Analysis and Synthesis of Interconnected Positive Systems

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Abstract

This paper is concerned with the analysis and synthesis of interconnected systems constructed from heterogeneous positive subsystems and a nonnegative interconnection matrix. We first show that the interconnected system is positive and stable if and only if a Metzler matrix, which is built from the coefficient matrices of the positive subsystems and the interconnection matrix, is Hurwitz stable. By means of this key result, we further provide several results that characterize the positivity and stability of interconnected systems in terms of the Frobenius eigenvalue of the interconnection matrix and the weighted L_1-induced norm of positive subsystems to be defined in this paper. Moreover, in the case where every subsystem is SISO, we provide explicit conditions under which the interconnected system has the property of persistence, i.e., the state of the interconnected system converges to a unique strictly positive vector (that is known in advance up to a strictly positive constant multiplicative factor) irrespective of nonnegative and nonzero initial states. We finally extend the persistence results to formation control of multi-agent positive systems. This result can be seen as a generalization of a well-known consensus algorithm that has been basically applied to interconnected systems constructed from integrators.

Index Terms

positive systems, interconnection, stability, persistence, multi-agent systems, formation control.

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I. INTRODUCTION

Recently, systems of interest in the field of engineering, biology, economics, etc., have become more complex and larger-scaled, and as such intensive research effort has been made for developing dedicated analysis and synthesis tools. The issue is how to derive sharpened analysis and synthesis conditions exploiting the properties of subsystems and interconnection structure [16], [23], [22], [12]. In this paper, we are particularly interested in the case where the subsystems are positive. A dynamical system is said to be (internally) positive if its state and output are nonnegative for any nonnegative initial state and nonnegative input [11], [19]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel connections are all positive. Even though their dynamics are very simple, the behavior of interconnected systems constructed from them is complicated and deserves investigation especially in the study area of multi-agent systems [23], [22]. This fact also strongly motivates us to focus on the interconnected positive systems. Nowadays the study on linear positive system is active and remarkable results have been obtained along with convex optimization theory [28], [15], [1], [25], [26], [29], [20], [4], [10]. Because of the positiveness, linear positive systems allow special type of Lyapunov functions for the analysis and synthesis, such as co-positive functions and quadratic in the state functions with diagonal Lyapunov matrices [2], [29], [26].

This paper is concerned with the analysis and synthesis of interconnected systems constructed from heterogeneous positive subsystems and a nonnegative interconnection matrix. We first show that the interconnected system is positive† and stable if and only if a Metzler matrix, which is built from the coefficient matrices of the positive subsystems and the interconnection matrix, is Hurwitz stable. By means of this key result, we further provide several results that characterize the positivity and stability of interconnected systems in terms of the Frobenius eigenvalue [17] of the interconnection matrix and the weighted L_1-induced norm of positive subsystems to be defined in this paper. Moreover, in the case where every subsystem is SISO, we provide explicit conditions under which the interconnected system has the property of persistence, i.e., the state of the interconnected system converges to a unique strictly positive vector (that is known in advance up to a strictly positive constant multiplicative factor) irrespective of nonnegative and

†More precisely, we replace the positivity by admissibility, whose definition is given later.

nonzero initial states. More precisely, we prove that the persistence is achieved if every positive subsystem shares identical unweighted L_1-induced norm γ (which is nothing but the steady-state gain) and if the interconnection matrix is irreducible [17] and has the Frobenius eigenvalue $1/\gamma$. We finally extend the persistence results to formation control of multi-agent systems [13], [22], [30], [31]. For multiple agents that move over a plane, the goal is to design a communication scheme over the agents with respect to each agent's position so that prescribed formation can be achieved. We show that such communication scheme synthesis is possible even if the agents have different dynamics (and hence heterogeneous) as long as they are positive, stable and share identical unweighted L_1-induced norm (steady-state gain). As illustrated later, this result can be seen as a generalization of a well-known consensus algorithm that has been basically applied to interconnected systems constructed from integrators [22]. Our results essentially concerns consensus-based *output control* of interconnected *heterogeneous* positive systems, and this is in stark contrast with recent results [10] on *state consensus* of interconnected *homogeneous* positive systems where homogeneousness drastically facilitates the treatment. We finally note that this paper assembles those results in [5], [9], [8] with explicit proofs for technical lemmas and theorems.

We use the following notations. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ $(A_{ij} \geq B_{ij})$ holds for all (i, j), where A_{ij} stands for the (i, j)-entry of A. In relation to this notation, we also define $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : \ x > 0\}$ and $\mathbb{R}^n_{+} := \{x \in \mathbb{R}^n : \ x \geq 0\}$. We also define $\mathbb{R}^{n\times m}_{++}$ and $\mathbb{R}^{n\times m}_{+}$ with obvious modifications. In addition, we denote by $\mathbb{D}^n_{++}$ the set of diagonal and strictly positive matrices of the size n. For $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ and $\rho(A)$ the set of the eigenvalues of A and the spectral radius of A, respectively. For $A \in \mathbb{R}^{n\times n}_{+}$, Theorem 8.3.1 in [17] states that there is an eigenvalue equal to $\rho(A)$. This eigenvalue is often called the Frobenius eigenvalue and denoted by $\lambda_{\rm F}(A)$ in this paper. For given vector $x \in \mathbb{R}^n$ we define its 1-norm by $\|x\|_1 := \sum_{i=1}^n |x_i|$. In addition, for $s(t) : \ \mathbb{R}_+ \to \mathbb{R}^n$, we define its L_1-norm by

$$\|s\|_1 := \int_0^\infty \|s(t)\|_1 dt.$$

Finally, we define the families of functions L_1^n, L_{1+}^n as follows:

$$L_1^n := \{s| \ s(t) : \ \mathbb{R}_+ \to \mathbb{R}^n, \|s\|_1 < \infty\}, \quad L_{1+}^n := \{s| \ s(t) : \ \mathbb{R}_+ \to \mathbb{R}^n_+, \ \|s\|_1 < \infty\}.$$

II. PRELIMINARIES

In this section, we gather basic definitions and fundamental results for positive systems.

Definition 1 (Metzler Ma~~r~~trix): [11] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ $(i \neq j)$.

In the sequel, we denote by $\mathbb{M}^n$ $(\mathbb{H}^n)$ the set of the Metzler (Hurwitz stable) matrices of ~~the~~ size n. Under these notations, the next lemmas hold. The proof of Lemma 2 is given in the appendix section, Subsection VII-A.

Lemma 1: [11], [19], [21] For given $A \in \mathbb{M}^n$, the following conditions are equivalent.

(i) The matrix A is Hurwitz stable, i.e., $A \in \mathbb{H}^n$.

(ii) The matrix A is nonsingular and $A^{-1} \leq 0$.

(iii) There exists $h \in \mathbb{R}^n_{++}$ such that $h^T A < 0$.

(iv) For any $g \in \mathbb{R}^n_+ \setminus \{0\}$, the vector Ag has at least one strictly negative entry.

Lemma 2: For given $P \in \mathbb{M}^{n_1}$, $Q \in \mathbb{R}^{n_1 \times n_2}_+$, $R \in \mathbb{R}^{n_2 \times n_1}_+$, and $S \in \mathbb{M}^{n_2}$, the following conditions are equivalent.

(i) $\Pi := \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathbb{H}^{n_1+n_2}$.

(ii) $P \in \mathbb{H}^{n_1}$, $S - RP^{-1}Q \in \mathbb{H}^{n_2}$.

(iii) $S \in \mathbb{H}^{n_2}$, $P - QS^{-1}R \in \mathbb{H}^{n_1}$.

To move on to the definition of positive systems, consider the linear system G described by

$$G: \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$, and $D \in \mathbb{R}^{n_z \times n_w}$. The definition ~~and a basic result~~ of positive systems are given in the following.

Definition 2 (Positive Linear System): [11] The linear system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Remark 1: In literature, a system satisfying the condition in Definition 2 is often called *internally* positive, to make a clear distinction from *externally* positive systems. Since we only deal with internally positive systems in this paper, we simply denote it by positive as in Definition 2.

Proposition 1: [11] The system (1) is positive if and only if $A \in \mathbb{M}^n$, $B \in \mathbb{R}^{n \times n_w}_+$, $C \in \mathbb{R}^{n_z \times n}_+$, and $D \in \mathbb{R}^{n_z \times n_w}_+$.

We next introduce the weighted L_1-induced norm of positive systems. It turns out in the next section that the weighted L_1-induced norm plays an important role in characterizing the stability of interconnected positive systems.

