LMI Approach to Linear Positive System Analysis and Synthesis

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March 29, 2013

Abstract

This paper is concerned with the analysis and synthesis of linear positive systems based on linear matrix inequalities (LMIs). We first show that the celebrated Perron-Frobenius theorem can be proved concisely by a duality-based argument. Again by duality, we next clarify a necessary and sufficient condition under which a Hurwitz stable Metzler matrix admits a diagonal Lyapunov matrix with some identical diagonal entries as the solution of the Lyapunov inequality. This new result leads to an alternative proof of the recent result by Tanaka and Langbort on the existence of a diagonal Lyapunov matrix for the LMI characterizing the H_∞ performance of continuous-time positive systems. In addition, we further derive a new LMI for the H_∞ performance analysis where the variable corresponding to the Lyapunov matrix is allowed to be non-symmetric. We readily extend these results to discrete-time positive systems and derive new LMIs for the H_∞ performance analysis and synthesis. We finally illustrate their effectiveness by numerical examples on robust state-feedback H_∞ controller synthesis for discrete-time positive systems affected by parametric uncertainties.
Keywords: positive system, diagonal Lyapunov matrix, LMI, duality.

1 Introduction

This paper is concerned with the analysis and synthesis of linear time-invariant (LTI) positive systems. A linear system is said to be positive (or more accurately, internally positive) if its state and output are both nonnegative for any nonnegative initial state and nonnegative input. Because of this strong property, there are remarkable, and very peculiar results that are valid only for positive systems. Among them, the existence of a diagonal Lyapunov matrix that characterizes stability is well known [6, 8].

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Recently, Shorten et al. showed that the peculiar "diagonal stability result" can be proved by means of the duality theory in convex optimization. They further obtained new results on the stability of switched positive systems [7, 9, 14, 15]. Along this line, Tanaka and Langbort proved that the KYP-type linear matrix inequality (LMI) characterizing the H_∞ performance of positive systems admits a diagonal Lyapunov matrix [16]. These recent results indicate that the duality theory is a powerful tool for positive system analysis.

Along the same line, in this paper, we develop duality-based arguments for positive system analysis. Our novel contribution can be summarized as follows:

1. We provide a duality-based concise proof of the Perron-Frobenius theorem [6, 8]. In addition to the existence of the Frobenius eigenvalue, we show the existence of the nonnegative eigenvector by duality.

2. Again by a duality-based argument, we clarify a necessary and sufficient condition under which a Hurwitz stable Metzler matrix admits a diagonal Lyapunov matrix with some identical diagonal entries. This condition leads to an alternative proof of the result in [16]. The analysis is partly motivated from the observation that the L_2 and L_1 induced norm analysis of positive systems can be transformed into the stability analysis of appropriately constructed positive systems [13, 5].

3. We derive new LMI conditions for the stability and H_∞ performance analysis of continuous-time positive systems, where the common positive definiteness constraint on the Lyapunov matrix P as in $P \succ 0$ can be relaxed to $P + P^T \succ 0$. This implies that P is not necessarily required to be symmetric.

4. We extend the above results to discrete-time positive systems and derive new LMIs for the H_∞ performance analysis and synthesis. We illustrate the effectiveness of these new LMIs by numerical examples on structured robust state-feedback H_∞ controller synthesis for discrete-time positive systems affected by parametric uncertainties.

Note that a conference version of this paper was presented in [4]. In the current paper we include new LMI results for discrete-time positive systems. In particular, we show that a given discrete-time positive system can be converted into a continuous-time positive system preserving the stability and the H_∞ norm. This enables us to derive new LMIs for the H_∞ performance analysis and synthesis of discrete-time positive systems.

We use the following notations in this paper. First, we denote by $\mathbb{S}_+^n$ the set of positive semidefinite matrices of size n. For a symmetric matrix $X \in \mathbb{R}^{n\times n}$, we also write $X \succ 0$ ($X \succeq 0$) to denote that X is positive (semi)definite. Similarly, we write $X \prec 0$ ($X \preceq 0$) to denote that X is negative (semi)definite. In addition, we denote by $\mathbb{D}_{++}^n$ the set of diagonal, and positive definite matrices of size n. For $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}\{A\} = A + A^T$. The notation $\lambda(A)$ stands for the set of the eigenvalues of A. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* stable if $\max_{\lambda\in\lambda(A)} \mathrm{Re}\ \lambda < 0$, and is said to be *Schur* stable if $\max_{\lambda\in\lambda(A)} |\lambda| < 1$. For two given matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i,j), where A_{ij} (B_{ij}) stands for the (i,j)-entry of A (B). We also define

$$\mathbb{R}_{++}^{n\times m} := \{A \in \mathbb{R}^{n\times m},\ A > 0\}, \quad \mathbb{R}_{+}^{n\times m} := \{A \in \mathbb{R}^{n\times m},\ A \geq 0\}.$$

Finally, for a given $A \in \mathbb{R}^{n\times n}$, we define by $\mathcal{D}(A) \in \mathbb{R}^n$ the vector composed of the diagonal entries, i.e., $\mathcal{D}(A) := [A_{11} \ \cdots \ A_{nn}]^T$.

2 Fundamentals of Positive Systems

In this brief section, we gather basic definitions and fundamental results for positive system analysis. See [6, 8] for a more complete treatment.

Definition 1 (Positive Linear System) [6] A linear system is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

A system satisfying the condition in Definition 1 is often called *internally* positive, to make a clear distinction from *externally* positive systems. Since we only deal with internally positive systems in this paper, we simply denote them by positive as in Definition 1.

Definition 2 (Metzler Martrix) [6] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ $(i \neq j)$.

Proposition 1 [6] Let us consider the continuous-time LTI system described by

$$G : \left\{ \begin{array}{rcrcl} \dot{x}(t) & = & Ax(t) & + & Bw(t), \\ z(t) & = & Cx(t) & + & Dw(t). \end{array} \right. \tag{1}$$

Then, this system is positive if and only if A is Metzler, $B \geq 0$, $C \geq 0$, and $D \geq 0$.

Proposition 2 [6] Let us consider the discrete-time LTI system described by

$$G_{\mathrm{d}} : \left\{ \begin{array}{rcrcl} x(k+1) & = & A_{\mathrm{d}}x(k) & + & B_{\mathrm{d}}w(k), \\ z(k) & = & C_{\mathrm{d}}x(k) & + & D_{\mathrm{d}}w(k). \end{array} \right. \tag{2}$$

Then, this system is positive if and only if $A_{\mathrm{d}} \geq 0$, $B_{\mathrm{d}} \geq 0$, $C_{\mathrm{d}} \geq 0$, and $D_{\mathrm{d}} \geq 0$.

