# COURSE ON LMI OPTIMIZATION WITH APPLICATIONS IN CONTROL

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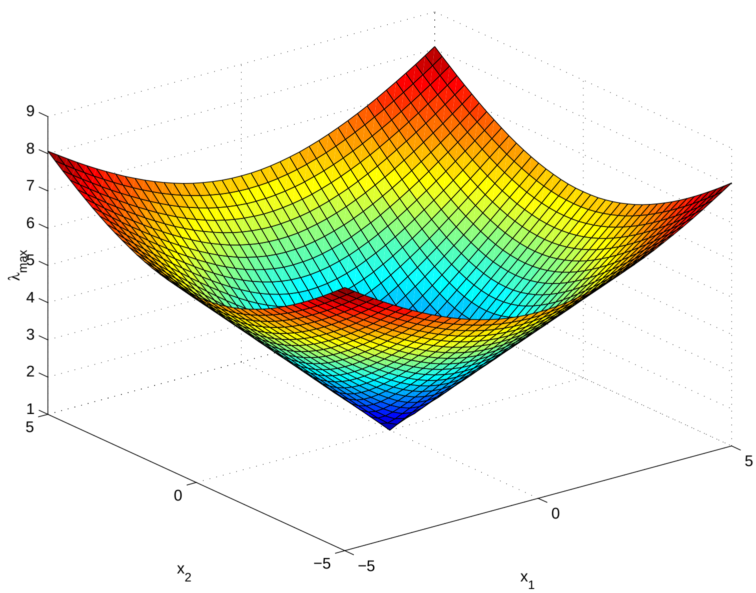


15 Octobre 2008

## Course outline

# I LMI optimization

I.1 Introduction: What is an LMI ? What is SDP ?

historical survey - applications - convexity - cones - polytopes

I.2 SDP duality

Lagrangian duality - SDP duality - KKT conditions

I.3 Solving LMIs

interior point methods - solvers - interfaces

# II LMIs in control

II.1 State-space analysis methods

Lyapunov stability - pole placement in LMI regions - robustness

II.2 State-space design methods

$H_2$, $H_\infty$, robust state-feedback and output-feedback design

# III Aerospace applications of LMIs

III.1 Interferometric cartwheel stationkeeping

Robust $\mathcal{D}/H_2$ performance via state-feedback

III.2 Robust pilot design for a flexible launcher

$H_2$, $H_\infty/H_2$ Multiobjective output-feedback design

# Course material

## Very good references on convex optimization:

- S. Boyd, L. Vandenberghe. Convex Optimization, Lecture Notes Stanford & UCLA, CA, 2002
- H. Wolkowicz, R. Saigal, L. Vandenberghe. Handbook of semidefinite programming, Kluwer, 2000
- A. Ben-Tal, A. Nemirovskii. Lectures on Modern Convex Optimization, SIAM, 2001

## Modern state-space LMI methods in control:

- C. Scherer, S. Weiland. Course on LMIs in Control, Lecture Notes Delft & Eindhoven Univ Tech, NL, 2002
- S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory, SIAM, 1994
- M. C. de Oliveira. Linear Systems Control and LMIs, Lecture Notes Univ Campinas, BR, 2002.

## Results on LMI and algebraic optimization in control:

- P. A. Parrilo, S. Lall. Mini-Course on SDP Relaxations and Algebraic Optimization in Control. European Control Conference, Cambridge, UK, 2003
- P. A. Parrilo, S. Lall. Semidefinite Programming Relaxations and Algebraic Optimization in Control, Workshop presented at the 42nd IEEE Conference on Decision and Control, Maui HI, USA, 2003

COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART I.1

# WHAT IS AN LMI ?
# WHAT IS SDP ?

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15 Octobre 2008

# LMI - Linear Matrix Inequality

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq \mathbf{0}$$

- $F_i \in \mathbb{S}^m$ given symmetric matrices
- $x_i \in \mathbb{R}^n$ decision variables

Fundamental property: feasible set is convex

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ F(x) \succeq \mathbf{0}\}$$

$\mathcal{S}$ is the Spectrahedron

Nota : $\succeq 0$ ($\succ 0$) means positive semidefinite (positive definite) e.g. real nonnegative eigenvalues (strictly positive eigenvalues) and defines generalized inequalities on PSD cone

Terminology coined out by Jan Willems in 1971

$$F(P) = \begin{bmatrix} A'P + PA + Q & PB + C' \\ B'P + C & R \end{bmatrix} \succeq \mathbf{0}$$

"The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms"

## Lyapunov's LMI

Historically, the first LMIs appeared around 1890 when Lyapunov showed that the autonomous system with LTI model:

$$\frac{d}{dt}x(t) = \dot{x}(t) = Ax(t)$$

is stable (all trajectories converge to zero) iff there exists a solution to the matrix inequalities

$$A'P + PA \prec \mathbf{0} \quad P = P' \succ \mathbf{0}$$

which are linear in unknown matrix $P$


Aleksandr Mikhailovich Lyapunov
(1857 Yaroslavl - 1918 Odessa)

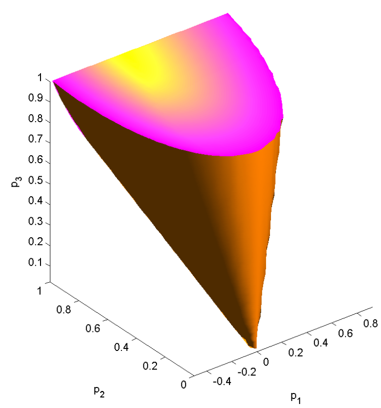
# Example of Lyapunov's LMI

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$A'P + PA \prec \mathbf{0} \qquad P \succ \mathbf{0}$$

$$\begin{bmatrix} -2p_1 & 2p_1 - 3p_2 \\ 2p_1 - 3p_2 & 4p_2 - 4p_3 \end{bmatrix} \prec \mathbf{0}$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ \mathbf{0}$$