Definition 3: Suppose G given by (1) is positive and $x(0) = 0$. Then, its weighted L_1-induced norm associated with weighting vectors $q_z \in \mathbb{R}^{n_z}_{++}$ and $q_w \in \mathbb{R}^{n_w}_{++}$ is defined by

$$\|G_{q_z,q_w}\|_{1+} := \sup_{\|q_w^T w\|_1=1,\ w\in L_{1+}^{n_w}} \|q_z^T z\|_1. \tag{2}$$

Remark 2: The standard L_1-induced norm of G given by (1) is defined as follows [14]:

$$\|G\|_1 := \sup_{\|w\|_1=1,\ w\in L_1^{n_w}} \|z\|_1. \tag{3}$$

From the positivity of G, we can easily confirm that the two L_1-induced norms given above can be linked by

$$\|G_{q_z,q_w}\|_{1+} = \|Q_z G Q_w^{-1}\|_1 \tag{4}$$

where $Q_z := \mathrm{diag}(q_{z,1}, \cdots, q_{z,n_z})$, $Q_w := \mathrm{diag}(q_{w,1}, \cdots, q_{w,n_w})$. Namely, as the denomination "weighted" L_1-induced norm stands, $\|G_{q_z,q_w}\|_{1+}$ coincides with the standard L_1-induced norm with weightings (or scalings) on the input and output signals. The vector representation of weightings as in q_z and q_w rather than the matrix representation as in Q_z and Q_w is useful in characterizing the weighted L_1-induced norm and the stability of interconnected positive systems by linear inequalities. This is firstly illustrated in the next theorem.

Theorem 1: Suppose G given by (1) is positive. Then, for given $q_z \in \mathbb{R}^{n_z}_{++}$, $q_w \in \mathbb{R}^{n_w}_{++}$, and $\gamma > 0$, the following conditions are equivalent.

(i) The matrix $A \in \mathbb{M}^n$ is Hurwitz stable and $\|G_{q_z,q_w}\|_{1+} < \gamma$.

(ii) There exists $h \in \mathbb{R}^n_{++}$ such that

$$\begin{bmatrix} h^T A + q_z^T C & h^T B + q_z^T D - \gamma q_w^T \end{bmatrix} < 0. \tag{5}$$

(iii) The matrix $A \in \mathbb{M}^n$ is Hurwitz stable and the following inequality holds:

$$q_z^T G(0) < \gamma q_w^T. \tag{6}$$

Here, $G(s)$ is the transfer matrix of the system G defined by $G(s) := C(sI - A)^{-1}B + D$.

If we let $q_z = \mathbf{1}^{n_z}$ and $q_w = \mathbf{1}^{n_w}$ where $\mathbf{1}^{n_z}$ stands for the all-ones vector of size n_z, the definition (2) essentially reduces to the standard L_1-induced norm as we noted in (4). This standard L_1-induced norm is employed as a performance index in recent studies on switched positive systems [32], [33]. Moreover, this standard L_1-induced norm is used in [4] as a useful tool for robust stability analysis of uncertain positive systems.

Even though related discussions on the proof of Theorem 1 can be found, for example, in [25], [4], we give a detailed proof of Theorem 1 in the appendix section, Subsection VII-B, for completeness. The next corollary directly follows from (iii) in Theorem 1.

Corollary 1: Suppose G given by (1) is positive and stable. Then, for given $q_z \in \mathbb{R}_{++}^{n_z}$, $q_w \in \mathbb{R}_{++}^{n_w}$, the weighted L_1-induced norm $\|G_{q_z,q_w}\|_{1+}$ is given by

$$\|G_{q_z,q_w}\|_{1+} = \min\ \gamma \text{ subject to } q_z^T G(0) \le \gamma q_w^T \tag{7}$$

or equivalently,

$$\|G_{q_z,q_w}\|_{1+} = \max_i \frac{(q_z^T G(0))_i}{q_{w,i}}. \tag{8}$$

This corollary implies that, if G given by (1) is stable and SISO, we have $\|G_{1,1}\|_{1+} = G(0)$. Namely, the unweighted L_1-induced norm coincides with the steady-state gain.

III. Stability Analysis of Interconnected Positive Systems

Let us consider the positive subsystem $G_i\ (i = 1, \cdots, N)$ represented by

$$G_i : \begin{cases} \dot{x}_i \ = \ A_i x_i \ + \ B_i w_i, \\ z_i \ = \ C_i x_i \ + \ D_i w_i, \end{cases} \tag{9}$$
$$A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\},\ B_i \in \mathbb{R}_+^{n_i \times n_{w_i}},\ C_i \in \mathbb{R}_+^{n_{z_i} \times n_i},\ D_i \in \mathbb{R}_+^{n_{z_i} \times n_{w_i}}.$$

As clearly shown in (9), we have assumed that $G_i\ (i = 1, \cdots, N)$ are all stable.

With these positive subsystems, let us define a positive and stable system $\mathcal{G}$ by

$$\mathcal{G} := \mathrm{diag}(G_1, \cdots, G). \tag{10}$$

The state space realization of $\mathcal{G}$ is given by

$$\mathcal{G} : \begin{cases} \dot{\widehat{x}} \ = \ \mathcal{A}\widehat{x} \ + \ \mathcal{B}\widehat{w}, \\ \widehat{z} \ = \ \mathcal{C}\widehat{x} \ + \ \mathcal{D}\widehat{w} \end{cases} \tag{11}$$

where

$$\begin{aligned} &\mathcal{A} := \mathrm{diag}(A_1, \cdots, A),\ \mathcal{B} := \mathrm{diag}(B_1, \cdots, B), \\ &\mathcal{C} := \mathrm{diag}(C_1, \cdots, C),\ \ \mathcal{D} := \mathrm{diag}(D_1, \cdots, D), \end{aligned} \tag{12}$$

$$\begin{aligned}
\widehat{x} &:= \begin{bmatrix} x_1^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{n_{\widehat{x}}}, \quad n_{\widehat{x}} := \sum_{i=1}^N n_i, \\
\widehat{w} &:= \begin{bmatrix} w_1^T & \cdots & w_N^T \end{bmatrix}^T \in \mathbb{R}^{n_{\widehat{w}}}, \quad n_{\widehat{w}} := \sum_{i=1}^N n_{w_i}, \\
\widehat{z} &:= \begin{bmatrix} z_1^T & \cdots & z_N^T \end{bmatrix}^T \in \mathbb{R}^{n_{\widehat{z}}}, \quad n_{\widehat{z}} := \sum_{i=1}^N n_{z_i}.
\end{aligned} \tag{13}$$

For a given interconnection matrix $\Omega \in \mathbb{R}_+^{n_{\widehat{w}} \times n_{\widehat{z}}}$, we are interested in the stability and the performance of the interconnected system $\mathcal{G} \star \Omega$ defined by (11) and $\widehat{w} = \Omega \widehat{z}$. In relation to the well-posedness of this interconnection, we make the next definition.

Definition 4: The interconnected system $\mathcal{G} \star \Omega$ is said to be *admissible* if the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz stable.

In the sequel, we require the admissibility of the interconnected system $\mathcal{G} \star \Omega$ whenever we analyze its stability and performance. The meaning of this presupposition, and its rationality as well, can be explained as follows. If $\det(\mathcal{D}\Omega - I) \neq 0$, then the interconnection is well-posed, and the state-space description of the interconnected system is represented by

$$\dot{\widehat{x}} = \mathcal{A}_{\text{cl}} \widehat{x}, \quad \mathcal{A}_{\text{cl}} := \mathcal{A} + \mathcal{B}\Omega(I - \mathcal{D}\Omega)^{-1}\mathcal{C}. \tag{14}$$

Thus, if the admissibility is ensured, we see that

(i) the interconnection $\mathcal{G} \star \Omega$ is well-posed;

(ii) the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz and hence $(I - \mathcal{D}\Omega)^{-1} \geq 0$ holds from (ii) of Lemma 1. Therefore the matrix A_{cl} is Metzler. It follows that the positive nature of the subsystems G_i $(i = 1, \cdots, N)$ is inherited to the interconnected system, i.e., the nonnegativity of the states x_i $(i = 1, \cdots, N)$ for any nonnegative initial states is still preserved under the interconnection.

We also note that admissibility is out of issue if $\mathcal{D} = 0$, since in this case we have $\mathcal{A}_{\text{cl}} = \mathcal{A} + \mathcal{B}\Omega\mathcal{C} \in \mathbb{M}^{n_{\widehat{x}}}$ and hence $\mathcal{G} \star \Omega$ is always (well-posed and) positive.

For the admissibility and stability of the interconnected system $\mathcal{G} \star \Omega$, we can obtain the next lemma that plays an important role in this paper.

Lemma 3: The interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix

$$\Pi := \begin{bmatrix} \mathcal{A} & \mathcal{B}\Omega \\ \mathcal{C} & \mathcal{D}\Omega - I \end{bmatrix} \tag{15}$$

is Hurwitz stable.

Proof of Lemma 3: From Definition 4, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrices $\mathcal{D}\Omega - I$ and $\mathcal{A}_{\rm cl} = \mathcal{A} + \mathcal{B}\Omega(I - \mathcal{D}\Omega)^{-1}\mathcal{C}$ are both Hurwitz stable. Thus the assertion readily follows from Lemma 2. ∎

From this key lemma, we can obtain various conditions for the admissibility and stability of the interconnected system according to the properties of the subsystems G_i $(i = 1, \cdots, N)$ and the interconnection matrix Ω. Typical examples are given in the following theorems. The first theorem, Theorem 2, concerns the interconnected system shown in Fig. 1 for the case $N = 3$.

