Steady-State Analysis of Delay Interconnected Positive Systems and Its Application to Formation Cntrol

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Abstract

This study is concerned with the analysis and synthesis of delay interconnected positive systems. For delay-free cases, it has been shown very recently that the output of the interconnected positive system converges to a positive scalar multiple of a prescribed positive vector under mild conditions on positive subsystems and a nonnegative interconnection matrix. This result is effectively used for formation control of multi-agent systems with positive dynamics. The goal of this paper is to prove that this steady-state property is essentially preserved under arbitrary (time-invariant) communication delays. In the context of formation control, this preservation indicates that the desired formation is achieved robustly against communication delays, even though the resulting formation is scaled depending upon initial conditions for the state. In showing the achievement of the steady-state property, the key mathematical issue is to prove that the delay interconnected positive system has stable poles only except for a pole of degree one at the origin, even though it has infinitely many poles in general. To this end, we develop frequency-domain (s-domain) analysis for delay interconnected positive systems, which has not been studied for delay-free cases.
Keywords: positive system, communication delay, formation control.

1 Introduction

A dynamical system is said to be positive if its state and output are both nonnegative for any nonnegative initial state and nonnegative input [13, 19]. This property can be seen naturally in biology, network communications, economics, and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel

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connections are all positive, and these are typical dynamics of moving objects. Even though their dynamics are pretty simple, large-scale systems constructed from those subsystems exhibit complicated behavior and deserve investigation in the study area of formation control of multi-agent systems [22, 28, 29]. We could say that positive system theory is deeply rooted in the theory of nonnegative matrices [4, 18], but recently, it has gained renewed interest from the viewpoint of convex optimization. Along this direction, excellent papers have been published, see, e.g., those by Rantzer [23, 24], Shorten et al. [14, 20, 26], Tanaka and Langbort [27], Blanchini et al. [5], Briat [6], and Najison [21]. We also emphasize that the study on consensus problems of multi-agent positive systems is a promising direction, and this issue is treated actively by Valcher and Misra [30] and Ebihara et al. [11]. On the other hand, study on the analysis and synthesis of retarded-type time-delay positive systems has also been active, and fruitful results have been obtained, e.g., by Haddad and Chellaboina [15], Ait Rami et al. [2], Shen and Lam [25], and Ebihara et al. [12]. Recently, these results are extended in part to neutral-type time-delay positive systems [8, 7].

Even though most of existing studies for (delay-free or delay) positive systems focus on stability and stabilization, it is important to bring the system of interest to a stability boundary in the consensus-based formation control of multi-agent systems [22, 28, 29]. In the context of positive systems, this issues is treated independently by Valcher and Misra [30] and Ebihara et al. [11], where in the latter paper we dealt with interconnected positive systems shown in Fig. 1. In Fig. 1, positive subsystems G_i $(i = 1, \cdots, N)$ are interconnected via an interconnection matrix Ω. Under mild conditions on positive subsystems G_i $(i = 1, \cdots, N)$ and the nonnegative interconnection matrix Ω, we showed that the interconnected positive system is on a stability boundary and its state converges to a positive scalar multiple of a prescribed positive vector. As a byproduct of this steady-state property, it turned out that the output $\widehat{z}$ converges to a positive right eigenvector of the interconnection matrix scaled by the steady-stage gains of the positive subsystems. As expected, this result is effectively used in the formation control of multi-agent positive systems [11]. The goal of this paper is to prove that the steady-state property is still preserved for delay interconnected positive systems shown in Fig. 2, where $h_{ij} \geq 0$ stands for the delay over the communication from subsystem G_j to G_i. In the context of formation control of multi-agent positive systems, this preservation indicates that the desired formation is achieved robustly against arbitrary (time-invariant) communication delays, even though the resulting formation is scaled depending upon initial conditions for the state. In showing the achievement of the steady-state property, the key mathematical issue is to prove that the delay interconnected positive system has stable poles only except for a pole of degree one at the origin, even though it has infinitely many poles in general. To this end, we develop frequency-domain (s-domain) analysis for delay interconnected positive systems, which has not been studied for delay-free cases [11].

A conference version of the present paper has been published in [10] but we completely rewrite this Introduction by citing latest papers published to this date. In addition, we provide a complete proof for the main result, Theorem 2, in the appendix section. We believe that the preliminary results for the proof, Lemmas 1-5, have their own significance in enriching linear positive system theory.

We use the following notations. We denote by $\mathbb{R}$ and $\mathbb{C}$ the set of real and complex numbers, respectively. We also use $\mathbb{R}_+$ $(\mathbb{R}_{++})$ and $\mathbb{C}_-$ $(\mathbb{C}_{--})$ for the set nonnegative (strictly positive) real numbers and complex numbers with nonpositive (strictly negative) real parts, respectively. The set of positive integers up to N is denoted by $\mathbb{Z}_N$, i.e., $\mathbb{Z}_N := \{1, \cdots, N\}$. For given two real matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i, j), where A_{ij} stands for the (i, j)-entry of A. In relation to this notation, we also define $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : \ x > 0\}$ and $\mathbb{R}^n_{+} := \{x \in \mathbb{R}^n : \ x \geq 0\}$. We also define $\mathbb{R}^{n\times m}_{++}$ and $\mathbb{R}^{n\times m}_{+}$ with obvious modifications. The set of positive definite and diagonal matrices of size n is denoted by $\mathbb{D}^{n\times n}_{++}$. The set of complex and diagonal matrices of size n is denoted by $\mathbb{C}^{n\times n}_{\mathrm{d}}$. For $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ and $\rho(A)$ the set of the eigenvalues of A and the spectral radius of A, respectively. For $A \in \mathbb{R}^{n\times n}_{+}$, Theorem 8.3.1 in [17] states that there is an eigenvalue equal to $\rho(A)$. This eigenvalue is related to the Perron-Frobenius Theorem and denoted by $\lambda_{\mathrm{F}}(A)$ in this paper. We finally define the set of n-vector-valued continuous functions over $[a, b]$ by $C([a, b], \mathbb{R}^n)$, and the set of nonnegative n-vector-valued continuous functions over $[a, b]$ by $C([a, b], \mathbb{R}^n_{+})$.

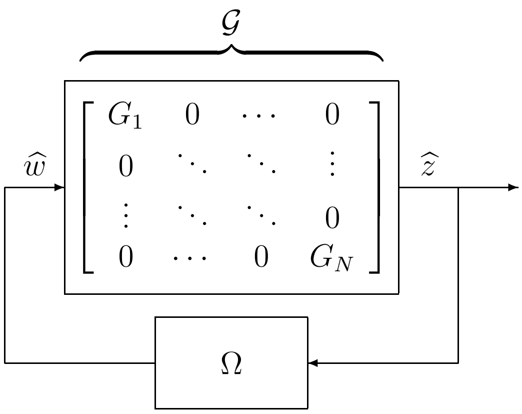

Figure 1: Interconnected Positive Systems in [11].

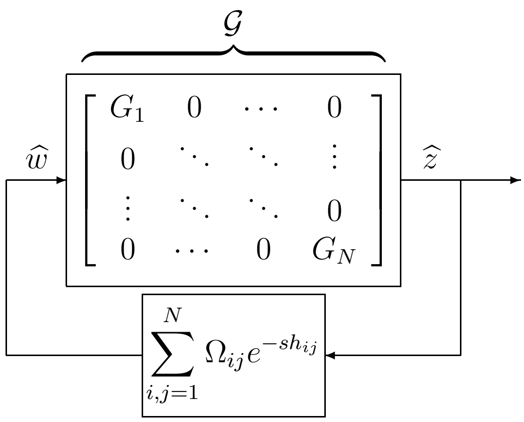

Figure 2: Interconnected Positive Systems with Communication Delays.

