Linearized Impulsive Fixed-Time Fuel-Optimal Space rendezvous: A New Numerical Approach

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Abstract: This paper focuses on the fixed-time minimum-fuel rendezvous between close elliptic orbits of an active spacecraft with a passive target spacecraft, assuming a linear impulsive setting and a Keplerian relative motion. Following earlier works developed in the 1960s, the original optimal control problem is transformed into a semi-infinite convex optimization problem using a relaxation scheme and duality theory in normed linear spaces. A new numerical convergent algorithm based on discretization methods is designed to solve this problem. Its solution is then used in a general simple procedure dedicated to the computation of the optimal velocity increments and optimal impulses locations. It is also shown that the semi-infinite convex programming has an analytical solution for the out-of-plane rendezvous problem. Different realistic numerical examples illustrate these results.

Keywords: Impulsive optimal control, elliptic rendezvous, primer vector, semi-infinite convex programming, discretization methods

1. INTRODUCTION

Since the first space missions (Gemini, Apollo, Vostok) involving more than one vehicle, space rendezvous between two spacecraft has become a key technology raising relevant open control issues. Formation flight (PRISMA), on-orbit satellite servicing or supply missions to the International Space Station (ISS) are all examples of projects that require adequate rendezvous planning tools. A main challenge is to achieve autonomous far range rendezvous on elliptical orbits while preserving optimality in terms of fuel consumption. In short, the far range rendezvous is an orbital transfer between an active chaser spacecraft and a passive target spacecraft, with specified initial and final conditions, over a fixed or a free time period. Searching for the guidance law that achieves the maneuver with the lowest possible fuel consumption leads to define a minimum-fuel optimal control problem.

In this article, the fixed-time linearized fuel-optimal impulsive space rendezvous problem as defined in Carter and Brient (1995), is studied assuming a linearized Keplerian relative motion. The impulsive approximation for the thrust means that instantaneous velocity increments are applied to the chaser whereas its position is continuous. Indirect approaches, based on the optimality conditions derived from the Pontryagin's maximum principle and leading to the so-called primer vector theory (Lawden (1963)), have been extensively studied. For a fixed number of impulses, necessary and sufficient conditions can be derived (Carter and Brient (1995)). However due to the nonconvex and polynomial nature of these conditions, a numerical solution is still difficult to compute and would only be suboptimal for the original rendezvous problem for which the number of possible maneuvers is free. In Arzelier et al. (2013), a mixed iterative algorithm combines variational tests with sophisticated numerical tools from algebraic geometry to solve these polynomial necessary and sufficient conditions of optimality and avoid the local optimization step. However, this algorithm remains heuristic with no proof of convergence in all cases and may exhibit only suboptimal solutions on some instances.

Neustadt (1964) proposed an important theoretical contribution for the optimal control problem: it is recast to a semi-infinite optimization problem, using a relaxation scheme and the duality theory in minimum-norm problems. Claeys et al. (2013) revisit his approach from the angle of generalized moment problems, by formulating it as a linear programming problem on measures. In this approach, the numerical solving is rather cumbersome since one needs high degree polynomial approximations for building hierarchies of linear-matrix inequalities (LMIs). Also, they consider only the case of ungimbaled identical thrusters, which gives a linear problem.

Following Neustadt (1964), we propose a new numerical algorithm to solve the fixed-time impulsive linear rendezvous without fixing *a priori* the number of impulses, and whose convergence is rigorously shown. Firstly, we focus on the moment problem formulation (Sec. 2) and recall topological duality theory results from Luenberger (1969) and Neustadt (1964), which allow for the moment problem to be transformed into a Semi-Infinite Convex Programming (SICP) Problem (Sec. 4). The novelty of our approach is to use discretization methods Reemtsen and Rückman (1998) to solve the SICP problem. A convergent numerical algorithm is designed in Sec. 4, whose solution is the optimal primer vector of the original rendezvous problem. An estimation of the numerical error made on the optimal cost of the original problem, is also provided. Then, the optimal impulses location and the optimal velocity increments are retrieved via a simple procedure fully exploiting results stated in Neustadt (1964). The efficiency of the proposed algorithm is illustrated with two different realistic numerical examples.