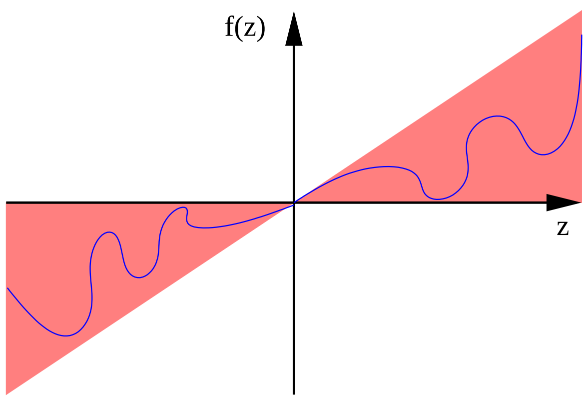

Matrices P satisfying Lyapunov LMI's

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} p_1 + \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} p_2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} p_3 \succ \mathbf{0}$$

# Some history (1)

1940s - Absolute stability problem: Lu're, Postnikov et al applied Lyapunov's approach to control problems with nonlinearity in the actuator

$$\dot{x} = Ax + b\sigma(x)$$




Sector-type nonlinearity

- Stability criteria in the form of LMIs solved analytically by hand

- Reduction to Polynomial (frequency dependent) inequalities (small size)

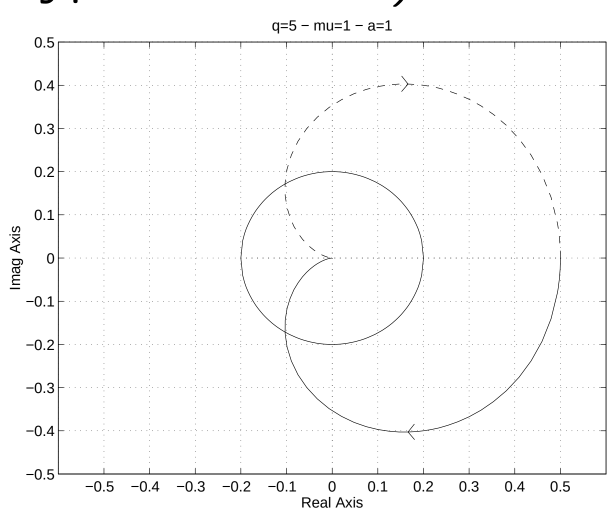
# Some history (2)

1960s: Yakubovich, Popov, Kalman, Anderson et al obtained the positive real lemma

The linear system $\dot{x} = Ax + Bu, \quad y = Cx + Du$ is passive $H(s) + H(s)^* \geq 0 \; \forall \; s + s^* > 0$ iff

$$P \succ \mathbf{0} \quad \begin{bmatrix} A'P + PA & PB - C' \\ B'P - C & -D - D' \end{bmatrix} \preceq \mathbf{0}$$

- Solution via a simple graphical criterion (Popov, circle and Tsypkin criteria)




Mathieu equation: $\ddot{y} + 2\mu\dot{y} + (\mu^2 + a^2 - q\cos\omega_0 t)y = 0$
$q < 2\mu a$

## Some history (3)

1971: Willems focused on solving algebraic Riccati equations (AREs)

$$A'P + PA - (PB + C')R^{-1}(B'P + C) + Q = \mathbf{0}$$

Numerical algebra

$$H = \begin{bmatrix} A - BR^{-1}C & BR^{-1}B' \\ -C'R^{-1}C & -A' + C'R^{-1}B' \end{bmatrix} \quad V = \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix}$$

$$P_{are} = V_2 V_1^{-1}$$

By 1971, methods for solving LMIs:

- Direct for small systems
- Graphical methods
- Solving Lyapunov or Riccati equations

# Some history (4)

1963: Bellman-Fan: infeasibility criteria for multiple Lyapunov inequalities (duality theory)

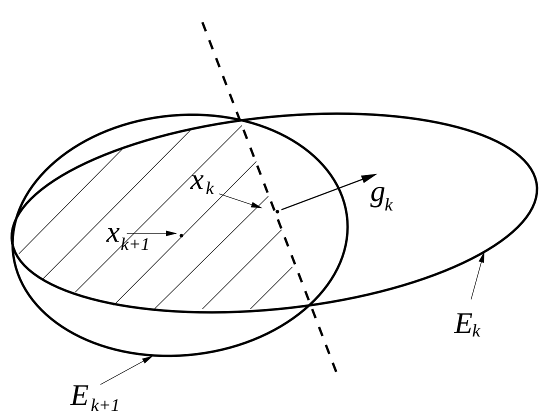
On Systems of Linear Inequalities in hermitian Matrix Variables

1975: Cullum-Donath-Wolfe: properties of criterion and algorithm for minimization of maximum eigenvalues

The minimization of certain nondifferentiable sums of eigenvalues of symmetric matrices

1979: Khachiyan: polynomial bound on worst case iteration count for LP ellipsoid algorithm

A polynomial algorithm in linear programming



## Some history (5)

1981: Craven-Mond: Duality theory

Linear Programming with Matrix variables

1984: Karmarkar introduces interior-point (IP) methods for LP: improved complexity bound and efficiency

1985: Fletcher: Optimality conditions for non-differentiable optimization

Semidefinite matrix constraints in optimization

1988: Overton: Nondifferentiable optimization

On minimizing the maximum eigenvalue of a symmetric matrix

1988: Nesterov, Nemirovski, Alizadeh extend IP methods for convex programming

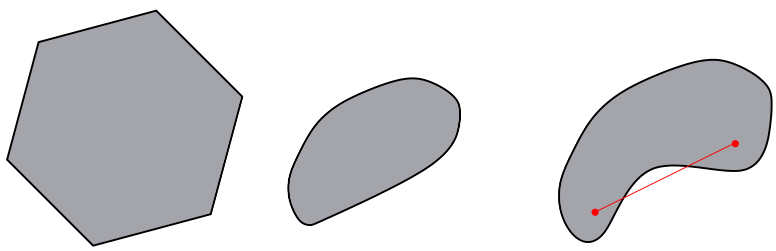
Interior-Point Polynomial Algorithms in Convex Programming

1990s: most papers on SDP are written (control theory, combinatorial optimization, approximation theory...)