2 Preliminaries

Consider the linear system described by

$$G : \left\{ \begin{array}{rcccl} \dot{x}(t) & = & Ax(t) & + & Bw(t), \\ z(t) & = & Cx(t) & + & Dw(t) \end{array} \right. \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_z\times n}$, and $D \in \mathbb{R}^{n_z\times n_w}$. The definition and a basic result of positive systems are given in the following.

Definition 1 (Positive Linear System) [13] The linear system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Definition 2 (Metzler Matrix) [13] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ $(i \neq j)$.

In this paper we denote by $\mathbb{M}^n$ ($\mathbb{H}^n$) the set of Metzler (Hurwitz) matrices of size n. Then the next result holds.

Proposition 1 [13] The system (1) is positive if and only if $A \in \mathbb{M}^n$, $B \in \mathbb{R}_+^{n \times n_w}$, $C \in \mathbb{R}_+^{n_z \times n}$, and $D \in \mathbb{R}_+^{n_z \times n_w}$.

3 Delay-Free Interconnected Positive Systems

In this section we quickly review our preceding results on the steady-state property of interconnected positive systems [11]. Consider the stable, SISO, strictly proper, and positive subsystem G_i ($i \in \mathbb{Z}_N$) represented by

$$
\begin{array}{l}
G_i : \left\{ \begin{array}{rcl} \dot{x}_i(t) &=& A_i x_i(t) + B_i w_i(t), \\ z_i(t) &=& C_i x_i(t), \end{array} \right. \\
A_i \in \mathbb{M}^{n_i} \cap \mathbb{H}^{n_i},\ B_i \in \mathbb{R}_+^{n_i \times 1},\ C_i \in \mathbb{R}_+^{1 \times n_i}.
\end{array}
\tag{2}
$$

The transfer function of G_i is defined by $G_i(s) = C_i(sI - A_i)^{-1}B_i$. With these positive subsystems, we define a positive and stable system $\mathcal{G}$ by

$$
\mathcal{G} := \mathrm{diag}(G_1, \cdots, G_N). \tag{3}
$$

The state space realization of $\mathcal{G}$ is given by

$$
\mathcal{G} : \left\{ \begin{array}{rcl} \dot{\widehat{x}}(t) &=& \mathcal{A}\widehat{x}(t) + \mathcal{B}\widehat{w}(t), \\ \widehat{z}(t) &=& \mathcal{C}\widehat{x}(t) \end{array} \right. \tag{4}
$$

where

$$
\mathcal{A} := \mathrm{diag}(A_1, \cdots, A_N),\ \mathcal{B} := \mathrm{diag}(B_1, \cdots, B_N),\ \mathcal{C} := \mathrm{diag}(C_1, \cdots, C_N), \tag{5}
$$

$$
\widehat{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_{\widehat{x}}}, \quad n_{\widehat{x}} := \sum_{i=1}^N n_i, \quad \widehat{z} := \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{R}^N,\ \widehat{w} := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in \mathbb{R}^N.
$$

The transfer function matrix of $\mathcal{G}$ is defined by $\mathcal{G}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$.

For given interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$, we investigated in [11] the interconnected system $\mathcal{G} \star \Omega$ defined by

$$
\widehat{w}(t) = \Omega \widehat{z}(t). \tag{6}
$$

The block-diagram of the interconnected system $\mathcal{G} \star \Omega$ is shown in Fig. 1. Note that $\mathcal{G} \star \Omega$ is positive since its state-space realization is given by

$$
\dot{\widehat{x}}(t) = \mathcal{A}_{\mathrm{cl}}\widehat{x}(t), \quad \mathcal{A}_{\mathrm{cl}} := \mathcal{A} + \mathcal{B}\Omega\mathcal{C} \in \mathbb{M}^{n_{\widehat{x}}}. \tag{7}
$$

On the steady-state property of $\mathcal{G} \star \Omega$, the next result has been shown in [11].

Theorem 1 [11] For given positive subsystems G_i ($i \in \mathbb{Z}_N$) represented by (2) and interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$, suppose the following conditions are satisfied:

(i) (A_i, B_i) is controllable and (A_i, C_i) is observable for all $i \in \mathbb{Z}_N$.
(ii) The interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ is irreducible (i.e., the directed graph $\Gamma(\Omega)$ is strongly connected).
(iii) $\lambda_{\rm F}(\mathcal{G}(0)\Omega) = 1$ holds.

Then, for the interconnected system $\mathcal{G} \star \Omega$, the next results hold.

(I) The matrix $\mathcal{A}_{\rm cl}$ given by (7) has an eigenvalue zero that is algebraically (and hence geometrically) simple. Moreover, $\mathcal{A}_{\rm cl}$ satisfies $\mathrm{Re}(\lambda) < 0$ $(\forall \lambda \in \sigma(\mathcal{A}_{\rm cl}) \setminus \{0\})$.
(II) If we denote the right and left eigenvectors of $\mathcal{G}(0)\Omega \in \mathbb{R}_+^{N\times N}$ associated with the Frobenius eigenvalue $\lambda_{\rm F}(\mathcal{G}(0)\Omega)$ by $v_{\rm R} \in \mathbb{R}_{++}^N$ and $v_{\rm L} \in \mathbb{R}_{++}^N$, respectively, we have $\mathcal{A}_{\rm cl}\xi_{\rm R} = 0$ and $\xi_{\rm L}^T\mathcal{A}_{\rm cl} = 0$ where
$$\xi_{\rm R} = -\mathcal{A}^{-1}\mathcal{B}\mathcal{G}(0)^{-1}v_{\rm R} \in \mathbb{R}_{++}^{n_{\widehat{x}}},\ \xi_{\rm L} = -\mathcal{A}^{-T}\mathcal{C}^T v_{\rm L} \in \mathbb{R}_{++}^{n_{\widehat{x}}},\ \xi_{\rm L}^T\xi_{\rm R} = 1. \quad (8)$$
Here the eigenvectors $v_{\rm R}, v_{\rm L} \in \mathbb{R}_{++}^N$ are appropriately scaled so that $\xi_{\rm L}^T\xi_{\rm R} = 1$ is satisfied.
(III) For any initial state $\widehat{x}(0) \in \mathbb{R}^{n_{\widehat{x}}}$, the state $\widehat{x}$ of $\mathcal{G} \star \Omega$ satisfies
$$\lim_{t\to\infty} \widehat{x}(t) = (\xi_L^T\widehat{x}(0))\xi_{\rm R} \in \mathbb{R}^{n_{\widehat{x}}}. \quad (9)$$
(IV) The output $\widehat{z}$ of $\mathcal{G} \star \Omega$ satisfies
$$\lim_{t\to\infty} \widehat{z}(t) = (\xi_L^T\widehat{x}(0))v_{\rm R} \in \mathbb{R}^N. \quad (10)$$
(V) If we define a linear function $V : \mathbb{R}^{n_{\widehat{x}}} \to \mathbb{R}$ by $V(\widehat{x}(t)) := \xi_L^T\widehat{x}(t)$, we have $V(\widehat{x}(t)) = V(\widehat{x}(0))$ $(\forall t \in \mathbb{R}_+)$. Namely, the quantity V serves as the first integral (conserved quantity) of the system $\mathcal{G} \star \Omega$.