Notations: a, e, ν are respectively the semi-major axis, the eccentricity and the true anomaly of the reference orbit. N is the number of velocity increments while ν_i, $i = 1, \cdots, N$, define impulses application locations. The velocity increment at ν_i will be denoted by $\Delta V(\nu_i)$. $\{b_i\}_{i=1,\cdots,N}$ is a sequence of variables b_i, $i = 1, \cdots, N$, and $\mathrm{sgn}(z)$ is the sign function of the variable z. The prime denotes differentiation with respect to the true anomaly ν. $\mathbf{O}_{p\times m}$ and $\mathbf{1}_m$ denote respectively the null matrix of dimensions $p \times m$ and the identity matrix of dimension m. Let $r \in \mathbb{N}^*$ and $(p, q) \in \mathbb{R}^2$ such that: $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Classically, $C([\nu_0, \nu_f], \mathbb{R}^r)$ is the Banach space of continuous functions $f : [\nu_0, \nu_f] \to \mathbb{R}^r$ equipped with the norm $\|f\|_q = \sup_{\nu_0 \leq \nu \leq \nu_f} \|f(\nu)\|_q$. Denote by $\mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ the normed linear space of Lebesgue integrable functions from $[\nu_0, \nu_f]$ to $\mathbb{R}^r$ with the norm given by: $\|u\|_{1,p} = \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p \mathrm{d}\nu$. Let $\mathrm{BV}([\nu_0, \nu_f], \mathbb{R}^r)$ be the space of functions of bounded variation over the interval $[\nu_0, \nu_f]$ with the norm: $\|g\|_{tv,p} = \sup_{P_\kappa} \sum_{i=1}^{\kappa} \|g(\nu_i) - g(\nu_{i-1})\|_p$, where the supremum is taken over all finite partitions $P_\kappa = (\nu_i)_{i=1,\ldots,\kappa}$ of $[\nu_0, \nu_f]$. For a symmetric real matrix $S \in \mathbb{R}^{n\times n}$, the notation $S \preceq 0$ ($S \succeq 0$) stands for the negative (positive) semi-definiteness of S. Finally, χ_A is the indicator function of the set A.

2. PROBLEM STATEMENT AND PRELIMINARIES

This section first introduces and reviews notations and assumptions for the minimum-fuel linearized fixed-time rendezvous problem. Then, adopting the approach of Neustadt (1964), the usual optimal control formulation of the rendezvous problem is recast as a moment problem defined on the functional space $\mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$.

2.1 Optimal control formulation of the rendezvous problem

Typically, in a rendezvous situation, a spacecraft is in sufficiently close proximity to allow for the linearization of the relative equations of motion. Their validity is guaranteed when the distance between the target and the chaser is assumed to be small compared to the radius of the target vehicle orbit. The equations of relative motion are written in a moving Local-Vertical-Local-Horizontal (LVLH) frame located at the center of gravity of a passive target and which rotates with its angular velocity. In this frame, the state vector $X^T = [\, p_x \ p_y \ p_z \ v_x \ v_y \ v_z \,]$ is composed of the positions and velocities of a chaser satellite in the in-track, cross-track and radial axes, respectively. Under the previous assumptions and using the true anomaly of the target-vehicle orbit as the independent variable, a system of linear differential equations with periodic coefficients is easily obtained and the considered minimum-fuel linearized rendezvous problem may be reformulated as the following optimal control problem:

Problem 1. (Optimal control problem)
Find $\bar{u} \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^3)$ solution of the optimal control problem:

$$\begin{aligned} &\inf_u \ \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p \mathrm{d}\nu \\ &\text{s.t. } X'(\nu) = A(\nu)X(\nu) + B(\nu)u(\nu), \ \forall\, \nu \in \ [\nu_0, \nu_f] \\ &\qquad X(\nu_0) = X_0, \ X(\nu_f) = X_f \in \mathbb{R}^6, \ \nu_0, \ \nu_f \text{ fixed}, \end{aligned} \tag{1}$$

where matrices $A(\nu)$ and $B(\nu)$ define the state-space model of relative dynamics given by Tschauner (1967):

$$A(\nu) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/(1+e\cos(\nu)) & -2 & 0 & 0 \end{bmatrix}, \ B(\nu) = \begin{bmatrix} \mathbf{O}_{3\times 3} \\ \dfrac{\mathbf{1}_3}{1+e\cos(\nu)} \end{bmatrix} \tag{2}$$

The form of these matrices shows that the equations describing motion in the plane of the target-vehicle orbit and those describing motion normal to the orbit plane can be decoupled and handled separately. Therefore, the out-of-plane and in-plane rendezvous can be dealt with independently: the state vector dimension and the number of inputs in (1) are denoted n and r, respectively with $n = 2$, $r = 1$ for the out-of-plane case and $n = 4$, $r = 2$ for the in-plane case. Due to space limitations, this paper focuses on the in-plane rendezvous.

Remark 1. In Problem 1, the 1-norm cost captures the consumption of fuel used. In fact, the performance index used in Problem 1 has been normalized to stick to the usual characteristic velocity expressed in m/s.

2.2 A minimum norm moment problem

Following the approach from Neustadt (1964), Problem 1 is now transformed into an equivalent problem of moment by integrating equation (1). As $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$, the equation (1) has a unique solution that exists for every $X_0 \in \mathbb{R}^n$ and for all $\nu \in \mathbb{R}$ and for $u(\nu) \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$, (Antsaklis and Michel (2003)):

$$X(\nu) = \Phi(\nu, \nu_0)X_0 + \int_{\nu_0}^{\nu} \Phi(\nu, \sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma, \tag{3}$$

where $\Phi(\nu, \nu_0) = \varphi(\nu)\varphi^{-1}(\nu_0)$ and $\varphi(\nu)$ are respectively the transition and Yamanaka-Ankersen fundamental matrices of Keplerian relative motion Yamanaka and Ankersen (2002). Let us define the matrix $Y(\nu) = \varphi^{-1}(\nu)B = [\, y_1(\nu) \cdots y_n(\nu) \,]^T \in \mathbb{R}^{n\times r}$, then:

$$\begin{aligned} c &= \varphi^{-1}(\nu_f)X(\nu_f) - \varphi^{-1}(\nu_0)X_0 \\ &= \int_{\nu_0}^{\nu_f} \varphi^{-1}(\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma = \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)\mathrm{d}\sigma. \end{aligned} \tag{4}$$