# Mathematical preliminaries (1)

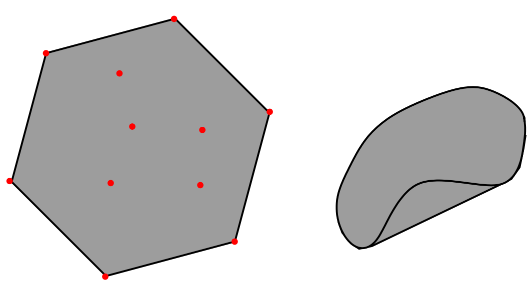
A set $\mathcal{C}$ is convex if the line segment between any two points in $\mathcal{C}$ lies in $\mathcal{C}$

$$\forall\, x_1, x_2 \in \mathcal{C} \quad \lambda x_1 + (1-\lambda) x_2 \in \mathcal{C} \quad \forall\, \lambda \quad 0 \leq \lambda \leq 1$$



The convex hull of a set $\mathcal{C}$ is the set of all convex combinations of points in $\mathcal{C}$

$$\mathrm{co}\,\mathcal{C} = \{\sum_i \lambda_i x_i \;:\; x_i \in \mathcal{C} \quad \lambda_i \geq 0 \quad \sum_i \lambda_i = 1\}$$



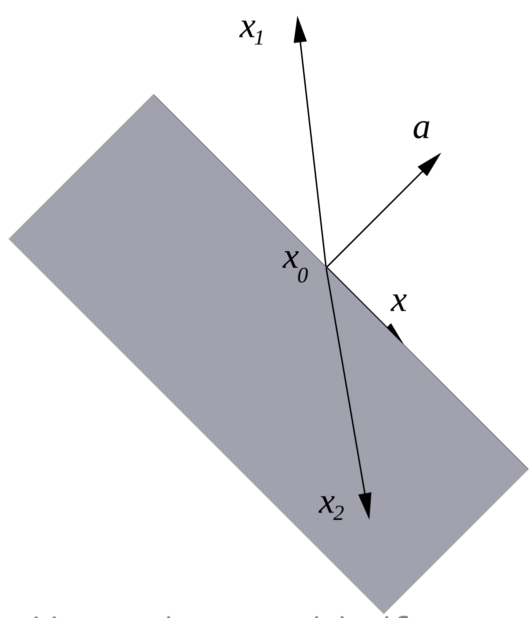
## Mathematical preliminaries (2)

A hyperplane is a set of the form:

$$\mathcal{H} = \left\{ x \in \mathbb{R}^n \mid a'(x - x_0) = 0 \right\} \quad a \neq 0 \in \mathbb{R}^n$$

A hyperplane divides $\mathbb{R}^n$ into two halfspaces:

$$\mathcal{H}_- = \left\{ x \in \mathbb{R}^n \mid a'(x - x_0) \leq 0 \right\} \quad a \neq 0 \in \mathbb{R}^n$$



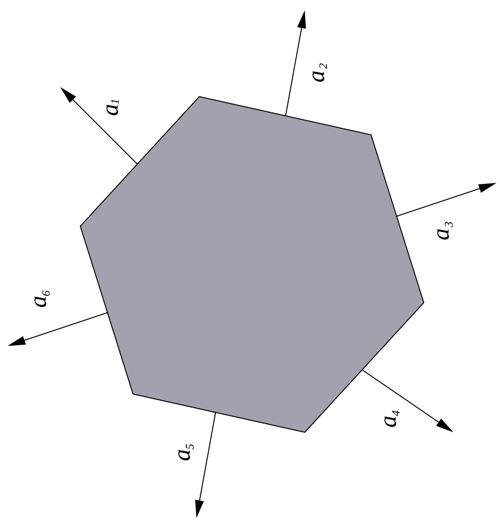

Hyperplane and halfspace
$x \in \mathcal{H}$, $x_1 \notin \mathcal{H}_-$, $x_2 \in \mathcal{H}_-$

# Mathematical preliminaries (3)

A polyhedron is defined by a finite number of linear equalities and inequalities

$$\begin{aligned}\mathcal{P} &= \left\{x \in \mathbb{R}^n : \ a_j' x \le b_j, j = 1, \cdots, m, c_i' x = d_i, i = 1, \cdots, p\right\} \\ &= \{x \in \mathbb{R}^n \ : \ Ax \preceq b, \ Cx = d\}\end{aligned}$$

A bounded polyhedron is a polytope



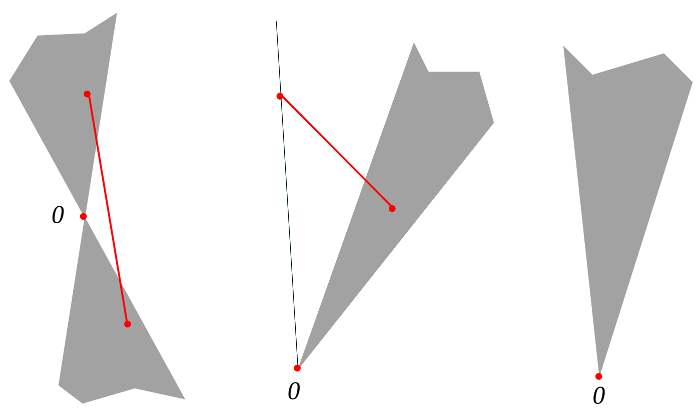

Polytope as an intersection of halfspaces

- positive orthant is a polyhedral cone
- k-dimensional simplexes in $\mathbb{R}^n$

$$\mathcal{X} = \mathrm{co}\{v_0, \cdots, v_k\} = \left\{\sum_{i=0}^{k} \lambda_i v_i \ \lambda_i \ge 0 \ \sum_{i=0}^{k} \lambda_i = 1\right\}$$

