stability:

> Quadratic stability $\Rightarrow$ Robust stability

but the converse is not always true

Quadratic stability for polytopic uncertainty

The system with polytopic uncertainty

$$\dot{x}(t) = Ax(t) \quad A \in \text{co}\,\{A_1, \ldots, A_N\}$$

is quadratically stable iff there exists a matrix P solving the LMIs

$$A_i'P + PA_i \prec \mathbf{0} \quad P \succ \mathbf{0}$$

Proof by convexity

$$\sum_{i=1}^{N} \lambda_i (A_i'P + PA_i) = A'(\lambda)P + PA(\lambda) \prec \mathbf{0}$$

for all $\lambda_i \geq 0$ such that $\sum_{i=1}^{N} \lambda_i = 1$

This is a vertex result: stability of a whole family of matrices is ensured by stability of the vertices of the family

Usually vertex results ensure computational tractability

Quadratic and robust stability: example

Consider the uncertain system matrix

$$A(\delta) = \begin{bmatrix} -1 & \delta_1 \\ \delta_2 & -1 \end{bmatrix} = -\mathbf{1}_2 + \delta_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

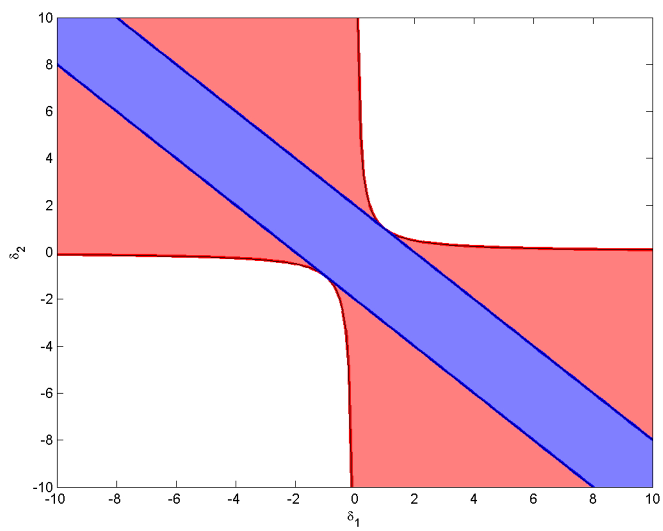
with real parameter (δ_1, δ_2)

Robust stability condition:

$$P(\lambda, \delta) = \lambda^2 + 2\lambda + 1 - \delta_1\delta_2 \qquad 1 - \delta_1\delta_2 > 0$$

Quadratic stability condition with $P = \mathbf{1}_2$:

$$\begin{bmatrix} -2 & \delta_1 + \delta_2 \\ \delta_1 + \delta_2 & -2 \end{bmatrix} \prec 0 \qquad \begin{matrix} \delta_1 + \delta_2 < 2 \\ \delta_1 + \delta_2 > -2 \end{matrix}$$



Quadratic stability for norm-bounded uncertainty

The system with norm-bounded uncertainty

$$\dot{x}(t) = (A + B\Delta C)x(t) \quad \|\Delta\|_2 \leq \rho$$

is quadratically stable iff there exists a matrix P solving the LMIs

$$\begin{bmatrix} A'P + PA + C'C & PB \\ B'P & -\gamma^2 I \end{bmatrix} \prec 0 \quad P \succ 0$$

with $\gamma^{-1} = \rho$

This is the bounded-real lemma

We can maximize the level of allowed uncertainty by minimizing scalar γ

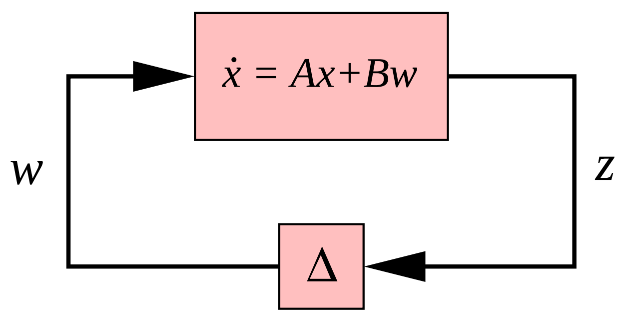
Norm-bounded uncertainty as feedback

Uncertain system

$$\dot{x} = (A + B\Delta C)x$$

can be written as the feedback system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bw \\ z & = & Cx \\ w & = & \Delta z \end{array}$$



so that for the Lyapunov function $V(x) = x^\star P x$ we have

$$\begin{array}{rcl} \dot{V}(x) & = & 2x^\star P \dot{x} \\ & = & 2x^\star P(Ax + Bw) \\ & = & x^\star(A'P + PA)x + 2x^\star PBw \\ & = & \begin{bmatrix} x \\ w \end{bmatrix}^\star \begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \end{array}$$

