Measures and LMIs for Impulsive Nonlinear Optimal Control

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***Abstract*—This note shows how to use semi-definite programming to find lower bounds on a large class of nonlinear optimal control problems with polynomial dynamics and convex semialgebraic state constraints and an affine dependence on the control. This is done by relaxing an optimal control problem into a linear programming problem on measures, also known as a generalized moment problem. The handling of measures by their moments reduces the problem to a convergent series of standard linear matrix inequality relaxations. When the optimal control consists of a finite number of impulses, we can recover simultaneously the actual impulse times and amplitudes by simple linear algebra. Finally, our approach can be readily implemented with standard software, as illustrated by a numerical example.**

I. Introduction

Optimal control is still an active area of current research despite the availability of powerful theoretical tools such as Pontryagin's maximum principle or the Hamilton-Jacobi-Bellman approach. However, numerical methods based on such optimality conditions rely on a certain number of assumptions that are often not met in practice, and state constraints are particularly hard to handle in the maximum principle framework.

On the other side, many numerical methods have been developed that deliver suboptimal solutions by restricting the search space and parametrizing it. However, the users of these methods are often left to wonder if a better solution exists. For example, in the particular case of impulsive controls, one could assume purely impulsive solutions of at most n impulses and obtain a static optimization problem with impulse times and amplitudes as unknowns, but it is often not known if more regular solutions could provide a better cost.

For a recent survey on impulsive control see e.g. [12] and the references therein. See also [8] for a recent application and more references. For historical works see e.g. [15], [16], [17] and also [3].

In this note, we describe and test a numerical method following ideas of [13], [10], but which addresses nonlinear optimal control problems whose optimal solution may now include impulsive controls. Our numerical scheme consists in solving a hierarchy of semi-definite relaxations in the form of Linear Matrix Inequalities (LMIs), whose associated sequence of optimal values provides a monotone nondecreasing sequence of lower bounds on the global minimum of affine-in-the control optimal control problems. In particular, the method may assert the global optimality of local solutions found by other methods, and as importantly, can also provide numerical certificates of infeasibility for ill-posed problems. Finally, in some cases, it is also possible to generate the globally optimal control law.

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A. Contributions

The note improves the model presented in [13], [10] in the following ways. First of all, the range of applications is much larger as impulsive controls can now be taken into account. Second, because controls are represented by measures and not by variables, the size of semi-definite programming (SDP) blocks composing the LMIs is significantly reduced. This allows to handle larger problems in terms of number of state variables as well as to reach higher LMI relaxations. Finally, total variation of controls can be handled very easily, as non-differentiable constraints or a cost to minimize.

Let us emphasize the fact that this note does not aim at a comprehensive mathematical treatment of impulsive control problems, in particular we do not investigate here the dual Hamilton-Jacobi-Bellman partial differential equation satisfied by the value function and its regularity properties. These developments will be reported elsewhere. We believe that a key contribution of our work is to provide a numerical method based on standard interfaces and solvers, relying on sophisticated, albeit by now relatively standard LMI formulations for measure/moment linear programming problems. As far as we know, this is the first time that a systematic, constructive and reproducible numerical approach is proposed for such constrained control problems. In non-trivial examples it has permitted to validate some results obtained by other local optimization methods and certify that the resulting solution was globally optimal.

B. Nomenclature

Integration of a function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to a measure μ on a set $X \subset \mathbb{R}^n$ is written $\int_X f(x)\, d\mu(x)$ or sometimes $\int_X f(x)\, \mu(dx)$ when more convenient. The Lebesgue or uniform measure on X is denoted by λ_X whereas the Dirac measure concentrated at point x^* is denoted by δ_{x^*}. A measure μ is a probability measure on the set X whenever $\int_X d\mu = 1$. The support of measure μ on X is the largest closed set B such that $\mu(X \setminus B) = 0$, and is denoted by $\operatorname{supp} \mu$. The indicator function of set X (equal to one in X and zero outside) is denoted by I_X.

$F(X)$ is the space of Borel measurable functions on X, whereas $BV(X)$ is the space of functions of bounded variation on X. $\mathcal{M}^+(X)$ is the space of finite positive measures on X whereas $\mathcal{M}(X)$ is the space of finite signed measures. $\mathbb{R}[x]$ is the ring of polynomials in the variable x. $\mathcal{B}(X)$ denotes the Borel σ-algebra associated with X.