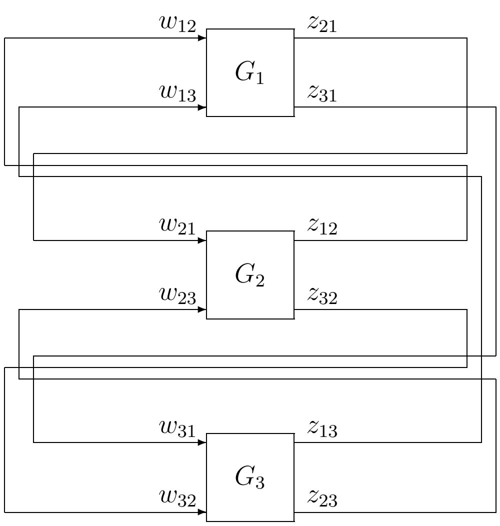

Fig. 1. Interconnected positive system ($N = 3$).

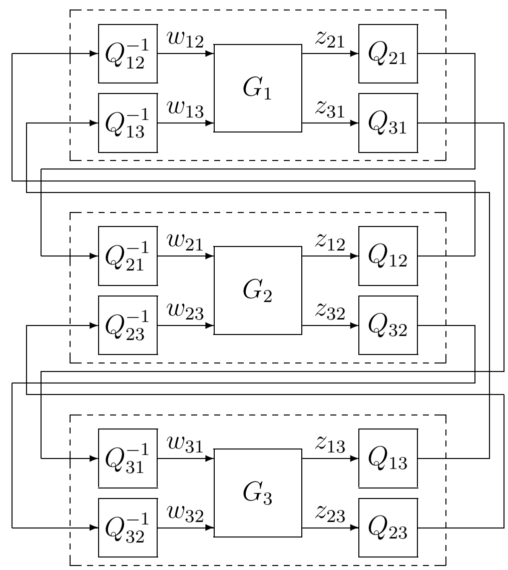

Fig. 2. Weightings on input and output signals.

Theorem 2: Let us consider the case where the i-th subsystem G_i represented by (9) has the following specific structure:

$$G_i : \begin{cases} \dot{x}_i = A_i x_i + \displaystyle\sum_{k=1, k\neq i}^{N} B_{ik} w_{ik}, \\ z_{ji} = C_{ji} x_i + \displaystyle\sum_{k=1, k\neq i}^{N} D_{jik} w_{ik} \quad (j \neq i) \end{cases} \tag{16}$$

$$A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\},\ B_{ik} \in \mathbb{R}_+^{n_i \times n_{w_{ik}}},\ C_{ji} \in \mathbb{R}_+^{n_{z_{ji}} \times n_i},\ D_{jik} \in \mathbb{R}_+^{n_{z_{ji}} \times n_{w_{ik}}}.$$

We assume that the size of w_{ij} and z_{ij} are identical, and N subsystems are interconnected by

$$w_{ij} = z_{ij} \ (i, j = 1, \cdots, N, i \neq j). \tag{17}$$

Then, the interconnected system is admissible and stable if and only if there exist weighting vectors $q_{ij} \in \mathbb{R}_{++}^{n_{w_{ij}}}$ $(i, j = 1, \cdots, N, i \neq j)$ such that

$$\begin{aligned} &\|G_{i,q_{z,i},q_{w,i}}\|_{1+} < 1, \\ q_{z,i} &= \begin{bmatrix} q_{1,i}^T & \cdots & q_{i-1,i}^T & q_{i+1,i}^T & \cdots & q_{N,i}^T \end{bmatrix}^T, \\ q_{w,i} &= \begin{bmatrix} q_{i,1}^T & \cdots & q_{i,i-1}^T & q_{i,i+1}^T & \cdots & q_{i,N}^T \end{bmatrix}^T \ (i = 1, \cdots, N). \end{aligned} \tag{18}$$

As noted, the interconnection structure assumed in Theorem 2 is illustrated in Fig. 1 for the case $N = 3$. The subscripts (i, j) of w_{ij} and z_{ij} indicate that these are the signals that flow from the subsystem j to the subsystem i. By defining

$$\begin{aligned} z_i &= \begin{bmatrix} z_{1,i}^T & \cdots & z_{i-1,i}^T & z_{i+1,i}^T & \cdots & z_{N,i}^T \end{bmatrix}^T, \\ w_i &= \begin{bmatrix} w_{i,1}^T & \cdots & w_{i,i-1}^T & w_{i,i+1}^T & \cdots & w_{i,N}^T \end{bmatrix}^T \ (i = 1, \cdots, N) \end{aligned} \tag{19}$$

and by representing the interconnection (17) by $\widehat{w} = \Omega \widehat{z}$, we can see that the interconnected system can be represented by $\mathcal{G} \star \Omega$. For $N = 3$, the interconnection matrix ~~can~~ be given by

$$\Omega = \left[\begin{array}{cc:cc:cc} 0 & 0 & I_{n_{w_{12}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_{w_{13}}} & 0 \\ \hdashline I_{n_{w_{21}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{w_{23}}} \\ \hdashline 0 & I_{n_{w_{31}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{w_{32}}} & 0 & 0 \end{array}\right]. \tag{20}$$

By applying Lemma 3 to the resulting interconnected system $\mathcal{G} \star \Omega$, we ~~can~~ obtain Theorem 2. The complete proof is given in the appendix section, Subsection VII-C.

The implication of the theorem is that the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if there exists a set of weighting vectors that renders the weighted L_1-induced norm of each positive subsystem less than unity. Namely, the condition for the admissibility and stability is separated into the L_1-induced norm conditions of each subsystem. In this sense, we could say that the weighting vectors work as *separators* that have played important roles for stability analysis of general linear systems [18], [27], [24]. Another interpretation is, as we usually do for separators as well, that weighting vectors serve as scalings for input and output signals. Indeed, from the link (4), we see that $\mathcal{G} \star \Omega$ is admissible and stable if and only if the standard L_1-induced norm of scaled systems (i.e., the systems encircled by dashed line in Fig. 2) are less than unity, where $Q_{ij} = \mathrm{diag}(q_{ij})$. What is interesting here is that such scaling-based stability condition is necessary and sufficient, which is hardly achievable for interconnected systems constructed from general (non-positive) linear systems.

In the case where $N = 3$, the condition (18) can be written concretely as

$$\begin{aligned} &\|G_{1,[q_{21}^T\ \ q_{31}^T]^T,[q_{12}^T\ \ q_{13}^T]^T}\|_{1+} < 1,\\ &\|G_{2,[q_{12}^T\ \ q_{32}^T]^T,[q_{21}^T\ \ q_{23}^T]^T}\|_{1+} < 1,\\ &\|G_{3,[q_{13}^T\ \ q_{23}^T]^T,[q_{31}^T\ \ q_{32}^T]^T}\|_{1+} < 1. \end{aligned} \tag{21}$$

As clearly shown in (21), the L_1-induced norm conditions are coupled with each other through weighting vectors. However, under the mild assumption that each subsystem provides exactly the same scalar output to the rest of subsystems, we can somehow decouple the condition and hence the admissibility and the stability can be determined by means of decoupled weighted L_1-induced norms of subsystems. This result leads to decentralized stabilizing controller synthesis for interconnected positive systems where the controller synthesis can be done in a distributed manner. See [7], [6] for details.

The results in Theorem 2 are valid for MIMO positive subsystems. On the other hand, in the case where every subsystem is SISO, conditions for the admissibility and the stability for the interconnected system $\mathcal{G} \star \Omega$ can be drastically simplified as we see in the next two theorems.

Theorem 3: Let us consider the case where the subsystems G_i $(i = 1, \cdots, N)$ represented by (9) are all SISO. Then, for given $\Omega \in \mathbb{R}_+^{N\times N}$, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix $\Psi\Omega - I$ is Hurwitz stable where $\Psi \in \mathbb{R}_+^{N\times N}$ is constructed from the unweighted L_1-induced norm (i.e., the steady-state gain) of each subsystem as in $\Psi := \mathrm{diag}(\|G_{1,1,1}\|_{1+}, \cdots, \|G_{N,1,1}\|_{1+}) = \mathrm{diag}(G_1(0), \cdots, G_N(0))$.

Proof of Theorem 3: From Lemma 3, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix Π defined by (15) is Hurwitz stable. From Lemma 2 and the fact that $\|G_{i,1,1}\|_{1+} = G_i(0) = -C_i A_i^{-1} B_i + D_i$ $(i = 1, \cdots, N)$, this condition holds if and only if both ~~of~~ the Metzler matrices $\mathcal{A}$ and $\mathcal{D}\Omega - I - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\Omega = \Psi\Omega - I$ are Hurwitz stable. Thus the assertion readily follows since $\mathcal{A}$ is Hurwitz stable from the assumption $A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\}$. ■

Theorem 4: Let us consider the case where the subsystems G_i $(i = 1, \cdots, N)$ represented by (9) are all SISO and share identical steady-state gain $\gamma > 0$, i.e., $G_1(0) = \cdots = G_N(0) = \gamma$.