In the sequel, we denote by $\mathbb{M}^n$ the set of the Metzler matrices of size n. The next theorem summarizes basic results for the Hurwitz stability of Metzler matrices.

Proposition 3 [6, 8] For a given $A \in \mathbb{M}^n$, the following conditions are equivalent.

- (i) The matrix A is Hurwitz stable.
- (ii) For any $h \in \mathbb{R}^n_+ \setminus \{0\}$, the row vector $h^T A$ has at least one strictly negative entry.
- (iii) There exists $h \in \mathbb{R}^n_{++}$ such that $h^T A < 0$.
- (iv) There exists $g \in \mathbb{R}^n_{++}$ such that $Ag < 0$.
- (v) The matrix A is nonsingular and satisfies $A^{-1} \leq 0$.

3 Preliminary Results

In this section, we introduce preliminary results that are effective for positive system analysis. For conciseness, with a slight abuse of notation, we first make the following definition.

Definition 3 For a given $H \in \mathbb{S}_+^n$, we define $\bar{h} \in \mathbb{R}_+^n$ by $\bar{h}_i = \sqrt{H_{ii}}$ $(i = 1, \cdots, n)$.

Under this definition, the following three lemmas hold.

Lemma 1 For a given $H \in \mathbb{S}_+^n$, we have

$$(\bar{h}\bar{h}^T)_{ii} = H_{ii}, \quad (\bar{h}\bar{h}^T)_{ij} \geq H_{ij} \ (i \neq j). \tag{3}$$

Proof: The first equality is obvious. On the other hand, since $H \succeq 0$, we have $H_{ii}H_{jj} \geq H_{ij}^2$ for $i \neq j$. It follows that $\sqrt{H_{ii}}\sqrt{H_{jj}} \geq H_{ij}$. Therefore, on the (i, j) entry of $\bar{h}\bar{h}^T - H$, we have $(\bar{h}\bar{h}^T)_{ij} - H_{ij} = \sqrt{H_{ii}}\sqrt{H_{jj}} - H_{ij} \geq 0$. This completes the proof. ■

Lemma 2 For given $A \in \mathbb{M}^n$ and $H \in \mathbb{S}_+^n$, we have $\mathcal{D}(\bar{h}\bar{h}^T A) \geq \mathcal{D}(HA)$.

Proof of Lemma 2: Since $A \in \mathbb{M}^n$ and hence $A_{ij} \geq 0$ $(i \neq j)$, we see from Lemma 1 that

$$\begin{array}{rcl} (\bar{h}\bar{h}^T A)_{ii} & = & (\bar{h}\bar{h}^T)_{ii}A_{ii} + \sum_{j=1,j\neq i}^n (\bar{h}\bar{h}^T)_{ij}A_{ji} \\ & \geq & H_{ii}A_{ii} + \sum_{j=1,j\neq i}^n H_{ij}A_{ji} \\ & = & (HA)_{ii}. \end{array}$$

This completes the proof. ■

This lemma in particular implies that if there exists $H \in \mathbb{S}_+^n$ that satisfies $\mathcal{D}(HA) \geq 0$ for a given $A \in \mathbb{M}^n$, then exactly the same property $\mathcal{D}(\bar{h}\bar{h}^T A) \geq 0$ holds with the rank-one matrix $\bar{h}\bar{h}^T$.

Lemma 3 For given $A \in \mathbb{M}^n$ and $h_1, h_2 \in \mathbb{R}_+^n$, the following conditions are equivalent.

(i) $\mathcal{D}\left(h_1(h_1^T A + h_2^T)\right) \geq 0$.

(ii) $h_1^T A + h_2^T \geq 0$.

Proof of Lemma 3: Since (ii) $\Rightarrow$ (i) is obvious, we prove (i) $\Rightarrow$ (ii) by contradiction. To this end, suppose $(h_1^T A + h_2^T)_i < 0$. Then, since A is Metzler and $h_1, h_2 \in \mathbb{R}_+^n$, we have $A_{ii} < 0$ and $h_{1,i} > 0$. Hence $h_{1,i}(h_1^T A + h_2^T)_i < 0$, which clearly contradicts (i). ■

4 Duality-based Proofs for Perron-Frobenius Theorem

The next theorem is widely known as the Perron-Frobenius Theorem. It states that, among all the eigenvalues of a nonnegative matrix, the one with the largest modulus is located on the right-hand side of the real axis.

Theorem 1 (Perron-Frobenius Theorem) [6, 8] Suppose $A \in \mathbb{R}_+^{n\times n}$ is given. Then, A has a nonnegative eigenvalue α such that $\alpha = \max_{\lambda\in\lambda(A)} |\lambda|$. Moreover, the eigenvector g corresponding to the eigenvalue α satisfies $g \geq 0^{\dagger}$.

This theorem is undoubtedly the central result in positive system analysis. It has a vast range of application areas such as biology, sociology and stochastic system analysis (see, ex., [11] and references cited therein). This theorem is proved, for example in [8], by means of the Collatz-Wielandt function. Our first contribution is to show that this celebrated theorem can be proved concisely by means of a duality-based argument.

Proof of Theorem 1: Let us denote by ρ the spectral radius of A, i.e., $\rho := \max_{\lambda\in\lambda(A)} |\lambda|$. Then, as a direct consequence of the discrete-time Lyapunov inequality, for all $\beta > \rho$ there exists $P \succ 0$ such that $\beta^2 P - APA^T \succ 0$. This property fails for $\beta = \rho$. Namely, there does not exist $P \succ 0$ such that $\rho^2 P - APA^T \succ 0$. From duality, or more specifically, from the strong alternatives for generalized inequalities [1, 3], this can be restated as: There exists a nonzero H such that $H \succeq 0$ and $A^T HA - \rho^2 H \succeq 0$. The latter inequality implies $\mathcal{D}(A^T HA) \geq \mathcal{D}(\rho^2 H)$. Here let us define $\bar{h} \in \mathbb{R}_+^n \setminus \{0\}$ from H as in Definition 3 and recall that $\bar{h}\bar{h}^T \geq H$ and $\mathcal{D}(\bar{h}\bar{h}^T) = \mathcal{D}(H)$. Since $A \in \mathbb{R}_+^{n\times n}$, we have $A^T\bar{h}\bar{h}^T A \geq A^T HA$. Hence, we see that $\mathcal{D}(A^T\bar{h}\bar{h}^T A) \geq \mathcal{D}(A^T HA) \geq \mathcal{D}(\rho^2 H) = \mathcal{D}(\rho^2\bar{h}\bar{h}^T)$ holds. Furthermore, since $\bar{h}^T A \geq 0$ and $\rho\bar{h}^T \geq 0$, the inequality implies that $\bar{h}^T A \geq \rho\bar{h}^T$, or equivalently, $\bar{h}^T(A-\rho I) \geq 0$. From (ii) of Proposition 3, this indicates that $A-\rho I$ is not Hurwitz stable. Since ρ is the spectral radius of A, this happens if and only if ρ is an eigenvalue of A.