If $k \in \mathbb{N}^n$ denotes a vector of indices then x^k with $x \in \mathbb{R}^n$ is the multi-index notation for $\prod x_i^{k_i}$. The degree of the index k is $\deg k = \sum k_i$. Finally, $\mathbb{N}^n_d$ is the set of all indices for which $\deg k \leq d,\ k \in \mathbb{N}^n$.

II. The optimal control problem

This note deals with the following nonlinear optimal control problem

$$\begin{aligned} V(x_0) &= \inf_{u(t) \in F([0,T])^m} J(x_0, u) \qquad (1)\\ &= \inf_{u(t)} \int_0^T (h(t, x(t)) + H(t)\, u(t))\, dt + \ h_T(x(T)) \end{aligned}$$

such that

$$\begin{aligned} &\dot{x}(t) = f(t, x(t)) + G(t)\, u(t), \quad \text{a.a. } t \in [0,T] \qquad (2)\\ &x(0) = x_0, \quad x(T) \in X_T,\\ &x(t) \in X \subset \mathbb{R}^n, \forall t \in [0,T], \end{aligned}$$

where the dot denotes differentiation with respect to time. Criterion $J(x_0, u)$ is *affine* in the controls u, and V is called the value function. It is assumed throughout this note that all problem data are polynomials, meaning that all functions are in $\mathbb{R}[t, x]$, and that all sets are compact basic semialgebraic. Recall that such sets are those which may be written as $\{x : p_i(x) \geq 0,\ i = 1, \dots, q\}$ for some family $\{p_i\}_{j=1}^q$, $p_i \subset \mathbb{R}[x]$. A mild technical condition (implying compactness of X) must be satisfied [14, Assumption 2.1]: There exists a r in the quadratic module generated by the family $\{p_i\}$ such that the level set $\{x \in \mathbb{R}^n : r(x) \geq 0\}$ is compact. In practice, this condition is often met, and adding a standard ball constraint $\sum x_i^2 \leq r^2$ to the state constraints will enforce the condition. The reason for making these assumptions will be apparent in later sections. Finally, set X is assumed convex, such that the construction proposed in §III is well defined.

Note that the hypotheses of polynomial dynamics, hence Lipschitz continuous on compact sets, implies uniqueness of the trajectory for a given pair (x_0, u), given this trajectory exists. This justifies the notation for $J(x_0, u)$.

Without additional assumptions, the infimum in problem (1)-(2) is not attained in general because of possible concentration (impulsive) effects [15]. A standard procedure to deal with this fact is to embed the controls $u(t)$ in the space of the weak derivatives of functions of bounded variation $w(t)$:

$$\begin{aligned} V_R(x_0) &= \inf_{w(t) \in BV([0,T])^m} J_R(x_0, w) \qquad (3)\\ &= \inf_{w(t)} \int_0^T h(t, x(t))\, dt + \int_0^T H(t)\, dw(t) + h_T(x(T)) \end{aligned}$$

such that

$$\begin{aligned} &dx(t) = f(t, x(t))\, dt + G(t)\, dw(t), \quad t \in [0,T] \qquad (4)\\ &x(0) = x_0, \quad x(T) \in X_T, \quad x(t) \in X \subset \mathbb{R}^n, \qquad (5) \end{aligned}$$

where V_R stands for the relaxed value function.

The relaxed controls above can be seen as a distribution of the first order, identified with measures, and it is therefore the (vector) distributional derivative $dw(t)$ of some (vector) function of bounded variation $w(t) \in BV([0,T])^m$, see e.g. [17], or [5, Prop. 8.3]. That is, for all compactly supported smooth test functions $v(t)$, it holds that $\int v(t)\, dw(t) = -\int w(t)\, \dot{v}(t)\, dt$. It can then be shown [17] that $x(t)$ satisfies (4) if and only if it is the solution of

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds + \int_0^t G(s) dw(s). \qquad (6)$$

Classic controls $u(t) \in F([0,T])^m$ are recovered as the derivative of the absolutely continuous parts of $w(t)$, whereas impulses arise at discontinuity points of $w(t)$. The possibility of handling both continuous and impulsive controls in a unified formalism is also a practical motivation for relaxation (3)-(4), besides the mathematical need for a well-posed problem.

