Dominant Pole Analysis of Stable Time-Delay Positive Systems

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Abstract

This paper is concerned with the dominant pole analysis of asymptotically stable time-delay positive systems (TDPSs). It is known that a TDPS is asymptotically stable if and only if its corresponding delay-free system is asymptotically stable, and this property holds irrespective of the length of delays. However, convergence performance (decay rate) should degrade according to the increase of delays and this intuition motivates us to analyze the dominant pole of TDPSs. As a preliminary result, in this paper, we show that the dominant pole of a TDPS is always real. We also construct a bisection search algorithm for the dominant pole computation, which readily follows from recent results on α-exponential stability of asymptotically stable TDPSs. Then, we next characterize a lower bound of the dominant pole as an explicit function of delays. On the basis of the lower bound characterization, we finally show that the dominant pole of an asymptotically stable TDPS is affected by delays if and only if associated coefficient matrices satisfy eigenvalue-sensitivity condition to be defined in this paper. Moreover, we clarify that the dominant pole goes to zero (from negative side) as time-delay goes to infinity if and only if the coefficient matrices are eigenvalue-sensitive.

Keywords: positive system, time-delay, dominant pole, decay-rate degradation.

1 Introduction

A linear time-invariant (LTI) system is said to be positive if its state and output are both nonnegative for any nonnegative initial state and nonnegative input [10, 13]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel connections are all positive. Even though their dynamics are pretty simple,

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large-scale systems constructed from those subsystems exhibit complicated behavior and deserve investigation in the area of multi-agent systems. Due to these reasons, intensive research efforts have been made for positive system (PS) analysis and synthesis. Nowadays, it is known that PSs admit extremely particular (strong) results for analysis and synthesis that are not valid for general systems; see, e.g., [15, 16, 18, 4, 6, 7, 8].

This particularity still holds for time-delay positive systems (TDPSs) and interesting results are available, see, e.g., [5, 2, 1, 3, 14, 17, 19]. Among them, it is known that an LTI TDPS is asymptotically stable if and only if its corresponding delay-free system is asymptotically stable, and this property holds irrespective of the length of delays [5, 2]. This property never holds for general systems, and ensure the robust stability of PSs against variations on delays. However, convergence performance (decay rate) should degrade according to the increase of delays and this intuition motivates us to analyze the dominant pole of TDPSs.

As a preliminary result, in this paper, we show that the dominant pole of TDPSs is always real. We also construct a bisection search algorithm for the dominant pole computation. These results are not necessarily new and readily follow from α-exponential stability analysis of TDPS in [19]. Still, time-domain discussion in this reference does not fully fit to the dominant-pole-analysis and dominant-pole-computation. Due to this reason, we validate those results rigorously via discussions in frequency-domain (s-domain). Then, we characterize lower bounds of the dominant pole in terms of Frobenius eigenvalue of nonnegative matrices associated with TDPSs. This enables us to derive an explicit lower bound of the dominant pole as a function of delays. On the basis of the lower bound characterization, we finally show that the dominant pole of an asymptotically stable TDPS is affected by delays if and only if associated coefficient matrices satisfy an *eigenvalue-sensitivity* condition to be defined in this paper. Moreover, we clarify that the dominant pole goes to zero (from negative side) as time-delay goes to infinity if and only if the coefficient matrices are eigenvalue-sensitive. We thus come to the conclusion that convergence performance (decay rate) of asymptotically stable PSs generically degrades due to time-delays.

We use the following notations. We denote by $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the set of integers, real, and complex numbers, respectively. We also use $\mathbb{Z}_+$ ($\mathbb{Z}_{++}$), $\mathbb{R}_+$ ($\mathbb{R}_{++}$), $\mathbb{C}_{--}$ for the set of nonnegative (strictly positive) integers, nonnegative (strictly positive) real numbers, and complex numbers with strictly negative real parts. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i,j), where A_{ij} stands for the (i,j)-entry of A. In relation to this notation, we also define $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : \ x > 0\}$ and $\mathbb{R}^n_{+} := \{x \in \mathbb{R}^n : \ x \geq 0\}$. We also define $\mathbb{R}^{n\times m}_{++}$ and $\mathbb{R}^{n\times m}_{+}$ with obvious modifications. For $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ and $\rho(A)$ the set of the eigenvalues of A and the spectral radius of A, respectively. For $A \in \mathbb{R}^{n\times n}_{+}$, Theorem 8.3.1 in [12] states that there is an eigenvalue equal to $\rho(A)$. This eigenvalue is related to the Perron-Frobenius Theorem [12] and denoted by $\lambda_{\rm F}(A)$ in this paper. It also happens to be the maximal real part of all eigenvalues of $A \in \mathbb{R}^{n\times n}_{+}$, i.e., $\lambda_{\rm F}(A) = \rho(A) = \max\{\mathrm{Re}(\lambda) : \ \lambda \in \sigma(A)\}$. We finally define the set of n-vector-valued continuous functions over $[a,b]$ by $C([a,b],\mathbb{R}^n)$, and the set of nonnegative n-vector-valued continuous functions over $[a,b]$ by $C([a,b],\mathbb{R}^n_{+})$.

A conference version of this paper will be presented in [9]. In this paper we include detailed proofs for the technical lemmas and theorems in Sections 4 and 5. They are the core to validate the results around lower-bound evaluation and eigenvalue-sensitivity condition.

2 Preliminaries

In this brief section, we gather basic definitions and fundamental results for positive system (PS) analysis.

Definition 1 (Metzler Martrix) [10] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \ge 0$ $(i \ne j)$.

In the sequel, we denote by $\mathbb{M}^n$ ($\mathbb{H}^n$) the set of Metzler (Hurwitz stable) matrices of size n. Under these notations, the next lemma holds.

Lemma 1 [10, 13] For given $A \in \mathbb{M}^n$, the following conditions are equivalent.

(i) The matrix A is Hurwitz stable, i.e., $A \in \mathbb{H}^n$.
(ii) The matrix A is nonsingular and $A^{-1} \le 0$.

We note that the condition (ii) is used repeatedly (but implicitly) in the following discussions.

By extending the Perron-Frobenius theorem [12] originally given for nonnegative matrices, we can easily confirm that a matrix $A \in \mathbb{M}_n$ has an eigenvalue that is equal to $\max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$. With a little abuse of notation, we also denote this eigenvalue for $A \in \mathbb{M}^n$ by $\lambda_{\mathrm{F}}(A)$. The next lemma is an extension of Theorem 8.1.18 of [12] and shown also in [19].

Lemma 2 For given $M_1, M_2 \in \mathbb{M}^n$, suppose $M_1 \le M_2$. Then, we have $\lambda_{\mathrm{F}}(M_1) \le \lambda_{\mathrm{F}}(M_2)$.

To move on to the definition of finite-dimensional linear time-invariant (FDLTI) PSs, consider

$$G: \left\{ \begin{array}{rcrcr} \dot{x}(t) &=& Ax(t) &+& Bw(t), \\ z(t) &=& Cx(t) &+& Dw(t) \end{array} \right. \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_z\times n}$, and $D \in \mathbb{R}^{n_z\times n_w}$. The definition and a basic result of FDLTI PSs are given in the following.

Definition 2 (FDLTI Positive System) [10] FDLTI system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Proposition 1 [10] FDLTI system (1) is positive if and only if $A \in \mathbb{M}^n$, $B \in \mathbb{R}_+^{n\times n_w}$, $C \in \mathbb{R}_+^{n_z\times n}$, and $D \in \mathbb{R}_+^{n_z\times n_w}$.