The latter part of the theorem can be proved again by a duality-based argument. By contradiction, suppose there exists no $g \in \mathbb{R}_+^n \setminus \{0\}$ such that $(A-\rho I)g = 0$. Then, from the strong alternative for linear inequalities [3], there exists $h \neq 0$ such that $h^T(A-\rho I) > 0$. Here, if $h \leq 0$, then we have $-h^T(A-\rho I) < 0$, and by perturbing h, we see that there exists $\widetilde{h} > 0$ such that $\widetilde{h}^T(A-\rho I) < 0$. From (iii) of Proposition 3, this contradicts the fact that $A-\rho I$ is not Hurwitz. Therefore, it suffices to consider the case where h has at least one strictly positive entry. With this in mind, let us denote by $h_+ \in \mathbb{R}_+^n \setminus \{0\}$ the projection of h onto $\mathbb{R}_+^n$. We also define h_- by $h_- := h - h_+ \leq 0$. Since $h^T(A-\rho I) > 0$, there exists $\varepsilon > 0$ such that $v := h^T(A-(\rho+\varepsilon)I) > 0$, and this implies $v_+ := h_+^T(A-(\rho+\varepsilon)I) \geq 0$ holds as well. This is because if v_+ has a negative entry, i.e., $v_{+,j} = (h_+^T A)_j - (\rho+\varepsilon)h_{+,j} < 0$, then it is obvious that $h_{+,j} > 0$ and hence $h_{-,j} = 0$. It follows that $v_- := h_-^T(A-(\rho+\varepsilon)I)$ satisfies $v_{-,j} = (h_-^T A)_j \leq 0$. This clearly contradicts the fact that $v_j = v_{+,j} + v_{-,j} > 0$. To summarize, we have established the existence of $h_+ \neq 0$ such that $h_+ \geq 0$ and $h_+^T(A-(\rho+\varepsilon)I) \geq 0$. From (ii) in Proposition 3, this contradicts the fact that $A-(\rho+\varepsilon)I$ is Hurwitz stable. This completes the proof. ■

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†Under the assumption that A is irreducible, the Perron-Frobenius Theorem ensures $g > 0$ that is stronger than $g \geq 0$. See [6, 8] for details.

In addition to the assertions in Theorem 1, it is known that the eigenvector for the maximal real eigenvalue $\alpha = \rho$ can be chosen to be strictly positive, whereas the one for $\alpha < \rho$ cannot (if any) [6]. This can be easily proved. Indeed, suppose $(A - \alpha I)g = 0$ holds for $\alpha < \rho$ and $g > 0$. Then, this implies $(A - \rho I)g = (A - \alpha I)g - (\rho - \alpha)g = -(\rho - \alpha)g < 0$. From (iv) of Proposition 3, this clearly contradicts the fact that $A - \rho I$ is not Hurwitz stable.

Before closing this section, we give the next lemma that is a direct consequence of the Perron-Frobenius theorem. The lemma states that, among all the eigenvalues of a Metzler matrix, the one with the largest real part is located on the real axis.

Lemma 4 Suppose $A \in \mathbb{M}^n$ is given. Then, A has a real eigenvalue α such that $\alpha = \max_{\lambda \in \lambda(A)} \mathrm{Re}\ \lambda$.

The proof of this lemma is as follows. Since $A \in \mathbb{M}^n$, there exists $\beta \geq 0$ such that $A + \beta I \in \mathbb{R}_+^{n \times n}$. Then, from Theorem 1, the matrix $A + \beta I$ has a nonnegative eigenvalue ν such that $\nu = \max_{\lambda \in \lambda(A+\beta I)} |\lambda|$ and therefore $\nu = \max_{\lambda \in \lambda(A+\beta I)} \mathrm{Re}\ \lambda$. This clearly shows the existence of α in Lemma 4 that is precisely given by $\alpha = \nu - \beta$.

5 Analysis of Continuous-Time Positive Systems

We next move to the analysis of continuous-time positive systems. The next theorem is the main result in this section.

Theorem 2 (New Results on Diagonal Stability of Metzler Matrix) For a given $\mathcal{A} \in \mathbb{M}^{n_1+n_2}$ with the partition

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}, \ \mathcal{A}_{11} \in \mathbb{M}^{n_1}, \ \mathcal{A}_{22} \in \mathbb{M}^{n_2}, \tag{4}$$

the following conditions are equivalent.

(i) The matrix $\mathcal{A}_{11} \in \mathbb{M}^{n_1}$ is Hurwitz stable and

$$\mathrm{He}\{-\mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12} + \mathcal{A}_{22}\} \prec 0. \tag{5}$$

(ii) There exists $\mathcal{X} \in \mathbb{D}_{++}^{n_1}$ such that

$$\mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X} & 0 \\ 0 & I_{n_2} \end{bmatrix} \right\} \prec 0. \tag{6}$$

(iii) There exists $W \in \mathbb{R}^{n_1 \times n_1}$ such that

$$\mathcal{W} + \mathcal{W}^T \succ 0, \tag{7a}$$

$$\mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{W} & 0 \\ 0 & I_{n_2} \end{bmatrix} \right\} \prec 0. \tag{7b}$$

Condition (ii) of Theorem 2 indicates that $\mathcal{A}$ is Hurwitz stable. Its stability is proved by means of a diagonal Lyapunov matrix where some of the diagonal entries are identical. The theorem hence formulates necessary and sufficient conditions for a stable Metzler matrix to admit such special type of Lyapunov matrices.

In the following, we give a proof of Theorem 2. We complete the proof of (i)⇒(ii), which is indeed the central part, by a duality-based argument.

Proof of Theorem 2: Since (ii)⇒(iii) is obvious, we prove (iii)⇒(i) and (i)⇒(ii).