Finally, also note that there may be a strict gap induced by this relaxation (i.e. $V_R < V$) in some non-generic degenerate cases. We therefore assume for the rest of this note that indeed, $V_R = V$.

III. The linear problem on measures

In this section, we formulate problem (3)-(4) as an equivalent infinite-dimensional linear programming problem on measures, a particular instance of the so-called *generalized moment problem* (see [14] for an introduction on the subject). This is a necessary intermediate step towards obtaining a tractable SDP problem for our method.

Problem (3)-(4) is control-affine, but states enter nonlinearly in dynamics, costs and constraints. State trajectories must also satisfy a differential equation, and further development is therefore needed to obtain a linear problem on measures. One way to work around this fact is to use so called *occupation measures* to encode the graph of both admissible trajectories and controls on measurable subsets of time and space.

Both elements of an admissible pair $(w(t), x(t))$ are vector of functions of bounded variation with respect to time, which may be identified with measures of time. By Lebesgue's decomposition theorem [11, §33.3], the control $w(dt)$ may be split into two parts:

$$w(dt) = \tilde{w}(dt) + w^j(dt) \qquad (7)$$

where, in keeping with the terminology of [1, Chap. 3], $\tilde{w}$ is the diffuse part, the sum of an absolutely continuous and a singular continuous (Cantor part) function, whereas $w^j(dt)$ is the jump part, supported on at most countably many points $t_i,\ i \in J$. That is, w is continuous almost everywhere on $[0,T]$, and w^j may be written as $w^j(dt) = \sum_{i \in J} U_i\, \delta_{t_i}(dt)$ with *jump amplitude* vectors $U_i \in \mathbb{R}^m$ supported at impulsive *jump times* $t_i,\ i \in J$. Obviously, $x(t)$ possesses the same decomposition, and has therefore left $x(t^-)$ and right $x(t^+)$ continuous limits for every t in $(0, T)$. At jump points of $x(t)$, which by (4) are the same as for $w(t)$, the following holds:

$$x(t_i^+) - x(t_i^-) = G(t_i)U_i, \quad \forall\, i \in J. \tag{8}$$

To encode those discontinuous trajectories, define the stochastic kernel ξ, also known as the conditional measure, as:

$$\xi(B|t) = \begin{cases} I_B(x(t)) = \int_B \delta_{x(t)}(dx), & \forall t \in [0,T] \setminus J \\ \frac{\lambda([x(t_i^-),x(t_i^+)] \cap B)}{\lambda([x(t_i^-),x(t_i^+)])}, & \forall t_i \in J. \end{cases} \tag{9}$$

That is, $\xi(\cdot \,|\, t)$ is the Dirac measure concentrated at state $x(t)$ along continuous trajectory arcs, while during jumps, it is uniformly distributed along the segment linking the state before and after the jump. This is represented by the Lebesgue measure defined on a line segment in X (not to be confused with the Lebesgue measure on the n-dimensional space X). The above denominator ensures that $\xi(\cdot \,|\, t)$ has unit mass for all t and therefore remains a probability measure during jumps. Readers familiar with the impulsive literature will recognize, behind this construct, the canonical graph completion introduced in [4]. Indeed, one can represent the trajectory "during" a jump by parametrizing the line segment $[x(t_i^-), x(t_i^+)]$ with $s \mapsto x(s) = x(t_i^-) + s\left(x(t_i^+) - x(t_i^-)\right)$, $s \in [0,1]$, such that $\xi(dx|t_i) = ds$. The fact that X is convex guarantees the existence of such a canonical completion.

The stochastic kernel ξ possesses by construction the following properties, for any continuously differentiable function $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $\forall t \in [0,T] \setminus J$:

$$v(x(t)) = \int_X v\, \xi(dx|t), \tag{10}$$

whereas, $\forall t \in J$,

$$v(x(t^+)) - v(x(t^-)) = \int_0^1 \frac{\partial v}{\partial x} \cdot \left(x(t^+) - x(t^-)\right) ds \tag{11}$$

$$= \int_X \frac{\partial v}{\partial x} \cdot \left(x(t^+) - x(t^-)\right) \xi(dx|t). \tag{12}$$