The result (10) clearly shows that the output $\widehat{z}(t) = [\ z_1(t)\ \cdots\ z_N(t)\ ]^T$ of the interconnected system $\mathcal{G} \star \Omega$ converges to $(\xi_L^T\widehat{x}(0))v_{\rm R} \in \mathbb{R}^N$. From the viewpoint of formation control, this result implies that, for given $v_{\rm obj} \in \mathbb{R}_{++}^N$ that represents the "shape" of the desired formation, we can achieve $\lim_{t\to\infty}\widehat{z}(t) = (\xi_L^T\widehat{x}(0))v_{\rm obj} \in \mathbb{R}^N$ by designing interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ satisfying (ii) and $(\mathcal{G}(0)\Omega)v_{\rm obj} = v_{\rm obj}$ [††]. Based on these ideas, a formation control of multi-agent positive systems is achieved in a sound way in [11].

4 Delay-Interconnected Systems: The Counterpart Results

Let us consider the case where the interconnected system $\mathcal{G}\star\Omega$ operates under communication delays. More precisely, we consider the case where (6) is replaced by
$$\widehat{w}(t) = \sum_{i,j=1}^N \Omega_{ij}\widehat{z}(t - h_{ij}). \quad (11)$$
Here, $h_{ij} \geq 0$ stands for the delay over the communication from subsystem G_j to G_i. Note that the entries of $\Omega_{ij} \in \mathbb{R}_+^{N\times N}$ are all zero except for the (i,j)-entry and $\sum_{i,j=1}^N \Omega_{ij} = \Omega$. We

[††] If $\Omega \in \mathbb{R}_+^{N\times N}$ and hence $\mathcal{G}(0)\Omega \in \mathbb{R}_+^{N\times N}$ is irreducible, only the Frobenius eigenvalue has associated eigenvector that is strictly positive [17]. Therefore $(\mathcal{G}(0)\Omega)v_{\rm obj} = v_{\rm obj}$ for $v_{\rm obj} \in \mathbb{R}_{++}^N$ ensures $\lambda_{\rm F}(\mathcal{G}(0)\Omega) = 1$.

denote by $\mathcal{G} \star \Omega_h$ the interconnected system constructed by $\mathcal{G}$ and (11). The block-diagram of $\mathcal{G} \star \Omega_h$ is shown in Fig. 2.

The state-space realization of $\mathcal{G} \star \Omega_h$ is given by

$$\dot{\widehat{x}}(t) = \mathcal{A}\widehat{x}(t) + \sum_{i,j=1}^{N} \mathcal{B}\Omega_{ij}\mathcal{C}\widehat{x}(t - h_{ij}), \quad \widehat{x}(t) = \phi(t)\ (t \in [-h, 0]). \tag{12}$$

Here, $h := \max_{i,j} h_{ij}$ and $\phi \in C([-h,0], \mathbb{R}^{n_{\widehat{x}}})$ is the initial state. We see from [15] that $\mathcal{G}\star\Omega_h$ is positive, in the sense that $\widehat{x}(t) \geq 0$ ($\forall t \in \mathbb{R}_+$) holds for any initial state $\phi \in C([-h,0], \mathbb{R}_+^{n_{\widehat{x}}})$. The positivity of $\mathcal{G}\star\Omega_h$ is ensured by the positivity of subsystems G_i ($i \in \mathbb{Z}_N$) and $\Omega \in \mathbb{R}_+^{N\times N}$.

The characteristic function of (7) is given by

$$\Delta(s) = \det(F(s)), \quad F(s) := sI - \mathcal{A} - \sum_{i,j=1}^{N} \mathcal{B}\Omega_{ij}\mathcal{C}e^{-sh_{ij}}. \tag{13}$$

A zero of $\Delta(s)$ is said to be a pole of $\mathcal{G} \star \Omega_h$. If we denote the set of poles of $\mathcal{G} \star \Omega_h$ by Λ, which is an infinite set in general, it is known that $\mathcal{G} \star \Omega_h$ is stable if and only if $\Lambda \subset \mathbb{C}_{--}$ holds [16].

Our goal in this paper is to prove that the steady-state property show in Theorem 1 is still preserved against arbitrary (time-invariant) communication delays. To this end, recall that the Lyapunov function $V(x(t)) = \xi_{\mathrm{L}}^T x(t)$ plays a key role in Theorem 1. Since $\xi_{\mathrm{L}} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$, this Lyapunov function in particular satisfies the positivity property $V(x(0)) > 0$ ($\forall x(0) \in \mathbb{R}_+ \setminus \{0\}$). With this in mind, let us define a linear functional $C([-h,0], \mathbb{R}^{n_{\widehat{x}}}) \to \mathbb{R}$ in the following.

Definition 3 We define the linear functional $\alpha_0 : C([-h,0], \mathbb{R}^{n_{\widehat{x}}}) \to \mathbb{R}$ by

$$\alpha_0(\phi) = \left\{1 + \xi_{\mathrm{L}}^{\mathrm{T}} \left(\sum_{i,j=1}^{N} h_{ij}\mathcal{B}\Omega_{ij}\mathcal{C}\right)\xi_{\mathrm{R}}\right\}^{-1} \xi_{\mathrm{L}}^{\mathrm{T}} \left(\phi(0) + \sum_{i,j=1}^{N} \mathcal{B}\Omega_{ij}\mathcal{C}\int_{-h_{ij}}^{0} \phi(\tau)d\tau\right). \tag{14}$$

Here, $\xi_{\mathrm{L}}, \xi_{\mathrm{R}} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$ are given by (8).

In this definition, note that

$$\alpha_0(\phi) \geq \left\{1 + \xi_{\mathrm{L}}^{\mathrm{T}} \left(\sum_{i,j=1}^{N} h_{ij}\mathcal{B}\Omega_{ij}\mathcal{C}\right)\xi_{\mathrm{R}}\right\}^{-1} \xi_{\mathrm{L}}^{T}\phi(0) > 0$$

holds for all $\phi \in C([-h,0], \mathbb{R}_+^{n_{\widehat{x}}})$ and $\phi(0) \neq 0$. Namely, this linear functional inherits the positivity property of $V(x(t)) = \xi_{\mathrm{L}}^T x(t)$. Using this linear functional, we can state the main results of this paper in the next theorem.

Theorem 2 For given positive subsystems G_i ($i \in \mathbb{Z}_N$) represented by (2), interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$, and time-invariant communication delays $h_{ij} \geq 0$ ($i, j \in \mathbb{Z}_N$), suppose the conditions (i)-(iii) in Theorem 1 are satisfied. Then, for the delay interconnected positive system $\mathcal{G} \star \Omega_h$ constructed from (4) and (11), the next results hold.

(I') The delay interconnected positive system $\mathcal{G} \star \Omega_h$ has stable poles only except for the pole of degree one at the origin.

(III') For any initial state $\phi \in C([-h, 0], \mathbb{R}^{n_{\widehat{x}}})$, the vector $\widehat{x}$ of $\mathcal{G} \star \Omega_h$ satisfies

$$\lim_{t\to\infty} \widehat{x}(t) = \alpha_0(\phi)\xi_{\mathrm{R}} \in \mathbb{R}^{n_{\widehat{x}}} \tag{15}$$

where the linear functional $\alpha_0 : C([-h, 0], \mathbb{R}^{n_{\widehat{x}}}) \to \mathbb{R}$ is given by (14) and $\xi_R \in \mathbb{R}^{n_{\widehat{x}}}_{++}$ is given by (8).