# Mathematical preliminaries (4)

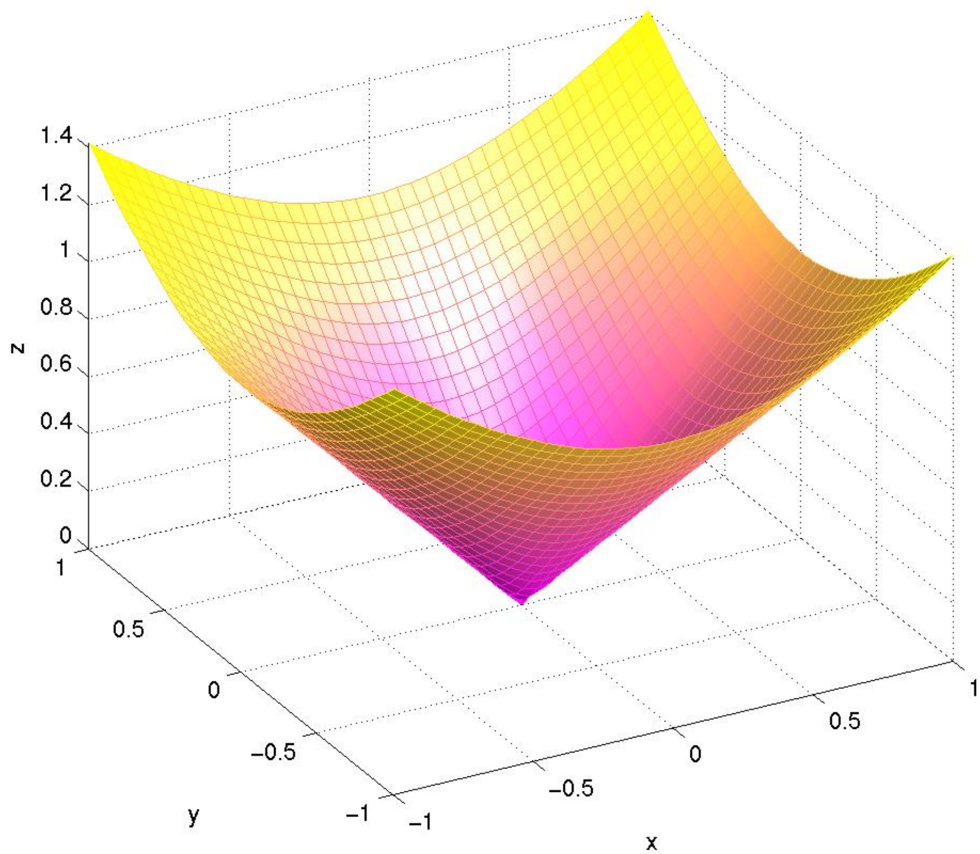
A set $\mathcal{K}$ is a cone if for every $x \in \mathcal{K}$ and $\lambda \geq 0$ we have $\lambda x \in \mathcal{K}$. A set $\mathcal{K}$ is a convex cone if it is convex and a cone




$\mathcal{K} \subseteq \mathbb{R}^n$ is called a proper cone if it is a closed solid pointed convex cone

$$a \in \mathcal{K} \quad \text{and} \quad -a \in \mathcal{K} \quad \Rightarrow \quad a = 0$$

# Lorentz cone $\mathbb{L}^n$

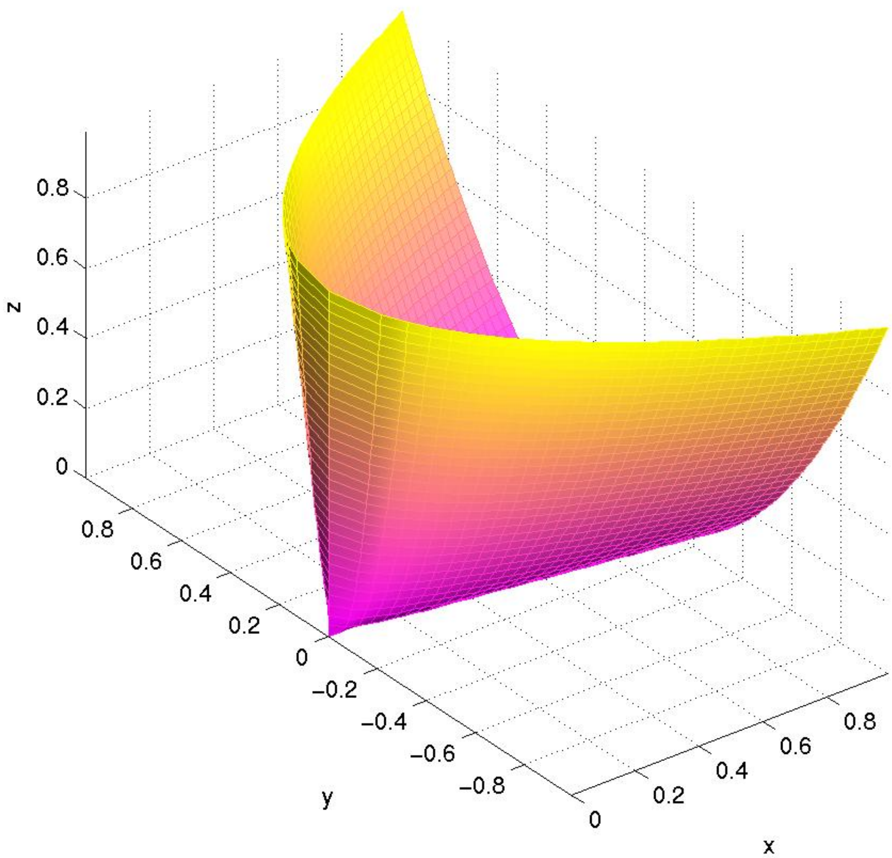



3D Lorentz cone or ice-cream cone

$$x^2 + y^2 \le z^2 \quad z \ge 0$$

arises in quadratic programming

## PSD cone $\mathbb{S}^n_+$




2D positive semidefinite cone

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0 \iff x \geq 0 \quad z \geq 0 \quad xz \geq y^2$$

arises in semidefinite programming

# Mathematical preliminaries (5)

Every proper cone $\mathcal{K}$ in $\mathbb{R}^n$ induces a partial ordering $\succeq_{\mathcal{K}}$ defining generalized inequalities on $\mathbb{R}^n$

$$a \succeq_{\mathcal{K}} b \quad \Leftrightarrow \quad a - b \in \mathcal{K}$$

The positive orthant, the Lorentz cone and the PSD cone are all proper cones

- positive orthant $\mathbb{R}^n_+$: standard coordinatewise ordering (LP)

$$x \succeq_{\mathbb{R}^n_+} y \;\Leftrightarrow\; x_i \geq y_i$$

- Lorentz cone $\mathbb{L}^n$

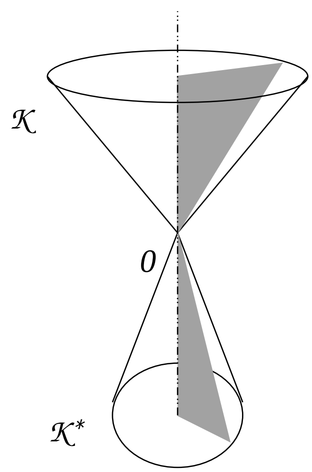
$$x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}$$