Norm-bounded uncertainty as feedback (2)

Since $\Delta^\star\Delta \preceq \rho^2 I$ it follows that

$$w^\star w = z^\star\Delta^\star\Delta z \preceq \rho^2 z^\star z$$

$$\Longleftrightarrow$$

$$w^\star w - \rho^2 z^\star z = \begin{bmatrix} x \\ w \end{bmatrix}^\star \begin{bmatrix} -C'C & 0 \\ 0 & \gamma^2\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$$

Combining with the quadratic inequality

$$\dot{V}(x) = \begin{bmatrix} x \\ w \end{bmatrix}^\star \begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

and using the S-procedure we obtain

$$\begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} \prec \begin{bmatrix} -C'C & 0 \\ 0 & \gamma^2\mathbf{1} \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} A'P + PA + C'C & PB \\ B'P & -\gamma^2\mathbf{1} \end{bmatrix} \prec \mathbf{0} \quad P \succ \mathbf{0}$$

Norm-bounded uncertainty: generalization

Now consider the feedback system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= \Delta z \end{aligned}$$

with additional feedthrough term Dw

We assume that matrix $\mathbf{1} - \Delta D$ is non-singular
$=$ well-posedness of feedback interconnection
so that we can write

$$w = \Delta z = \Delta(Cx + Dw)$$
$$(\mathbf{1} - \Delta D)w = \Delta Cx$$
$$w = (\mathbf{1} - \Delta D)^{-1}\Delta Cx$$

and derive the linear fractional transformation
(LFT) uncertainty description

$$\dot{x} = Ax + Bw = (A + B(\mathbf{1} - \Delta D)^{-1}\Delta C)x$$

Norm-bounded LFT uncertainty

The system with norm-bounded LFT uncertainty

$$\dot{x} = \left(A + B(\mathbf{1} - \Delta D)^{-1}\right) x \quad \|\Delta\|_2 \leq \rho$$

is quadratically stable iff there exists a matrix P solving the LMIs

$$\begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2\mathbf{1} \end{bmatrix} \prec \mathbf{0} \quad P \succ \mathbf{0}$$

Notice the lower right block $D'D - \gamma^2\mathbf{1} \prec \mathbf{0}$ which ensures non-singularity of $\mathbf{1} - \Delta D$ hence well-posedness

LFT modeling can be used more generally to cope with rational functions of uncertain parameters, but this is not covered in this course..

Sector-bounded uncertainty

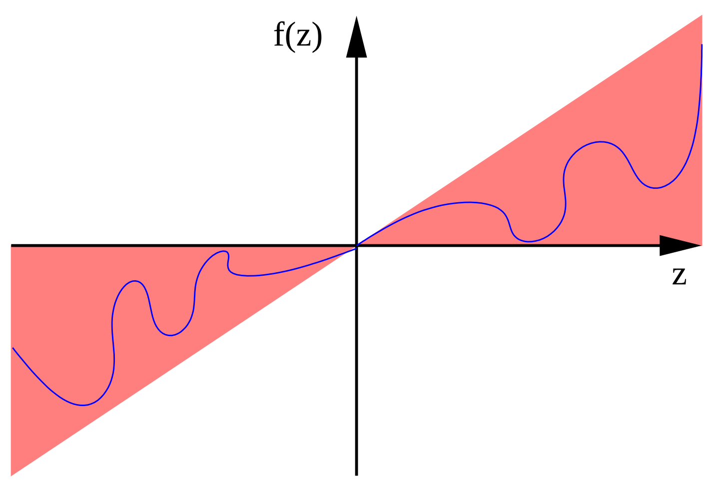
Consider the feedback system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= f(z) \end{aligned}$$