(IV') The output $\widehat{z}$ of $\mathcal{G} \star \Omega_h$ satisfies

$$\lim_{t\to\infty} \widehat{z}(t) = \alpha(\phi)v_{\mathrm{R}} \in \mathbb{R}^N \tag{16}$$

where $v_{\mathrm{R}} \in \mathbb{R}^N_{++}$ is the right eigenvector of $\mathcal{G}(0)\Omega \in \mathbb{R}^{N\times N}_+$ associated with the Frobenius eigenvalue $\lambda_{\mathrm{F}}(\mathcal{G}(0)\Omega)$.

(V') Let us define a linear functional $V_{\mathrm{d}} : C([t-h, t], \mathbb{R}^{n_{\widehat{x}}}) \to \mathbb{R}$ by

$$V_{\mathrm{d}}(\widehat{x}_t) := \xi_{\mathrm{L}}^T \left(\widehat{x}(t) + \sum_{i,j=1}^{N} \mathcal{B}\Omega_{ij}\mathcal{C} \int_{-h_{ij}}^{0} \widehat{x}(t+\tau)d\tau \right) \quad (t \geq 0), \tag{17}$$

where $\widehat{x}_t(\tau) = \widehat{x}(t+\tau)$ $(-h \leq \tau \leq 0)$. Then, we have

$$V_{\mathrm{d}}(\widehat{x}_t) = V_{\mathrm{d}}(\widehat{x}_0) \quad (\forall t \in \mathbb{R}_+).$$

Namely, the quantity V_{d} serves as the first integral (the conserved quantity) of the system $\mathcal{G} \star \Omega_h$.

Even though the proof of this theorem is the theoretical core of this paper, we defer it for the appendix section and give the next important remarks on the results (I')-(V').

Remark 1

- The result (I') implies that the delay-free interconnected positive system $\mathcal{G} \star \Omega$ that is on stability boundary still remains to be on stability boundary against arbitrary (time-invariant) communication delays. This is a novel result for the stability of retarded-type time-delay positive systems (TDPSs), where an available result is that if a delay-free positive system is stable, then this system remains to be stable against arbitrary (time-invariant) delays [15]. For the proof of (I'), the standard time-domain arguments using Lyapunov-Krasovskii functionals do not work fine since we have to prove rigorously the "degree-one property" of the pole at the origin. For this reason, in the appendix section, we give the proof of (I') by the arguments in frequency-domain (s-domain). We believe that the preliminary results for the proof, Lemmas 1-5 in the appendix section, have their own significance in enriching linear positive system theory.
- If we let $h = 0$ in Theorem 2, the linear functional α_0 given by (14) reduces to $\alpha_0(\phi) = \xi_{\mathrm{L}}^T\phi(0) = \xi_{\mathrm{L}}^T\widehat{x}(0)$. It follows that the results (15) and (16) reduce to (9) and (10), respectively. In this sense, the results (III') and (IV') are natural extension of (III) and (IV) in Theorem 1 to communication-delay cases. Exactly the same comment applies also to the result (V') on the first integral of $\mathcal{G} \star \Omega_h$.
- The result (IV') clearly shows that the output $\widehat{z}$ converges to a scalar multiple of $v_{\mathrm{R}} \in \mathbb{R}^N_{++}$ that is equal to the delay-free case. In the context of formation control of multi-agent positive systems [11] that is briefly reviewed in the preceding section,

this result ensures that the desired formation for an appropriately constructed interconnected positive system $\mathcal{G} \star \Omega$ is achieved robustly against arbitrary (time-invariant) communication delays, even though the resulting formation is scaled depending upon the initial state ϕ. Note that the dependence on the initial state, which is observed also in the delay-free case [11], is unavoidable in the current problem setting where we do not allow to equip external inputs for the interconnected systems.

To summarize, we have shown in Theorem 2 the counterpart results of Theorem 1 for delay interconnected positive systems. In particular, we derived explicit closed-form formulas (15), (16), and (17) by introducing a linear functional given by (14).

5 Numerical Examples

Let us consider the formation control problem of a multi-agent positive system constructed from N agents. The problem setup is borrowed in part from [11]. The i-th agent ($i \in \mathbb{Z}_N$) can move over the (x, y)-plane with independent (interference-free) dynamics $Z_{i,x}(s)$ and $Z_{i,y}(s)$ along x and y axes, respectively. Assume

$$Z_{i,j}(s) = \frac{k_{i,j}}{s(s + a_{i,j})} U_{i,j}(s) \ (i = 1, \cdots, N, \ j = x, y)$$

where $k_{i,j}, a_{i,j} > 0$. Applying a local feedback

$$U_{i,j}(s) = -f_{i,j}(Z_{i,j}(s) - W_{i,j}(s))$$

with $0 < f_{i,j} < a_{i,j}^2/4k_{i,j}$, we have

$$Z_{i,j}(s) = G_{i,j}(s)W_{i,j}(s), \ G_{i,j}(s) = \left[\begin{array}{cc|c} -p_{i,j} & 1 & 0 \\ 0 & -q_{i,j} & p_{i,j}q_{i,j} \\ \hline 1 & 0 & 0 \end{array}\right],$$
$$p_{i,j} + q_{i,j} = a_{i,j}, \ p_{i,j}q_{i,j} = f_{i,j}k_{i,j}.$$

It turns out that each subsystem $G_{i,j}$ ($i \in \mathbb{Z}_N, \ j = x, y$) is stable, SISO, strictly proper, positive and satisfies $G_{i,j}(0) = 1$. For simplicity, we consider the case $k_{i,j} = k = 1$, $a_{i,j} = a = 10$, and $f_{i,j} = 0.8 \times a^2/4k = 20$ ($i \in \mathbb{Z}_N, \ j = x, y$).

Assuming that the agent i can communicate with agent $i - 1$ and $i + 1$ (agent 0 and $N + 1$ should be regarded as agent N and 1, respectively), our goal here is to design a communication scheme (i.e., interconnection matrices Ω_x and Ω_y along x- and y-axes) over the agents with respect to each agent's position so that prescribed formation can be achieved. To form a circle, we let $[v_{\mathrm{obj},x} \ v_{\mathrm{obj},y}]_i = [2 + \cos(2\pi i/N) \quad 2 + \sin(2\pi i/N)]$ and constructed two interconnection matrices Ω_x, Ω_y by following [11].

In Figs. 3 and 4 given at the last page, we show the simulation results for delay-free case, where we let the initial states of two interconnected systems along x- and y- axes as $x_{i,j}(0) = [z_{i,j}(0) \ 0]^T (i \in \mathbb{Z}_N, \ j = x, y)$. In these figures, the blue dots show the terminal positions of agents computed beforehand from (10). We can confirm that the desired formation has been achieved around 5 [sec]. On the other hand, In Figs. 5-7, we show the simulation results under uniform communication delay 1 [sec]. We let the initial condition as $\phi_{i,j}(t) = x_{i,j}(0) = [z_{i,j}(0)\ 0]^T$ $(-1 \le t \le 0,\ i \in \mathbb{Z}_N,\ j = x, y)$. The blue dots in these figures show the terminal positions of agents computed from (16). Again we can confirm that the desired formation has been achieved eventually. However, due to the communication delays, the convergence is definitely slower than the delay-free case.