- PSD cone $\mathbb{S}^n_+$: Löwner partial order

# Mathematical preliminaries (6)

The set $\mathcal{K}^* = \{y \in \mathbb{R}^n \mid x'y \leq 0 \ \ \forall\ x \in \mathcal{K}\}$ is called the dual cone of the cone $\mathcal{K}$

- Revolution cone $\mathcal{K}_s(\theta) = \{x \in \mathbb{R}^n : \ s'x \leq ||x|| \cos\theta\}$




$\mathcal{K}_s(\theta)^* = \mathcal{K}_{-s}(\frac{\pi}{2} - \theta)$

- $(\mathbb{R}^n_+)^* = \mathbb{R}^n_-$

$\mathcal{K}^*$ is closed and convex, $\mathcal{K}_1 \subseteq \mathcal{K}_2 \ \Rightarrow \ \mathcal{K}_2^* \subseteq \mathcal{K}_1^*$

$\preceq_{\mathcal{K}^*}$ is a dual generalized inequality

$$x \preceq_{\mathcal{K}} y \quad \Leftrightarrow \quad \lambda' x \leq \lambda' y \ \ \forall\ \lambda \succeq_{\mathcal{K}^*} 0$$

## Mathematical preliminaries (7)

$f\ :\ \mathbb{R}^n \to \mathbb{R}$ is convex if dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$ and $0 \le \lambda \le 1$

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

If $f$ is differentiable: dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$

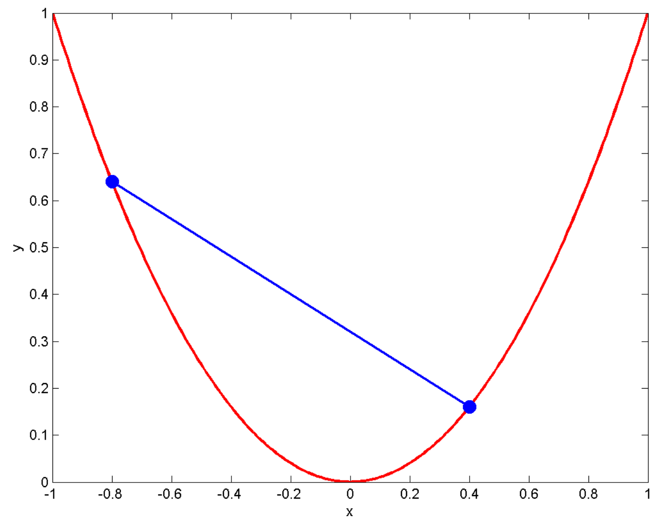
$$f(y) \ge f(x) + \nabla f(x)'(y-x)$$

If $f$ is twice differentiable: dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$

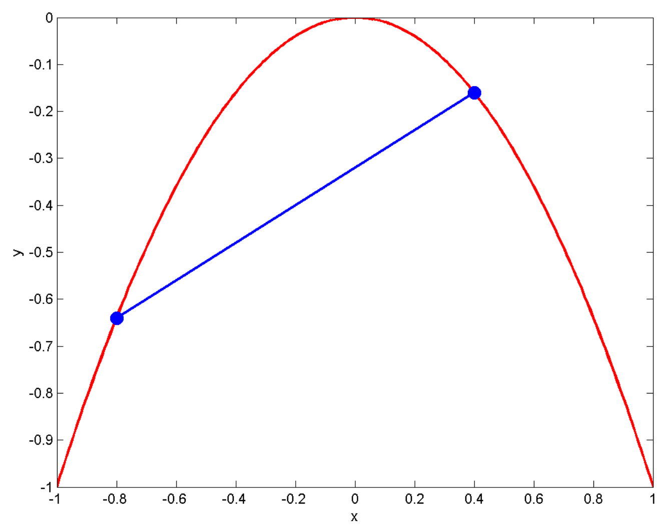
$$\nabla^2 f(x) \succeq \mathbf{0}$$

Quadratic functions:
$f(x) = (1/2)x'Px + q'x + r$ is convex if and only if $P \succeq \mathbf{0}$

Convex function $y = x^2$




Nonconvex function $y = -x^2$




Mind the sign !

# LMI and SDP formalisms (1)

In mathematical programming terminology
LMI optimization = semidefinite programming
(SDP)

**LMI** (SDP dual)

$$\begin{array}{ll} \text{min} & c'x \\ \text{under} & F_0 + \sum_{i=1}^{n} x_i F_i \prec \mathbf{0} \end{array}$$

**SDP** (primal)

$$\begin{array}{ll} \text{min} & -\text{Tr}(F_0 Z) \\ \text{under} & -\text{Tr}(F_i Z) = c_i \\ & Z \succeq \mathbf{0} \end{array}$$

$x \in \mathbb{R}^n$, $Z \in \mathbb{S}^m$, $F_i \in \mathbb{S}^m$, $c \in \mathbb{R}^n$, $\quad i = 1, \cdots, n$

Nota:
In a typical control LMI

$$A'P + PA = F_0 + \sum_{i=1}^{n} x_i F_i \prec \mathbf{0}$$

individual matrix entries are decision variables

# LMI and SDP formalisms (2)

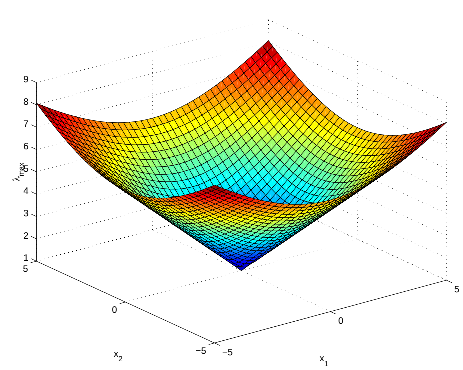
$$\exists\, x \in \mathbb{R}^n \mid \underbrace{F_0 + \sum_{i=1}^{n} x_i F_i}_{F(x)} \prec \mathbf{0} \quad \Leftrightarrow \quad \min_{x \in \mathbb{R}^n} \lambda_{max}(F(x))$$

The LMI feasibility problem is a convex and non differentiable optimization problem.