where vector function $f(z)$ satisfies

$$z^\star f(z) \geq 0 \qquad f(0) = 0$$

which is a sector condition



$f(z)$ can also be considered as an uncertainty but also as a non-linearity

Quadratic stability for sector-bounded uncertainty

We want to establish quadratic stability with the quadratic Lyapunov matrix $V(x) = x^\star P x$ whose derivative

$$\begin{aligned} \dot{V}(x) &= 2x^\star P(Ax + Bf(z)) \\ &= \begin{bmatrix} x \\ f(z) \end{bmatrix}^\star \begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix} \end{aligned}$$

must be negative when

$$\begin{aligned} 2z^\star f(z) &= 2(Cx + Df(z))^\star f(z) \\ &= \begin{bmatrix} x \\ f(z) \end{bmatrix}^\star \begin{bmatrix} 0 & C' \\ C & D + D' \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix} \end{aligned}$$

is non-negative, so we invoke the S-procedure to derive the LMIs

$$\begin{bmatrix} A'P + PA & PB + C' \\ B'P + C & D + D' \end{bmatrix} \prec \mathbf{0} \qquad P \succ \mathbf{0}$$

This is called the positive-real lemma

Beyond quadratic stability: PDLF

Quadratic stability:

- arbitrary fast variation of parameters
- computationally tractable
- conservative or pessimistic (worst-case)

Robust stability:

- very slow variation of parameters
- computationally difficult (in general)
- exact (is it really relevant ?)

Conservatism stems from single Lyapunov function for the whole uncertainty set

For example, given an LTI system affected by box, or interval uncertainty

$$\dot{x}(t) = A(\lambda)x(t) = \sum_{i=1}^{N} \lambda_i A_i x(t)$$

where

$$\lambda \in \Lambda = \{\lambda_i \in [\underline{\lambda_i},\ \overline{\lambda_i}]\}$$

we may consider parameter-dependent Lyapunov matrices, such as

$$P(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$$

Polytopic Lyapunov certificates

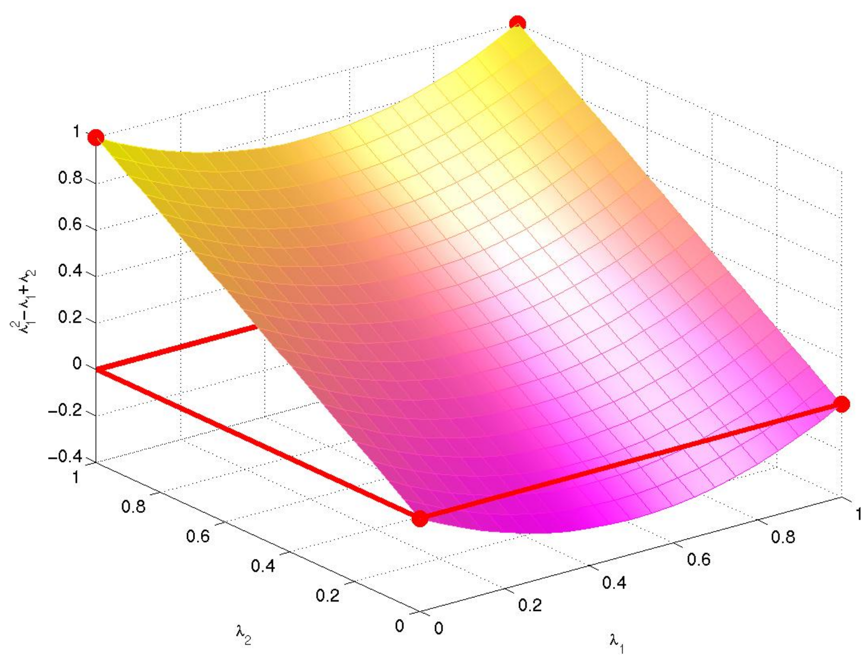
Quadratic Lyapunov function $V(x) = x^\star P(\lambda)x$ must be positive with negative derivative along system trajectories hence

$$P(\lambda) = \sum_{i=1}^{N} \lambda_i P_i \quad P(\lambda) \succ \mathbf{0} \quad \forall\ \lambda \in \Lambda$$

and we have to solve parameterized LMIs

$$A'(\lambda)P(\lambda) + P(\lambda)A(\lambda) \prec \mathbf{0} \quad \forall\ \lambda \in \Lambda$$

