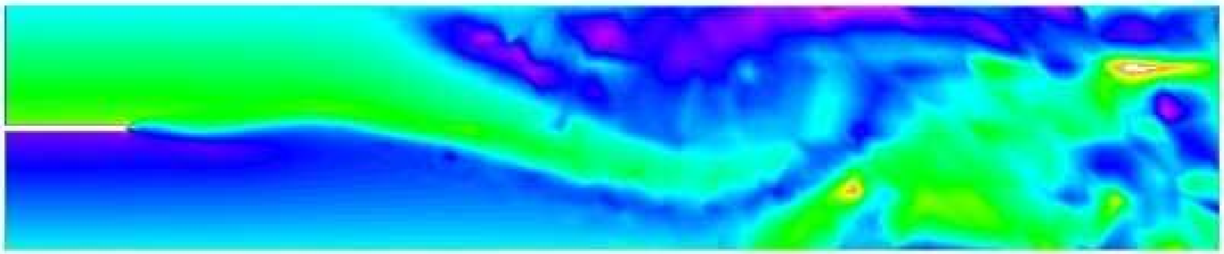
Robust control theory for flow control via reduced order models

D. Arzelier

LAAS-CNRS



✍ **Procedure 1** *[Kasnakoglu 2007]*

- *NSNL* $\xrightarrow{Lin.}$ *NSL* $\xrightarrow{Red.}$ *Linear synthesis on LROM*
- *NSNL* $\xrightarrow{Red.}$ *NLROM* $\xrightarrow{Lin.}$ *Linear synthesis on LROM*

✔ *Proper Orthogonal Decomposition (POD modes + actuation mode)*

✔ *Snapshots*

✔ *Galerkin Projection (GP)*

The nonlinear ROM:

$$\begin{aligned}\dot{x}(t) &= C + Lx(t) + x^T Qx + \hat{C}u(t) + \hat{L}x(t)u(t) + \hat{Q}u^2(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $x(t) \in \mathbb{R}^6$, $y(t) \in \mathbb{R}^6$ and $u(t) \in \mathbb{R}$

Linearization around an equilibrium state x_{eq} (or not) to get the LTI ROM:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $x \in \mathbb{R}^6$, $y \in \mathbb{R}^6$ and $u = f(x,t) \in \mathbb{R}$?

➥ Closed-form solutions ≠ Numerical solutions (LAAS-CNRS + S. Boyd 1990-1995)

Control pb. = Mathematical programming pb. + efficient numerical method

➦ Weak results ≠ Strong results

Complexity : P = NP ? and global proof : convexity

❶ LMI formalism emerging (Willems 1971)

❷ Developments of theoretical and numerical SDP tools:

Interior points [Nesterov et Nemirovskii 1994] and cutting planes [Kelley 1960]

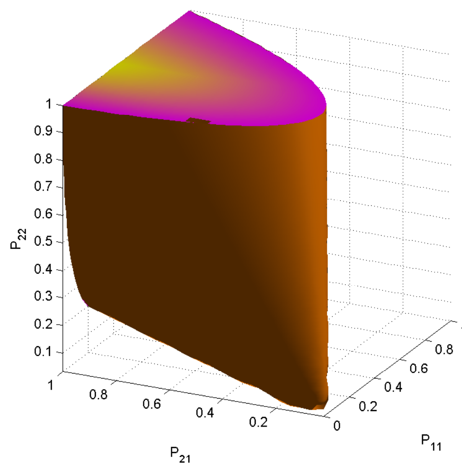
- "Easy" problems: numerical alternative to usual closed-form solutions
- "Tough" problems: convex relaxations

$$\min_{x \in \mathbb{R}^m} \quad \sum_{i=1}^{m} c_i x_i = c' x$$

$$\text{sous} \quad F_0 + \sum_{i=1}^{m} F_i x_i \succ \mathbf{0}$$

$$A' P + P A \prec \mathbf{0}$$

$$P \succ \mathbf{0}$$



Nonconvexity and/or nonpolynomial complexity obstruction

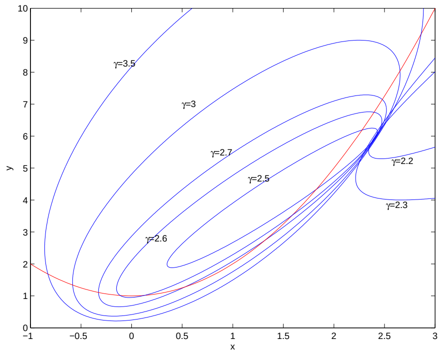
$$\min_{x\in\mathbb{R}^m} \quad \sum_{i=1}^{m} c_i x_i = c' x$$

$$\text{s.t.} \quad F_0 + \sum_{i=1}^{m} F_i x_i + \sum_{i=1}^{m}\sum_{j=i}^{m} G_{ij} x_i x_j \succ \mathbf{0}$$

$$M'PM < \gamma^2 P$$

$$\text{rank}(UPV) = 1$$

$$P = \begin{bmatrix} 1 & x & 0 \\ x & y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Convex relaxations (Lagrangian duality)
- Algorithmic Approach (nonsmooth optimization)

✔ Take advantage of the theoretical and practical structure of the system in the algorithm

✔ Large scale and reliability (guaranteed computations) are still open problems for SDP (sensitivity theory)