(iii)⇒(i) From Lemma 4, we see that $\mathcal{A}_{11}$ has a real eigenvalue α such that $\alpha = \max_{\lambda\in\lambda(\mathcal{A}_{11})} \mathrm{Re}\,\lambda$. If we denote by $\xi \in \mathbb{R}^{n_1}$ the corresponding left eigenvector satisfying $\xi^T\mathcal{A}_{11} = \alpha\xi^T$, we see from $\mathrm{He}\{\mathcal{A}_{11}\mathcal{W}\} \prec 0$, implied by (7b), that $\alpha(\xi^T(\mathcal{W}+\mathcal{W}^T)\xi) < 0$. Since $\mathcal{W}+\mathcal{W}^T \succ 0$, this implies $\alpha < 0$ and hence $\mathcal{A}_{11}$ is Hurwitz stable. In addition, if we multiply (7b) by $[\ -\mathcal{A}_{21}\mathcal{A}_{11}^{-1}\ I_{n_2}]$ from the left and its transpose from the right, we readily obtain (5).

(i)⇒(ii) By contradiction, suppose that there is no $\mathcal{X} \in \mathbb{D}_{++}^{n_1}$ such that (6) holds. Then, from duality or more specifically from the separating hyper-plane theorem [3], there exists a matrix $H \in \mathbb{S}_+^{n_1+n_2} \setminus \{0\}$ such that

$$\mathrm{trace}\left(H\mathrm{He}\left\{\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}\begin{bmatrix}\mathcal{X} & 0\\ 0 & I_{n_2}\end{bmatrix}\right\}\right) \geq 0 \quad \forall \mathcal{X} \in \mathbb{D}_{++}^{n_1}$$

or equivalently,

$$\mathrm{trace}\left(H\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}\begin{bmatrix}\mathcal{X} & 0\\ 0 & I_{n_2}\end{bmatrix}\right) \geq 0 \quad \forall \mathcal{X} \in \mathbb{D}_{++}^{n_1}.$$

This can be rewritten as

$$\mathrm{trace}\left(H\begin{bmatrix}\mathcal{A}_{11} & 0\\ \mathcal{A}_{21} & 0\end{bmatrix}\begin{bmatrix}\mathcal{X} & 0\\ 0 & 0\end{bmatrix}\right) + \mathrm{trace}\left(H\begin{bmatrix}0 & \mathcal{A}_{12}\\ 0 & \mathcal{A}_{22}\end{bmatrix}\right) \geq 0 \quad \forall \mathcal{X} \in \mathbb{D}_{++}^{n_1}.$$

It follows that we have

$$\mathcal{D}\left(H\begin{bmatrix}\mathcal{A}_{11} & 0\\ \mathcal{A}_{21} & 0\end{bmatrix}\right) \geq 0, \quad \mathrm{trace}\left(H\begin{bmatrix}0 & \mathcal{A}_{12}\\ 0 & \mathcal{A}_{22}\end{bmatrix}\right) \geq 0.$$

Then, since both

$$\begin{bmatrix}\mathcal{A}_{11} & 0\\ \mathcal{A}_{21} & 0\end{bmatrix}, \quad \begin{bmatrix}0 & \mathcal{A}_{12}\\ 0 & \mathcal{A}_{22}\end{bmatrix}$$

are Metzler, we see from Lemma 2 that $\bar{h} \in \mathbb{R}_+^{n_1+n_2} \setminus \{0\}$ defined from H as in Definition 3 satisfies

$$\mathcal{D}\left(\bar{h}\bar{h}^T\begin{bmatrix}\mathcal{A}_{11} & 0\\ \mathcal{A}_{21} & 0\end{bmatrix}\right) \geq 0, \quad \mathrm{trace}\left(\bar{h}\bar{h}^T\begin{bmatrix}0 & \mathcal{A}_{12}\\ 0 & \mathcal{A}_{22}\end{bmatrix}\right) \geq 0.$$

If we further define $\bar{h} =: [\ \bar{h}_1^T\ \bar{h}_2^T\ ]^T$ where $\bar{h}_1 \in \mathbb{R}_+^{n_1}$ and $\bar{h}_2 \in \mathbb{R}_+^{n_2}$, it follows that

(a) $\mathcal{D}(\bar{h}_1(\bar{h}_1^T\mathcal{A}_{11} + \bar{h}_2^T\mathcal{A}_{21})) \geq 0$, and

(b) $(\bar{h}_1^T\mathcal{A}_{12} + \bar{h}_2^T\mathcal{A}_{22})\bar{h}_2 \geq 0$.

From (a) and Lemma 3, we have $\bar{h}_1^T\mathcal{A}_{11} + \bar{h}_2^T\mathcal{A}_{21} \geq 0$. If $\mathcal{A}_{11}^{-1} \leq 0$ does not hold, we see from (iv) in Proposition 3 that $\mathcal{A}_{11}$ is not Hurwitz stable and hence the condition (i) never holds. If $\mathcal{A}_{11}^{-1} \leq 0$, then we have $0 \leq \bar{h}_1^T \leq -\bar{h}_2^T\mathcal{A}_{21}\mathcal{A}_{11}^{-1}$. This clearly shows that $\bar{h}_2 \neq 0$ because $\bar{h} = [\ \bar{h}_1^T\ \bar{h}_2^T\ ]^T \neq 0$. By substituting this inequality to (b), we have $\bar{h}_2^T(-\mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12} + \mathcal{A}_{22})\bar{h}_2 \geq 0$. Since $\bar{h}_2 \neq 0$, this contradicts (5) in (i). ■

We next show how existing results for positive system analysis can be reproduced by this theorem. First, by specializing the matrix $\mathcal{A}$ to $\mathcal{A} = \mathrm{diag}(A, -1)$ with $A \in \mathbb{M}^n$, $n_1 = n$ and $n_2 = 1$, we can readily obtain the next corollary. The condition (ii) given below is nothing but the "diagonal stability condition" for Metzler matrices [6, 8, 13]. It should be noted that for this particular choice of $\mathcal{A}$ the condition (5) resumes to $-2 < 0$ and is trivially satisfied.

Corollary 1 For a given $A \in \mathbb{M}^n$, the following conditions are equivalent.

(i) The matrix $A \in \mathbb{M}^n$ is Hurwitz stable.

(ii) There exists $X \in \mathbb{D}_{++}^n$ such that $\mathrm{He}\{AX\} \prec 0$.

(iii) There exists $W \in \mathbb{R}^{n\times n}$ such that $W + W^T \succ 0$, $\mathrm{He}\{AW\} \prec 0$.