These properties allow to rigorously define the various occupation measures that will be the decision variables of a generalized problem of moments. For this purpose, the *time occupation measure*, measuring the occupation of $A \times B$ by the pair $(t, x(t))$ all along the trajectory, is defined as:

$$\mu[x_0, w(t)](A \times B) = \int_A \xi(B \,|\, t)\, dt \tag{13}$$

$\forall A \in \mathcal{B}([0,T])$, $\forall B \in \mathcal{B}(X)$. In particular, $\mu[x_0, w(t)](A, B)$ is the time spent by the trajectory $x(t)$ on set $A \times B$ (whence the name "occupation measure"). Note that we write $\mu[x_0, w(t)]$ to emphasize the dependence of μ on initial state x_0 and relaxed control $w(t)$. However, for notational simplicity, we may use the notation μ.

The *control occupation measure* can then be defined as:

$$\omega[x_0, w(t)](A \times B) = \int_A \xi(B \,|\, t)\, dw(t) \tag{14}$$

$\forall A \in \mathcal{B}([0,T])$ and $\forall B \in \mathcal{B}(X)$, and with $\xi(\cdot \,|\, t)$ defined as in (9). In the same way as for the time occupation measure, $\omega(A, B)$ measures the amount of control delivered by the trajectory $x(t)$ on set $A \times B$.

Finally, the *final state occupation measure* is defined as

$$\mu_T[x_0, w(t)](B) = I_B(x(T)), \qquad \forall B \in \mathcal{B}(X). \tag{15}$$

By construction, for any continuous function $v : \mathbb{R}^n \to \mathbb{R}$, the following holds :

$$\int_{X_T} v\, d\mu_T[x_0, w(t)](x) = v(x(T)). \tag{16}$$

The next theorem show how these measures are related:

Theorem 1. If $w(t)$ is an admissible relaxed control for a trajectory starting at x_0 and satisfying relaxed dynamics (4) on $[0,T] \times X$, then its corresponding occupation measure $\mu[x_0, w(t)]$, final state measure $\mu_T[x_0, w(t)]$ and control measure $\omega[x_0, w(t)]$ satisfy the linear equation

$$\int_{X_T} v(T, x)\, d\mu_T(x) - v(0, x_0) = \int_{[0,T]\times X} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f\right) d\mu + \int_{[0,T]\times X} \frac{\partial v}{\partial x} G\, d\omega \tag{17}$$

for all continuously differentiable test functions $v(t, x)$ on $[0,T] \times X$.

Proof: Evaluating such a test function along an admissible trajectory yields, by the chain rule on BV functions [1, Th. 3.96]:

$$v(T, x(T^+)) - v(0, x(0^-)) = \int_0^T dv(t, x(t)) = \\ = \underbrace{\int_0^T \frac{\partial v}{\partial t} dt}_{A_1} + \underbrace{\int_0^T \frac{\partial v}{\partial x} d\tilde{x}}_{A_2} + \underbrace{\sum_{i \in J} v(t_i, x(t_i^+)) - v(t_i, x(t_i^-))}_{A_3} . \tag{18}$$

The aim is to express the above temporal integration as a spatial integration with respect to the previously defined occupation measures.

Given (16), the left hand side of (17) is equal to the left hand side of (18). For the right-hand side, observe that by (10) and (13):

$$A_1 = \int_0^T \int_X \frac{\partial v}{\partial t} \xi(dx|t)\, dt = \int_{[0,T]\times X} \frac{\partial v}{\partial t} d\mu(t, x). \tag{19}$$

Using (4) and the fact that $x(t)$ is diffuse when $w(t)$ is, A_2 is given by

$$A_2 = \int_0^T \frac{\partial v}{\partial x} f(t, x(t))\, dt + \int_0^T \frac{\partial v}{\partial x}(t, x(t))\, G\, d\tilde{w}(t) \tag{20}$$

$$= \int_{[0,T]\times X} \frac{\partial v}{\partial x} f(t, x)\, d\mu + \int_0^T \int_X \frac{\partial v}{\partial x} G\, d\xi(x|t)\, d\tilde{w}(t), \tag{21}$$

where the last relation uses again (10) and the definitions (13) and (14). Finally, injecting (12) in A_3 gives

$$A_3 = \sum_{i\in J} \int_X \frac{\partial v}{\partial x} \left(x(t_i^+) - x(t_i^-)\right) \xi(dx|t_i) \quad (22)$$

$$= \sum_{i\in J} \int_0^T \int_X \frac{\partial v}{\partial x} G(t)\, \xi(dx|t) \underbrace{U_i \delta_{t_i}(dt)}_{=dw^j(t)}, \quad (23)$$

where the last relation uses (8). Since $w(dt) = \tilde{w}(dt) + w^j(dt)$, using (14) leads to the desired result. ■