3 Dominant Pole of Stable Time-Delay Positive Systems

3.1 Stability and Positivity of LTI Time-Delay Systems

For given $A_0 \in \mathbb{M}^n$, $A_i \in \mathbb{R}_+^{n\times n}$ and $h_i \in \mathbb{R}_{++}$ $(i = 1, \cdots, N)$ with $h := \max\limits_{i=1,\cdots,N} h_i$, define

$$
\Sigma : \begin{cases} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{N} A_i x(t-h_i) \ (t \geq 0), \\ x(\theta) &= \phi(\theta) \ (-h \leq \theta \leq 0) \end{cases} \tag{2}
$$

where $\phi \in C([-h, 0], \mathbb{R}^n)$ is the function of initial condition. We first state definitions and related results on positivity and stability of the (retarded) time-delay system (TDS) Σ given by (2).

Definition 3 [5, 2] The TDS (2) is said to be *positive* if for every $\phi \in C([-h, 0], \mathbb{R}_+^n)$ the solution x satisfies $x(t) \in \mathbb{R}_+^n$ for all $t \geq 0$.

Proposition 2 [5, 2] The TDS (2) is positive if and only if $A_0 \in \mathbb{M}^n$ and $A_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \cdots, N)$.

From this proposition and the underlying assumption we made on A_i $(i = 0, \cdots, N)$ in Σ, it is valid to say that Σ is a (LTI) time-delay positive system (TDPS).

Definition 4 The TDPS (2) is said to be *asymptotically stable* if for every $\phi \in C([-h, 0], \mathbb{R}^n)$ the solution x satisfies $\|x(t)\| \to 0 \quad (t \to \infty)$.

Proposition 3 [5, 2] The TDPS (2) is asymptotically stable if and only if $A := \sum_{i=0}^{N} A_i \in \mathbb{M}^n$ is Hurwitz stable.

Associated with the TDPS Σ, let us define an FDLTI PS by $\Sigma_0 : \ \dot{\xi}(t) = A\xi(t)$. Then, Proposition 3 says that TDPS Σ is stable if and only if the delay-free PS Σ_0 is stable.

For the TDPS Σ represented by (2), its associated characteristic equation is given by

$$
\det \left(\lambda I - A_0 - \sum_{i=1}^{N} A_i e^{-\lambda h_i} \right) = 0. \tag{3}
$$

We denote by $\Lambda_\Sigma \subset \mathbb{C}$ the set of solutions of (3) (i.e., the set of *poles* of TDPS Σ). Note that the set Λ_Σ is infinite in general [11]. From general result for LTI-TDSs [11], we see that Σ is asymptotically stable if and only if $\Lambda_\Sigma \subset \mathbb{C}_{--}$.

3.2 Dominant Pole Analysis

Our goal in this paper is to analyze the *dominant pole* of the TDPS Σ defined by

$$
\kappa(\Sigma) := \arg \max_{\lambda \in \Lambda_\Sigma} \mathrm{Re}(\lambda). \tag{4}
$$

Even though Λ_Σ is infinite in general, we can confirm that $\max_{\lambda \in \Lambda_\Sigma} \mathrm{Re}(\lambda)$ is achieved for some $\lambda \in \Lambda_\Sigma$ (and therefore we can write "max" rather than "sup"); see, for example, p. 18 (Lemma 4.1) of [11]. The background and motivation for the dominant pole analysis is as follows.

(i) For general LTI-TDSs (that are not necessarily positive), the exact computation of the dominant pole is hard and hence studies on the stability analysis of LTI-TDSs are still in progress. If we confine ourselves to PSs, however, it has been shown that we can obtain novel (strong) results that cannot hold true for general LTI systems [15, 16, 18, 4, 6, 7, 8]. We reveal that this is again true for the analysis of TDPSs and show that exact (or more precisely, arbitrarily high accuracy) computation of the dominant pole is indeed possible.

(ii) As we have seen in Proposition 3, TDPS Σ is stable if and only if delay-free PS Σ_0 is stable and this result holds true irrespective of $h_i \in \mathbb{R}_{++}$ $(i = 1, \cdots, N)$. This ensures the robust stability of TDPSs against variations on the time-delays. However, this qualitative result does not allow us to deduce how the performance (such as decay rate) deteriorates due to the time-delays. In this paper, we prove that the dominant pole $\kappa(\Sigma)$ is always real and $\kappa(\Sigma) \to 0$ as $h_i \to \infty$ holds (under the mild condition of *eigenvalue-sensitivity* on the pair (A_0, A_i) discussed in the following). It follows that the convergence performance generically deteriorates as the time-delays increase.

The next theorem gives a characterization of the dominant pole of TDPSs. We note that this theorem readily follows from [19]. The relationship with the result in [19] will be given after the theorem.

Theorem 1 Consider the TDPS Σ described by (2) and assume Σ is asymptotically stable, i.e.,

$$A = \sum_{i=0}^{N} A_i \in \mathbb{M}^n \cap \mathbb{H}^n. \tag{5}$$

Then, the dominant pole $\kappa(\Sigma)$ defined by (4) is given by

$$\kappa(\Sigma) = -\alpha^\star < 0 \tag{6}$$

where $\alpha^\star \in \mathbb{R}_{++}$ is a unique solution of

$$\lambda_{\mathrm{F}}\left(\alpha^\star I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha^\star h_i}\right) = 0. \tag{7}$$

Moreover, we have

$$\kappa(\Sigma) \geq \kappa(\Sigma_0) := \lambda_{\mathrm{F}}(A) \tag{8}$$

irrespective of $h_i \in \mathbb{R}_{++}$ $(i = 1, \cdots, N)$.

The implications of this theorem are as follows.

(i) For the asymptotically stable TDPS Σ, its dominant pole $\kappa(\Sigma)$ is real, and its computation can be done by solving (7). In the latter part of this section, we build an efficient bisection search algorithm so that we can solve (7) as accurately as desired.

(ii) The relation (8) reveals that the dominant pole of TDPS Σ is larger than or equal to that of delay-free PS Σ_0. Since dominant pole is a reasonable measure for the rate of convergence, we could say that convergence performance of asymptotically stable PSs deteriorates due to time-delays. More quantitative analysis on the deterioration of the convergence performance (i.e., how fast the dominant pole increases due to the increase of time-delays) is given in the next section.

We now discuss the relationship with the result in [19]. It is shown in [19] that if TDPS Σ is α-exponentially stable (see [19] for its definition), then $\lambda_{\rm F}\left(\alpha I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha h_i}\right) \leq 0$. On the other hand, if $\lambda_{\rm F}\left(\alpha I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha h_i}\right) < 0$, then TDPS Σ is α-exponentially stable. From these results, it is almost obvious that (6) and (7) hold. Still, since the discussion in [19] has been done in time-domain, it does not fully fit to the dominant-pole-analysis and dominant-pole-computation. Due to this reason, we will give a rigorous proof for Theorem 1 in frequency-domain (s-domain), and construct an explicit computation method of $\kappa(\Sigma)$ based on the proof.

Proof of Theorem 1: We first prove that $\alpha^\star \in \mathbb{R}_{++}$ satisfying (7) is unique. To this end, define

$$\mathcal{A}(\alpha) := \alpha I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha h_i}. \tag{9}$$