We can also obtain the next corollary on the H_∞ performance analysis of positive systems. The condition (ii) given below is the recent result of Tanaka and Langbort [16], which shows the existence of a diagonal Lyapunov matrix for the H_∞ performance LMI.

Corollary 2 Let us consider the continuous-time positive system described by (1) where $A \in \mathbb{M}^n$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{l\times n}$, $D \in \mathbb{R}_+^{l\times m}$. Then, for a given $\gamma > 0$, the following conditions are equivalent.

(i) The matrix A is Hurwitz stable and $\|G\|_\infty < \gamma$.

(i)' The matrix A is Hurwitz stable and $\|G(0)\| = \|-CA^{-1}B + D\| < \gamma$.

(ii) There exists $X \in \mathbb{D}_{++}^n$ such that

$$\begin{bmatrix} AX + XA^T & XC^T & B \\ CX & -\gamma I_l & D \\ B^T & D^T & -\gamma I_m \end{bmatrix} \prec 0. \tag{8}$$

(iii) There exists $W \in \mathbb{R}^{n\times n}$ such that

$$W + W^T \succ 0, \tag{9a}$$

$$\begin{bmatrix} AW + W^TA^T & W^TC^T & B \\ CW & -\gamma I_l & D \\ B^T & D^T & -\gamma I_m \end{bmatrix} \prec 0. \tag{9b}$$

Proof of Corollary 2: From the definition of the H_∞ norm, (i)$\Rightarrow$(i)' is obvious. The implication (ii)$\Rightarrow$(i) also follows from the standard KYP lemma [12]. Therefore, for the validity of Corollary 2, it suffices to show (i)'$\Leftrightarrow$(ii)$\Leftrightarrow$(iii). To this end, we note that the condition (i)' can be restated equivalently in the form of (i) in Theorem 2 as follows:

(i)" The matrix $\mathcal{A}_{11} \in \mathbb{M}^n$ is Hurwitz stable and $\mathrm{He}\{-\mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12} + \mathcal{A}_{22}\} \prec 0$ holds where

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \left[\begin{array}{c:cc} A & 0 & B \\ \hdashline C & -\frac{1}{2}\gamma I_l & D \\ 0 & 0 & -\frac{1}{2}\gamma I_m \end{array}\right]. \tag{10}$$

Indeed, for any $\gamma > 0$, the condition

$$\mathrm{He}\{\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12}\} = \begin{bmatrix} -\gamma I_l & G(0) \\ G(0)^T & -\gamma I_m \end{bmatrix} \prec 0$$

holds iff $\|G(0)\| < \gamma$. Therefore (i)' and (i)" are equivalent. For the matrix $\mathcal{A}$ given by (10), the conditions (6) and (7) in Theorem 2 reduce respectively to (8) and (9). Therefore (i)'$\Leftrightarrow$(i)"$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) holds. ■

As noted, the condition (ii) in Corollary 1 is the diagonal stability condition for stable Metzler matrices. The diagonal stability condition for general matrices was first studied by Barker et al. [2] from the view point of duality theory and recently refined by Shorten et al. to deal with positive systems [14]. From this known result, it is obvious from the outset that there exists $\mathcal{X}_a \in \mathbb{D}_{++}^{n_1+n_2}$ satisfying $\mathrm{He}\{\mathcal{A}\mathcal{X}_a\} \prec 0$ in Theorem 2. Beyond that, what we have shown is that we can let $\mathcal{X}_a = \mathrm{diag}(\mathcal{X}, I_{n_2})$ ($\mathcal{X} \in \mathbb{D}_{++}^{n_1}$) under the additional condition (5). Namely, the condition (5) is necessary and sufficient for the existence of $\mathcal{X}_a \in \mathbb{D}_{++}^{n_1+n_2}$ with the identical latter n_2 diagonal entries. As a side effect of this theorem, the recent result by Tanaka and Langbort [16] has been recovered as we have seen in Corollary 2. We note that in [16] they directly worked on the dual of the LMI (8) and proved its validity. They showed that the LMI (8) is useful for structured static state-feedback H_∞ controller synthesis (see related discussions in Section 7).

On the other hand, the condition (iii) in Propositions 1 and 2 are new conditions for the stability and H_∞ performance analysis of positive systems. In these conditions, the usual positive definite (and hence symmetric) constraint on the Lyapunov matrix has been relaxed to $W \in \mathbb{R}^{n\times n}$ with $W + W^T \succ 0$. The extra freedom introduced by relaxing W to be a non-symmetric matrix becomes effective, for example, when we deal with robust stability and robust H_∞ performance analysis problems of positive systems affected by parametric uncertainties. For a more concrete discussion, see Section 7.

6 Analysis of Discrete-Time Positive Systems

The next result, Theorem 3, is a counterpart result of Theorem 2 for discrete-time system analysis. The proof of Theorem 3 is almost the same as that of Theorem 2 and hence omitted to avoid duplicated arguments.

Theorem 3 (New Results on Schur Stability of Nonnegative Matrix) For a given $\mathcal{A} \in \mathbb{R}_+^{n_1+n_2}$ with the partition

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}, \ \mathcal{A}_{11} \in \mathbb{R}_+^{n_1 \times n_1}, \ \mathcal{A}_{22} \in \mathbb{R}_+^{n_2 \times n_2}, \tag{11}$$

the following conditions are equivalent.

(i) The matrix $\mathcal{A}_{11} \in \mathbb{R}_+^{n_1+n_2}$ is Schur stable and

$$\|\mathcal{A}_{21}(I - \mathcal{A}_{11})^{-1}\mathcal{A}_{12} + \mathcal{A}_{22}\| < 1. \tag{12}$$

(ii) There exists $\mathcal{X} \in \mathbb{D}_{++}^{n_1}$ such that

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X} & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}^T \prec \begin{bmatrix} \mathcal{X} & 0 \\ 0 & I_{n_2} \end{bmatrix}. \tag{13}$$

This theorem is a slight extension of the diagonal stability result [6, 8] saying that a matrix $A \in \mathbb{R}_+^{n\times n}$ is Schur stable if and only if there exists $X \in \mathbb{D}_{++}^n$ such that $AXA^T \prec X$. This existing result readily follows from Theorem 3 by letting $\mathcal{A} = \mathrm{diag}(A, 0)$. On the other hand, similarly to the continuous-time system case, we see from Theorem 3 that the H_∞ performance of discrete-time positive systems can be characterized by an LMI with diagonal Lyapunov matrices.