The following obvious lemma provides a characterization of infeasible impulsive optimal control problems:

Lemma 1. If no triplet of arbitrary finite measures $(\mu, \mu_T, \omega) \in (\mathcal{M}^+, \mathcal{M}^+, \mathcal{M})$ satisfies (17), neither original problem (1)-(2) or relaxed problem (3)-(4) are feasible.

Turning now to feasible problems, the standard measure relaxation that is the topic of this section consists in enlarging the search among all triplets of measure satisfying (17), instead of only considering the occupation measures as defined above:

Lemma 2 (Measure relaxation)**.** Consider the minimization problem on arbitrary finite measures:

$$\begin{aligned} V_M(x_0) &= \inf_{\mu,\mu_T\in\mathcal{M}^+,\ \omega\in\mathcal{M}} J_M(x_0,\mu,\mu_T,\omega) \\ &= \inf_{\mu,\mu_T,\omega} \int_{[0,T]\times X} h\, d\mu + \int_{[0,T]\times X} H\, d\omega + \int_{X_T} h_T\, d\mu_T \end{aligned} \quad (24)$$

under constraints (17). Then $V(x_0) \geq V_R(x_0) \geq V_M(x_0)$.

Proof: First, by construction, problem (3)-(4) is a relaxation of problem (1)-(2), since we optimize over derivatives of functions of bounded variations, i.e. measures, instead of measurable functions. This proves the first inequality.

Secondly, by Theorem 1, every admissible trajectory for problem (24) generates an occupation and control measure satisfying (17). This proves the second inequality. ■

It should be noted that for a well-posed control problem (3)-(4), one expects that in fact $V_M(x_0) = V_R(x_0)$ and that an optimal solution of the relaxed problem will be a triplet of occupation measures corresponding to an optimal trajectory of relaxed problem (3)-(4) with given initial state x_0 and relaxed control $w(t)$. However, this will be proved in subsequent works. Note that for the standard polynomial optimal control problem (1)-(2), without impulsive controls, and under additional convexity assumptions, it has been proved in [13] that indeed $V_M(x_0) = V_R(x_0) = V(x_0)$. See also [7].

We provide now a few extensions of our canonical problem that are easily captured by the occupation measure formalism.

A. Free initial state

We consider now the case where x_0 is itself a decision variable of our optimization problem instead of being given, taking its values in the compact set X_0. This initial state incurs an additional initial cost of $h_0(x_0)$ to the total cost. This can be handled by introducing the *initial state* occupation measure μ_0 as

$$\mu_0[x_0, w(t)](B) = I_B(x_0), \qquad \forall B \in \mathcal{B}(X) \quad (25)$$

which is the obvious analogue of the final-state occupation measure μ_T. Measure μ_0 can be seen as an *unknown* probability measure on X_0. The appropriate modifications for problem (24) can then easily be deduced by analogy with μ_T. It is however necessary to add the constraint that one of the end-point measures is a probability measure, i.e. $\int d\mu_0 = 1$, to avoid trivial solutions.

B. Decomposition of control measures and handling of total variation

All measures in problem (24) are positive measures, except for the signed measures ω which deserve special treatment. In fact, in the next section and in the rest of the note, only positive measures will be considered. One way to proceed is to relax $u(t) = u^+(t) - u^-(t)$ in original problem (1)-(2), with $u^+(t)$ and $u^-(t)$ both positive functions, and embedding this new problem into measure relaxations as above. This leads in problem (24) to substitute ω by $\omega^+ - \omega^-$, with those signed control-state occupation measure associated to respectively $u^+(t)$ and $u^-(t)$. It should be kept in mind that this decomposition is not unique; Adding an arbitrary finite positive measure ν to both ω^+ and ω^- yields the same ω.