It is important to notice for the remainder of the analysis that for the specific matrices $Y(\nu)$ encountered in the rendezvous problem, $y_1(\nu) \cdots y_n(\nu)$ are linearly independent elements of $\mathcal{C}([\nu_0, \nu_f], \mathbb{R}^r)$. This will be assumed in the rest of the paper. It follows from (4) that Problem 1 can be equivalently written as:

Problem 2. (Minimum norm moment problem) Find $\bar{u}(t) \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ solution of the minimum norm moment problem:

$$\begin{aligned} &\inf_u \ \|u\|_{1,p} = \inf_u \int_{\nu_0}^{\nu_f} \|u(\nu)\|_p \mathrm{d}\nu \\ &\text{s.t. } \int_{\nu_0}^{\nu_f} Y(\sigma)u(\sigma)\mathrm{d}\sigma = c, \ \nu_0, \ \nu_f \text{ fixed}. \end{aligned} \tag{5}$$

It is well-known that Problem 2 may not reach its optimal solution due to concentration effects (see the reference

Roubíček (2006)). This is mainly due to the fact that the functional space $\mathcal{L}_{1,p}([\nu_0,\nu_f],\mathbb{R}^r)$ in which the optimal solution is sought, is not the topological dual of any other functional space Luenberger (1969). It is then necessary to resort to a relaxation scheme by embedding the space $\mathcal{L}_{1,p}([\nu_0,\nu_f],\mathbb{R}^r)$ in the dual space $\mathcal{C}^*([\nu_0,\nu_f],\mathbb{R}^r)$ of the Banach space $\mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$.

3. A CLASSICAL APPROACH REVISITED

In this section, the theoretical framework used to transform the original optimal control problem into a semi-infinite optimization program is recalled.

3.1 Relaxation of the original problem

A so-called relaxed problem is considered, whose solutions are thought of as generalized solutions of the original Problem 2.

Problem 3. (Relaxed problem)
Determine $\bar{g} \in \mathrm{BV}([\nu_0,\nu_f],\mathbb{R}^r)$ solution of the following problem:

$$\begin{aligned} &\inf_g \ \|g\|_{tv,p} = \inf_g \sup_{P_\kappa} \sum_{i=1}^{\kappa} \|g(\nu_i) - g(\nu_{i-1})\|_p, \\ &\text{s.t.} \int_{\nu_0}^{\nu_f} Y(\nu)\mathrm{d}g(\nu) = c. \end{aligned} \tag{6}$$

$P_\kappa = \{\nu_0 = \nu_1 < \nu_2, \cdots, < \nu_\kappa = \nu_f\}$ is any finite partition of $[\nu_0,\nu_f]$. It is shown in Neustadt (1964) that the infimum of Problem 3 is reached and that it is equal to the infimum of Problem 2, denoted by $\bar{\eta}$ in what follows.

In addition, a unique association between the space $\mathrm{BV}([\nu_0,\nu_f],\mathbb{R}^r)$ and the dual $\mathcal{C}^*([\nu_0,\nu_f],\mathbb{R}^r)$ of the space $\mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$ is defined by the Riesz Representation Theorem, Luenberger (1969). Defining the bilinear form pairing $\mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$ and $\mathcal{C}^*([\nu_0,\nu_f],\mathbb{R}^r)$ by the duality bracket:

$$l(y_i) = \langle y_i(\cdot), l\rangle = \int_{\nu_0}^{\nu_f} y_i(\nu)^T \mathrm{d}g(\nu), \tag{7}$$

Problem 3 may equivalently rewritten as:

Problem 4. (Linear minimum norm problem)
Find a linear functional $\bar{l} \in \mathcal{C}^*([\nu_0,\nu_f],\mathbb{R}^r)$ solution of the linear minimum norm problem:

$$\begin{aligned} \bar{\eta} = \inf_l \ \|l\| &= \inf_l \sup_{\|y(\cdot)\|_q \le 1} |l(y)| \\ \text{s.t.} \quad l(y_i) &= \langle y_i(\cdot), l\rangle = c_i, \ \forall\ i = 1, \cdots, n. \end{aligned} \tag{8}$$

Despite the fact that Problem 4 is an infinite-dimensional optimization problem, it is particularly appealing due to its simplicity and the possibility to use a duality principle based on the extension form of the Hahn-Banach theorem. This establishes the equivalence between two optimization problems respectively defined in a Banach space and its dual. The result is summarized in the next subsection.

3.2 A semi-infinite programming problem

The following seminal and important result has been originally given in Neustadt (1964) in its complete form and partially in Krasovskii (1957) for particular optimization problems. Here, we follow the lines developed in the textbook of (Luenberger, 1969, Chapter 5).