Given the model:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) \qquad \text{where } x \in \mathbb{R}^n \ , \ y \in \mathbb{R}^r \text{ and } u \in \mathbb{R}^m$$

☛ **Problem 1** : *SOF*

Find $K \in \mathbb{R}^{r\times m}$ *s.t.* $A + BKC$ *asymptotically stable (closed-loop spectrum in* $\mathbb{C}^-$*)*

Given the uncertain model :

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t)$$

$$\begin{aligned} M &= \begin{bmatrix} A & B & C \end{bmatrix} \in \Omega \\ &= \mathrm{co}\left\{\begin{bmatrix} A^1 & B^1 & C^1 \end{bmatrix}, \cdots, \begin{bmatrix} A^N & B^N & C^N \end{bmatrix}\right\} \end{aligned}$$

$$M(\xi) = \begin{bmatrix} A(\xi) & B(\xi) & C(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A^i & B^i & C^i \end{bmatrix}, \ \xi \in \Xi = \left\{ \xi = \sum_{i=1}^{N} \xi_i = 1 \quad \xi_i \geq 0 \right\}$$

☛ **Problem 2** : *Robust SOF*

Find a robustly stabilizing SOF $u(t) = Ky(t)$ *i.e. find a single matrix* $K \in \mathbb{R}^{r\times m}$ *s.t.*

$$\Omega_{bf} = \mathrm{co}\left\{A^1 + B^1KC, \cdots, A^N + B^NKC\right\}$$

is asymptotically stable $\forall \ \xi \in \Xi$

- Existence problem is a decidable (algorithmic) problem [Anderson 1975]
- Different approaches based on Lyapunov theory
  - ✔ Quantifier elimination procedure of Tarski-Seidenberg [Anderson 1975]
  - ✔ Nonlinear Programming methods [Levine-Athans 1970], [Overton 2000]
  - ✔ $\mathcal{BMI}$ approaches

$$P > \mathbf{0} \qquad (A + BKC)'P + P(A + BKC) < \mathbf{0}$$

  - ✔ $\mathcal{LMI}$ optimization with rank constraint:

$$B^{\perp}(AX + XA')B^{\perp'} < \mathbf{0} \qquad X = Y^{-1} \text{ or rank} \begin{bmatrix} X & \mathbf{1} \\ \mathbf{1} & Y \end{bmatrix} = n$$

$$C'^{\perp}(A'Y + YA)C'^{\perp'} < \mathbf{0}$$

  - ✔ Lagrangian relaxations + coordinate descent-type algorithm

The triplet (A, B, C) is stabilizable via SOF iff

$$\exists\ X \in \mathbb{S}_n^+,\ K_{sf} \in \mathbb{R}^{m\times n},\ Z \in \mathbb{R}^{m\times r},\ F \in \mathbb{R}^{m\times m} \text{ s.t.:}$$

$$\mathcal{L}(A,B,C) = \begin{bmatrix} A & B \\ \mathbf{1} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} \begin{bmatrix} A & B \\ \mathbf{1} & \mathbf{0} \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} {K_{sf}}' \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} \right\} < \mathbf{0}$$

$$K = -F^{-1}Z$$

- First part of $\mathcal{L}(A, B, C)$ reminds us of KYP lemma
- Additional variables F, Z, allow a decoupling between the computation of the Lyapunov certificate X and of the SOF K
- The additional variable K_{sf} is necessarily a stabilizing SF for (A, B) (application of elimination lemma)

Robust SOF Problem

- Testing stability of a polytope of matrices is an $\mathcal{NP}$-hard problem [Coxson and DeMarco 1999])

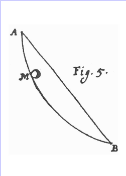
- Quadratic stabilizability framework: $\exists\ X \in \mathbb{S}_n^+$, and $\exists\ K \in \mathbb{R}^{r\times n}$

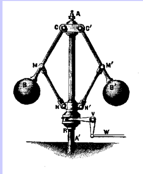
$$\begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix}' \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix} < \mathbf{0} \quad \forall\ \xi \in\ \Xi$$

- Robust stabilizability: For each $M = \begin{bmatrix} A & B \end{bmatrix} \in \Omega$, $\exists\ X \in \mathbb{S}_n^+$ and $\exists\ K \in \mathbb{R}^{r\times m}$ s.t.:

$$\begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix}' \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} & X(\xi) \\ X(\xi) & \mathbf{0} \end{bmatrix} \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix} < \mathbf{0}$$

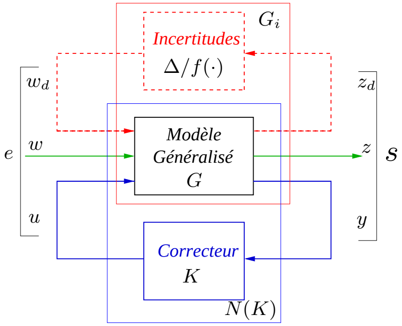
Quadratic stabilizability $\Rightarrow$ robust stabilizability





- Internal stability (Lyapunov)
- Heterogeneous performances (time and frequency responses)

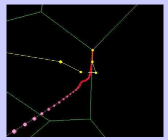
✔ Modern linear framework



- Standard model = central artefact [Doyle 1983]
- Robust stability and performance
- Optimality

CONSTRUCTIVE approach for C/S

- Lyapunov theory
- Optimization (Duality)



Some Preliminary Results