One obvious advantage of this decomposition is that problems of minimization of the $\mathcal{L}_1$ norm of the original controls $u(t)$, such as those taken from the orbital rendezvous literature, can now be handled. Indeed, consider the new problem

$$V = \inf_{u(t)} \sum_i \int_0^T |u_i(t)|\, dt. \quad (26)$$

under dynamical constraints (2). Its occupation measure relaxation then reads

$$V_M = \inf_\omega \sum_i \int_{[0,T]\times X} d|\omega_i| = \inf_\omega \sum_i \int_{[0,T]\times X} d\left(\omega_i^+ + \omega_i^-\right). \quad (27)$$

Non-unicity of the decomposition is not an issue here, as the cost to be minimized is the sum of masses of positive measures. Hence, the solutions ω^+ and ω^- will naturally tend to the standard Hahn-Jordan decomposition of measure ω, which is unique except on sets of null Lebesgue measure [2].

IV. The associated moment problem

So far, the hypothesis of polynomial data has not been used, but this crucial assumption is necessary for this section, where measures are manipulated through their moments. This leads to a semi-definite programming (SDP) problem with countably many linear constraints.

Define the moments of a measure $\mu(dz)$ on $Z \subset \mathbb{R}^n$ as

$$y_k^\mu = \int_Z z^k\, d\mu(z) := \int_Z z_1^{k_1} \cdots z_n^{k_n}\, d\mu(z). \quad (28)$$

Then, with a sequence $y = (y_k)$, $k \in \mathbb{N}^n$, let $L_y : \mathbb{R}[z] \to \mathbb{R}$ be the (Riesz) linear functional of $f = \sum_k f_k z^k$

$$f \quad \mapsto \quad L_y(f) = \sum_k f_k y_k, \quad f \in \mathbb{R}[z]. \quad (29)$$

Define the moment matrix of order $d \in \mathbb{N}$ associated with y as the real symmetric matrix $M_d(y)$ whose (i,j)th entry reads

$$M_d(y)[i,j] = L_y\left(z^{i+j}\right) = y_{i+j}, \quad \forall i,j \in \mathbb{N}^n_d. \tag{30}$$

Similarly, define the localizing matrix of order d associated with y and $h \in \mathbb{R}[z]$ as the real symmetric matrix $M_d(h\,y)$ whose (i,j)th entry reads

$$\begin{aligned} M_d(h\,y)[i,j] &= L_y\left(h(z)\,z^{i+j}\right) && (31)\\ &= \sum_k h_k\, y_{i+j+k}, \quad \forall i,j \in \mathbb{N}^n_d. && (32)\end{aligned}$$

As a last definition, a sequence $y^\mu = (y^\mu_k)$ is said to have a representing measure if there exists a finite Borel measure μ on X, such that relation (28) holds for every $k \in \mathbb{N}^n$.

The construction of the SDP program associated with generalized moment problem (24) can now be stated. Its decision variable is the sequence of moments y, the aggregate of sequences y^μ, $y^{\omega^\pm}$ and y^{μ_T}. As cost (24) is polynomial, it can be rewritten as the scalar product $b'y$, with b identified term-by-term by:

$$b'y = L^\mu_y(h) + L^{\mu^+}_y(H) - L^{\mu^-}_y(H) + L^{\mu_T}_y(h_T). \tag{33}$$

That is, b is the coefficients of polynomials h, H and h_T expressed in monomial basis (28).

Similarly, constraint (17) needs only be verified for the countably many polynomial test functions $v \in \mathbb{R}[t,x]$, since the measures are supported on compact subsets of $\mathbb{R}^{1+n}$. Therefore, for the k-th such test function v_k, (17) defines a linear constraint between moments of the form $a'_k y = c_k$. Scalar c_k and sequence a_k can be deduced by identification with

$$\begin{aligned} a'_k y - c_k = \; & L^{\mu_T}_y(v_k(T,\cdot)) - v_k(0,x(0)) \\ & -L^\mu_y\left(\tfrac{\partial v_k}{\partial t} + \tfrac{\partial v_k}{\partial x} f\right) \\ & -L^{\mu^+}_y\left(\tfrac{\partial v_k}{\partial x} G\right) + L^{\mu^-}_y\left(\tfrac{\partial v_k}{\partial x} G\right) \end{aligned}. \tag{34}$$

Finally, the only nonlinear constraints are the convex SDP constraints for measure representativeness. Indeed, it follows from [14, Theorem 3.8] that a sequence of moments y^μ has a representing measure defined on an Archimedean, basic semi-algebraic set $X^\mu = \{x : p^\mu_i(x) \geq 0,\ i = 1,2,\ldots\}$ if and only if $M_d(y^\mu) \succeq 0,\ \forall\, d \in \mathbb{N}$ and $M_d(p^\mu_i\, y^\mu) \succeq 0,\ \forall\, d \in \mathbb{N}$ and $\forall p^\mu_i$ defining set X^μ.