6 Conclusion

In this paper, we analyzed steady-state properties of interconnected positive systems under communication delays. We clarified that the particular steady-state property is still preserved against arbitrary communication delays. In the context of formation control of multi-agent positive systems, this result indicates that the formation of an appropriately constructed interconnected positive system is achieved robustly against arbitrary communication delays, even though the resulting formation is scaled depending upon the initial state. We verified all of the theoretical results by numerical examples on the formation control of multi-agent positive systems.

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A Proof of Theorem 2

In the following proof, we note that the properties (I)-(V) in Theorem 1 are satisfied under the assumptions of Theorem 2.

A.1 Proof of (I')

For the proof of (I'), we have a long way to go. In the following, we denote by Λ the (infinite) set of poles of $\mathcal{G} \star \Omega_h$. The proof consists of the following three parts:

(a) $\Lambda \subset \mathbb{C}_-$.
(b) $\mathcal{G} \star \Omega_h$ has a pole of degree one at the origin.
(c) $\mathcal{G} \star \Omega_h$ has no poles on the imaginary axis except for the origin.

A.1.1 Proof of (a)

We see from [15, 1] that $\mathcal{G} \star \Omega_h$ is asymptotically stable (i.e., $\Lambda \subset \mathbb{C}_{--}$) if and only if its delay-free counterpart $\mathcal{G} \star \Omega$ is asymptotically stable. In the current situation, $\mathcal{G} \star \Omega$ is on the stability boundary: the assertion (I) of Theorem 1 ensures that the corresponding coefficient matrix $\mathcal{A}_{\rm cl}$ satisfies $\mathrm{Re}(\lambda) < 0$ ($\forall \lambda \in \sigma(\mathcal{A}_{\rm cl}) \setminus \{0\}$). Therefore $\mathcal{G} \star \Omega_h$ cannot have a pole on the open right half plane and hence we can readily conclude that $\Lambda \subset \mathbb{C}_-$ holds.

A.1.2 Proof of (b)

Since (I) and (II) in Theorem 1 hold, it suffices to prove the following proposition.

Proposition 2 For given $\mathsf{A}_0 \in \mathbb{M}^n$ and $\mathsf{A}_i \in \mathbb{R}_+^{n\times n}$ ($i \in \mathbb{Z}_L$), suppose the following conditions hold.

(I) The matrix $\mathsf{A} := \sum_{i=0}^{L} \mathsf{A}_i$ has an eigenvalue zero that is algebraically (and hence geometrically) simple. Moreover, A satisfies $\mathrm{Re}(\lambda) < 0$ ($\forall \lambda \in \sigma(\mathsf{A}) \setminus \{0\}$).
(II) $\exists \xi_{\mathrm{R}}, \xi_{\mathrm{L}} \in \mathbb{R}_{++}^n$ such that $\mathsf{A}\xi_{\mathrm{R}} = 0$, $\xi_{\mathrm{L}}^T \mathsf{A} = 0$, and $\xi_{\mathrm{L}}^T \xi_{\mathrm{R}} = 1$.

Then, for any $h_i \in \mathbb{R}_+$ ($i = 1, \cdots, L$), the entire function

$$\Delta_0(s) := \det\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)$$

has a zero of degree one at the origin. Moreover, the residue of $\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)^{-1}$ at the origin is given by

$$\left\{1 + \xi_{\mathrm{L}}^T \left(\sum_{i=1}^{L} h_i \mathsf{A}_i\right) \xi_{\mathrm{R}}\right\}^{-1} \xi_{\mathrm{R}} \xi_{\mathrm{L}}^T. \tag{18}$$

Note that A_0, A_i, and A in Proposition 2 correspond to $\mathcal{A}$, $\mathcal{B}\Omega_{ij}\mathcal{C}$, and $\mathcal{A}_{\mathrm{cl}}$ in Theorem 2, respectively.
Proof of the First Part: For the proof, we first define

$$J_1 = \begin{bmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & 0_{n-1,n-1} \end{bmatrix}, \ J_2 = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}.$$

From (I) and (II), there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$T = \begin{bmatrix} \xi_{\mathrm{R}} & * \end{bmatrix}, \ T^{-1} = \begin{bmatrix} \xi_{\mathrm{L}}^T \\ * \end{bmatrix}, \quad T^{-1}\mathsf{A}T = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & \Lambda_0 \end{bmatrix}$$

for some $\Lambda_0 \in \mathbb{H}^{n-1}$. Note that the Taylor series expansion centered at the origin of the (matrix-valued) entire function $F_0(s) := sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}$ is given by

$$F_0(s) = \sum_{n=0}^{\infty} \mathcal{A}_n s^n$$

where

$$\mathcal{A}_0 := -\mathsf{A}, \quad \mathcal{A}_1 := I + \sum_{i=1}^{L} h_i \mathsf{A}_i, \quad \mathcal{A}_n := -\sum_{i=1}^{L} \frac{(-h_i)^n}{n!} \mathsf{A}_i \ (n \geq 2).$$

If we define $\widehat{\mathcal{A}}_i := T^{-1}\mathcal{A}_i T$ ($i = 0, 1, 2, \cdots$), we have

$$
\begin{aligned}
F_0(s) &= \sum_{n=0}^{\infty} \mathcal{A}_n s^n \\
&= T\left(\sum_{n=0}^{\infty} \widehat{\mathcal{A}}_n s^n\right) T^{-1} \\
&= T\left(\begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & -\Lambda_0 \end{bmatrix} + \sum_{n=1}^{\infty} \widehat{\mathcal{A}}_n s^n\right) T^{-1} \\
&= T\left\{\begin{bmatrix} s & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}\left(\begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & -\Lambda_0 \end{bmatrix} + J_1\left(\sum_{n=1}^{\infty} \widehat{\mathcal{A}}_n s^{n-1}\right) + J_2\left(\sum_{n=1}^{\infty} \widehat{\mathcal{A}}_n s^n\right)\right)\right\} T^{-1} \\
&= T\left(\begin{bmatrix} s & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix} H(s)\right) T^{-1}
\end{aligned}
\tag{19}
$$

where

$$
H(s) := \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & -\Lambda_0 \end{bmatrix} + J_1\left(\sum_{n=1}^{\infty} \widehat{\mathcal{A}}_n s^{n-1}\right) + J_2\left(\sum_{n=1}^{\infty} \widehat{\mathcal{A}}_n s^n\right). \tag{20}
$$

From (19), it is clear that

$$
\Delta_0(s) = \det(F_0(s)) = s\det(H(s)). \tag{21}
$$

By definition, $\Delta_0(s)$ has a zero of degree one at the origin if and only if its Taylor expansion centered at the origin is given by

$$
\Delta_0(s) = \sum_{i=1}^{\infty} a_i s^i, \quad a_1 \neq 0. \tag{22}
$$

On the other hand, since $H(s)$ in (20) is a matrix valued entire function, its determinant $\det(H(s))$ is an entire function as well, and hence it is possible to take its Taylor series expansion centered at the origin as in

$$
\det(H(s)) = \sum_{i=0}^{\infty} b_i s^i. \tag{23}
$$

Therefore, from (21), (22), and (23), it suffices to prove that $b_0(= a_1) \neq 0$. This is equivalent to showing $\det(H(0)) \neq 0$ that is proved below.