Theorem 1. (Luenberger (1969))

Let $y_i(\cdot) \in \mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$, $\forall\ i = 1, \cdots, n$ and suppose that

$$D = \{l \in \mathcal{C}^* : \langle y_i(\cdot), l\rangle = c_i,\ i = 1, \cdots, n\} \neq \emptyset, \tag{9}$$

then

$$\bar{\eta} = \min_{l \in D} \|l\| = \max_{\|Y^T(\nu)\lambda\|_q \le 1} c^T\lambda. \tag{10}$$

In addition, let $\bar{l}$ and $\bar{\lambda}$ be optimal solutions of (10), $\bar{\lambda} = \mathrm{Arg}[\max_{\|Y^T(\nu)\lambda\|_q \le 1} c^T\lambda]$ and let $\bar{y}(\nu) = \sum_{i=1}^{n} \bar{\lambda}_i y_i(\nu) = Y^T(\nu)\bar{\lambda} \in \mathbb{R}^r$. Then the optimal $\bar{l}$ is aligned with the optimal $\bar{y}$:

$$\begin{aligned} \langle \bar{y}(\cdot), \bar{l}\rangle &= \int_{\nu_0}^{\nu_f} \bar{\lambda}^T Y(\nu)\mathrm{d}\bar{g}(\nu) = \|\bar{y}(\cdot)\|_q \|\bar{l}\| \\ &= \sup_{\nu_0 \le \nu \le \nu_f} (\|\bar{y}(\cdot)\|_q)\ \|\bar{g}\|_{tv,p} \end{aligned} \tag{11}$$

The two problems defined in eq. (10) may be considered as dual through the equality of the optimal values of their respective objectives and the relation between their solutions thanks to the alignment condition in eq. (11). This results in a significant simplification: The infinite-dimensional optimization Problem 4 has been converted to a search of an optimal vector $\bar{\lambda}$ in a finite-dimensional vector space submitted to a continuum of constraints, yielding a semi-infinite convex problem (SICP):

Problem 5. (SICP problem) Find $\bar{\lambda} \in \mathbb{R}^n$ solution of

$$\begin{aligned} \bar{\mu} = &\min_{\lambda \in \mathbb{R}^n} \ -c^T\lambda \\ &\text{s.t.} \quad \|Y^T(\nu)\lambda\|_q \le 1. \end{aligned} \tag{12}$$

Note that $\bar{\mu} = -\bar{\eta}$. An efficient numerical method for solving Problem 5 is given in Sec. 4. Once its solution is obtained, the alignment relation between the function $\bar{y}(\cdot)$ element of the Banach space $\mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$ and the functional $\bar{l}$ belonging to its dual space $\mathcal{C}^*([\nu_0,\nu_f],\mathbb{R}^r)$ is particularly important to get back to the optimal bounded variation solution of the relaxed Problem 3.

Theorem 2. (Neustadt (1964))

Let $y_i(\cdot) \in \mathcal{C}([\nu_0,\nu_f],\mathbb{R}^r)$, $i = 1, \ldots, n$ and $\bar{\lambda} \in \mathbb{R}^n$ be an optimal solution of Problem (12). Define the sets $\Gamma_s = \{\hat{\nu} \in [\nu_0,\nu_f] : |\bar{y}_s(\hat{\nu})| = 1\}$ and $\Gamma = \left\{\hat{\nu} \in [\nu_0,\nu_f],\ \|\bar{y}(\hat{\nu})\|_q = \max_{\nu_0 \le \nu \le \nu_f} \|\bar{y}(\nu)\|_q = 1\right\}$. Note that $\Gamma = \cup_s \Gamma_s$ for $p = 1$. There is an optimal solution $\bar{g}(\cdot) \in \mathrm{BV}([\nu_0,\nu_f],\mathbb{R}^r)$ of the relaxed Problem 3, which is a step function with at most n points of discontinuity $\hat{\nu}_j \in \Gamma$, $j = 1, \cdots, N \le n$. Its jumps are given by:

$$\begin{aligned} &\bar{g}_s(\hat{\nu}_j) - \bar{g}_s(\hat{\nu}_j^-) = \alpha_{\hat{\nu}_j}\mathrm{sgn}(\bar{y}_s(\hat{\nu}_j))\chi_{\Gamma_j},\ \alpha_{\hat{\nu}_j} > 0,\ p = 1, \text{ or} \\ &\bar{g}_s(\hat{\nu}_j) - \bar{g}_s(\hat{\nu}_j^-) = \alpha_{\hat{\nu}_j}|\bar{y}_s(\hat{\nu}_j)|^{q-1}\mathrm{sgn}(\bar{y}_s(\hat{\nu}_j)),\ 1 < p < \infty, \end{aligned} \tag{13}$$

for $s = 1, \cdots, r$ and $\alpha_{\hat{\nu}_j}$ solutions of the linear system:

$$\sum_{j=1}^{N} \beta_i(\hat{\nu}_j)\alpha_{\hat{\nu}_j} = c_i,\ i = 1, \cdots, n \tag{14}$$

where $\beta_i(\hat{\nu}_j)$ are given by:

$$\begin{aligned} &\beta_i(\hat{\nu}_j) = \sum_{s=1}^{r} y_{i,s}(\hat{\nu}_j)\mathrm{sgn}(\bar{y}_s(\hat{\nu}_j)),\ p = 1, \text{ or} \\ &\beta_i(\hat{\nu}_j) = \sum_{s=1}^{r} y_{i,s}(\hat{\nu}_j)|\bar{y}_s(\hat{\nu}_j)|^{q-1}\mathrm{sgn}(\bar{y}_s(\hat{\nu}_j)),\ 1 < p < \infty, \end{aligned} \tag{15}$$

for all $j = 1, \cdots, N$.