Example :

$$F(x) = \begin{bmatrix} -x_1 - 1 & -x_2 \\ -x_2 & -1 + x_1 \end{bmatrix}$$

$$\lambda_{max}(F(x)) = 1 + \sqrt{(x_1^2 + x_2^2)}$$



# LMI and SDP formalisms (3)

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & \\ & b - A'x \in \mathcal{K} \end{array}$$

$$\begin{array}{ll} \min & b'y \\ \text{s.t.} & \\ & Ay = c \\ & y \in \mathcal{K} \end{array}$$

Conic programming in cone $\mathcal{K}$

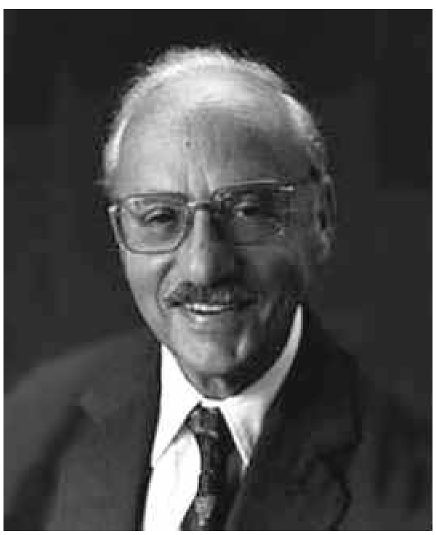
- positive orthant (LP)
- Lorentz (second-order) cone (SOCP)
- positive semidefinite cone (SDP)

Hierarchy: LP cone $\subset$ SOCP cone $\subset$ SDP cone

## LMI and SDP formalisms (4)

LMI optimization = generalization of linear programming (LP) to cone of positive semidefinite matrices = semidefinite programming (SDP)

Linear programming pioneered by
- Dantzig and its simplex algorithm (1947, ranked in the top 10 algorithms by SIAM Review in 2000)
- Kantorovich (co-winner of the 1975 Nobel prize in economics)



George Dantzig
(1914 Portland, Oregon)

Leonid V Kantorovich
(1921 St Petersburg - 1986)

Unfortunately, SDP has not reached maturity of LP or SOCP so far..

## Applications of SDP

- control systems (part II of the course)
- robust optimization
- signal processing
- synthesis of antennae arrays
- design of chips
- structural design (trusses)
- geometry (ellipsoids)
- graph theory and combinatorics (MAXCUT, Shannon capacity)

and many others...

See Helmberg's page on SDP

`www-user.tu-chemnitz.de/~helmberg/semidef.html`

# Robust optimization (1)

In many real-life applications of optimization problems, exact values of input data (constraints) are seldom known

- Uncertainty about the future
- Approximations of complexity by uncertainty
- Errors in the data
- variables may be implemented with errors

$$\begin{array}{ll} \text{min} & f_0(x,u) \\ \text{under} & f_i(x,u) \leq 0 \quad i = 1, \cdots, m \end{array}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables and $u \in \mathbb{R}^p$ is the parameters vector.

- Stochastic programming
- Sensitivity analysis
- Interval arithmetic
- Worst-case analysis

$$\begin{array}{lll} \min\limits_{x} & \sup\limits_{u \in \mathcal{U}} & f_0(x,u) \\ \text{under} & & \sup\limits_{u \in \mathcal{U}} f_i(x,u) \leq 0 \quad i = 1, \cdots, m \end{array}$$

## Robust optimization (2)

Case study by Ben Tal and Nemirovski:
[Math. Programm. 2000]
90 LP problems from NETLIB + uncertainty
*quite small (just 0.1%) perturbations of "obviously uncertain" data coefficients can make the "nominal" optimal solution $x^*$ heavily infeasible*
Remedy: robust optimization, with robustly feasible solutions guaranteed to remain feasible at the expense of possible conservatism
Robust conic problem: [Ben Tal Nemirovski 96]

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c'x \\ \text{s.t.} & Ax - b \in \mathcal{K}, \quad \forall\ (A, b) \in \mathcal{U} \end{array}$$

This last problem, the so-called robust counterpart is still convex, but depending on the structure of $\mathcal{U}$, can be much harder that original conic problem

# Robust optimization (3)

| Uncertainty | Problem | Optimization Problem |
|---|---|---|
| polytopic | LP | LP |
| ellipsoid | | SOCP |
| LMI | | SDP |
| polytopic | SOCP | SOCP |
| ellipsoid | | SDP |
| LMI | | NP-hard |

Examples of applications:

Robust LP: Robust portfolio design in finance [Lobo 98], discrete-time optimal control [Boyd 97], robust synthesis of antennae arrays [Lebret 94], FIR filter design [Wu 96]

Robust SOCP: robust least-squares in identification [El Ghaoui 97], robust synthesis of antennae arrays and FIR filter synthesis

# Robust optimization (4)
# Robust LP as a SOCP

**Robust counterpart of robust LP**

$$\min_{x \in \mathbb{R}^n} \quad c'x$$

s.t.

$$a_i' x \leq b_i, \quad i = 1, \cdots m,$$
$$\forall \ a_i \in \mathcal{E}_i$$
$$\mathcal{E}_i = \{\overline{a}_i + P_i u \mid ||u||_2 \leq 1 \text{ and } P_i \succeq 0\}$$

Note that

$$\max_{a_i \in \mathcal{E}_i} \ a_i' x = \overline{a}_i' x + ||P_i x||_2 \leq b_i$$

**SOCP formulation**

$$\min_{x \in \mathbb{R}^n} \quad c'x$$

s.t.