From (20), we have

$$
H(0) = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & -\Lambda_0 \end{bmatrix} + J_1\widehat{\mathcal{A}}_1 = \begin{bmatrix} (\widehat{\mathcal{A}}_1)_{1,1} & * \\ 0_{n-1,1} & -\Lambda_0 \end{bmatrix}. \tag{24}
$$

It follows that $\det(H(0)) = (\widehat{\mathcal{A}}_1)_{1,1}\det(-\Lambda_0)$. Since $\Lambda_0 \in \mathbb{H}^{n-1}$, we have $\det(-\Lambda_0) \neq 0$. On the other hand, since $\widehat{\mathcal{A}}_1 = T^{-1}(I + \sum_{i=1}^{L} h_i \mathsf{A}_i)T$, we have

$$
(\widehat{\mathcal{A}}_1)_{1,1} = \xi_{\mathrm{L}}^T(I + \sum_{i=1}^{L} h_i \mathsf{A}_i)\xi_{\mathrm{R}} = 1 + \sum_{i=1}^{L} h_i \xi_{\mathrm{L}}^T \mathsf{A}_i \xi_{\mathrm{R}} \geq 1. \tag{25}
$$

Therefore we can conclude that $\det(H(0)) \neq 0$. This completes the proof. ■

Proof of the Second Part: First note that

$$F_0(s)^{-1} = \left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)^{-1} = \frac{\mathrm{adj}\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)}{\det\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)}.$$

Since

- $\mathrm{adj}\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)$ is a matrix valued entire function,
- $\det\left(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}\right)$ has a zero of degree one at the origin,
- $\lim_{s\to 0} \mathrm{adj}(sI - \mathsf{A}_0 - \sum_{i=1}^{L} \mathsf{A}_i e^{-sh_i}) = \mathrm{adj}(-\mathsf{A}) \neq 0$, which stems from the fact that $\mathrm{rank}(\mathsf{A}) = n - 1$ (see p. 13 of [17]),

we can conclude that $F_0(s)^{-1}$ has a pole of degree one at the origin. Therefore, its residue at the origin is given by $\lim_{s\to 0} sF_0(s)^{-1}$. From (19), (20), (24) and (25), we have

$$\begin{aligned}
\lim_{s\to 0} sF_0(s)^{-1} &= \lim_{s\to 0} sT \left(H(s)^{-1} \begin{bmatrix} s^{-1} & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}\right) T^{-1} \\
&= \lim_{s\to 0} T \left(H(s)^{-1} \begin{bmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & sI_{n-1} \end{bmatrix}\right) T^{-1} \\
&= T \left(H(0)^{-1} \begin{bmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & 0_{n-1} \end{bmatrix}\right) T^{-1} \\
&= \left(1 + \sum_{i=1}^{L} h_i \xi_{\mathrm{L}}^T \mathsf{A}_i \xi_{\mathrm{R}}\right)^{-1} \xi_{\mathrm{R}} \xi_{\mathrm{L}}^T.
\end{aligned}$$

This completes the proof. ■

A.1.3 Proof of (c)

The assertion readily follows if we can prove

$$\det\left(j\omega I - \mathcal{A} - \mathcal{B}\left(\sum_{l=1}^{L} e^{-j\omega h_l}\Omega_l\right)\mathcal{C}\right) \neq 0 \ (\forall \omega \in \mathbb{R} \setminus \{0\})$$

where $h_l \in \mathbb{R}_+$ and $\Omega_l \in \mathbb{R}_+^{N\times N}$ ($l \in \mathbb{Z}_L$) are arbitrary such that $\sum_{l=1}^{L} \Omega_l = \Omega$ and $\lambda_{\mathrm{F}}(\mathcal{G}(0)\Omega) = \lambda_{\mathrm{F}}(\Omega\mathcal{G}(0)) = 1$. To this end, we introduce a sequence of lemmas. The first lemma concerns the frequency response of positive systems and has its own interesting aspect in linear system theory.

Lemma 1 Suppose the positive system G given by (1) is stable ($A \in \mathbb{M}^n \cap \mathbb{H}^n$), strictly proper ($D = 0$), and $G(0) = -CA^{-1}B \neq 0$. Then, G is *strictly DC-dominant*, i.e.,

$$\|G(0)\| > \|G(j\omega)\| \quad (\forall \omega \in (\mathbb{R} \cup \{\infty\}) \setminus \{0\}).$$

Proof of Lemma 1: For $L(s)$ given by $L(s) = 1 + Ts$ $(T > 0)$, define $G_L(s) := L(s)G(s)$. Then we have

$$\begin{aligned} G_L(s) &= (1+Ts)C(sI-A)^{-1}B \\ &= C(I+TA)(sI-A)^{-1}B + TCB \\ &= \left[\begin{array}{c|c} A & B \\ \hline C(I+TA) & TCB \end{array}\right]. \end{aligned}$$

If we choose $T \in \mathbb{R}_{++}$ such that $I + TA \geq 0$, which is always possible since $A \in \mathbb{M}^n$, we see that G_L is a stable positive system. It follows from [24, 9] that $\|G_L(0)\| \geq \|G_L(j\omega)\|$ $(\forall \omega \in \mathbb{R})$. From this fact and $\|G(0)\| = \|G_L(0)\|$, we have for all $\omega \in (\mathbb{R} \cup \{\infty\}) \setminus \{0\}$ that

$$\|G(j\omega)\| \leq \|G_L(0)\|/|L(j\omega)| \leq \|G(0)\|/|L(j\omega)| < \|G(0)\|$$

since $|L(j\omega)| > 1$ $(\forall \omega \in (\mathbb{R}\cup\{\infty\})\setminus\{0\})$. This clearly shows that G is strictly DC-dominant. ■

The next four lemmas, Lemmas 2-5, concerns the frequency response $\sum_{l=1}^{L} e^{j\omega_l}\Omega_l$ $(\omega_l \in [0, 2\pi],\ l \in \mathbb{Z}_L)$.

Lemma 2 For given $\Omega_l \in \mathbb{R}_+^{N\times N}$ $(l \in \mathbb{Z}_L)$, define

$$\Omega := \sum_{l=1}^{L} \Omega_l, \quad \widehat{\Omega} := \begin{bmatrix} I_N \\ \vdots \\ I_N \end{bmatrix} \begin{bmatrix} \Omega_1 & \cdots & \Omega_L \end{bmatrix} \in \mathbb{R}^{LN\times LN}.$$

Then the following conditions are equivalent.

(i) $\rho(\Omega)(= \lambda_{\mathrm{F}}(\Omega)) < 1$.