This theorem states important results that have been known for a while in the aerospace community but whose value has not been completely exploited to derive efficient numerical algorithms for impulsive maneuvers design. First, it says that the optimal controlled trajectory for the minimum-fuel Keplerian linearized elliptic rendezvous problem is purely impulsive and that the number of impulses is upper-limited by n which is the dimension of the fixed final conditions of the optimal control problem.

Remark 2. It is also shown in Neustadt (1964) that a sequence of functions $u_\epsilon(\cdot) \in \mathcal{L}_{1,p}([\nu_0, \nu_f], \mathbb{R}^r)$ converges to a linear combination of $\delta(\cdot)$ functions corresponding to the function $\bar{g}(\cdot)$ with equal norms. Let $\Delta V(\hat{\nu}_j) = \bar{g}(\hat{\nu}_j) - \bar{g}(\hat{\nu}_j^-)$, then roughly speaking, this may be described by:

$$\bar{u}_\epsilon(\nu) \to \sum_{j=1}^{N} \Delta V(\hat{\nu}_j)\delta(\hat{\nu}_j - \nu), \ \epsilon \to 0. \tag{16}$$

Indeed, the initial optimal control problem amounts to find the sequences of optimal impulse locations $\{\hat{\nu}_i\}_{i=1,\cdots,N}$ and optimal impulse vectors $\{\Delta V(\hat{\nu}_i)\}_{i=1,\cdots,N}$ verifying the boundary equation:

$$c = \sum_{i=1}^{N} Y(\hat{\nu}_i)\Delta V(\hat{\nu}_i). \tag{17}$$

3.3 Primer-vector interpretation and relation with the mixed algorithm in Arzelier et al. (2013)

The vector $y(\nu) = Y^T(\nu)\lambda$ involved in (12) is nothing but the primer vector initially defined in the seminal work of Lawden (Lawden (1963)). In this reference, the primer vector $y(\nu)$ is defined as the velocity adjoint vector arising from applying the Pontryagin Maximum Principle to optimal trajectory problems or Lagrangian duality as in Carter and Brient (1995) where the vector $\bar{\lambda}$ is the optimal Lagrange multiplier. For an optimal impulsive trajectory, the primer vector $y(\nu)$ must satisfy the well-known Lawden's necessary and sufficient optimality conditions recalled in Carter and Brient (1995) or in Arzelier et al. (2013). In this last reference, a mixed iterative algorithm aiming at converging to the minimum-fuel solution over the number of impulses via an iterative process is designed by taking advantage of the polynomial nature of the underlying optimality conditions. Although efficient in practice on some instances, this last algorithm suffers from the lack of proof of convergence of the iterative procedure based on simple heuristic rules. As will be shown in the Section 5 dedicated to numerical examples, this algorithm may fail and may only exhibit a suboptimal solution. The next section proposes a new procedure based on a discretization algorithm for the solution of the semi-infinite programming Problem 5 whose convergence may be rigorously established.

4. A CONVERGENT DISCRETIZATION APPROACH

4.1 General solving procedure

Based on Problem 5 and Theorem 2, a convergent numerical method is presented. Firstly, the SICP Problem 5 is solved using Algorithm 1 given in Section 4.2 together with its convergence proof. Algorithm 1 provides a numerical value for the optimal cost. Secondly, one identifies the impulse locations and velocity increments based on Theorem 2 in Algorithm 2 in Section 4.3.

4.2 Convergent discretization algorithms for SICP

Consider the general formulation of Problem 5 as a semi-infinite programming problem $\mathcal{P}(\Theta)$:

$$\begin{array}{ll} \text{Minimize} & f(\lambda) \\ \text{subject to} & g(\lambda, \ \nu) \leq 0, \ \nu \in \Theta \end{array} \tag{18}$$

Note that in our case f is a linear function of λ, $g(\cdot, \nu)$, $\nu \in \Theta$ is convex and Θ is a compact set (a closed interval). Efficient discretization methods have been developed for such problems (Reemtsen and Rückman, 1998, Chap.7). They consider a sequence of finite subsets $\Theta_i \subseteq \Theta$ and solve $\mathcal{P}(\Theta_i)$ respectively. Let $M(\Theta_i)$ be the set of feasible points for problem $\mathcal{P}(\Theta_i)$: $M(\Theta_i) = \{\lambda : g(\lambda, \nu) \leq 0, \nu \in \Theta_i\}$. The advantage is that for finite programs $\mathcal{P}(\Theta_i)$, feasibility can usually be checked easily and accurately.