Corollary 3 Let us consider the discrete-time positive system described by (2) where $A_\mathrm{d} \in \mathbb{R}_+^{n\times n}$, $B_\mathrm{d} \in \mathbb{R}_+^{n\times m}$, $C_\mathrm{d} \in \mathbb{R}_+^{l\times n}$, $D_\mathrm{d} \in \mathbb{R}_+^{l\times m}$. Then, for a given $\gamma > 0$, the following conditions are equivalent.

(i) The matrix A_d is Schur stable and $\|G_\mathrm{d}\|_\infty < \gamma$.

(i)' The matrix A_d is Schur stable and $\|G_\mathrm{d}(1)\| = \|C_\mathrm{d}(I - A_\mathrm{d})^{-1}B_\mathrm{d} + D_\mathrm{d}\| < \gamma$.

(ii) There exists $X \in \mathbb{D}_{++}^n$ such that

$$\begin{bmatrix} A_\mathrm{d}XA_\mathrm{d}^T - X & A_\mathrm{d}XC_\mathrm{d}^T & B_\mathrm{d} \\ C_\mathrm{d}XA_\mathrm{d}^T & C_\mathrm{d}XC_\mathrm{d}^T - \gamma I_l & D_\mathrm{d} \\ B_\mathrm{d}^T & D_\mathrm{d}^T & -\gamma I_m \end{bmatrix} \prec 0. \tag{14}$$

Proof: From the definition of the H_∞ norm, (i)⇒(i)' is obvious. On the other hand, note that the condition (i)' can be restated equivalently in the form of (i) in Theorem 3 as follows:

(i)" The matrix $\mathcal{A}_{11} \in \mathbb{R}_+^n$ is Schur stable and $\|\mathcal{A}_{21}(I - \mathcal{A}_{11})^{-1}\mathcal{A}_{12} + \mathcal{A}_{22}\| < 1$ holds where

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} A_\mathrm{d} & B_\mathrm{d} \\ \frac{1}{\gamma}C_\mathrm{d} & \frac{1}{\gamma}D_\mathrm{d} \end{bmatrix}. \tag{15}$$

Thus (i')⇔(ii) readily follows from Theorem 3 and Schur complement. We finally note that (ii)⇒(i) is obvious from the KYP lemma [12]. ■

In addition to the condition (ii) in Corollary 3, we can derive new LMI characterizations for the H_∞ performance of discrete-time positive systems by paying attention to the similarity between the conditions (i)' of Corollaries 2 and 3. Indeed, from the Perron-Frobenius theorem, we see that a matrix $A_\mathrm{d} \in \mathbb{R}_+^{n\times n}$ is Schur stable if and only if $A_\mathrm{d} - I \in \mathbb{M}_n$ is Hurwitz stable. It follows that the discrete-time system G_d with coefficient matrices $\{A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d}, D_\mathrm{d}\}$ is stable and satisfies $\|G_\mathrm{d}\|_\infty < \gamma$ if and only if the continuous-time system G with coefficient matrices $\{A_\mathrm{d} - I, B_\mathrm{d}, C_\mathrm{d}, D_\mathrm{d}\}$ is stable and satisfies $\|G\|_\infty < \gamma$. Applying (ii) and (iii) in Corollary 2 to the latter continuous-time system, we can readily obtain the next result.

Corollary 4 The next conditions are equivalent to (i), (i)', and (ii) in Corollary 3.

(iii) There exists $X \in \mathbb{D}_{++}^n$ such that

$$\begin{bmatrix} (A_\mathrm{d} - I)X + X(A_\mathrm{d} - I)^T & XC_\mathrm{d}^T & B_\mathrm{d} \\ C_\mathrm{d}X & -\gamma I_l & D_\mathrm{d} \\ B_\mathrm{d}^T & D_\mathrm{d}^T & -\gamma I_m \end{bmatrix} \prec 0. \tag{16}$$

(iv) There exists $W \in \mathbb{R}^{n\times n}$ such that

$$W + W^T \succ 0, \tag{17a}$$

$$\begin{bmatrix} (A_\mathrm{d} - I)W + W^T(A_\mathrm{d} - I)^T & W^TC_\mathrm{d}^T & B_\mathrm{d} \\ C_\mathrm{d}W & -\gamma I_l & D_\mathrm{d} \\ B_\mathrm{d}^T & D_\mathrm{d}^T & -\gamma I_m \end{bmatrix} \prec 0. \tag{17b}$$

Remark 1 The conditions (iii) and (iv) have been shown also in [10] by the similarity between the conditions (i)' of Corollaries 2 and 3. However, in [10], the proof of the validity of (i)' of Corollary 3 has been done by directly establishing the equivalence of (i)' and (ii). The proof is rather involved. In the current paper, this difficulty is successfully circumvented by finding out the connection to the stability result of Theorem 3.

7 Robust State-Feedback H_∞ Controller Synthesis for Discrete-Time Positive Systems

In this section, we discuss state-feedback H_∞ controller synthesis for discrete-time positive systems using semidefinite programming (SDP).

7.1 SDPs for Robust State-Feedback Synthesis

Let us consider the discrete-time uncertain positive system described by

$$G_{\mathrm{d},\theta}\ \left\{\begin{array}{rcllll} x(k+1) & = & A(\theta)x(k) & + & B_1(\theta)w(k) & + & B_2(\theta)u(k), \\ z(k) & = & C_1(\theta)x(k) & + & D_{11}(\theta)w(k) & + & D_{12}(\theta)u(k) \end{array}\right. \tag{18}$$

where

$$\begin{array}{l} \begin{bmatrix} A(\theta) & B_1(\theta) & B_2(\theta) \\ C_1(\theta) & D_{11}(\theta) & D_{12}(\theta) \end{bmatrix} = \sum_{i=1}^N \theta_i \begin{bmatrix} A^{[i]} & B_1^{[i]} & B_2^{[i]} \\ C_1^{[i]} & D_{11}^{[i]} & D_{12}^{[i]} \end{bmatrix}, \\ \Theta = \left\{\theta \in \mathbb{R}_+^N :\ \sum_{i=1}^N \theta_i = 1\right\}. \end{array} \tag{19}$$

Here, $A^{[i]} \in \mathbb{R}_+^{n\times n}$, $B_1^{[i]} \in \mathbb{R}_+^{n\times n_w}$, $B_2^{[i]} \in \mathbb{R}_+^{n\times n_u}$, $C_1^{[i]} \in \mathbb{R}_+^{n_z\times n}$, $D_{11}^{[i]} \in \mathbb{R}_+^{n_z\times n_w}$, and $D_{12}^{[i]} \in \mathbb{R}_+^{n_z\times n_u}$ $(i = 1, \cdots, N)$ are known matrices. On the other hand, $\theta \in \mathbb{R}_+^N$ is an uncertain time-invariant parameter whose only available information is $\theta \in \Theta$.