Then, for given $\Omega \in \mathbb{R}_+^{N\times N}$, the interconnected system $\mathcal{G}\star\Omega$ is admissible and stable if and only if $\gamma\lambda_{\rm F}(\Omega) < 1$.

Proof of Theorem 4: From Theorem 3, we see that the interconnected system $\mathcal{G}\star\Omega$ is admissible and stable if and only if $\gamma\Omega - I \in \mathbb{M}^N$ is Hurwitz stable. This condition can be restated equivalently as in $\gamma\lambda_{\rm F}(\Omega) < 1$. This completes the proof. ■

These three theorems clearly show that the admissibility and stability of interconnected positive systems can be fully characterized in terms of weighted L_1-induced norms of subsystems. Moreover, if all subsystems are SISO and share identical steady-state gain $\gamma > 0$, we see from Theorem 4 that the interconnected system $\mathcal{G}\star\Omega$ is on the stability boundary if $\gamma\lambda_{\rm F}(\Omega) = 1$. This idea leads us to the persistence analysis of $\mathcal{G}\star\Omega$ as detailed in the next section.

IV. Persistence Analysis of Interconnected Positive Systems

In this section, we are interested in the *persistence* of the interconnected system $\mathcal{G}\star\Omega$. After giving our main results on the persistence of $\mathcal{G}\star\Omega$, we show that the persistence results can be applied to formation control of multi-agent systems.

A. Persistence Analysis

We first give the precise definition of what we call persistence.

Definition 5: For given positive subsystems G_i $(i \in \mathbb{Z}_N)$ represented by (9) and interconnection matrix $\Omega \in \mathbb{R}_+^{n_{\widehat{w}}\times n_{\widehat{z}}}$, consider the interconnected system $\mathcal{G}\star\Omega$. Then, the interconnected system $\mathcal{G}\star\Omega$ is said to have the property of *persistence* if it is admissible and if there exist $\xi_0, \xi_\infty \in \mathbb{R}_{++}^{n_{\widehat{x}}}$ such that

$$\lim_{t\to\infty} \widehat{x}(t) = (\xi_0^T \widehat{x}(0))\xi_\infty \tag{22}$$

for any initial state $\widehat{x}(0) \in \mathbb{R}^{n_{\widehat{x}}}$.

This definition requires that the state $\widehat{x}$ of $\mathcal{G}\star\Omega$ converges to a strictly positive scalar multiple of a strictly positive vector as long as $\widehat{x}(0) \in \mathbb{R}_+^{n_{\widehat{x}}} \setminus \{0\}$, i.e., all the states $\widehat{x}_i$ $(i = 1, \cdots, n_{\widehat{x}})$ become strictly positive and hence "excited" eventually. This is the reason why we call the property persistence. It is also clear that persistence requires that the interconnected system $\mathcal{G}\star\Omega$ is on the stability boundary.

To state our main result on the persistence of $\mathcal{G}\star\Omega$, we first need to review the definition and related results on *irreducible* matrices. Similarly to [10], it turns out that the irreducibility of interconnection matrix plays a crucial role in achieving persistence.

Definition 6: [Reducible Matrix [17] (p. 360)] A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *reducible* if either

(a) $n = 1$ and $M = 0$ or

(b) $n \geq 2$ and there exist a permutation matrix $P \in \mathbb{R}^{n\times n}$ and r with $1 \leq r \leq n-1$ such that
$$P^T M P = \begin{bmatrix} Q & R \\ 0_{n-r,r} & S \end{bmatrix},\ Q \in \mathbb{R}^{r\times r},\ S \in \mathbb{R}^{(n-r)\times(n-r)}.$$

Definition 7: [Irreducible Matrix [17] (p. 361)] A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *irreducible* if it is not reducible.

Definition 8: [Directed Graph of Matrices [17] (p. 357)] The directed graph of $M \in \mathbb{R}^{n\times n}$, denoted by $\Gamma(M)$, is the directed graph on n nodes $P_1,\ P_2,\ \cdots,\ P_n$ such that there is a directed arc in $\Gamma(M)$ from P_i to P_j if and only if $M_{ij} \neq 0$ or equivalently, $\mathrm{In}(M)_{ij} \neq 0$. Here, $\mathrm{In}(M)$ stands for the indicator matrix of M.

Definition 9: [Strongly Connected Graph [17] (p. 358)] A directed graph Γ is said to be *strongly connected* if between every pair of distinct nodes P_i, P_j in Γ there is a directed path of finite length that begins at P_i and ends at P_j.

Under these definitions, it is known that the next results hold.

Proposition 2: [17] (p. 362) For given $M \in \mathbb{R}^{n\times n}$, the following conditions are equivalent.

(a) M is irreducible.

(b) $(I_n + \mathrm{In}(M))^{n-1} > 0$.

(c) $\Gamma(M)$ is strongly connected.

Proposition 3: [17] (p. 508) Suppose $M \in \mathbb{R}_+^{n\times n}$ is irreducible. Then the following conditions hold.

(i) $\rho(M) > 0$ and $\rho(M)$ is an eigenvalue of M.

(ii) There is a vector $v \in \mathbb{R}_{++}^n$ such that $Mv = \rho(M)v$.

(iii) $\rho(M)$ is an algebraically (and hence geometrically) simple eigenvalue of M.

The next corollary directly follows from Proposition 3, where (iii) is particularly important.

Corollary 2: Suppose $M \in \mathbb{M}^n$ is irreducible. Then the following conditions hold where $\alpha := \max_{\lambda\in\sigma(M)} \mathrm{Re}(\lambda)$.

(i) $\alpha \in \mathbb{R}$ is an algebraically (and hence geometrically) simple eigenvalue of M.

(ii) There is a vector $v \in \mathbb{R}_{++}^n$ such that $Mv = \alpha v$.

(iii) $\mathrm{Re}(\lambda) < \alpha$ $(\forall \lambda \in \sigma(M) \setminus \{\alpha\})$.

We are now ready to state our main result on the persistence of $\mathcal{G} \star \Omega$ and give its proof.

Theorem 5: Let us consider the case where every subsystem G_i represented by (9) is SISO-. Suppose G_i $(i = 1, \cdots, N)$ and a given interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ satisfy the following conditions.

(i) (A_i, B_i) is controllable and (A_i, C_i) is observable for all $i = 1, \cdots, N$.

(ii) $G_1(0) = \cdots = G_N(0) =: \gamma(> 0)$ holds.

(iii) The interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ is irreducible, i.e., the directed graph $\Gamma(\Omega)$ is strongly connected.

(iv) $\lambda_{\rm F}(\Omega) = 1/\gamma$ holds.

Then, for the interconnected system $\mathcal{G} \star \Omega$, the next results hold.

(I) The interconnected system $\mathcal{G} \star \Omega$ is admissible, i.e., the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz stable.

(II) The matrix $\mathcal{A}_{\rm cl}$ given by (14) satisfies $\sigma(\mathcal{A}_{\rm cl}) \subset \overline{\mathbb{C}_-}$, i.e., $\mathrm{Re}(\lambda) \le 0$ $(\forall \lambda \in \sigma(\mathcal{A}_{\rm cl}))$.

(III) If we denote the right and left eigenvectors of Ω associated with the Frobenius eigenvalue $\lambda_{\rm F}(\Omega)$ by $v_{\rm R} \in \mathbb{R}_{++}^N$ and $v_{\rm L} \in \mathbb{R}_{++}^N$, respectively, we have $\mathcal{A}_{\rm cl}\xi_{\rm R} = 0$ and $\xi_{\rm L}^T \mathcal{A}_{\rm cl} = 0$ where

$$\xi_{\rm R} = -\mathcal{A}^{-1}\mathcal{B}v_{\rm R} \in \mathbb{R}_{++}^{n_{\widehat{x}}}, \quad \xi_{\rm L} = -\mathcal{A}^{-T}\mathcal{C}^T v_{\rm L} \in \mathbb{R}_{++}^{n_{\widehat{x}}}, \quad \xi_{\rm L}^T \xi_{\rm R} = 1. \tag{23}$$

Here the eigenvectors $v_{\rm R}, v_{\rm L} \in \mathbb{R}_{++}^N$ are appropriately scaled so that $\xi_{\rm L}^T\xi_{\rm R} = 1$ is satisfied.

(IV) The matrix $\mathcal{A}_{\rm cl}$ has eigenvalue 0 that is algebraically (and hence geometrically) simple. Moreover, we have $\mathrm{Re}(\lambda) < 0$ $(\forall \lambda \in \sigma(\mathcal{A}_{\rm cl}) \setminus \{0\})$.

(V) We have

$$\lim_{t\to\infty} \widehat{x}(t) = f(\widehat{x}(0))\xi_{\rm R}, \quad f(\widehat{x}(0)) = \xi_{\rm L}^T \widehat{x}(0) \tag{24}$$

for any initial state $\widehat{x}(0) \in \mathbb{R}^{n_{\widehat{x}}}$.

The results (I), (III) and (V) of Theorem 5 clearly show that, under the conditions (i)-(iv), the interconnected system $\mathcal{G} \star \Omega$ has the property of persistence, and (22) in Definition 5 is satisfied with $\xi_0 = \xi_{\rm L} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$ and $\xi_\infty = \xi_{\rm R} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$.