$$\overline{a}_i' x + ||P_i x|| \leq b_i, \quad i = 1, \cdots m,$$

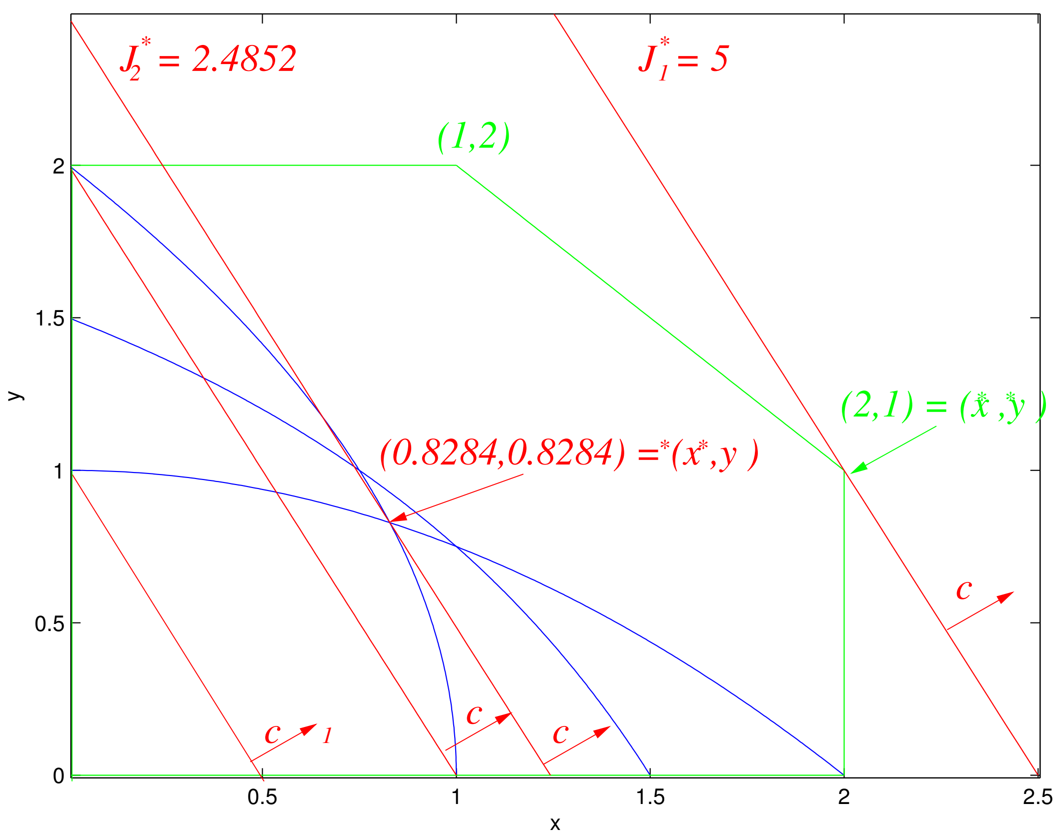
# Robust optimization (5)
# Example of Robust LP

$$J_1^* = \max_{x,y} \quad 2x + y$$
$$\text{s.t.} \quad x \geq 0, \; y \geq 0$$
$$x \leq 2$$
$$y \leq 2$$
$$x + y \leq 3$$

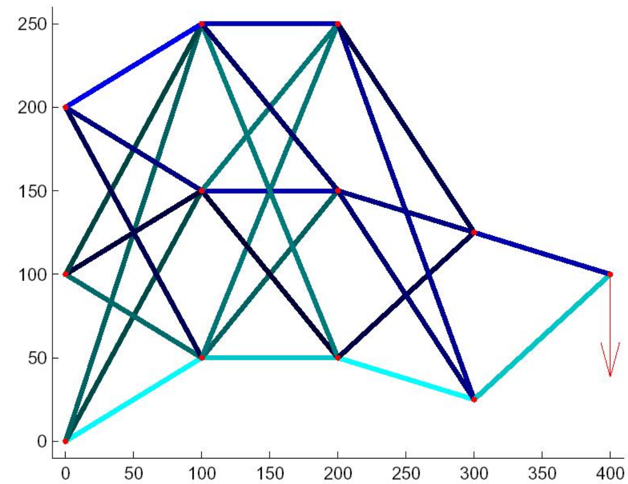
$$J_2^* = \max_{x,y} \quad 2x + y$$
$$\text{s.t.} \quad x \geq 0, \; y \geq 0$$
$$\sqrt{x^2 + y^2} \leq 3 - x - y$$
$$\sqrt{x^2 + y^2} \leq 2 - x$$
$$\sqrt{x^2 + y^2} \leq 2 - y$$

$(x^*, y^*) = (2, 1)$
$J_1^* = 5$

$(x^*, y^*) = (0.8284, 0.8284)$
$J_2^* = 2.4852$



# Truss Topology Design (TTD)




A truss is a network of $N$ nodes connected by elastic bars of length $l_i$ (fixed) and cross-sections $s_i$ (to be designed)

When subjected to a given load, the truss is deformed and the distorted truss stores potential energy (compliance) measuring stiffness of the truss.

Standard TTD:
For given initial nodes set $N$, external nominal load $f$ and total volume of bars $v$, allocate this resource to the bars i.o.t. minimize the compliance (maximize the stiffness) of the resulting truss
The compliance of the truss w.r.t. a load $f$ is:

$$C = \frac{1}{2} f' d$$

where $d$ is the displacement vector

# Truss Topology Design (2)

Construction reacts to external force $f$ on each node with displacement vector $d$ satisfying equilibrium displacement equations:

$$A(t)d = f$$

where $A(t)$ is the stiffness matrix, $t = l's$ is the volume of the truss.
Linearity assumption: stiffness matrix $A(s)$ affine in $s$ and positive definite.

$$A(s) = \sum_{i=1}^{N} l_i s_i b_i b_i'$$

Constraints on decision variables:
- Bounds on cross-sections:

$$a \leq s \leq b$$

- Bound on total volume (weight)