Then, since $\mathcal{A}(\alpha) \in \mathbb{M}^n$ holds irrespective of $\alpha \in \mathbb{R}_+$, we can define $\lambda_{\rm F}(\mathcal{A}(\alpha))$ over $\alpha \in \mathbb{R}_+$. It is obvious that $\lambda_{\rm F}(\mathcal{A}(\alpha))$ is strictly monotonically increasing over $\alpha \in \mathbb{R}_+$ because for $0 \leq \alpha_1 < \alpha_2$ we have

$$\begin{array}{rcl}
\lambda_{\rm F}(\mathcal{A}(\alpha_1)) & = & \lambda_{\rm F}(\alpha_1 I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha_1 h_i}) \\
& = & \lambda_{\rm F}(A_0 + \sum_{i=1}^{N} A_i e^{\alpha_1 h_i}) + \alpha_1 \\
& \leq & \lambda_{\rm F}(A_0 + \sum_{i=1}^{N} A_i e^{\alpha_2 h_i}) + \alpha_1 \\
& < & \lambda_{\rm F}(A_0 + \sum_{i=1}^{N} A_i e^{\alpha_2 h_i}) + \alpha_2 \\
& = & \lambda_{\rm F}(\alpha_2 I + A_0 + \sum_{i=1}^{N} A_i e^{\alpha_2 h_i}) \\
& = & \lambda_{\rm F}(\mathcal{A}(\alpha_2))
\end{array}$$

where we have used Lemma 2 to validate the third nonstrict inequality. From this strictly monotonically increasing property, the continuity of $\lambda_{\rm F}(\mathcal{A}(\alpha))$ with respect to $\alpha \in \mathbb{R}_+$, $\lambda_{\rm F}(\mathcal{A}(0)) = \lambda_{\rm F}(A) < 0$ that follows from $A \in \mathbb{M}^n \cap \mathbb{H}^n$, and $\lambda_{\rm F}(\mathcal{A}(\alpha)) \to \infty$ $(\alpha \to \infty)$, it is clear that there exists a unique $\alpha^\star \in \mathbb{R}_{++}$ such that $\lambda_{\rm F}(\mathcal{A}(\alpha^\star)) = 0$.

We next move on to the proof of (6). In view of the characteristic equation (3), it is obvious that $-\alpha^\star \in \Lambda_\Sigma$ since

$$\det\left(-\alpha^\star I - A_0 - \sum_{i=1}^{N} A_i e^{\alpha^\star h_i}\right) = 0.$$

To prove $\kappa(\Sigma) = -\alpha^\star$ by contradiction, suppose there exist $\alpha \in \mathbb{R}_{++}$ and $\omega \in \mathbb{R}$ such that

$$-\alpha + j\omega \in \Lambda_\Sigma,\ \ \alpha < \alpha^\star.$$

Then, from $-\alpha + j\omega \in \Lambda_\Sigma$, we have

$$\det\left((-\alpha + j\omega)I - A_0 - \sum_{i=1}^{N} A_i e^{-(-\alpha+j\omega)h_i}\right) = 0.$$

This can be rewritten equivalently as

$$\det\left(j\omega I - A_{0,\alpha} - \sum_{i=1}^{N} A_{i,\alpha} e^{-j\omega h_i}\right) = 0 \tag{10}$$

where $A_{0,\alpha} := \alpha I + A_0$, $A_{i,\alpha} := A_i e^{\alpha h_i}$ $(i = 1, \cdots, N)$. From $\alpha < \alpha^\star$ and the strictly monotonically increasing property of $\lambda_{\mathrm{F}}(\mathcal{A}(\alpha))$ with respect to $\alpha \in \mathbb{R}_+$, we have

$$\lambda_{\mathrm{F}}\left(A_{0,\alpha} + \sum_{i=1}^{N} A_{i,\alpha}\right) = \lambda_{\mathrm{F}}(\mathcal{A}(\alpha)) < \lambda_{\mathrm{F}}(\mathcal{A}(\alpha^\star)) = 0.$$

Therefore, from Proposition 3, the TDPS Σ_α described by

$$\Sigma_\alpha :\ \ \dot{x}(t) = A_{0,\alpha} x(t) + \sum_{i=1}^{N} A_{i,\alpha} x(t - h_i)\ (t \geq 0)$$

is asymptotically stable and hence its associated characteristic equation described by

$$\det\left(\lambda I - A_{0,\alpha} - \sum_{i=1}^{N} A_{i,\alpha} e^{-\lambda h_i}\right) = 0$$

has no solution on $j\mathbb{R}$. This contradicts (10).

The proof for (8) directly follows from the (strictly) monotonically increasing property of $\lambda_{\mathrm{F}}(\mathcal{A}(\alpha))$ if we note

$$\begin{aligned}
&\lambda_{\mathrm{F}}\left(\mathcal{A}(-\lambda_{\mathrm{F}}(A))\right) \\
&= \lambda_{\mathrm{F}}\left(-\lambda_{\mathrm{F}}(A)I + A_0 + \sum_{i=1}^{N} A_i e^{-\lambda_{\mathrm{F}}(A)h_i}\right) \\
&\geq \lambda_{\mathrm{F}}\left(-\lambda_{\mathrm{F}}(A)I + A_0 + \sum_{i=1}^{N} A_i\right) \\
&= \lambda_{\mathrm{F}}\left(-\lambda_{\mathrm{F}}(A)I + A\right) \\
&= 0 \\
&= \lambda_{\mathrm{F}}(\mathcal{A}(\alpha^\star)).
\end{aligned}$$

This clearly shows that $-\lambda_{\rm F}(A) \geq \alpha^\star$. Therefore $\kappa(\Sigma_0) = \lambda_{\rm F}(A) \leq -\alpha^\star = \kappa(\Sigma)$. This completes the proof. ∎

Seemingly, for the computation of $\alpha^\star \in \mathbb{R}_{++}$ satisfying (7), we need to solve a transcendental equation. However, from the proof of Theorem 1, we see that $\alpha^\star \in \mathbb{R}_{++}$ can be characterized by $\lambda_{\rm F}(\mathcal{A}(\alpha^\star)) = 0$, i.e., the unique value that renders $\mathcal{A}(\alpha) \in \mathbb{M}^n$ on the stability boundary. In view of this fact, we can build an efficient bisection search algorithm for the computation of $\kappa(\Sigma) = -\alpha^\star$.

Bisection Search Algorithm for the Computation of $\kappa(\Sigma)$

Step 0: Define $\overline{\alpha} = -\lambda_{\rm F}(A) > 0$ and $\underline{\alpha} = 0$ and choose $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.
Step 1: Compute $\lambda_{\rm F}(A(\alpha))$. If $\lambda_{\rm F}(A(\alpha)) > 0$, then let $\overline{\alpha} := \alpha$ and $\alpha := (\alpha + \underline{\alpha})/2$. Else if $\lambda_{\rm F}(A(\alpha)) \leq 0$, then let $\underline{\alpha} := \alpha$ and $\alpha := (\alpha + \overline{\alpha})/2$. Go to Step 2.
Step 2: If $\overline{\alpha} - \underline{\alpha} < \varepsilon$, then let $\kappa(\Sigma) = -\underline{\alpha}$ and exit. Else, go to Step 1.

By specifying $\varepsilon > 0$ sufficiently small in this algorithm, we can compute $\alpha^\star$ as accurately as desired. The value resulting from the bisection is ensured to be an upper bound of $\kappa(\Sigma)$ since we exit by specifying $\kappa(\Sigma) = -\underline{\alpha}$ in Step 2.

3.3 Monotonicity of $\kappa(\Sigma)$

Consider asymptotically stable TDPSs $\Sigma^{[1]}$ and $\Sigma^{[2]}$ described by (2) with coefficient matrices and time-delays given by

$$\begin{array}{l} \Sigma^{[1]}: \ A_i = A_i^{[1]} \ (i = 0, \cdots, N), \ h_i = h_i^{[1]} \ (i = 1, \cdots, N) \\ \Sigma^{[2]}: \ A_i = A_i^{[2]} \ (i = 0, \cdots, N), \ h_i = h_i^{[2]} \ (i = 1, \cdots, N). \end{array} \tag{11}$$

For $\alpha \in \mathbb{R}_+$, let

$$\mathcal{A}_{\Sigma^{[1]}}(\alpha) = \alpha I + A_0^{[1]} + \sum_{i=1}^{N} A_i^{[1]} e^{\alpha h_i^{[1]}} \in \mathbb{M}^n,$$
$$\mathcal{A}_{\Sigma^{[2]}}(\alpha) = \alpha I + A_0^{[2]} + \sum_{i=1}^{N} A_i^{[2]} e^{\alpha h_i^{[2]}} \in \mathbb{M}^n.$$

Then, if $A_i^{[1]} \leq A_i^{[2]}$ $(i = 0, \cdots, N)$ and $h_i^{[1]} \leq h_i^{[2]}$ $(i = 1, \cdots, N)$, it is obvious that $\mathcal{A}_{\Sigma^{[1]}}(\alpha) \leq \mathcal{A}_{\Sigma^{[2]}}(\alpha)$ and hence $\lambda_{\rm F}(\mathcal{A}_{\Sigma^{[1]}}(\alpha)) \leq \lambda_{\rm F}(\mathcal{A}_{\Sigma^{[2]}}(\alpha))$ $(\forall \alpha \in \mathbb{R}_+)$. Since the dominant pole of TDPS Σ is given by $\kappa(\Sigma) = -\alpha^\star$ and $\alpha^\star$ is characterized by $\lambda_{\rm F}(\mathcal{A}(\alpha^\star)) = 0$, and since $\lambda_{\rm F}(\mathcal{A}(\alpha))$ is (strictly) monotonically increasing over $\alpha \in \mathbb{R}_+$, we can readily obtain the next result.