(ii) There exists $X \in \mathbb{D}_{++}^{LN\times LN}$ such that
$$\widehat{\Omega}^T X \widehat{\Omega} \prec X. \tag{26}$$

(iii) There exist $X_l \in \mathbb{D}_{++}^{N\times N}$ $(l \in \mathbb{Z}_L)$ such that
$$\begin{bmatrix} \Omega_1^T \\ \vdots \\ \Omega_L^T \end{bmatrix} \sum_{l=1}^{L} X_l \begin{bmatrix}\Omega_1 \cdots \Omega_L\end{bmatrix} \prec \mathrm{diag}(X_1, \cdots, X_L). \tag{27}$$

Proof of Lemma 2: It is clear that $\rho(\Omega) = \rho(\widehat{\Omega})$ and hence the condition $\rho(\Omega) < 1$ holds if and only if $\rho(\widehat{\Omega}) < 1$ holds. The latter condition is equivalent to (ii) from the diagonal stability result for nonnegative matrices [13]. Hence we have (i) ⇔ (ii). By partitioning $X \in \mathbb{D}_{++}^{LN\times LN}$ in (26) as in $X = \mathrm{diag}(X_1, \cdots, X_L)$ $(X_i \in \mathbb{D}_{++}^{N\times N},\ i \in \mathbb{Z}_L)$, it is obvious that (26) can be rewritten as (27). Therefore (ii) ⇔ (iii). ■

Lemma 3 For given $\Omega_l \in \mathbb{R}_+^{N\times N}$ $(l \in \mathbb{Z}_L)$, define $\Omega := \sum_{l=1}^{L} \Omega_l$. Suppose $\rho(\Omega)(= \lambda_{\mathrm{F}}(\Omega)) < 1$. Then we have

$$\rho\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) < 1 \quad \forall \omega_l \in [0, 2\pi]\ (l \in \mathbb{Z}_L). \tag{28}$$

Moreover, there exists $X \in \mathbb{D}_{++}^{N\times N}$ such that

$$\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)^* X \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) \prec X \quad \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L). \tag{29}$$

Proof of Lemma 3: From Lemma 2, we see that there exist $X_l \in \mathbb{D}_{++}^{N\times N}$ $(l \in \mathbb{Z}_L)$ such that (27) holds. Multiplying $[\ e^{-j\omega_1} I_N \ \cdots \ e^{-j\omega_L} I_N\ ]$ from the left and its complex conjugate transpose from the right, we have

$$\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)^* \left(\sum_{l=1}^{L} X_l\right) \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) \prec \sum_{l=1}^{L} X_l \quad \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L).$$

This clearly shows that (29) holds with $X = \sum_{l=1}^{L} X_l \in \mathbb{D}_{++}^{N\times N}$. The condition (28) readily follows from (29). ■

Lemma 4 For given $\Omega_l \in \mathbb{R}_{+}^{N\times N}$ $(l \in \mathbb{Z}_L)$, define $\Omega := \sum_{l=1}^{L} \Omega_l$. Suppose $\rho(\Omega)(= \lambda_{\mathrm{F}}(\Omega)) = 1$. Then, for any $\varepsilon > 0$, there exists $X_\varepsilon \in \mathbb{D}_{++}^{N\times N}$ such that

$$\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)^* X_\varepsilon \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) \prec (1+\varepsilon) X_\varepsilon \quad \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L). \tag{30}$$

Proof of Lemma 4: The result readily follows if we apply Lemma 3 to the case where $\Omega_l' := \Omega_l/\sqrt{1+\varepsilon}$ $(l \in \mathbb{Z}_L)$. ■

Lemma 5 For given $\Omega_l \in \mathbb{R}_{+}^{N\times N}$ $(l \in \mathbb{Z}_L)$, define $\Omega := \sum_{l=1}^{L} \Omega_l$. Suppose $\rho(\Omega)(= \lambda_{\mathrm{F}}(\Omega)) = 1$. Then, for given $\Gamma \in \mathbb{C}_{\mathrm{d}}^{N\times N}$ with $\|\Gamma\| < 1$, we have

$$\rho\left(\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)\Gamma\right) < 1 \quad \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L). \tag{31}$$

Proof of Lemma 5: From Lemma 4 and the assumption $\|\Gamma\| < 1$, there exists $\varepsilon > 0$ and associated $X_\varepsilon \in \mathbb{D}_{++}^{N\times N}$ such that

$$\left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)^* X_\varepsilon \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) \prec (1+\varepsilon) X_\varepsilon \ \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L), \quad (1+\varepsilon)\Gamma^* X_\varepsilon \Gamma \prec X_\varepsilon.$$

It follows that

$$\Gamma^* \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right)^* X_\varepsilon \left(\sum_{l=1}^{L} e^{j\omega_l}\Omega_l\right) \Gamma \prec X_\varepsilon \quad \forall \omega_l \in [0, 2\pi] \ (l \in \mathbb{Z}_L).$$

This completes the proof. ■

Now we are ready to state the proof of (c).

Proof of (c): Since G_i ($i \in \mathbb{Z}_N$) are strictly DC-dominant from Lemma 1, we see $\|\mathcal{G}(0)^{-1}\mathcal{G}(j\omega)\| < 1$ ($\forall \omega \in \mathbb{R} \setminus \{0\}$). Therefore, from Lemma 5 and $\lambda_{\mathrm{F}}(\Omega\mathcal{G}(0)) = 1$, we have

$$\det\left(I - \left(\sum_{i,j=1}^{N} e^{-j\omega h_{ij}}\Omega_{ij}\mathcal{G}(0)\right)\mathcal{G}(0)^{-1}\mathcal{G}(j\omega)\right) \neq 0 \quad \forall \omega \in \mathbb{R} \setminus \{0\}.$$

It follows that

$$\begin{aligned}
&\det\left(I - \left(\sum_{i,j=1}^{N} e^{-j\omega h_{ij}}\Omega_{ij}\right)\mathcal{G}(j\omega)\right) \neq 0 \quad \forall \omega \in \mathbb{R} \setminus \{0\} \\
&\Leftrightarrow \det\left(I - \left(\sum_{i,j=1}^{N} e^{-j\omega h_{ij}}\Omega_{ij}\right)\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}\right) \neq 0 \quad \forall \omega \in \mathbb{R} \setminus \{0\} \\
&\Leftrightarrow \det\left(I - \mathcal{B}\left(\sum_{i,j=1}^{N} e^{-j\omega h_{ij}}\Omega_{ij}\right)\mathcal{C}(j\omega I - \mathcal{A})^{-1}\right) \neq 0 \quad \forall \omega \in \mathbb{R} \setminus \{0\} \\
&\Leftrightarrow \det\left(j\omega I - \mathcal{A} - \mathcal{B}\left(\sum_{i,j=1}^{N} e^{-j\omega h_{ij}}\Omega_{ij}\right)\mathcal{C}\right) \neq 0 \quad \forall \omega \in \mathbb{R} \setminus \{0\}.
\end{aligned}$$

This completes the proof. ■

A.2 Proof of (III'), (IV') and (V')

Once (I') is proved, the proof of (III'), (IV'), and (V') can be done straightforwardly by following the standard routine for dealing with time-delay systems [16, 3]. By applying the Laplace transform to (12) and denote by $\widehat{X}(s)$ the Laplace transform of $\widehat{x}(t)$, we have

$$\widehat{X}(s) = F(s)^{-1}p(s) \tag{32}$$

where $F(s)$ is given by (13) and $p(s)$ is given by

$$p(s) = \phi(0) + \sum_{i,j=1}^{N} \mathcal{B}\Omega_{ij}\mathcal{C}e^{-sh_{ij}} \int_{-h_{ij}}^{0} \phi(\tau)e^{-s\tau}d\tau. \tag{33}$$

By applying the inverse Laplace transform to (32), we can obtain the next lemma.

Lemma 6 [16] For sufficiently large c, we have

$$\widehat{x}(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{ts}F(s)^{-1}p(s)ds \ (t > 0) \tag{34}$$

where $F(s)$ and $p(s)$ are given by (13) and (33), respectively.

We next expand the solution in (34) as an infinite series. Intuitively, such expansion can be done by replacing the integration to a counter integration that includes the poles of $e^{ts}F(s)^{-1}p(s)$ (i.e., the zeros of $\Delta(s)$) and applying the residue theorem. Such intuition can indeed be justified and we can obtain the next lemma.