If we apply the state-feedback $u = Kx$ with $K \in \mathbb{R}^{n_u\times n}$ to this uncertain system, the closed-loop system $T(G_{\mathrm{d},\theta}, K)$ can be described by

$$T(G_{\mathrm{d},\theta}, K)\ \left\{\begin{array}{rclll} x(k+1) & = & (A(\theta) + B_2(\theta)K)x(k) & + & B_1(\theta)w(k), \\ z(k) & = & (C_1(\theta) + D_{12}(\theta)K)x(k) & + & D_{11}(\theta)w(k). \end{array}\right. \tag{20}$$

Our goal in this section is to find $K \in \mathbb{R}^{n_u\times n}$ such that $\inf_K \max_{\theta\in\Theta} \|T(G_{\mathrm{d},\theta}, K)\|_\infty$ under the constraint that

(i) the closed-loop system $T(G_{\mathrm{d},\theta}, K)$ remains to be positive for all $\theta \in \Theta$,

(ii) the feedback gain K satisfies a sparsity constraint specified by a given index set $\mathcal{Z} \in \{1, \cdots, n_u\} \times \{1, \cdots, n\}$ as in

$$K \in \mathcal{K}_{\mathrm{st},\mathcal{Z}}, \quad \mathcal{K}_{\mathrm{st},\mathcal{Z}} := \left\{K \in \mathbb{R}^{n_u\times n} : K_{i,j} = 0 \quad \forall (i,j) \in \mathcal{Z}\right\}. \tag{21}$$

The positivity constraint (i) is natural in some applications where preserving the positive nature of the open-loop system is necessary. This constraint is equivalent to

$$K \in \mathcal{K}_{\mathrm{pos}}, \tag{22}$$

$$\mathcal{K}_{\mathrm{pos}} := \left\{K \in \mathbb{R}^{n_u\times n} : A(\theta) + B_2(\theta)K \in \mathbb{R}_+^{n\times n},\ C_1(\theta) + D_{12}(\theta)K \in \mathbb{R}_+^{n_z\times n} \quad \forall\theta \in \Theta.\right\}.$$

It follows that our problem can be restated concisely as

$$\gamma_{\mathrm{opt}} = \inf_{K\in\mathcal{K}_{\mathrm{st},\mathcal{Z}}\cap\mathcal{K}_{\mathrm{pos}}} \max_{\theta\in\Theta} \|T(G_{\mathrm{d},\theta}, K)\|_\infty. \tag{23}$$

To compute the upper bound of γ_{opt} and obtain suboptimal controllers, we can derive the following SDPs by means of (14) and (16).

$$\gamma_{\rm d} = \inf_{X\in\mathbb{D}^n_{++}, Y\in\mathcal{K}_{{\rm st},\mathcal{Z}}} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} -X & 0 & B_1^{[i]} & A^{[i]}X + B_2^{[i]}Y \\ 0 & -\gamma I_l & D_{11}^{[i]} & C_1^{[i]}X + D_{12}^{[i]}Y \\ B_1^{[i]T} & D_{11}^{[i]T} & -\gamma I_m & 0 \\ (A^{[i]}X + B_2^{[i]}Y)^T & (C_1^{[i]}X + D_{12}^{[i]}Y)^T & 0 & -X \end{bmatrix} \prec 0, \tag{24}$$

$$A^{[i]}X + B_2^{[i]}Y \in \mathbb{R}_+^{n\times n},\ C_{12}^{[i]}X + D_{12}^{[i]}Y \in \mathbb{R}_+^{n_z\times n} \quad (i = 1, \cdots, N).$$

$$\gamma_{\rm c} = \inf_{X\in\mathbb{D}^n_{++}, Y\in\mathcal{K}_{{\rm st},\mathcal{Z}}} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} {\rm He}(A^{[i]}X + B_2^{[i]}Y - X) & (C_1^{[i]}X + D_{12}^{[i]}Y)^T & B_1^{[i]} \\ C_1^{[i]}X + D_{12}^{[i]}Y & -\gamma I_l & D_{11}^{[i]} \\ B_1^{[i]T} & D_{11}^{[i]T} & -\gamma I_m \end{bmatrix} \prec 0, \tag{25}$$

$$A^{[i]}X + B_2^{[i]}Y \in \mathbb{R}_+^{n\times n},\ C_{12}^{[i]}X + D_{12}^{[i]}Y \in \mathbb{R}_+^{n_z\times n} \quad (i = 1, \cdots, N).$$

If the LMI (24) or (25) is feasible, the desired state-feedback gain can be reconstructed by $K = YX^{-1}$. As noticed in [16], we can observe that the diagonal Lyapunov matrix works fine to yield a state-feedback gain K satisfying $K \in \mathcal{K}_{{\rm st},\mathcal{Z}} \cap \mathcal{K}_{\rm pos}$.

Now, we are ready to state the main result of this section. The next theorem shows that the SDP (25), which is originated from the new LMI characterization (16), is better (not worse) than the SDP (24) obtained from (14).

Theorem 4 For the SDPs (24) and (25), we have $\gamma_{\rm d} \geq \gamma_{\rm c}$.

Proof: For the proof, it suffices to show that if (14) is feasible with $X = X_0 \in \mathbb{D}^n_{++}$, then (16) is feasible with exactly the same $X = X_0$. To this end, we first note that (14) and (16) can be rewritten equivalently as

$$L_{\rm d}(X) := \begin{bmatrix} A_{\rm d}XA_{\rm d}^T - X & A_{\rm d}XC_{\rm d}^T \\ C_{\rm d}XA_{\rm d}^T & C_{\rm d}XC_{\rm d}^T - \gamma I_l \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} B_{\rm d} \\ D_{\rm d} \end{bmatrix}\begin{bmatrix} B_{\rm d} \\ D_{\rm d} \end{bmatrix}^T \prec 0,$$

$$L_{\rm c}(X) := \begin{bmatrix} (A_{\rm d} - I)X + X(A_{\rm d} - I)^T & XC_{\rm d}^T \\ C_{\rm d}X & -\gamma I_l \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} B_{\rm d} \\ D_{\rm d} \end{bmatrix}\begin{bmatrix} B_{\rm d} \\ D_{\rm d} \end{bmatrix}^T \prec 0.$$