Theorem 2 Consider asymptotically stable TDPSs $\Sigma^{[1]}$ and $\Sigma^{[2]}$ described by (2) with coefficient matrices and time-delays given by (11). Suppose $A_i^{[1]} \leq A_i^{[2]}$ $(i = 0, \cdots, N)$ and $h_i^{[1]} \leq h_i^{[2]}$ $(i = 1, \cdots, N)$ hold. Then, we have

$$\kappa(\Sigma^{[1]}) \leq \kappa(\Sigma^{[2]}). \tag{12}$$

This theorem implies that the dominant pole of TDPS Σ is monotonically non-decreasing with respect to the increase of A_i $(i = 0, \cdots, N)$ and h_i $(i = 1, \cdots, N)$.

3.4 Numerical Examples

Consider the case where $N = 1$ in (2). As for the matrices A_0 and A_1, we consider the following two cases:
Case I

$$A_0 = \begin{bmatrix} -3.5 & 0.2 & 0.9 \\ 0.3 & -3.9 & 0.5 \\ 0.5 & 0.5 & -3.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.8 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}$$

Case II

$$A_0 = \begin{bmatrix} -3.9 & 0.1 & 0.0 \\ 0.8 & -3.9 & 0.8 \\ 0.1 & 0.1 & -4.0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.7 & 0.0 & 0.7 \\ 0.2 & 0.8 & 0.9 \end{bmatrix}$$

We denote by $\Sigma_{\rm I}$ and $\Sigma_{\rm II}$ the corresponding TDPSs.

By letting $\varepsilon = 10^{-8}$, we carried out the bisection search algorithm for each $h_1 \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ and obtained the results shown in Table 1. As we have shown in Theorem 2, we see from Table 1 that $\kappa(\Sigma_{\rm I})$ and $\kappa(\Sigma_{\rm II})$ monotonically increase as h_1 increases. In this example, the dominant poles of delay-free PSs $\Sigma_{\rm I,0}$ and $\Sigma_{\rm II,0}$ are $\kappa(\Sigma_{\rm I,0}) = -1.7731$ and $\kappa(\Sigma_{\rm II,0}) = -2.0378$, respectively, and hence $\kappa(\Sigma_{\rm I,0}) > \kappa(\Sigma_{\rm II,0})$. However, the dominant poles of TDPSs denoted by $\kappa(\Sigma_{\rm I})$ and $\kappa(\Sigma_{\rm II})$ happen to coincide numerically at $h_1 = 0.2$ and become $\kappa(\Sigma_{\rm I}) < \kappa(\Sigma_{\rm II})$ for $h_1 > 0.2$. This result leads to the natural consequence that the convergence performance of a TDPS is not necessarily relevant to the corresponding delay-free PS.

Table 1: Computation Results of $\kappa(\Sigma)$ for Cases I and II.

h_1	$\kappa(\Sigma_{\rm I})$	$\kappa(\Sigma_{\rm II})$
0	-1.7731	-2.0378
0.2	-1.5162	-1.5162
0.4	-1.2860	-1.1646
0.6	-1.0983	-0.9333
0.8	-0.9502	-0.7748
1.0	-0.8334	-0.6608

4 Explicit Lower Bounds for $\kappa(\Sigma)$

In the preceding section, we showed an efficient bisection search algorithm for the computation of the dominant pole of asymptotically stable TDPSs. In spite of this achievement, in this section, we analyze lower bounds of the dominant pole. The lower bound analysis is well motivated due to the following reasons.

(i) The bisection algorithm and the underlying result, Theorem 1, do not allow us to see clearly the relationship between $\kappa(\Sigma)$ and the time-delays h_i $(i = 1, \cdots, N)$. Intuitively, it is expected that $\kappa(\Sigma)$ increases (and hence convergence performance deteriorates) as h_i $(i = 1, \cdots, N)$ increase and this is indeed proved in Theorem 2. Still, we have not obtained any definite quantitative results on this issue (i.e., how fast $\kappa(\Sigma)$ increases as h_i $(i = 1, \cdots, N)$ increase).

(ii) If we can obtain good lower bounds by relatively cheap computation, we can use them in place of $\lambda_{\mathrm{F}}(A)$ in Step 0 of the bisection algorithm so that we can accelerate its convergence.

In this section, we characterize lower bounds of $\kappa(\Sigma)$ by means of Frobenius eigenvalues of nonnegative matrices associated with TDPS Σ. This enables us to derive an explicit lower bound of $\kappa(\Sigma)$ as a function of h_i $(i = 1, \cdots, N)$. The lower bound computation is also motivated from theoretical interest on how the treatment by Taylor-series expansion and finite-degree truncation of exponential functions work fine in dealing with TDPSs.

4.1 Lower Bound Analysis of $\kappa(\Sigma)$

Let us revisit $\mathcal{A}(\alpha)$ defined by (9). From the strictly monotonically increasing property of $\lambda_{\mathrm{F}}(\mathcal{A}(\alpha))$ with respect to $\alpha \in \mathbb{R}_+$, the following alternative representation of $\alpha^\star$ can be obtained:

$$\alpha^\star := \min \boldsymbol{\alpha}, \quad \boldsymbol{\alpha} := \{\alpha \in \mathbb{R}_+ : \ \det(\mathcal{A}(\alpha)) = 0\}. \tag{13}$$

For the lower bound evaluation of the dominant pole given by $\kappa(\Sigma) = -\alpha^\star$, we consider upper bounds of $\alpha^\star$. To this end, we apply Taylor-series expansion of degree $K \in \mathbb{Z}_+$ to exponential functions $e^{\alpha h_i}$ $(i = 1, \cdots, N)$ in $\mathcal{A}(\alpha)$ and define

$$\mathcal{A}^{[K]}(\alpha) \ := \ \alpha I + A_0 + \sum_{i=1}^{N} A_i \sum_{j=0}^{K} \frac{(\alpha h_i)^j}{j!}.$$

Note that $\mathcal{A}^{[0]}(\alpha) = \alpha I + A$ and for $K \in \mathbb{Z}_{++}$ we have

$$\mathcal{A}^{[K]}(\alpha) = \sum_{j=0}^{K} \mathcal{A}_j \alpha^j$$

where

$$\mathcal{A}_0 = A \in \mathbb{M}^n \cap \mathbb{H}^n, \ \mathcal{A}_1 = I + \sum_{i=1}^{N} h_i A_i \in \mathbb{R}_+^{n\times n}, \ \mathcal{A}_j = \sum_{i=1}^{N} \frac{h_i^j}{j!} A_i \in \mathbb{R}_+^{n\times n} \ (j = 2, \cdots, K).$$