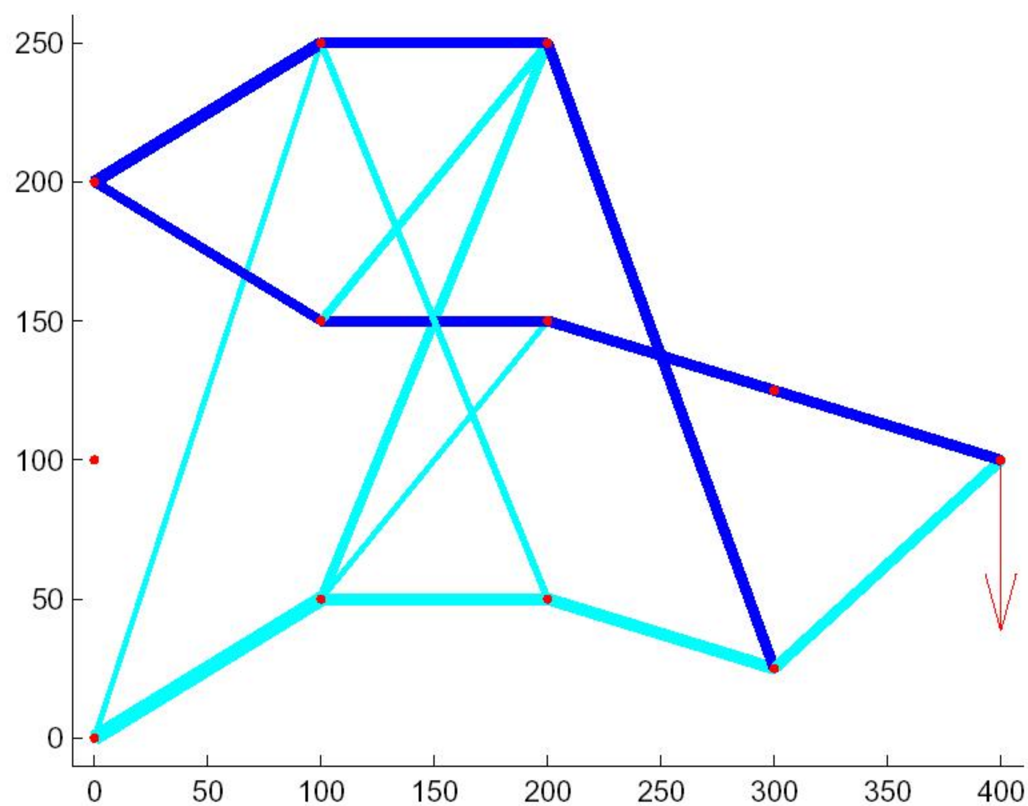
$$l's = \sum_{i=1}^{N} l_i s_i \leq v$$

# Truss topology design (3)

TTD an be formulated as an LMI optimization problem:

$$\min_{\tau,s} \tau \quad \text{s.t.} \quad \begin{bmatrix} \tau & f' \\ f & A(s) \end{bmatrix} \succeq 0 \quad l's \leq v \quad a \preceq s \preceq b$$

Optimal truss [Scherer 04]



## Combinatorial optimization (1)

Combinatorics: Graph theory, polyhedral combinatorics, combinatorial optimization, enumerative combinatorics...

Definition: Optimization problems in which the solution space is discrete (finite collection of objects) or a decision-making problem in which each decision has a finite (possibly many) number of feasibilities

Depending upon the formalism

- 0-1 Linear Programming problems: 0-1 Knapsack problem,...
- Propositional logic: Maximum satisfiability problems...
- Constraints satisfaction problems: Airline crew assignment
- Graph problems: Max-Cut, Shannon capacity of a graph,...

## Combinatorial optimization (2)

- Many CO problems are NP-complete
- Combinatorial explosion (the number of objects may be huge and grows exponentially in the size of the representation)
- Scanning all feasibilities (objects) one by one and choosing the best one is not an option

Two strategies:

- Exact algorithms (not guaranteed to run in polynomial time)
- Polynomial-time algorithms (guaranteed to give an optimal solution)

Fundamental concept in CO: Relaxations (combinatorial, linear, Lagrangian relaxations)

Optimize over larger easy convex space instead of optimizing over hard genuine feasible set

- Relaxed solution should be easy to get
- Relaxed solution should be "close" to the original

# Combinatorial optimization (3)

## SDP relaxation of QP in binary variables

$$(BQP) \quad \max_{x \in \{-1,1\}} \quad x'Qx$$

Noticing that $x'Qx = \text{trace}(Qxx')$
we get the equivalent form

$$(BQP) \quad \max_{X} \quad \text{trace}(QX)$$
$$s.t. \quad \begin{array}{l} \text{diag}(X_{ii}) = e = \left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]' \\ X \succeq 0 \\ \text{rank}(X) = 1 \end{array}$$

Dropping the non convex rank constraint leads to the SDP relaxation:

$$(SDP) \quad \max_{X} \quad \text{trace}(QX)$$
$$s.t. \quad \begin{array}{l} \text{diag}(X_{ii}) = e = \left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right]' \\ X \succeq 0 \end{array}$$

Interpretation: lift from $\mathbb{R}^n$ to $\mathbb{S}^n$

## Combinatorial optimization (4)

Example

$$(BQP) \quad \min_{x \in \{-1,1\}} \quad x'Qx = x_1x_2 - 2x_1x_3 + 3x_2x_3$$

with $Q = \begin{bmatrix} 0 & 0.5 & -1 \\ 0.5 & 0 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix}$

SDP relaxation

$$(SDP) \quad \min_{X} \quad \text{trace}(QX) = X_1 - 2X_2 + 3X_3$$

$$s.t. \quad X = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & X_3 \\ X_2 & X_3 & 1 \end{bmatrix} \succeq 0$$

$$X^* = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{rank}(X^*) = 1$$

From $X^* = x^* x^{*'}$, we recover the optimal solution of (BQP)

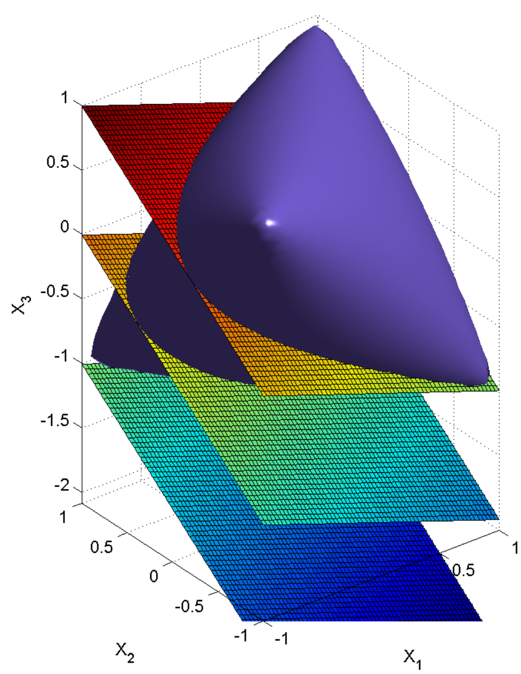
$$x^* = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}'$$

## Combinatorial optimization (4)

Example (continued)

Visualization of the feasible set of (SDP) in $(X_1, X_2, X_3)$ space :

$$X = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & X_3 \\ X_2 & X_3 & 1 \end{bmatrix} \succeq 0$$




Optimal vertex is $\begin{bmatrix} -1 & 1 & -1 \end{bmatrix}$