Under certain conditions, one chooses an initial set Θ_0, and obtains an initial solution λ_0 of $\mathcal{P}(\Theta_0)$. Then Θ_i is chosen as: $\Theta_i = \Theta_{i-1} \cup \left\{ \arg \left[\max_{\nu \in \Theta} g(\lambda_{i-1}, \nu) \right] \right\}$. One has to ensure that the sequence of solutions of $\mathcal{P}(\Theta_i)$ converges to the solution of $\mathcal{P}(\Theta)$. In the following, we summarize results from (Reemtsen and Rückman, 1998, Lemma 2.4, Chap.7, Theorem 2.8, Chap.7, Corollary 2.9, Chap.7), which prove that this procedure is convergent. Algorithm 1 details the implementation for our particular case.

For each feasible point $\lambda_\Theta \in M(\Theta)$ (if such point exists) and $\Theta_i \subseteq \Theta$, define the level set

$$L(\lambda_\Theta, \Theta_i) = M(\Theta_i) \cap \{\lambda : f(\lambda) \leq f(\lambda_\Theta)\}. \tag{19}$$

Theorem 3. (Reemtsen and Rückman, 1998, Chap.7) Let f and $g(., \nu)$, $\nu \in \Theta$, be convex. Let a sequence of compact sets $(\Theta_i)_{i \in \mathbb{N}}$ s.t. Θ_0 is finite, $\Theta_i \subseteq \Theta_{i+1} \subseteq \Theta$ and $\lim_{i \to \infty} \text{dist}(\Theta_i, \Theta) = 0$ where dist is the classical Hausdorff distance.

(Assumption A1.) Suppose there exists $\lambda_\Theta \in M(\Theta)$ s.t. $L(\lambda_\Theta, \Theta_0)$ is bounded.

Then the set of solutions of $\mathcal{P}(\Theta_i)$ is nonempty and compact. Algorithm 1 generates an infinite sequence λ_i such that λ_i has an accumulation point and each such point solves $\mathcal{P}(\Theta)$. Moreover the sequence $\inf_{\lambda \in M(\Theta_i)} f(\lambda)$ converges monotonically increasingly to $\inf_{\lambda \in M(\Theta)} f(\lambda)$ when $i \to \infty$.

In what follows, we consider two cases which arise in practice and which specify the norms for Problem 5:

– for a gimbaled single thruster one has $p = q = 2$, which gives a semi-infinite positive semi-definite (SDP) problem:

$$\begin{array}{ll} \inf_{\lambda \in \mathbb{R}^n} & -c^T\lambda \\ \text{s.t.} & \begin{bmatrix} -1 & \lambda^T Y(\nu) \\ Y^T(\nu)\lambda & -\mathbf{1} \end{bmatrix} \preceq \mathbf{0}, \ \forall \ \nu \in [\nu_0, \nu_f]; \end{array} \tag{20}$$

– for 6 ungimbaled identical thrusters, one has $p = 1$, $q = \infty$ which gives a semi-infinite linear programing (LP) problem:

$$\begin{array}{ll} \inf_{\lambda \in \mathbb{R}^n} & -\ c^T\lambda \\ \text{s.t.} & \left| \sum_{i=1}^{n} \lambda_i y_{i,s}(\nu) \right| \leq 1, \forall \ \nu \in [\nu_0, \nu_f], \ s = 1, \ldots, r. \end{array} \tag{21}$$

Both problems defined by (21) and (20) are particular instances of $\mathcal{P}(\Theta)$ for which discretized versions can be efficiently numerically solved. For the convergence proof, Assumption A1 in Theorem 3 is verified in what follows.

Lemma 1. Let $\Theta_0 = \{\theta_0, \theta_1\} \subseteq [\nu_0, \nu_f]$, $\theta_1 - \theta_0 \neq k\pi,\ k \in \mathbb{N}$. Assumption A1 holds for both Problems in eqs. (20) and (21) for $L(0, \Theta_0)$.

Thus, Algorithm 1 is initialized based on Lemma 1 and an initial $\lambda^{(0)}$ (and primer vector $Y^T(\nu)\lambda^{(0)}$) is computed by solving eq. (17) for $\nu \in \{\theta_0, \theta_1\}$.

Input: interval $\Theta = [\nu_0, \nu_f]$, matrix $Y(\nu)$, initial condition c, accuracy ε
Output: $\mu^{(i)}$ and $\lambda^{(i)}$ numerical solution of Pb. 5
Init:
$i \leftarrow 0$;
$\Theta_0 \leftarrow \{\theta_0; \theta_1\} \subset \Theta$ s.t. $\theta_0 - \theta_1 \neq k\pi$;
Solve eq. (17) for ΔV_0 and ΔV_1;
Solve for $\lambda^{(0)}$ the system $Y^T(\theta_k)\lambda^{(0)} = \Delta V_k / \|\Delta V_k\|_q$, $k = 0, 1$.
while $\max\limits_{\theta \in \Theta} \|Y(\theta)^T \lambda^{(i)}\|_q - 1 > \varepsilon$ **do**

$$i \leftarrow i + 1;\ \Theta_i \leftarrow \Theta_{i-1} \cup \left\{ \arg \left[\max_{\theta \in \Theta} \|Y^T(\theta)\lambda^{(i)}\|_q \right] \right\};$$