Obviously $\mathcal{A}^{[K]}(\alpha) \in \mathbb{M}^n$ holds irrespective of $K \in \mathbb{Z}_+$ and $\alpha \in \mathbb{R}_+$. Moreover, for each $K \in \mathbb{Z}_+$, it is easy to confirm that $\lambda_{\mathrm{F}}(\mathcal{A}^{[K]}(0)) = \lambda_{\mathrm{F}}(A) < 0$ and $\lambda_{\mathrm{F}}(\mathcal{A}^{[K]}(\alpha))$ is strictly monotonically increasing with respect to $\alpha \in \mathbb{R}_+$. It is also true that $\lambda_{\mathrm{F}}(\mathcal{A}^{[K]}(\alpha)) \to \infty$ as $\alpha \to \infty$. Therefore, we can uniquely define $\alpha_K^\star \in \mathbb{R}_{++}$ by

$$\lambda_{\mathrm{F}}(\mathcal{A}^{[K]}(\alpha_K^\star)) = 0. \tag{14}$$

Again, $\alpha_K^\star$ has an alternative representation of the form

$$\alpha_K^\star := \min \boldsymbol{\alpha}_K, \quad \boldsymbol{\alpha}_K := \left\{\alpha \in \mathbb{R}_+ : \ \det(\mathcal{A}^{[K]}(\alpha)) = 0\right\}. \tag{15}$$

Since $\lambda_{\rm F}(\mathcal{A}^{[K_1]}(\alpha)) \leq \lambda_{\rm F}(\mathcal{A}^{[K_2]}(\alpha))$ holds for all $\alpha \in \mathbb{R}_+$ if $K_1 \leq K_2$ and $\lambda_{\rm F}(\mathcal{A}^{[K]}(\alpha)) \leq \lambda_{\rm F}(\mathcal{A}(\alpha))$ holds for all $\alpha \in \mathbb{R}_+$ and $K \in \mathbb{Z}_+$, we see that

$$\alpha_{K_1}^\star \geq \alpha_{K_2}^\star \ (K_1 \leq K_2), \quad \alpha_K^\star \geq \alpha^\star \ (\forall K \in \mathbb{Z}_+). \tag{16}$$

Moreover, it is obvious that $\alpha_K^\star \to \alpha^\star$ $(K \to \infty)$. It follows that $\{\alpha_K^\star\}$ $(K \in \mathbb{Z}_+)$ is a sequence of monotonically non-increasing upper bounds that converges to $\alpha^\star$. This can be restated equivalently that $\{-\alpha_K^\star\}$ $(K \in \mathbb{Z}_+)$ is a sequence of monotonically non-decreasing lower bounds that converges to $\kappa(\Sigma)(= -\alpha^\star)$. In the following theorem, we show that $\alpha_K^\star$ $(K \in \mathbb{Z}_{++})$ can be characterized by the Frobenius eigenvalue of a nonnegative matrix of size $Kn \times Kn$.

Theorem 3 Consider the TDPS Σ described by (2) and assume Σ is asymptotically stable, i.e., (5) holds. Define $\alpha_K^\star$ $(K \in \mathbb{Z}_+)$ by (14). Then, we have

$$\alpha_0^\star = -\lambda_{\rm F}(A) \tag{17}$$

and

$$\alpha_K^\star = \left(\lambda_{\rm F}\left(\mathcal{A}_{\rm aug}^{[K]}\right)\right)^{-1} \ (K \in \mathbb{Z}_{++}) \tag{18}$$

where

$$\mathcal{A}_{\rm aug}^{[K]} = \begin{bmatrix} -\mathcal{A}_0^{-1}\mathcal{A}_1 & -\mathcal{A}_0^{-1}\mathcal{A}_2 & \cdots & \cdots & \cdots & -\mathcal{A}_0^{-1}\mathcal{A}_K \\ I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & I & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I & 0 \end{bmatrix}. \tag{19}$$

Proof of Theorem 3: It is obvious that (17) holds true and hence we prove (18). To this end, we first note that $\mathcal{A}^{[K]}(\alpha)$ can be rewritten equivalently as

$$\mathcal{A}^{[K]}(\alpha) = \mathcal{A}_0 + \mathcal{B}\mathcal{D}(\alpha)^{-1}\mathcal{C}(\alpha), \quad \mathcal{B} := \begin{bmatrix} \mathcal{A}_1 & \cdots & \cdots & \cdots & \mathcal{A}_K \end{bmatrix},$$
$$\mathcal{C}(\alpha) := \begin{bmatrix} \alpha I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \mathcal{D}(\alpha) := \begin{bmatrix} I & 0 & \cdots & \cdots & 0 \\ -\alpha I & I & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & I & 0 \\ 0 & \cdots & 0 & -\alpha I & I \end{bmatrix}.$$

Therefore we can obtain the following sequence of equivalent reformulations:

$$
\begin{aligned}
& \det\left(\mathcal{A}^{[K]}(\alpha)\right)=0 \\
\Leftrightarrow\ & \det\left(\begin{bmatrix} \mathcal{A}_0 & \mathcal{B} \\ -\mathcal{C}(\alpha) & \mathcal{D}(\alpha) \end{bmatrix}\right)=0 \\
\Leftrightarrow\ & \det\left(I-\alpha \begin{bmatrix} -\mathcal{A}_0^{-1}\mathcal{A}_1 & -\mathcal{A}_0^{-1}\mathcal{A}_2 & \cdots & \cdots & \cdots & -\mathcal{A}_0^{-1}\mathcal{A}_K \\ I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & I & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I & 0 \end{bmatrix} \right)=0 \\
\Leftrightarrow\ & \det\left(I-\alpha\mathcal{A}_{\rm aug}^{[K]}\right)=0.
\end{aligned}
$$

From $\mathcal{A}_{\rm aug}^{[K]}\in\mathbb{R}_+^{Kn\times Kn}$ and the representation of $\alpha_K^\star$ given by (15), we can easily confirm that $\alpha_K^\star$ is given by (18). It should be noted that $\lambda_{\rm F}(\mathcal{A}_{\rm aug}^{[K]})>0$ ($\forall K\in\mathbb{Z}_{++}$) since we have from Lemma 2 that $\lambda_{\rm F}(\mathcal{A}_{\rm aug}^{[K]})\geq\lambda_{\rm F}(\mathcal{A}_{\rm aug}^{[1]})$ ($\forall K\in\mathbb{Z}_{++}$) and again from Lemma 2 that

$$
\begin{aligned}
\lambda_{\rm F}\left(\mathcal{A}_{\rm aug}^{[1]}\right) &= \lambda_{\rm F}\left(-\mathcal{A}_0^{-1}\mathcal{A}_1\right) \\
&= \lambda_{\rm F}\left(-\mathcal{A}_0^{-1}\left(I+\sum_{i=1}^N h_iA_i\right)\right) \\
&\geq \lambda_{\rm F}\left(-\mathcal{A}_0^{-1}\right) \\
&= \lambda_{\rm F}\left(-A^{-1}\right) \\
&= -\lambda_{\rm F}\left(A\right)^{-1} \\
&> 0
\end{aligned}
$$

holds. This completes the proof. ■

As a byproduct of Theorem 3, we can represent a lower bound of $\kappa(\Sigma)$ as a function of $h_i\in\mathbb{R}_{++}$ $(i=1,\cdots,N)$. Just for simplicity consider the case where $N=1$. Then, *under the assumption that* $\lambda_{\rm F}(-A^{-1}A_1)>0$ *that becomes a very important issue in the sequel*, we see that $\alpha_1^\star$ given by (18) satisfies

$$
\begin{aligned}
\alpha_1^\star &= \lambda_{\rm F}\left(-\mathcal{A}_0^{-1}\mathcal{A}_1\right)^{-1} \\
&= \lambda_{\rm F}\left(-A^{-1}(I+h_1A_1)\right)^{-1} \\
&\leq \lambda_{\rm F}\left(-h_1A^{-1}A_1\right)^{-1} \\
&= \frac{1}{h_1}\lambda_{\rm F}\left(-A^{-1}A_1\right)^{-1}.
\end{aligned}
$$

It follows that $\kappa(\Sigma)\geq-\alpha_1^\star\geq-\frac{1}{h_1}\lambda_{\rm F}\left(-A^{-1}A_1\right)^{-1}$. We thus obtain the next theorem.