We need the following lemma for the proof of Theorem 5. The proof of this lemma is given in the appendix section, Subsection VII-D.

Lemma 4: For given $A \in \{\mathbb{M}^n \cap \mathbb{H}^n\}$, $B \in \mathbb{R}_+^{n\times 1}$, and $C \in \mathbb{R}_+^{1\times n}$, we have $A^{-1}B < 0$ if (A, B) is controllable. Similarly, we have $CA^{-1} < 0$ if (A, C) is observable.

Proof of Theorem 5:

Proof of (I): From (i) and Lemma 4, it is clear that $-C_iA_i^{-1}B_i > 0$ $(i = 1, \cdots, N)$. If we define $\mathcal{S} := \mathrm{diag}(-C_1A_1^{-1}B_1, \cdots, -C_NA_N^{-1}B_N) \in \mathbb{D}_{++}^N$, we have $\mathcal{D} = \gamma I - \mathcal{S}$ from (ii). On the other hand, from (iii) and Proposition 3, we see that there exists $v_{\mathrm{R}} \in \mathbb{R}_{++}^N$ such that $\Omega v_{\mathrm{R}} = \lambda_{\mathrm{F}}(\Omega)v_{\mathrm{R}}$. This implies $(\gamma\Omega - I)v_{\mathrm{R}} = 0$ since (iv) holds. Therefore we have $(\mathcal{D}\Omega - I)v_{\mathrm{R}}$ $= ((\gamma I - \mathcal{S})\Omega - I)\,v_{\mathrm{R}} = -\mathcal{S}\Omega v_{\mathrm{R}} = -\lambda_{\mathrm{F}}(\Omega)\mathcal{S}v_{\mathrm{R}} < 0$. It follows from (the dual version of) (iii) of Lemma 1 that $\mathcal{D}\Omega - I$ is Hurwitz stable.

Proof of (II): From Theorem 4, we see that $\sigma(\mathcal{A}_{\mathrm{cl}}) \subset \mathbb{C}_-$ if and only if $\gamma\lambda_{\mathrm{F}}(\Omega) < 1$. Since at present $\gamma\lambda_{\mathrm{F}}(\Omega) = 1$ holds from (iv), we see that $\sigma(\mathcal{A}_{\mathrm{cl}}) \subset \overline{\mathbb{C}_-}$ holds from the continuity of the eigenvalue of $\mathcal{A}_{\mathrm{cl}}$ with respect to perturbations on it.

Proof of (III): By defining $\Omega_{\mathcal{D}} := \Omega(I - \mathcal{D}\Omega)^{-1}$, we readily see

$$
\begin{aligned}
\mathcal{A}_{\mathrm{cl}}\xi_{\mathrm{R}} &= -\left(\mathcal{A} + \mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}\right)\mathcal{A}^{-1}\mathcal{B}v_{\mathrm{R}} \\
&= -\mathcal{B}v_{\mathrm{R}} - \mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v_{\mathrm{R}} \\
&= -\mathcal{B}v_{\mathrm{R}} - \mathcal{B}\Omega_{\mathcal{D}}(\mathcal{D} - \gamma I)v_{\mathrm{R}} \\
&= -\mathcal{B}\left(I + \Omega(I - \mathcal{D}\Omega)^{-1}(\mathcal{D} - \gamma I)\right)v_{\mathrm{R}} \\
&= -\mathcal{B}\left(I + (I - \Omega\mathcal{D})^{-1}\Omega(\mathcal{D} - \gamma I)\right)v_{\mathrm{R}} \\
&= -\mathcal{B}(I - \Omega\mathcal{D})^{-1}\left(I - \gamma\Omega\right)v_{\mathrm{R}} \\
&= -\mathcal{B}(I - \Omega\mathcal{D})^{-1}\left(1 - \gamma\lambda_{\mathrm{F}}(\Omega)\right)v_{\mathrm{R}} \\
&= 0.
\end{aligned}
$$

The equality $\xi_{\mathrm{L}}^T\mathcal{A}_{cl} = 0$ follows similarly. On the other hand, since (A_i, B_i) is controllable and (A_i, C_i) is observable, we see $-A_i^{-1}B_i > 0$ and $-C_iA_i^{-1} > 0$ $(i = 1, \cdots, N)$ from Lemma 4. Moreover, since Ω is irreducible, we have $v_{\mathrm{R}} > 0$ and $v_{\mathrm{L}} > 0$ from Proposition 3. Therefore we have $\xi_{\mathrm{R}} = -\mathcal{A}^{-1}\mathcal{B}v_{\mathrm{R}} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$ and $\xi_{\mathrm{L}} = -\mathcal{A}^{-T}\mathcal{C}^Tv_{\mathrm{L}} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$.

Proof of (IV): We can prove that $\mathcal{A}_{\mathrm{cl}}$ is irreducible and hence the assertion readily follows from (II), (III) and Corollary 2. The proof for the irreducibility of $\mathcal{A}_{\mathrm{cl}}$, which is indeed the core of the proof of Theorem 5, is given in the appendix section, Subsection VII-E.

Proof of (V): Since (IV) holds and ~~and~~ since $\xi_{\mathrm{R}} \in \mathbb{R}_{++}^{n_{\widehat{x}}}$ is the right eigenvector of $\mathcal{A}_{\mathrm{cl}}$ corresponding to the eigenvalue 0, it is an elementary fact that the state $\widehat{x}$ of the interconnected system $\mathcal{G} \star \Omega$ converges to $f(\widehat{x}(0))\xi_{\mathrm{R}}$ for some linear function $f :\ \mathbb{R}^{n_{\widehat{x}}} \to \mathbb{R}$. Furthermore, for the dynamics of the interconnected system represented by $\dot{\widehat{x}} = \mathcal{A}_{\mathrm{cl}}\widehat{x}$, we can readily see that $\xi_{\rm L}^T\dot{\widehat{x}} = 0$. Therefore we have $\xi_{\rm L}^T\widehat{x}(0) = f(\widehat{x}(0))\xi_{\rm L}^T\xi_{\rm R}$. Since $\xi_{\rm L}^T\xi_{\rm R} = 1$ from (III), we have $f(\widehat{x}(0)) = \xi_{\rm L}^T\widehat{x}(0)$. This completes the proof. ■

We have a strong prospect that the persistence result in Theorem 5 has a wide range of applications. The scope includes analysis of coexistence of competitive species in systems biology, dynamic resource allocation in social system design, and analysis and synthesis of multi-agent systems. The last issue is pursued in the next section, Section V, after giving several preliminary results in the next subsections.

B. Analysis of Steady-State Output

The next result concerns the steady state output of $\mathcal{G} \star \Omega$. This is a direct consequence of Theorem 5 and illustrates its usefulness in the application to formation control of multi-agent systems with positive dynamics.

Corollary 3: Consider the case where every subsystem G_i represented by (9) is SISO and satisfies conditions (i) and (ii) in Theorem 5. Then, for given $v_{\rm obj} \in \mathbb{R}_{++}^N$, the output of interconnected system $\mathcal{G} \star \Omega$ satisfies

$$\lim_{t\to\infty} \widehat{z}(t) = \gamma f(\widehat{x}(0)) v_{\rm obj} (= \gamma(\xi_{\rm L}^T\widehat{x}(0)) v_{\rm obj}) \tag{25}$$

if $\Omega \in \mathbb{R}_+^{N\times N}$ has the following property in addition to (iii) of Theorem 5:

(iv') $\Omega v_{\rm obj} = (1/\gamma) v_{\rm obj}$ holds.

Namely, for any initial state $\widehat{x}(0) \in \mathbb{R}_+^{n_{\widehat{x}}} \setminus \{0\}$, we can achieve the convergence of the output $\widehat{z}(t) = [\ z_1(t)\ \ \cdots\ \ z_N(t)\ ]^T$ to $\gamma f(\widehat{x}(0)) v_{\rm obj} \in \mathbb{R}_{++}^N$ by the interconnection with $\Omega \in \mathbb{R}_+^{N\times N}$ having properties (iii) and (iv').

This corollary implies that, for given $v_{\rm obj} \in \mathbb{R}_{++}^N$ that represents the output position of each agent in a "desired formation," we can achieve the convergence (25) as long as we design Ω satisfying (iii) and (iv'). This is the basic idea to use the results in Theorem 5 and Corollary 3 for the formation control of multi-agent systems.