Lemma 7 ([3], p. 109) Let $\{s_r\}$ be the sequence of zeros of $\Delta(s)$ given by (13) arranged in order of decreasing real parts. Then

$$\widehat{x}(t) = \sum_{r=1}^{\infty} e^{s_r t} p_r(t)(t > 0) \tag{35}$$

where $e^{s_r t}p_r(t)$ is the residue of $e^{ts}F(s)^{-1}p(s)$ at s_r and $p_r(t)$ is a polynomial of degree less than the degree of s_r.

In (35), we see from the results of (I)' that $s_1 = 0$ whose degree is one and $\mathrm{Re}(s_r) < 0$ $(i = 2, 3, \cdots)$. Since $p_r(t)$ is a polynomial, $\lim_{t\to\infty} \sum_{r=2}^{\infty} e^{s_r t}p_r(t) = 0$. It follows that

$$\lim_{t\to\infty} \widehat{x}(t) = \lim_{t\to\infty} p_1(t) + \lim_{t\to\infty} \sum_{r=2}^{\infty} e^{s_r t} p_r(t) = p_1 \tag{36}$$

where $p_1(t) = p_1 \in \mathbb{R}^{n_{\widehat{x}}}$ is a constant. For the computation of p_1 that is the residue of $e^{ts}F(s)^{-1}p(s)$ at $s = s_1 = 0$, we see from (18) that

$$\lim_{s\to 0} sF(s)^{-1} = \left\{ 1 + \xi_{\mathrm{L}}^T \left(\sum_{i,j=1}^{N} h_{ij}\mathcal{B}\Omega_{ij}\mathcal{C} \right) \xi_{\mathrm{R}} \right\}^{-1} \xi_{\mathrm{R}}\xi_{\mathrm{L}}^T.$$

Therefore we arrive at

$$p_1 = \lim_{s\to 0} se^{ts}F(s)^{-1}p(s) = \left\{ 1 + \xi_{\mathrm{L}}^T \left(\sum_{i,j=1}^{N} h_{ij}\mathcal{B}\Omega_{ij}\mathcal{C} \right) \xi_{\mathrm{R}} \right\}^{-1} \xi_{\mathrm{R}}\xi_{\mathrm{L}}^T p(0) = \alpha_0(\phi)\xi_{\mathrm{R}}.$$

Here, $\alpha_0 : C([-h, 0], \mathbb{R}^n) \to \mathbb{R}$ is given by (14). This completes the proof of (III'). The validity of (IV') and (V') follows via elementary mathematics.

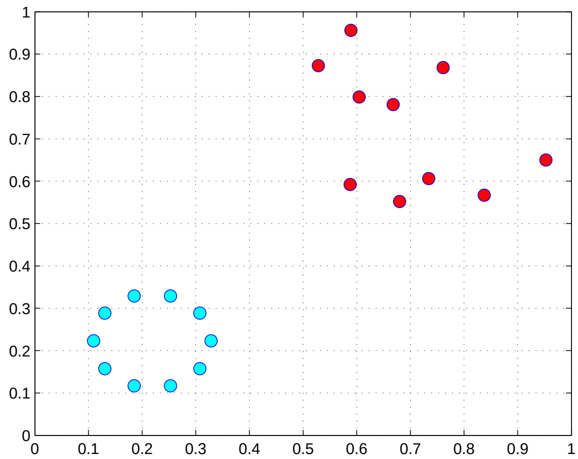

Figure 3: Positions at $t = 0$[sec] (delay-free case).

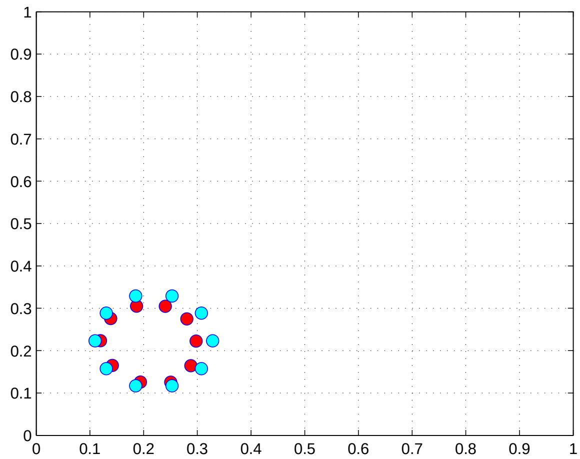

Figure 4: Positions at $t = 5$[sec] (delay-free case).

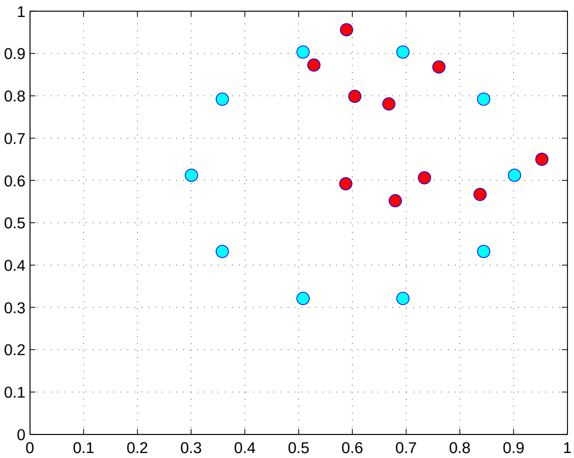

Figure 5: Positions at $t = 0$[sec] (under communication delays).

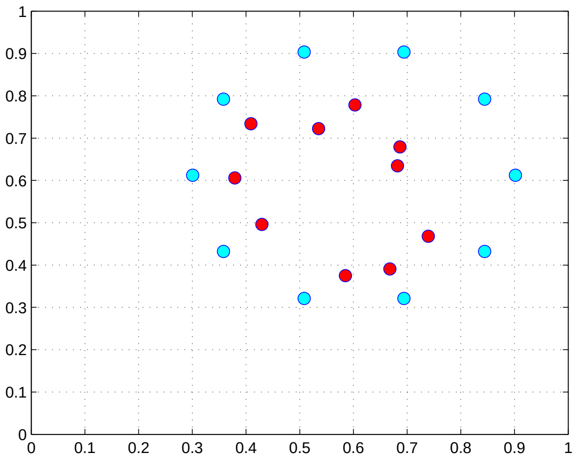

Figure 6: Positions at $t = 5$[sec] (under communication delays).

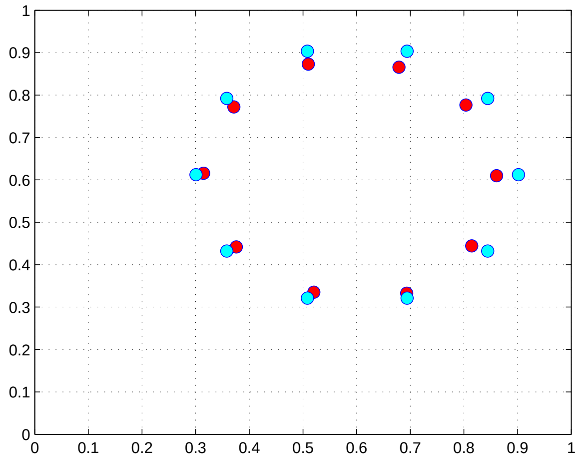

Figure 7: Positions at $t = 16$[sec] (under communication delays).