Theorem 4 Consider the TDPS Σ described by (2) and assume Σ is asymptotically stable, i.e., (5) holds. We further assume that for a subset $J\subset\{1,\cdots,N\}$ the condition $\lambda_{\rm F}(-A^{-1}A_j)>0$ $(j\in J)$ holds and define

$$
\underline{\kappa}_j(h_j)=-\frac{1}{h_j}\lambda_{\rm F}\left(-A^{-1}A_j\right)^{-1}.
$$

Then we have $\kappa(\Sigma) \geq \underline{\kappa}_j(h_j)$ $(\forall j \in J)$. In particular, $\kappa(\Sigma) \to 0$ as $h_j \to \infty$ for each $j \in J$.

This theorem shows that $\kappa(\Sigma)$ goes to 0 faster (no later) than the order of $1/h_j$ provided that $\lambda_{\mathrm{F}}(-A^{-1}A_j) > 0$.

4.2 Numerical Examples

Consider the asymptotically stable TDPS Σ_{I} dealt with in Subsection 3.4. For this TDPS, we first computed $-\alpha_K^\star$ $(K = 1, 3, 5)$, the lower bounds of $\kappa(\Sigma_{\mathrm{I}})$, by means of Theorem 3. The results are shown in Table 2. In this example, the convergence of the sequence $\{\alpha_K\}$ is rather fast and we see that the lower bound $-\alpha_5^\star$ almost coincides with $\kappa(\Sigma_{\mathrm{I}})$ in all tested cases.

In principal, we can accelerate the convergence of the bisection algorithm for the computation of $\kappa(\Sigma_{\mathrm{I}})$ by letting $\overline{\alpha} = \alpha_5^\star$ in Step 0. However, the effect was insignificant in this (rather small size) example since the bisection algorithm with $\overline{\alpha} = -\lambda_{\mathrm{F}}(A)$ in Step 0 already runs very fast.

Table 2: Results for Lower Bounds Computation of $\kappa(\Sigma_{\mathrm{I}})$.

h_1	$-\alpha_1^\star$	$-\alpha_3^\star$	$-\alpha_5^\star$	$\kappa(\Sigma_{\mathrm{I}})$
0.2	-1.5484	-1.5164	-1.5162	-1.5162
0.4	-1.3747	-1.2876	-1.2860	-1.2860
0.6	-1.2361	-1.1019	-1.0984	-1.0983
0.8	-1.1231	-0.9556	-0.9503	-0.9502
1.0	-1.0290	-0.8401	-0.8335	-0.8334

In this numerical example, $\lambda_{\mathrm{F}}(-A^{-1}A_1) \approx 0.4057 > 0$ and hence we can apply the results in Theorem 4. In Fig. 1, we show the plots of $\kappa(\Sigma)$ and $\underline{\kappa}_1(h_1)$ for $h_1 \in [1\ 20]$. As we have shown in Theorem 4, we can confirm that $\kappa(\Sigma)$ goes to 0 faster than $\underline{\kappa}_1(h_1)$ as h_1 increases.

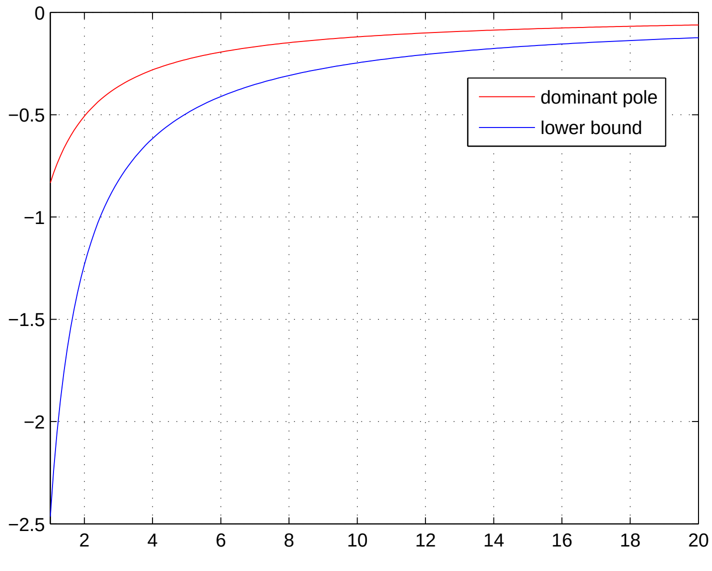

Figure 1: Plots of $\kappa(\Sigma)$ for Cases I and II.

Now we return to the assumption we made in Theorem 4. Again, just for simplicity, consider the case $N = 1$. Theorem 4 says that, if $\lambda_{\rm F}(-A^{-1}A_1) = \lambda_{\rm F}(-(A_0 + A_1)^{-1}A_1) > 0$, then Λ_Σ is sensitive to time-delay h_1 and in particular $\kappa(\Sigma) \to 0$ $(h_1 \to \infty)$ holds. This result asks a simple question: if $\lambda_{\rm F}(-(A_0 + A_1)^{-1}A_1) = 0$, what will happen?

In relation to this question, consider the case where

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

In this case, the condition $\lambda_{\rm F}(-(A_0 + A_1)^{-1}A_1) = 0$ surely holds. On the other hand, the characteristic equation (3) associated with the corresponding TDPS is nothing but the algebraic equation given by $(\lambda+1)(\lambda+2) = 0$ and hence $\Lambda_\Sigma = \{-1, -2\} = \sigma(A_0)$ irrespective of $h_1 \in \mathbb{R}_{++}$. Namely, Λ_Σ is insensitive to $h_1 \in \mathbb{R}_{++}$.

These observations motivate us to reveal the relationship between the insensitivity of Λ_Σ to $h_1 \in \mathbb{R}_{++}$ and the Frobenius eigenvalue $\lambda_{\rm F}(-(A_0 + A_1)^{-1}A_1)$. This topic is pursued in the next section, where we clarify that Λ_Σ is insensitive to $h_1 \in \mathbb{R}_{++}$ *if and only if* $\lambda_{\rm F}(-A_0^{-1}A_1) = 0$.

5 Pole-Sensitivity on Time-Delay

For given two matrices $M_0, M_1 \in \mathbb{R}^{n\times n}$, we first make the definition of *eigenvalue-sensitivity (insensitivity)* of M_0 relative to M_1.

Definition 5 For given $M_0, M_1 \in \mathbb{R}^{n\times n}$, M_0 is said to be *eigenvalue-insensitive relative to* M_1 if

$$\sigma(M_0 + \nu M_1) = \sigma(M_0) \quad \forall \nu \in \mathbb{C}. \tag{21}$$

Definition 6 For given $M_0, M_1 \in \mathbb{R}^{n\times n}$, M_0 is said to be *eigenvalue-sensitive relative to* M_1 if M_0 is not eigenvalue-insensitive relative to M_1.

These definitions play a crucial role to describe clearly under what condition $\Lambda_\Sigma = \Lambda_{\Sigma_0} = \sigma(A_0)$ holds irrespective of time-delays. On the other hand, the next lemma concerns the condition under which $\lambda_{\rm F}(M) = 0$ holds for $M \in \mathbb{R}_+^{n\times n}$.

Lemma 3 For given $M \in \mathbb{R}_+^{n\times n}$, the following conditions are equivalent:

(i) $\lambda_{\rm F}(M) = 0$.