A brief sketch of the proof of Corollary 3 is as follows. Since $\Omega \in \mathbb{R}_+^{N\times N}$ satisfies $\Omega v_{\rm obj} = (1/\gamma) v_{\rm obj}$ for $v_{\rm obj} \in \mathbb{R}_{++}^N$, we can see from Corollary 8.1.30 of [17] that $\lambda_{\rm F}(\Omega) = 1/\gamma$. Namely, a matrix Ω satisfying the condition (iv') satisfies the condition (iv) in Theorem 5 as well. It follows from Theorem 5 that $\widehat{x}_\infty = -f(\widehat{x}(0))\mathcal{A}^{-1}\mathcal{B} v_{\rm obj}$ where $\widehat{x}_\infty := \lim_{t\to\infty}\widehat{x}(t)$. Therefore, for $\widehat{z}_\infty := \lim_{t\to\infty}\widehat{z}(t)$, we obtain

$$
\begin{aligned}
\widehat{z}_\infty &= (I-\mathcal{D}\Omega)^{-1}\mathcal{C}\widehat{x}_\infty \\
&= -f(\widehat{x}(0))(I-\mathcal{D}\Omega)^{-1}\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v_{\rm obj} \\
&= -f(\widehat{x}(0))(I-\mathcal{D}\Omega)^{-1}(\mathcal{D}-\gamma I)v_{\rm obj} \\
&= -\frac{f(\widehat{x}(0))}{\lambda_{\rm F}(\Omega)}(I-\mathcal{D}\Omega)^{-1}(\lambda_{\rm F}(\Omega)\mathcal{D}-\gamma\lambda_{\rm F}(\Omega)I)v_{\rm obj} \\
&= -\frac{f(\widehat{x}(0))}{\lambda_{\rm F}(\Omega)}(I-\mathcal{D}\Omega)^{-1}(\mathcal{D}\Omega-I)v_{\rm obj} \\
&= \gamma f(\widehat{x}(0))v_{\rm obj}.
\end{aligned}
$$

This validates the assertion in Corollary 3.

C. Relationship with the f-Consensus Protocol [22], [13]

Theorem 5 and Corollary 3 are closely related to (and meaningful extensions of) the results already obtained in the study area of multi-agent systems [13], [22], [30], [31]. In this section, we show that the f-consensus protocol shown in [22], [13] can readily be obtained along with Theorem 5.

The communication over multi-agents in [22], [13] is determined by the directed graph $G(\mathcal{I},\mathcal{E})$ with the set of nodes $\mathcal{I} := \{1,\cdots,N\}$ and edges $\mathcal{E}\subseteq\mathcal{I}\times\mathcal{I}$. The dynamics of the agents are assumed to be identical integrator as in

$$P_i:\ \dot{x}_i(t) = w_i(t),\quad x_i(t)\in\mathbb{R}. \tag{26}$$

The goal is to determine the input w_i $(i=1,\cdots,N)$ by the communication with other agents over network so that we can achieve

$$\lim_{t\to\infty}\widehat{x}(t) = f(\widehat{x}(0))\mathbf{1}^N,\quad \widehat{x} := [\ x_1,\ \cdots,\ x_N\ ]^T\in\mathbb{R}^N. \tag{27}$$

If (27) is achieved for some $f:\mathbb{R}^N\to\mathbb{R}$, we say that f-consensus is achieved. In order to achieve an f-consensus, the following protocol is presented in [22], [13]:

$$w_i(t) = \sum_{j\in\mathcal{N}_i}(x_j(t)-x_i(t)). \tag{28}$$

Here, $\mathcal{N}_i$ is the neighbors of the node i defined by $\mathcal{N}_i := \{j\in\mathcal{I}:\ (j,i)\in\mathcal{E}\}$. The interconnected system constructed from (26) and (28) can be represented by

$$\dot{\widehat{x}}(t) = -L\widehat{x}(t),\quad L = D-A \tag{29}$$

where $L \in \mathbb{R}^{N\times N}$ is the graph Laplacian of G defined by

$$\begin{aligned} &L := D - A, \\ &D := \mathrm{diag}(d_1, \cdots, d_N),\ d_i = |\mathcal{N}_i|, \\ &A := [A_{i,j}],\ A_{i,j} = 1\ (j \in \mathcal{N}_i),\ A_{i,j} = 0\ (j \notin \mathcal{N}_i). \end{aligned} \tag{30}$$

It is easy to see that $L\mathbf{1}^N = 0$ holds (i.e., $\mathbf{1}^N \in \mathbb{R}^N_{++}$ is the right-eigenvector of L with respect to the eigenvalue 0). It is shown in [22], [13] that, if the graph $G(\mathcal{I}, \mathcal{E})$ is strongly connected, an f-consensus is achieved by (28) as in

$$\lim_{t\to\infty} \widehat{x}(t) = f(\widehat{x}(0))\mathbf{1}^N, \quad f(\widehat{x}(0)) = \xi_0^T \widehat{x}(0). \tag{31}$$

Here, $\xi_0 \in \mathbb{R}^N$ is the left-eigenvector of L with respect to the eigenvalue 0 satisfying $\xi_0^T \mathbf{1}^N = 1$.

In the following, we will show that (31) follows directly from Theorem 5. To this end, we first note that ~~that~~ (29) is a positive system since $-L \in \mathbb{M}^N$. Moreover, (29) can be rewritten as

$$\dot{\widehat{x}}(t) = -D\widehat{x}(t) + \widehat{w}(t), \quad \widehat{z}(t) := D\widehat{x}(t), \quad \widehat{w}(t) = AD^{-1}\widehat{z}(t). \tag{32}$$

From this expression, we can regard (29) as an interconnected system constructed from N positive, SISO and stable subsystems $G_i\ (i = 1, \cdots, N)$ given by

$$G_i : \left\{ \begin{array}{rcl} \dot{x}_i(t) & = & -d_i x_i(t) + w_i(t), \\ z_i(t) & = & d_i x_i(t) \end{array} \right. \tag{33}$$

and the interconnection matrix

$$\Omega = AD^{-1} \in \mathbb{R}^{N\times N}_+. \tag{34}$$

It is clear that $G_i\ (i = 1, \cdots, N)$ in the form of (33) satisfies (i) and (ii) of Theorem 5 with $\gamma = 1$. On the other hand, the interconnection matrix $\Omega \in \mathbb{R}^{N\times N}_+$ given in (34) is irreducible if (and only if) the graph $G(\mathcal{I}, \mathcal{E})$ is strongly connected, and its Frobenius eigenvalue is 1 with the right-eigenvector $v_{\mathrm{R}} = D\mathbf{1}^N \in \mathbb{R}^N_{++}$ and the left-eigenvector $v_{\mathrm{L}} = \xi_0$. Therefore $\Omega \in \mathbb{R}^{N\times N}_+$ satisfies the condition (iii) and (iv) of Theorem 5 with $\gamma = 1$. Moreover, it is easy to see from (23) that $\xi_{\mathrm{R}} = \mathbf{1}^N$ and $\xi_{\mathrm{L}} = \xi_0$ in this case. It follows that (24) in Theorem 5 coincides with (31).

To summarize, Theorem 5 turns out to be an intriguing extension of f-consensus protocol shown in [22], [13]. Theorem 5 shows that, under certain conditions, we can achieve f-consensus (with respect to the output of each subsystem) even if we generalize the dynamics of each agent from integrators to positive systems, and interconnection matrix from graph-Laplacian matrices to nonnegative matrices.

D. Parametrization of Interconnection Matrices

For the preparation of the formation control of multi-agent systems based on Theorem 5 and Corollary 3, it is meaningful to show a concrete way to construct a desired $\Omega \in \mathbb{R}_+^{N\times N}$ that satisfies $\Omega v_{\rm obj} = (1/\gamma)v_{\rm obj}$ and $\Gamma(\Omega) = \Gamma$ for prescribed $v_{\rm obj} \in \mathbb{R}_{++}^N$ and graph structure Γ. For illustration, consider the cases where Γ is schematically shown in Figs.3 and 4 for $N = 3$.

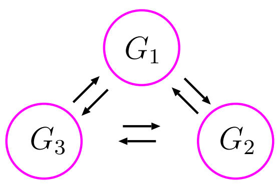

Fig. 3. Graph structure $\Gamma_{\rm A}$.

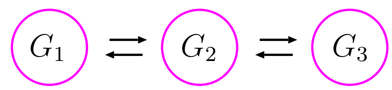

Fig. 4. Graph structure $\Gamma_{\rm B}$.