This leads to the problem:

$$V^\infty_M = \inf_y b'y \tag{35}$$

such that

$$\begin{array}{ll} & Ay = c, \\ \forall d \in \mathbb{N}: & M_d(y^\mu) \succeq 0, \qquad M_d(p^\mu_i\, y^\mu) \succeq 0, \\ & M_d(y^{\omega^\pm}) \succeq 0, \quad M_d(p^{\omega^\pm}_i\, y^{\omega^\pm}) \succeq 0, \\ & M_d(y^{\mu_T}) \succeq 0, \quad M_d(p^{\mu_T}_i\, y^{\mu_T}) \succeq 0, \end{array} \tag{36}$$

where operator A and sequence c are built from (34).

We have proved the following result:

Theorem 2. Relaxed problem on measures (24) and the infinite-dimensional SDP moment problem (35) share the same optimum:

$$V_M = V^\infty_M. \tag{37}$$

For the rest of the note, we will therefore use V_M to denote the cost of the Measure LP problem (24) or Moment SDP problem (35) indifferently.

V. LMI relaxations

The final step to reach a tractable problem is relatively obvious: We simply truncate the problem to its first few moments. Let $d_0 \in \mathbb{N}$ be the smallest integer such that all criterion, dynamics and constraint monomials belong to $\mathbb{N}^{n+1}_{2d_0}$. This is the degree of the so called *first relaxation*. For each relaxation, we get a standard LMI problem that can be solved numerically by off-the-shelf software by simply truncating in problem (35) to involve only moments in $\mathbb{N}^{n+1}_{2d}$, with $d \geq d_1$ the relaxation order.

Theorem 3. Let us denote by $V^{d_0+i}_M$ the optimum obtained by solving the finite-dimensional truncation to moments of degree up to $2(d_0+i)$ of SDP problem (35), for $i = 0,1,\ldots$ Then

$$V^{d_0}_M \leq V^{d_0+1}_M \leq \cdots V^\infty_M = V_M. \tag{38}$$

Proof: By construction, observe that $j > i \Rightarrow V^{d_0+j}_M \geq V^{d_0+i}_M$, i.e. the sequence $V^{d_0+i}_M$ is monotonically non-decreasing. Asymptotic convergence to V_M follows from [14, Theorem 3.8] as in the proof of Theorem 2. ■

Therefore, by solving the truncated problem for ever greater relaxation orders, we will obtain a monotonically non-decreasing sequence of lower bounds to the true optimal cost. In the example below, we will see that in practice, the optimal cost is usually reached after a few relaxations.

Remark 1. The GloptiPoly software [9] completely automates the construction of truncated moment problems from high level definitions such as those defining problem (24)–(17). This is why sections IV-V have been kept to a bare minimum. See [14] for more details.

VI. Example

In this section, a basic example is worked out to illustrate how the method works. The example uses GloptiPoly [9] for building the truncated LMI moment problems and SeDuMi [18] for its numerical solution. For other examples, see [6].

Consider

$$V = \inf_{u(t)} \int_0^2 x^2(t)\, dt \tag{39}$$

such that

$$\begin{array}{l} \dot{x}(t) = u(t), \\ x(0) = 1, \quad x(2) = \frac{1}{2}, \\ x^2(t) \leq 1. \end{array} \tag{40}$$

The optimal solution for this problem consists in reaching the turnpike $x(t) = 0$ by an impulse at initial time $t = 0$, and likewise, departing from it by an impulse at final time $t = T = 2$.