Then, we have

$$L_{\rm d}(X_0) - L_{\rm c}(X_0) = \begin{bmatrix} A_{\rm d} - I \\ C_{\rm d} \end{bmatrix} X_0 \begin{bmatrix} A_{\rm d} - I \\ C_{\rm d} \end{bmatrix}^T \succeq 0.$$

This clearly shows that if $L_{\rm d}(X_0) \prec 0$, then we have $L_{\rm c}(X_0) \prec 0$. This completes the proof. ■

We finally note that, once we have obtained state-feedback gains $K_{\rm d}, K_{\rm c} \in \mathcal{K}_{{\rm st},\mathcal{Z}} \cap \mathcal{K}_{\rm pos}$ by solving the SDP (24) and (25), respectively, it is reasonable to solve the following SDP to evaluate the closed-loop system performance in a less conservative fashion:

$$\gamma_{\rm ana}(K) = \inf_{W\in\mathbb{R}^{n\times n}} \gamma \quad \text{subject to}$$

$$W + W^T \succ 0,$$

$$\begin{bmatrix} {\rm He}\{(A^{[i]}X + B_2^{[i]}K - I)W\} & W^T(C_1^{[i]} + D_{12}^{[i]}K)^T & B_1^{[i]} \\ (C_1^{[i]} + D_{12}^{[i]}K)W & -\gamma I_l & D_{11}^{[i]} \\ B_1^{[i]T} & D_{11}^{[i]T} & -\gamma I_m \end{bmatrix} \prec 0 \quad (i = 1, \cdots, N). \tag{26}$$

This SDP is based on the LMI (17). Since the variable $X \in \mathbb{D}_{++}^n$ in (25) is relaxed to $W \in \mathbb{R}^{n\times n}$ with $W + W^T \succ 0$ in (26), it is obvious that $\gamma_{\rm c} \ge \gamma_{\rm ana}(K_{\rm c})$. It is also easy to see that $\gamma_{\rm d} \ge \gamma_{\rm ana}(K_{\rm d})$. In this way, we can evaluate the close-loop system performance in a less conservative way by using the SDP (26).

7.2 Numerical Example

Let us consider the case where $N = 2$ in (18) and (19) where the coefficient matrices are

$$\begin{aligned}
\left[\begin{array}{ccc} A^{[1]} & B_1^{[1]} & B_2^{[1]} \\ C_1^{[1]} & D_{11}^{[1]} & D_{12}^{[1]} \end{array}\right] &= \left[\begin{array}{cccc:c:cc} 0.4 & 0.5 & 0.1 & 0.2 & 0.9 & 0.1 & 0.5 \\ 0.4 & 0.1 & 0.1 & 0.5 & 0.1 & 0.3 & 0.7 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.9 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0 & 0.3 & 0.4 & 0.3 & 0.8 \\ \hdashline 0.1 & 0.2 & 0.2 & 0.5 & 0.1 & 0.8 & 0.6 \end{array}\right], \\
\left[\begin{array}{ccc} A^{[2]} & B_1^{[2]} & B_2^{[2]} \\ C_1^{[2]} & D_{11}^{[2]} & D_{12}^{[2]} \end{array}\right] &= \left[\begin{array}{cccc:c:cc} 0.3 & 0.2 & 0.4 & 0.1 & 0.9 & 0.8 & 0.4 \\ 0.3 & 0.3 & 0.3 & 0.1 & 0.7 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.1 & 0.1 & 0.7 & 0.9 & 1.0 \\ 0.2 & 0.3 & 0.5 & 0.5 & 0.4 & 0.3 & 1.0 \\ \hdashline 0.5 & 0.7 & 0.4 & 0 & 0.1 & 0.3 & 0.7 \end{array}\right].
\end{aligned} \tag{27}$$

For this plant, we solve the state-feedback robust H_∞ control problem (23) with $\mathcal{Z} = \{(3,3),(3,4),(4,3),(4,4)\}$. This structural constraint implies that we actually look for static output-feedback controllers $u = Ky$ with $y = [x_1\ x_2]^T$. By solving the SDPs (24), (25), and (26), we obtained the following results:

$$\begin{aligned}
K_{\rm d} &= \begin{bmatrix} 0.1667 & -0.0140 & 0 & 0 \\ -0.2500 & -0.1368 & 0 & 0 \end{bmatrix}, \quad \gamma_{\rm d} = 33.0912, \quad \gamma_{\rm ana}(K_{\rm d}) = 7.3878, \\
K_{\rm c} &= \begin{bmatrix} 0.1667 & -0.2105 & 0 & 0 \\ -0.2500 & -0.0526 & 0 & 0 \end{bmatrix}, \quad \gamma_{\rm c} = 6.6884, \quad \gamma_{\rm ana}(K_{\rm c}) = 6.3178.
\end{aligned} \tag{28}$$

We can see that $\gamma_{\rm d} \ge \gamma_{\rm c}$, $\gamma_{\rm d} \ge \gamma_{\rm ana}(K_{\rm d})$, and $\gamma_{\rm c} \ge \gamma_{\rm ana}(K_{\rm c})$ hold. We thus confirmed the effectiveness of the LMIs (16) and (17) derived in this paper.

8 Conclusion

In this paper, we showed that several remarkable and peculiar results for positive system analysis can be proved concisely by duality-based arguments. We clarified a necessary and sufficient condition under which a Hurwitz (Schur) stable Metzler (nonnegative) matrix admits a diagonal Lyapunov matrix with some identical diagonal entries as the solution of the Lyapunov inequality. This result leads us to an alternative and concise proof for the fact that the KYP-type LMI characterizing the H_∞ performance of positive systems admits diagonal Lyapunov matrices as well. On the other hand, we also showed that the Lyapunov matrix in the Lyapunov inequalities and the KYP-type LMIs can be relaxed to non-symmetric in the case of positive systems. Moreover, for the H_∞ performance of discrete-time positive systems, we derived new LMIs that are structurally different from the KYP-type LMI. We illustrated the effectiveness of these new LMIs by numerical examples on structurally-constrained robust state-feedback H_∞ controller synthesis.

9 Acknowledgements

The authors are grateful to Professor Tomomichi Hagiwara, Kyoto University, Japan, and Professor Yasuaki Oishi, Nanzan University, Japan, for the fruitful discussion on LMI-based positive system analysis.

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