Find $\lambda^{(i)}$ solution of discretized problem:

$$\begin{aligned} \mu^{(i)} = & \inf_{\lambda \in \mathbb{R}^n} -c^T \lambda \\ & \text{s.t. } \|Y^T(\theta_k)\lambda\|_q \leq 1 \quad \text{for all } \theta_k \in \Theta_i \end{aligned}$$

end
return $\mu^{(i)}, \lambda^{(i)}$.
Algorithm 1: Numerical procedure for solving Problem 5

We give in what follows an estimation of the accuracy of the obtained numerical value $\mu^{(i)}$ with respect to the optimal cost η in Problem 4. The discretization method produces outer approximations of a solution of the SIP problem, i.e. the approximate solutions of $\mathcal{P}(\Theta_i)$ are not feasible for $\mathcal{P}(\Theta)$, but provide increasing lower bounds for its solution. A global solution $\bar{\lambda}^{(i)}$ of $\mathcal{P}(\Theta_i)$ which is feasible for $\mathcal{P}(\Theta)$, solves $\mathcal{P}(\Theta)$, since: $f(\bar{\lambda}^{(i)}) = \inf\limits_{\lambda \in M(\Theta_i)} f(\lambda) \leq \inf\limits_{\lambda \in M(\Theta)} f(\lambda) \leq f(\bar{\lambda}^{(i)})$.

Thus, if the discretized problem $\mathcal{P}(\Theta_i)$ is accurately solved, one has: $\mu^{(i)} \leq \inf\limits_{\lambda \in M(\Theta)} f(\lambda)$. This gives an upper bound for $\bar{\eta}$, using equation (10):

$$\bar{\eta} = \max_{\|Y^T(\nu)\lambda\|_q \leq 1} c^T \lambda = - \min_{\|Y^T(\nu)\lambda\|_q \leq 1} -c^T \lambda \leq -\mu^{(i)}. \tag{22}$$

A lower bound can also be obtained.

Lemma 2. Suppose one can rigorously check that when Algorithm 1 stops,

$$\max_{\theta \in \Theta} \|Y(\theta)^T \lambda^{(i)}\|_q \leq 1 + \varepsilon,$$

where ε is a user defined input parameter. Then

$$\frac{-\mu^{(i)}}{1 + \varepsilon} \leq \bar{\eta}. \tag{23}$$

Thus, given ε, the output $\mu^{(i)}, \lambda^{(i)}$ of Algorithm 1 provides a good numerical approximation for the optimal cost of the original problem, $\bar{\eta}$. The impulse locations and impulse vectors are recovered as follows.

4.3 Reconstruction of the solution

The impulse locations can be identified based on Theorem 2 i.e., by finding $\Gamma = \{\hat{\nu}_k \in [\nu_0, \nu_f] :\ \|Y(\hat{\nu}_k)^T \lambda^{(i)}\|_q = 1\}$. This is done numerically on a grid of $[\nu_0, \nu_f]$. Then one solves the system given in eq. (17). This is always possible, since, according to Neustadt, the following holds: if at most n locations are found in Γ, the system is underdetermined/determined and it has at least one solution; if more than n locations are found in Γ, one can select n among them such that the system has a solution. The detailed numerical procedure is given in Algorithm 2.

Input: interval $\Theta = [\nu_0, \nu_f]$, matrix $Y(\nu)$, initial condition c, accuracy ε, numerical solution $\lambda^{(i)} \in \mathbb{R}^n$ of Pb. 5
Output: impulse locations and impulse vectors $\Gamma_{imp}, \{\Delta V_i\}$
$\Gamma_d \leftarrow$ discretized grid of $[\nu_0, \nu_f]$
$\Gamma \leftarrow \{\hat{\nu}_k \in \Gamma_d : \|Y(\hat{\nu}_k)^T \lambda^{(i)}\|_q - 1 \in [-\varepsilon, \varepsilon]\}$
$N \leftarrow$ size(Γ)
if *($N \leq n$)* **then**
$\Gamma_{imp} \leftarrow \Gamma$
Solve for $\Delta V_i, i = 1, \ldots, N$, the linear system $c = \sum\limits_{\hat{\nu}_i \in \Gamma_{imp}} Y(\hat{\nu}_i)\Delta V_i$.
else
$\Gamma_{imp} \leftarrow$ **Choose** n points in Γ s.t. the linear system $c = \sum\limits_{\hat{\nu}_i \in \Gamma_{imp}} Y(\hat{\nu}_i)\Delta V_i$ has a solution.
end
return $\Gamma_{imp}, \{\Delta V_i\}$.
Algorithm 2: Numerical Reconstruction of impulse locations and vectors

5. NUMERICAL EXAMPLE

The algorithms were implemented in C language and the discretized SDP problems are solved with SDPA developed by Yamashita et al. (2011). The numerical example is dedicated to the in-plane motion case and based on some example of the Automated Transfer Vehicle (ATV) setup, (Labourdette et al. (2008)). The parameters of the rendezvous are given in Table 1.

Semi-major axis	a = 6763 km.
Inclination	i=52 deg.
Argument of perigee	ω=0 deg.
R.A. of node	Ω= 0 deg.
Eccentricity	e = 0.0052
Initial time	$\nu_0 = 0$ rad.
X_0^T	[-30 0.5 8.514 0] km. - m/s.
Final anomaly	$\nu_f = 8.1832$ rad.
Duration	$t_f - t_0 = 7200$ s. (1.3 orbits)
X_f^T	[-100 0 0 0] m. - m/s.