By injecting $h = x^2$, $f = 0$ and $G = 1$ in (24)-(17), and using the fact that the fixed-end problem has $\mu_T = \delta_{\frac{1}{2}}$, the associated problem on measure reads:

$$V_M = \inf_{\mu,\omega^+,\omega^- \in \mathcal{M}^+} \int_K x^2\, d\mu \tag{41}$$

such that, $\forall\, v \in \mathbb{R}[t,x]$,

$$v(2,\frac{1}{2}) - v(0,1) = \int_K \frac{\partial v}{\partial t}\, d\mu + \int_K \frac{\partial v}{\partial x}\, d\left(\omega^+ - \omega^-\right), \quad (42)$$

$$K = \left\{(t,x) \in \mathbb{R}\times\mathbb{R} : \begin{array}{l} 1-x^2 \geq 0 \\ t(2-t) \geq 0 \end{array}\right\}. \quad (43)$$

The last equation defining K is one possible choice for a semialgebraic representation of $t \in [0,2]$.

We consider now the construction of the first LMI relaxation of (41)-(42)-(43), defining moments by

$$y^\mu_{ij} = \int t^i x^j\, d\mu \qquad i,j \in \mathbb{N}. \quad (44)$$

This relaxation will involve moments y_{ij} such that $i + j \leq 2$ since cost (41) and constraints (43) are quadratic. Given (44), cost (41) is simply:

$$\int_K x^2\, d\mu = y^\mu_{02}. \quad (45)$$

For dynamical constraints (42), we need to consider polynomial test functions of the form $v(t,x) = t^i x^j$ with $i + j \leq 2$. For instance, for the fifth test function in graded lexicographic order $v_5 = tx$, (42) reduces to

$$2\cdot\frac{1}{2} - 0\cdot 1 = \int_K x\, d\mu + \int_K t\, d\left(\omega^+ - \omega^-\right), \quad (46)$$

hence to moment constraint

$$1 = y^\mu_{01} + y^{\omega^+}_{10} - y^{\omega^-}_{10}. \quad (47)$$

Finally, the representativeness constraint for each moment sequence associated with measure $\tau = \{\mu, \omega^+, \omega^-\}$ are as follows. Moment matrix $M_1(y^\tau)$, whose lines and columns are referenced by multi-indexes $\{00, 10, 01\}$, is constructed from (30) to give the SDP constraint:

$$\begin{bmatrix} y^\tau_{00} & y^\tau_{10} & y^\tau_{01} \\ y^\tau_{10} & y^\tau_{20} & y^\tau_{11} \\ y^\tau_{01} & y^\tau_{11} & y^\tau_{02} \end{bmatrix} \succeq 0. \quad (48)$$

Localizing matrices $M_0(p^\tau_i y^\tau)$ associated with each constraint p^τ_i defining set (43) are simply scalars for the first relaxation. Using (31) thus yields the positivity constraints:

$$\begin{aligned} M_0((1-x^2)y^\tau)[00,00] &= L_{y^\tau}((1-x^2)t^0x^0) \\ &= y^\tau_{00} - y^\tau_{02} \geq 0 \end{aligned} \quad (49)$$

for the constraint $1 - x^2 \geq 0$, and likewise for $t(2-t) \geq 0$,

$$M_0(t(2-t)y^\tau) = 2y^\tau_{10} - y^\tau_{02} \geq 0. \quad (50)$$

Solving the LMI problem just constructed yields $V^1_M = 0$, the true cost to (39)-(40). In addition, using the extraction routine defined in [14] at the second relaxation, the optimal control and trajectory can be recovered from the value of the optimal measures:

$$\mu(dt,dx) = \lambda_{[0,2]}(dt)\,\delta_0(dx) \quad (51)$$

$$\omega(dt,dx) = -\delta_0(dt)\,\lambda_{[0,1]}(dx) + \delta_2(dt)\,\lambda_{[0,\frac{1}{2}]}(dx). \quad (52)$$

VII. Conclusion

The focus of this work is on actual computation of optimal impulsive controls for systems described by ordinary differential equations with polynomial dynamics and polynomial (semi-algebraic) constraints on the state. State trajectory and controls are captured by measures which are linearly constrained, resulting in an infinite-dimensional Linear Programming (LP) problem consistent with the formalism of our GloptiPoly software [9]. This LP problem on measures can then be solved numerically via a hierarchy of Linear Matrix Inequality (LMI) relaxations, for which off-the-shelf Semi-Definite Programming (SDP) solvers can be used. If the solution is purely impulsive, the optimal impulse sequence can then be retrieved by simple linear algebra, and the tightness of the bounds obtained with our approach can be assessed by *a posteriori* simulation or comparison with suboptimal control sequences computed by alternative techniques.

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