(ii) There exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that $P^T M P \in \mathbb{U}_+$ where $\mathbb{U}_+$ is a set of nonnegative and strictly upper triangular matrices defined by
$\mathbb{U}_+^n := \left\{ U \in \mathbb{R}_+^{n\times n} : \ U_{i,j} = 0 \ (i \geq j) \right\}.$

Proof of Lemma 3: (ii) $\Rightarrow$ (i) is obvious and hence we prove (i) $\Rightarrow$ (ii). Since $\lambda_{\rm F}(M) = 0$, there exists $v \in \mathbb{R}_+^n \setminus \{0\}$ such that $Mv = 0$ (see Theorem 8.3.1 of [12]). Since $v \in \mathbb{R}_+^n \setminus \{0\}$, there exists a permutation matrix $P_1 \in \mathbb{R}^{n\times n}$ such that $(P_1^T v)_1 > 0$. Moreover, we have

$$Mv = 0 \;\Leftrightarrow\; (P_1^T M P_1) P_1^T v = 0 \Rightarrow P_1^T M P_1 = \begin{bmatrix} 0 & M_{11} \\ 0_{n-1} & M_{12} \end{bmatrix}$$

where $M_{12} \in \mathbb{R}_+^{(n-1)\times(n-1)}$ and $\lambda_{\mathrm{F}}(M_{12}) = 0$. Repeating the same procedure for $M_{12} \in \mathbb{R}_+^{(n-1)\times(n-1)}$, we see that there exists a permutation matrix $P_{2,s} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$P_{2,s}^T M_{12} P_{2,s} = \begin{bmatrix} 0 & M_{21} \\ 0_{n-2} & M_{22} \end{bmatrix}$$

where $M_{22} \in \mathbb{R}_+^{(n-2)\times(n-2)}$ and $\lambda_{\mathrm{F}}(M_{22}) = 0$. This implies that

$$P_2^T P_1^T M P_1 P_2 = \begin{bmatrix} 0 & M_{11,1} & M_{11,2} \\ 0 & 0 & M_{21} \\ 0_{n-2} & 0_{n-2} & M_{22} \end{bmatrix},$$

$$P_2 := \begin{bmatrix} 1 & 0 \\ 0 & P_{2,s} \end{bmatrix}.$$

Repeating the same procedure for the rest up to $n-3$ times, we arrive at the desired conclusion that there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that $P^T M P \in \mathbb{U}_+$. ■

By means of Lemma 3, we can obtain the next lemma saying that $A_0 \in \mathbb{M}^n \cap \mathbb{H}^n$ is eigenvalue-insensitive relative to $A_1 \in \mathbb{R}_+^{n\times n}$ if and only if $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) = 0$.

Lemma 4 For given $A_0 \in \mathbb{M}^n \cap \mathbb{H}^n$ and $A_1 \in \mathbb{R}_+^{n\times n}$, the following conditions are equivalent:

(i) A_0 is eigenvalue-insensitive relative to A_1.
(ii) $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) = 0$.

This lemma gives a concrete way to check the eigenvalue-insensitivity of A_0 relative to A_1, which is by no means obvious from Definition 5. The proof of this lemma is rather involved and hence given in the appendix.

Now we are ready to state the main result of this section.

Theorem 5 Consider the TDPS described by (2) where $N = 1$ and assume Σ is asymptotically stable, i.e., (5) holds. Then, the following conditions are equivalent:

(i) $\Lambda_\Sigma = \sigma(A_0),\ \forall h_1 \in \mathbb{R}_+$.
(ii) A_0 is eigenvalue-insensitive relative to A_1.
(iii) $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) = 0$.

Proof of Theorem 5: We have shown in Lemma 4 that (ii) $\Leftrightarrow$ (iii) holds. Therefore, to complete the proof, it suffices to show that (i) $\Rightarrow$ (iii) and (i) $\Leftarrow$ (ii).
<u>(i) $\Rightarrow$ (iii):</u> By contradiction, suppose $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) > 0$. Since $A = A_0 + A_1 \geq A_0$ and $A, A_0 \in \mathbb{M}^n \cap \mathbb{H}^n$, we can easily confirm from (ii) of Lemma 1 that $-A^{-1} \geq -A_0^{-1} \geq 0$ and hence $-A^{-1}A_1 \geq -A_0^{-1}A_1 \geq 0$. It follows from Lemma 2 that $\lambda_{\mathrm{F}}(-A^{-1}A_1) \geq \lambda_{\mathrm{F}}(-A_0^{-1}A_1) > 0$. Therefore from Theorem 4 we have $\kappa(\Sigma) \to 0$ as $h_1 \to \infty$. This shows that $\Lambda_\Sigma \neq \sigma(A_0)$ under the variation of $h_1 \in \mathbb{R}_+$.
<u>(i)$\Leftarrow$(ii):</u> Suppose A_0 is eigenvalue-insensitive relative to A_1. Then by definition (21) we have $\sigma(A_0 + \nu A_1) = \sigma(A_0)$ $(\forall \nu \in \mathbb{C})$. It follows that $\det\left(\lambda I - A_0 - A_1 e^{-\lambda h_1}\right) = 0$ holds if and only if $\lambda \in \sigma(A_0)$ irrespective of $h_1 \in \mathbb{R}_+$. This clearly shows that (i) holds. ■

The next corollary is a direct consequence of Theorem 5.

Corollary 1 Consider the TDPS described by (2) where $N = 1$ and assume Σ is asymptotically stable, i.e., (5) holds. Then, $\kappa(\Sigma) \to 0$ as $h_1 \to \infty$ *if and only if* A_0 is eigenvalue-sensitive relative to A_1.

Proof of Corollary 1: "if" part: Suppose A_0 is eigenvalue-sensitive relative to A_1. Then we have already shown in the proof of Theorem 5 that $\kappa(\Sigma) \to 0$ as $h_1 \to \infty$.
"only if" part: By contradiction, suppose A_0 is eigenvalue-insensitive relative to A_1. Then, again from Theorem 5, we see that $\Lambda_\Sigma = \sigma(A_0)$ $(\forall h_1 \in \mathbb{R}_+)$ and hence $\kappa(\Sigma) = \lambda_{\mathrm{F}}(A_0)$ $(\forall h_1 \in \mathbb{R}_+)$. Therefore $\kappa(\Sigma) \to 0$ never happens by letting $h_1 \to \infty$. ■

Before closing this section, we summarize important results we have obtained in sections 4 and 5 for the asymptotically stable TDPS (2) with $N = 1$.

- $A_0 \in \mathbb{M}^n \cap \mathbb{H}^n$ is eigenvalue insensitive relative to $A_1 \in \mathbb{R}_+^{n\times n}$ if and only if $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) = 0$ (Lemma 4).
- $\Lambda_\Sigma = \sigma(A_0)$ $(\forall h_1 \in \mathbb{R}_+)$ holds if and only if A_0 is eigenvalue insensitive relative to A_1, i.e., $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) = 0$ (Theorem 5).
- As for the dominant pole, $\kappa(\Sigma) \to 0$ as $h_1 \to \infty$ if and only if A_0 is eigenvalue sensitive relative to A_1, i.e., $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) > 0$ (Corollary 1). In particular, if $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) > 0$ then $\lambda_{\mathrm{F}}(-A^{-1}A_1) > 0$ holds and we have $\kappa(\Sigma) \geq -\frac{1}{h_1}\lambda_{\mathrm{F}}\left(-A^{-1}A_1\right)^{-1}$ (Theorem 4).

Since $\lambda_{\mathrm{F}}(-A_0^{-1}A_1) > 0$ generically holds, we thus conclude that convergence performance of asymptotically stable PSs generically deteriorates by the introduction of time-delays.

6 Conclusion

In this paper, we analyzed dominant pole of asymptotically stable time-delay positive systems. Even though a time-delay positive system is stable if and only if its corresponding delay-free system is stable, we have shown that the dominant pole is (or poles as a whole are) affected by delays if and only if associated coefficient matrices satisfy eigenvalue-sensitivity condition. Moreover, we clarified that the dominant pole goes to zero as time-delay goes to infinity if and only if the coefficient matrices are eigenvalue-sensitive. We thus obtained solid quantitative evaluation on the dominant pole of time-delay positive systems in terms of delays.