For graph structure $\Gamma_{\rm A}$, any interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ satisfying $\Omega v_{\rm obj} = (1/\gamma)v_{\rm obj}$ and $\Gamma(\Omega) = \Gamma_{\rm A}$ can be parametrized by

$$\Omega = \frac{1}{\gamma}\Omega(v_{\rm obj}, p) \in \mathbb{R}_{++}^{N\times N} \tag{35}$$

where

$$\Omega(v_{\rm obj}, p)_{i,j} = \begin{cases} (1-p_1)\dfrac{v_{{\rm obj},1}}{v_{{\rm obj},N}} & (i,j) = 1, N, \\ p_i \dfrac{v_{{\rm obj},i}}{v_{{\rm obj},j}} & (1 \le i \le N,\ j = i+1), \\ (1-p_i)\dfrac{v_{{\rm obj},i}}{v_{{\rm obj},j}} & (1 \le i \le N,\ j = i-1), \\ p_N \dfrac{v_{{\rm obj},N}}{v_{{\rm obj},1}} & (i,j) = (N,1), \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Here, parameter $p \in \mathbb{R}_{++}^N$ can be chosen arbitrarily among $0 < p < \mathbf{1}^N$. On the other hand, for graph structure $\Gamma_{\rm B}$, any interconnection matrix $\Omega \in \mathbb{R}_+^{N\times N}$ satisfying $\Omega v_{\rm obj} = (1/\gamma)v_{\rm obj}$ and $\Gamma(\Omega) = \Gamma_{\rm B}$ can be parametrized again by (35) and (36) where parameter $p \in \mathbb{R}_{++}^N$ can be chosen such that $p_1 = 1$, $p_N = 0$, and $0 < p_i < 1$ $(i \in \mathbb{Z}_N \setminus \{1, N\})$. In both cases, we can confirm that resulting interconnection matrix Ω is irreducible (since $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are both strongly connected). When $N = 3$, the matrix $\Omega(v_{\rm obj}, p)$ can be respectively illustrated for $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ as follows:

$$\underline{\Gamma(\Omega) = \Gamma_{\rm A}}$$
$$\Omega(v_{\rm obj}, p) = \begin{bmatrix} 0 & p_1 \dfrac{v_{\rm obj,1}}{v_{\rm obj,2}} & (1-p_1)\dfrac{v_{\rm obj,1}}{v_{\rm obj,3}} \\ (1-p_2)\dfrac{v_{\rm obj,2}}{v_{\rm obj,1}} & 0 & p_2 \dfrac{v_{\rm obj,2}}{v_{\rm obj,3}} \\ p_3 \dfrac{v_{\rm obj,3}}{v_{\rm obj,1}} & (1-p_3)\dfrac{v_{\rm obj,3}}{v_{\rm obj,2}} & 0 \end{bmatrix},$$

$$\underline{\Gamma(\Omega) = \Gamma_{\rm B}}$$
$$\Omega(v_{\rm obj}, p) = \begin{bmatrix} 0 & \dfrac{v_{\rm obj,1}}{v_{\rm obj,2}} & 0 \\ (1-p_2)\dfrac{v_{\rm obj,2}}{v_{\rm obj,1}} & 0 & p_2 \dfrac{v_{\rm obj,2}}{v_{\rm obj,3}} \\ 0 & \dfrac{v_{\rm obj,3}}{v_{\rm obj,2}} & 0 \end{bmatrix}.$$

V. Formation Control of Multi-Agent Positive Systems

In this section, we apply the results in Section IV to formation control of multi-agent systems.

A. Problem Setting and Consensus-based Formation Control

Let us consider a multi-agent system with N agents, where the i-th agent $(i = 1, \cdots, N)$ can move on the (x, y)-plane. We denote by $(z_{i,x}(t), z_{i,y}(t))$ the position of agent i. Furthermore, we define $\widehat{z}_j := [z_{1,j} \ \cdots \ z_{N,j}]^T \ (j = x, y)$ by stacking the coordinates of the agents.

We assume that agent i has independent dynamics along the x- and y-axes, denoted by $P_{i,x}(s)$ and $P_{i,y}(s)$, respectively, and independent control inputs $u_{i,x}(t)$ and $u_{i,y}(t)$. We further assume that, as typical dynamics of moving agents, $P_{i,j}(s)$ are given by

$$Z_{i,j}(s) = P_{i,j}(s)U_{i,j}(s), \ P_{i,j}(s) = \frac{k_{i,j}}{s(s+a_{i,j})}, \ k_{i,j} > 0, \ a_{i,j} > 0 \ (i = 1, \cdots, N, \ j = x, y).$$

Since $P_{i,j}(s)$ is not stable (or say, on the stability boundary), we cannot apply directly the results in Theorem 5. To get around this difficulty, we apply a minor feedback as in

$$u_{i,j}(t) = -f_{i,j}(z_{i,j}(t) - w_{i,j}(t)) \ (i \in \mathbb{Z}_N, \ j = x, y)$$

with $0 < f_{i,j} \le a_{i,j}^2/4k_{i,j}$, where $w_{i,j} \ (i \in \mathbb{Z}_N, \ j = x, y)$ is the exogenous input kept for the interconnection. Then we have

$$Z_{i,j}(s) = G_{i,j}(s)W_{i,j}(s),$$
$$G_{i,j}(s) = \left[\begin{array}{cc|c} -b_{i,j} & 1 & 0 \\ 0 & -c_{i,j} & b_{i,j}c_{i,j} \\ \hline 1 & 0 & 0 \end{array}\right], \quad b_{i,j}+c_{i,j}=a_{i,j},\ b_{i,j}c_{i,j}=f_{i,j}k_{i,j}. \tag{37}$$

It follows from Proposition 1 that $G_{i,j}$ $(i=1,\cdots,N,\ j=x,y)$ are positive (with respect to the minimal realizations (37)), SISO, and stable systems with $G_{i,j}(0)=1$ $(i=1,\cdots,N,\ j=x,y)$. The last property is a natural consequence from the fact that each open-loop transfer function $P_{i,j}(s)$ $(i=1,\cdots,N,\ j=x,y)$ includes an integrator. We emphasize that the properties of $G_{i,j}(s)$ mentioned above robustly hold against "small" perturbations on the plant parameters and the minor-feedback gains. For description simplicity, we define $\widehat{w}_j := [w_{1,j}\ \cdots\ w_{N,j}]^T$ $(j=x,y)$.

We assume that N-agents independently communicate their x and y positions each other. Our goal here is to design interconnection matrices Ω_x and Ω_y such that, under the interconnection with Ω_x and Ω_y for $(\widehat{z}_x,\widehat{w}_x)$ and $(\widehat{z}_y,\widehat{w}_y)$, respectively, the following formation can be achieved:

$$\lim_{t\to\infty}[\ \widehat{z}_x(t)\ \widehat{z}_y(t))\ ] = [\ f_x(\widehat{x}_x(0))v_{\mathrm{obj},x}\quad f_y(\widehat{x}_y(0))v_{\mathrm{obj},y}\ ]. \tag{38}$$

Here, $v_{\mathrm{obj},j}\in\mathbb{R}^N_{++}$ $(j=x,y)$ are given vectors that specify the desired formation, and $\widehat{x}_j(0)$ $(j=x,y)$ stand for the initial states of the corresponding interconnected systems. On the other hand, $f_j:\mathbb{R}^{2N}\to\mathbb{R}$ $(j=x,y)$ stand for the scaling factors that depend upon the initial states. It is obvious that we can readily solve this problem by following Theorem 5 and Corollary 3.

Remark 3: Since the synthesis method of interconnection matrices proposed in Theorem 5 and Corollary 3 are based on the idea of consensus, and since we do not allow to incorporate any external signals to the interconnected systems, we cannot exclude the effect of initial states at the limits of the outputs. The problem setting (38) has been done along with this fact. Similar problem setting can be found, for example, in [10].

Remark 4: To illustrate our results in Theorem 5 and Corollary 3 in a realistic situation, we assume typical second-order dynamics of moving agents (i.e., integrator plus first-order lag) and showed that we can make them satisfy the conditions (i) and (ii) in Theorem 5 by applying minor-feedbacks. It is of course possible to apply the results in Theorem 5 and Corollary 3 to the agents with other dynamics as long as they satisfy (i) and (ii) of Theorem 5.

B. Numerical Examples

Along with the basic problem settings stated in Subsection V-A, we generated $a_{i,j}$ and $k_{i,j}$ randomly over the closed interval $[10\ \ 20]$ and $[1\ \ 2]$, respectively, and then let $f_{i,j}$ as $f_{i,j} = 0.8 \times a_{i,j}^2/4k_{i,j}$. We thus constructed $G_{i,j}$ $(i = 1, \cdots, N,\ j = x, y)$. We let $[v_{\mathrm{obj},x}\ \ v_{\mathrm{obj},y}]_i = [2 + \cos(2\pi i/N) \quad 2 + \sin(2\pi i/N)]$ so that the agents can form a (scaled) circle.

As for the graph structures of the interconnection matrices, we consider the two cases Γ_{A} and Γ_{B} (see Figs. 3, 4). Namely, we designed two sets of interconnection matrices $(\Omega_x^{[\mathrm{A}]}, \Omega_y^{[\mathrm{A}]})$ and $(\Omega_x^{[\mathrm{B}]}, \Omega_y^{[\mathrm{B}]})$ with $\Gamma(\Omega_j^{[k]}) = \Gamma_k$ $(j = x, y,\ k = \mathrm{A}, \mathrm{B})$ following the parametrizations shown in (36). Here, by using the freedom of parameter p in (36), we designed $(\Omega_x^{[\mathrm{A}]}, \Omega_x^{[\mathrm{B}]})$ so that they share identical left-eigenvector $v_{\mathrm{L},x}$ with respect to the Frobenius eigenvalue 1. Similarly for $(\Omega_y^{[\mathrm{A}]}, \Omega_y^{[\mathrm{B}]})$. From (25) and (23), these allow us to have identical scaling factors along x- and y-axes between the interconnection with $(\Omega_x^{[\mathrm{A}]}, \Omega_y^{[\mathrm{A}]})$ and $(\Omega_x^{[\mathrm{B}]}, \Omega_y^{[\mathrm{B}]})$. Therefore formations to be achieved are exactly the same as long as the initial states are the same. The synthesis of interconnection matrices can be done by solving linear equalities as detailed in [8]. We thus carried out simulation for the case $N = 20$, under the same initial conditions.