Table 1. Parameters of the ATV example

For the in-plane rendezvous, two different examples are studied: I- a single gimbaled thruster using $\mathcal{L}_{1,2}$ norm and II- 6 ungimbaled thrusters with $\mathcal{L}_{1,1}$ norm.

I: $\mathcal{L}_{1,2}$ norm For $\varepsilon = 10^{-4}$, Algorithm 1 gives the optimal solution $\bar{\lambda} = [-1.177\ 1.132\ -1.571\ 14.36]^T .10^{-4}$ after 6 iterations (Fig. 1, 2 and 3 show the initialization step, the first and the last iterations). Algorithm 2 builds a 4-impulse minimum-fuel solution with an optimal cost of 10.7989 m/s. The optimal impulse locations are given by $\Gamma_{imp} = \{0,\ 1.3872,\ 6.6639,\ 8.1832\}$ [rad]. The optimal trajectory in the plane (x, z) is depicted by Fig. 4. It is important to remark that the mixed algorithm proposed in Arzelier et al. (2013) fails to converge to the optimal solution on this particular instance and stops at a 4-impulse suboptimal solution whose consumption is given by 11.01 m/s.

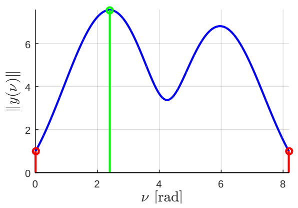

Fig. 1. Primer vector norm: Initialization.

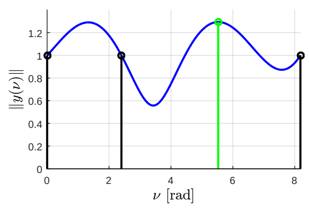

Fig. 2. Primer vector norm: First iteration.

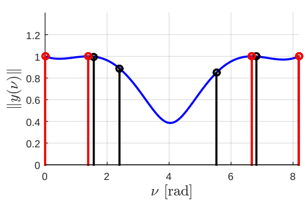

Fig. 3. Primer vector norm: Final iteration.

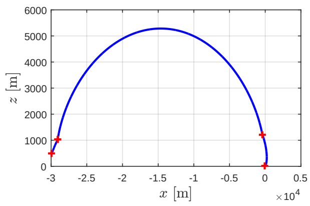

Fig. 4. Optimal trajectory in (x, z) plane.

II: $\mathcal{L}_{1,1}$ norm The $\mathcal{L}_{1,1}$ case is run considering a tolerance parameter $\varepsilon = 10^{-4}$. The optimal solution $\bar{\lambda} = [0.1041\ -0.1083\ 0.1373\ 1.2679]^T$ is obtained after 5 iterations of Algorithm 1. Then, the optimal impulse locations are given by $\Gamma_{imp} = \{0,\ 1.3352,\ 6.7087,\ 8.1832\}$ [rad]. The total fuel-consumption for this in-plane maneuver is of 10.8415 m/s. The norm of the primer vector history is proposed after the two-impulse initialization of Algorithm 1 on Fig. 5 while the second and the final iterations are on Fig. 6 and 7. Finally, the optimal trajectory is exposed on Fig. 8. The comparisons of $\mathcal{L}_{1,2}$ and $\mathcal{L}_{1,1}$ fuel-minimum solutions show a minor difference with respect to the optimal locations and overall consumption.

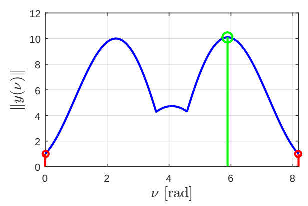

Fig. 5. Primer vector norm: Initialization.

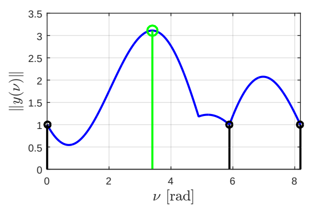

Fig. 6. Primer vector norm: First iteration.

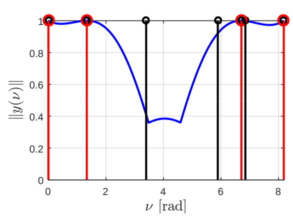

Fig. 7. Primer vector norm: Final iteration.

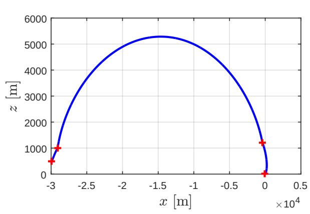

Fig. 8. Optimal trajectory in (x, z) plane.

6. CONCLUSIONS

A new convergent numerical algorithm has been proposed to solve the linearized impulsive fixed-time fuel-optimal space rendezvous problem. Beside its convergence proof, the algorithm features simplicity, speed and reliability: it makes use of state of the art linear/SDP solvers; on classical rendezvous mission examples, for accuracies of $\varepsilon = 10^{-4}$, no more than 10 iterations are necessary, which accounts for few milliseconds on a modern computer; the numerical error bounds provide guarantees that the input accuracy ε is met at algorithm's output.

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