In the future work, it is interesting to extend Theorem 5 and Corollary 1 to multiple delay cases. The behaviour of poles might be complicated if multiple delays vary independently. This topic is currently under investigation.

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Appendix

Proof of Lemma 4: (i)⇒(ii): By contradiction, suppose $\lambda_{\rm F}(-A_0^{-1}A_1) > 0$. Then we see that

$$\det(A_0 + \nu A_1) = \det(A_0)\det(I + \nu A_0^{-1}A_1) = 0$$

holds for $\nu = \lambda_{\rm F}(-A_0^{-1}A_1)^{-1} > 0$. It follows that $0 \in \sigma(A_0 + \nu A_1)$ whereas $0 \notin \sigma(A_0)$ and hence A_0 is eigenvalue-sensitive relative to A_1 by definition. This contradicts (i).
(i)⇐(ii): Suppose $\lambda_{\rm F}(-A_0^{-1}A_1) = 0$. Then, from Lemma 3, there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{U}_+$ such that

$$-\widetilde{A}_0^{-1}\widetilde{A}_1 = U,\ \widetilde{A}_0 := P^T A_0 P,\ \widetilde{A}_1 := P^T A_1 P.$$

Since $\widetilde{A}_0 \in \mathbb{M}^n \cap \mathbb{H}^n$ and $\widetilde{A}_1 \in \mathbb{R}_+^{n\times n}$, we can assume without generality that $-A_0^{-1}A_1 = U$ holds for some $U \in \mathbb{U}_+$ from the outset. Then we have

$$-A_0 U = A_1,\ U \in \mathbb{U}_+. \tag{22}$$

Note that $U \in \mathbb{U}_+$ has the structure of the form

$$U =: \begin{bmatrix} u_1 \cdots u_n \end{bmatrix}, \quad u_i = \begin{bmatrix} u_{i,\rm sub} \\ 0_{n-i+1,1} \end{bmatrix}, u_{i,\rm sub} \in \mathbb{R}_+^{i-1}. \tag{23}$$

In particular, $u_1 = 0$. If $U = 0$, then (22) implies $A_1 = 0$ and hence (i) trivially holds. Therefore we assume $U \neq 0$ in the sequel.

For m_1 such that $1 \le m_1 \le n-1$, suppose $u_k = 0$ $(k = 1, \cdots, m_1)$ and $u_{m_1+1} \neq 0$. Then we have

$$\begin{aligned} U &= \begin{bmatrix} 0_{n,m_1} & u_{m_1+1} & \cdots & u_n \end{bmatrix} \in \mathbb{U}_+, \\ A_1 &= \begin{bmatrix} 0_{n,m_1} & a_{m_1+1} & \cdots & a_n \end{bmatrix} \in \mathbb{R}_+^{n\times n}. \end{aligned}$$

Assume the number of strictly positive elements of u_{m_1+1} is μ_1 $(1 \le \mu_1 \le m_1)$. Then, there exists a permutation matrix $P^{[1]} \in \mathbb{R}^{n\times n}$ of the form

$$P^{[1]} = \begin{bmatrix} P_{\rm sub}^{[1]} & 0 \\ 0 & I_{n-m_1} \end{bmatrix},\ P_{\rm sub}^{[1]} \in \mathbb{R}^{m_1\times m_1}$$

such that

$$\begin{aligned} U^{[1]} &:= P^{[1]T} U P^{[1]} \\ &= \begin{bmatrix} 0_{\mu_1,m_1} & \widehat{u}_{m_1} & * \\ 0_{n-\mu_1,m_1} & 0 & * \end{bmatrix} \in \mathbb{U}_+ \end{aligned} \tag{24}$$

where $\widehat{u}_{m_1} \in \mathbb{R}^{\mu_1}_{++}$. We note that, by slightly loosening the structure shown above, we can write $U^{[1]}$ in the following form as well since $\mu_1 \leq m_1$:

$$U^{[1]} = \begin{bmatrix} 0_{\mu_1,\mu_1} & U^{[1]}_{12} \\ 0_{n-\mu_1,\mu_1} & U^{[2]} \end{bmatrix} \in \mathbb{U}_+. \tag{25}$$

If we define $A^{[1]}_0 := P^{[1]T} A_0 P^{[1]}$ and $A^{[1]}_1 := P^{[1]T} A_1 P^{[1]}$, we see that these have the structure of the form

$$\begin{aligned} A^{[1]}_0 &= \begin{bmatrix} A^{[1]}_{0,11} & A^{[1]}_{0,12} \\ 0_{n-\mu_1,\mu_1} & A^{[2]}_0 \end{bmatrix}, \quad A^{[2]}_0 \in \mathbb{M}^{n-\mu_1}, \\ A^{[1]}_1 &= \begin{bmatrix} 0_{\mu_1,\mu_1} & A^{[1]}_{1,12} \\ 0 & A^{[2]}_1 \end{bmatrix}, \quad A^{[2]}_1 \in \mathbb{R}^{(n-\mu_1)\times(n-\mu_1)}_+. \end{aligned}$$

The structure of $A^{[1]}_0$ stems from $A^{[1]}_0 \in \mathbb{M}^n$, $-A^{[1]}_0 U^{[1]} = A^{[1]}_1 \geq 0$, and the structure of $U^{[1]}$ given in (24). The above result tells us that

$$\begin{aligned} \sigma(A_0 + \nu A_1) &= \sigma(A^{[1]}_0 + \nu A^{[1]}_1) \\ &= \sigma(A^{[1]}_{0,11}) \cup \sigma(A^{[2]}_0 + \nu A^{[2]}_1). \end{aligned}$$

We note that

(a) $A^{[1]}_{0,11} \in \mathbb{M}^{\mu_1}$ and $A^{[2]}_0 \in \mathbb{M}^{n-\mu_1}$ are submatrices of $A^{[1]}_0 = P^{[1]T} A_0 P^{[1]}$ and hence $\sigma(A^{[1]}_{0,11}) \cup \sigma(A^{[2]}_0) = \sigma(A_0)$,

(b) the size of $A^{[2]}_0$ and $A^{[2]}_1$ are $n - \mu_1$ and has been reduced from those of A_0 and A_1 at least $\mu_1 \geq 1$,

(c) the matrix pair $(A^{[2]}_0, A^{[2]}_1)$ satisfies exactly the same condition as that of (A_0, A_1), i.e., $A^{[2]}_0 \in \mathbb{M}^{n-\mu_1} \cap \mathbb{H}^{n-\mu_1}$, $A^{[2]}_1 \in \mathbb{R}^{(n-\mu_1)\times(n-\mu_1)}_+$, and $\lambda_{\mathrm{F}}\left(-(A^{[2]}_0)^{-1} A^{[2]}_1\right) = 0$.

Therefore, by repeating the same procedure detailed above, we see that there exists $\mu_2 \geq 1$ (unless $A^{[2]}_1$ is 0 since in this case the proof is done) such that

$$\sigma(A^{[2]}_0 + \nu A^{[2]}_1) = \sigma(A^{[2]}_{0,11}) \cup \sigma(A^{[3]}_0 + \nu A^{[3]}_1)$$

where $A^{[2]}_{0,11} \in \mathbb{M}^{\mu_2}$, $A^{[3]}_0 \in \mathbb{M}^{n-\mu_1-\mu_2}$, $A^{[3]}_1 \in \mathbb{R}^{(n-\mu_1-\mu_2)\times(n-\mu_1-\mu_2)}_+$, and $\lambda_{\mathrm{F}}\left(-(A^{[3]}_0)^{-1} A^{[3]}_1\right) = 0$. By repeating this procedure L times such that $A^{[L+1]}_1 = 0$ where it is clear that $L \leq n-1$, we arrive at the desired conclusion that $\sigma(A_0 + \nu A_1) = \sigma(A_0)$ irrespective of $\nu \in \mathbb{C}$. This completes the proof. ■