Parameterized LMIs feature non-linear terms in λ so it is not enough to check vertices of Λ, denoted by vertΛ



$\lambda_1^2 - \lambda_1 + \lambda_2 \geq 0$ on vert Δ
but not everywhere on $\Delta = [0,\ 1] \times [0,\ 1]$

Time-invariant uncertainty and PDLF

We must find $x \in \mathbb{R}^{n\times(n+1)/2}$ s.t.

$$F(x,\lambda) = \begin{bmatrix} P(\lambda) & \mathbf{0} \\ \mathbf{0} & -A'(\lambda)P(\lambda) - P(\lambda)A(\lambda) \end{bmatrix} \succ \mathbf{0}$$

for all $\lambda \in \Lambda$ = infinite number of LMIs

Lagrangian duality or projection lemma leads to the sufficient condition
$\exists\ N$ matrices $P_i \in \mathbb{S}_n$ and a matrix $H \in \mathbb{R}^{2n\times n}$

$$P_i \succ \mathbf{0} \qquad \forall\ i = 1,\cdots,N$$

$$\begin{bmatrix} \mathbf{0} & P_i \\ P_i & \mathbf{0} \end{bmatrix} + \begin{bmatrix} A_i' \\ -\mathbf{1} \end{bmatrix} H' + H \begin{bmatrix} A_i & -\mathbf{1} \end{bmatrix} \prec \mathbf{0}$$

- Parameter-dependent Lyapunov function $P(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$
- Slack variable $H' = \begin{bmatrix} F' & G' \end{bmatrix}$

A general relaxation procedure

Objective:

Solving a finite number of LMIs
instead of an infinite number of LMIs

A sufficient condition to ensure feasibility of the parameter-dependent LMI $F(x, \lambda)$ is

$$F(x, \lambda) \succ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & h(\lambda)\mathbf{1} \end{bmatrix}$$

$$h(\lambda) \geq 0$$

for all $\lambda \in \Lambda$ and $h(\lambda) \in \mathbb{R}[\lambda_1, \cdots, \lambda_N]$

- $h(\lambda)$ is chosen to get LMIs conditions independent from λ
- coefficients of $h(\lambda)$ may be considered as additional variables

Multiconvexity

For

$$h(\lambda) = \sum_{i=1}^{N} \lambda_i^2$$

we get the following sufficient conditions

$$\exists\ N\ \ P_i \succ \mathbf{0} \text{ and } \exists\ N\ \ \lambda_i \in \mathbb{R}$$

$$A_i' P_i + P_i A_i \prec -\lambda_i \mathbf{1} \quad \forall\ i = 1, \cdots, N$$

$$\begin{gathered} A_i' P_i + P_i A_i + A_j' P_j + P_j A_j \\ -(A_i' P_j + P_j A_i + A_j' P_i + P_i A_j) \succeq -(\lambda_i + \lambda_j)\mathbf{1} \\ \forall\ 1 \leq i < j \leq N \end{gathered}$$

which is a finite set of vertex LMIs. Proof is based on multiconvexity of quadratic functions

Nota: multiconvexity of h is ensured if

$$\frac{\partial^2 h(x)}{\partial x_i^2} \geq 0 \quad \forall\ i = 1, \cdots, n$$

Another sufficient condition

For

$$h(\lambda) = \sum_{i=1}^{N} \sum_{j>i} (\lambda_i - \lambda_j)^2$$

we get the following sufficient conditions

$$\exists \; N \;\; P_i \succ \mathbf{0}$$

$$A_i' P_i + P_i A_i \prec -\mathbf{1} \quad \forall \; i = 1, \cdots, N$$

$$A_i' P_j + P_j A_i + A_j' P_i + P_i A_j \prec \frac{2}{N-1} \mathbf{1}$$
$$\forall \; 1 \leq i < j \leq N$$

Nota: identical procedures are possible with

$$F(\lambda) = \sum_{i=1}^{N} \lambda_i F_i \quad G(\lambda) = \sum_{i=1}^{N} \lambda_i G_i$$