Figs 5-8 are the simulation results for the case $(\Omega_x^{[\mathrm{A}]}, \Omega_y^{[\mathrm{A}]})$, and Figs 9-12 are the simulation results for the case $(\Omega_x^{[\mathrm{B}]}, \Omega_y^{[\mathrm{B}]})$. In both cases, we see that the agents gradually form a (scaled) circle and converge to the position shown by blue dot which is computed in advance from (25). The convergence is rather slow for the graph structure Γ_{B} in comparison with the graph structure Γ_{A}, and this slower convergence is often observed in other simulations. This would be partially because the graph structure Γ_{B} is more sparse than Γ_{A}.

VI. Conclusion

In this paper, we presented several novel results on the stability and persistence of interconnected heterogeneous positive systems. We showed that the stability and persistence can be characterized completely in terms of the (weighted) L_1-induced norm of each positive subsystem and the Frobenius eigenvalue of the interconnection matrix. We illustrated the usefulness of the persistence results by applying them to formation control of multi-agent systems. By noting the fact that typical dynamics of moving agents are positive, we showed that efficient synthesis of communication scheme over the agents can be done as long as the dynamics of the agents are all positive, stable, SISO and ~~and~~ share an identical steady-state gain.

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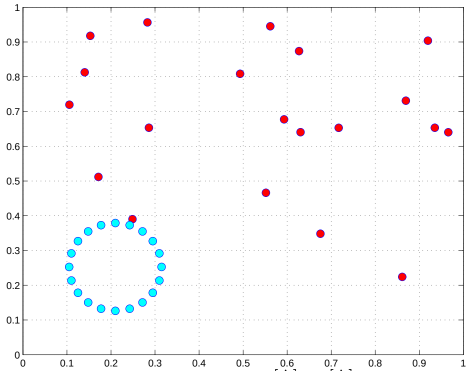

Fig. 5. Agent position under $(\Omega_x^{[A]}, \Omega_y^{[A]})$. ($t = 0$ [sec]).

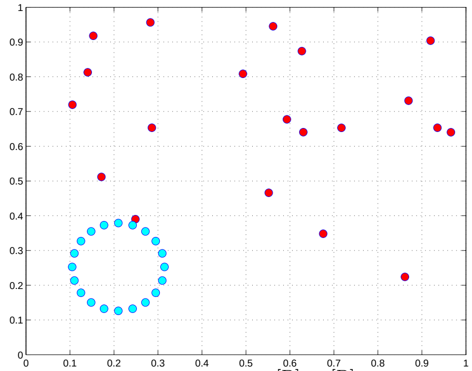

Fig. 9. Agent position under $(\Omega_x^{[B]}, \Omega_y^{[B]})$. ($t = 0$ [sec]).

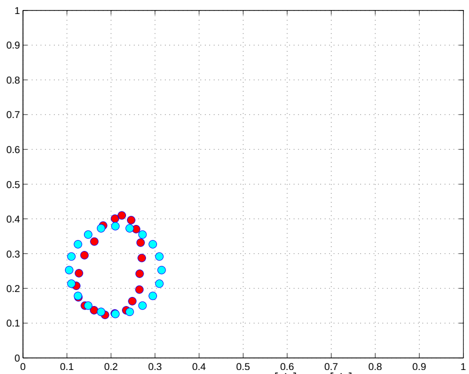

Fig. 6. Agent position under $(\Omega_x^{[A]}, \Omega_y^{[A]})$. ($t = 10$ [sec]).

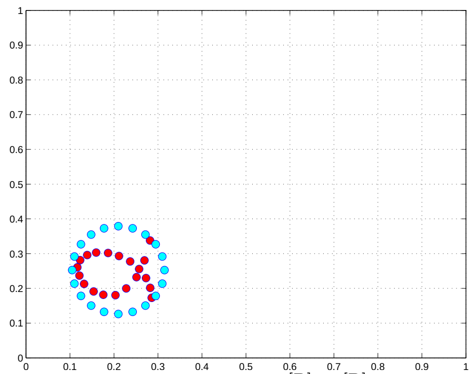

Fig. 10. Agent position under $(\Omega_x^{[B]}, \Omega_y^{[B]})$. ($t = 10$ [sec]).

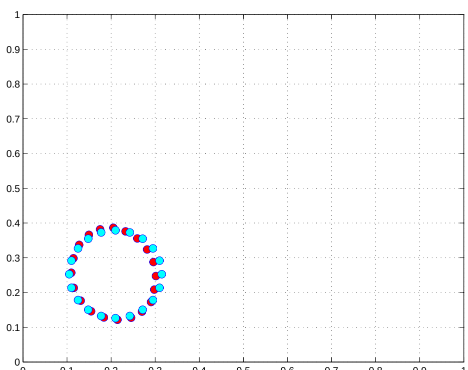

Fig. 7. Agent position under $(\Omega_x^{[A]}, \Omega_y^{[A]})$. ($t = 20$ [sec]).

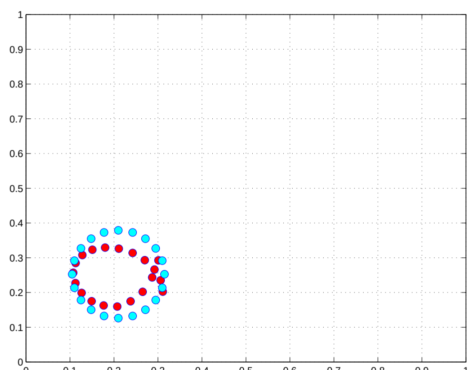

Fig. 11. Agent position under $(\Omega_x^{[B]}, \Omega_y^{[B]})$. ($t = 20$ [sec]).

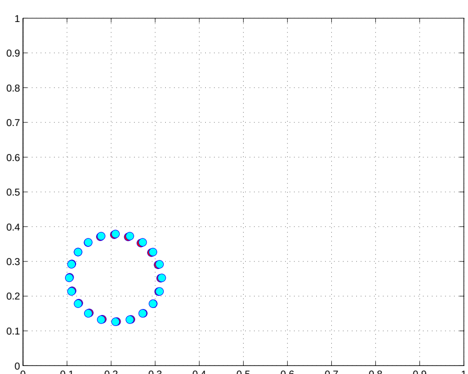

Fig. 8. Agent position under $(\Omega_x^{[A]}, \Omega_y^{[A]})$. ($t = 30$ [sec]).

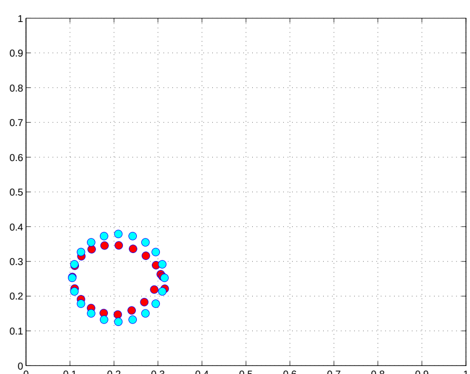

Fig. 12. Agent position under $(\Omega_x^{[B]}, \Omega_y^{[B]})$. ($t = 30$ [sec]).

VII. APPENDICES

A. Proof of Lemma 2

Proof of Lemma 2: We will prove the equivalence of (i) and (ii). The equivalence of (i) and (iii) follows similarly.

(i) $\Rightarrow$ (ii) Suppose (i) holds. Then, from (iii) of Lemma 1, there exist $h_1 \in \mathbb{R}^{n_1}_{++}$ and $h_2 \in \mathbb{R}^{n_2}_{++}$ such that

$$h_1^T P + h_2^T R < 0, \quad h_1^T Q + h_2^T S < 0. \tag{39}$$

The first inequality clearly shows that P is Hurwitz stable again from (iii) of Lemma 1. Since P is Metzler and Hurwitz and hence $P^{-1} \leq 0$ holds from (ii) of Lemma 1, the first inequality in (39) implies $h_1^T > -h_2^T R P^{-1}$. From this and the second inequality, we have

$$h_2^T (S - RP^{-1}Q) < 0. \tag{40}$$

It is obvious that $S - RP^{-1}Q$ is Metzler since $P^{-1} \leq 0$ and hence, again from (iii) of Lemma 1, we conclude that $S - RP^{-1}Q$ is Hurwitz stable.

(ii) $\Rightarrow$ (i) Suppose (ii) holds. Then, from (iii) of Lemma 1, there exists $h_2 \in \mathbb{R}^{n_2}_{++}$ such that (40) holds. It follows that there exists $\varepsilon > 0$ such that

$$h_2^T S - (h_2^T R + \varepsilon \mathbf{1}^{n_1 T}) P^{-1} Q < 0$$

where $\mathbf{1}^{n_1} \in \mathbb{R}^{n_1}$ stands for the all-ones vector. If we define $h_1 := -((h_2^T R + \varepsilon \mathbf{1}^{n_1 T}) P^{-1})^T$, we have $h_1 > 0$ since P is Hurwitz and hence $P^{-1} \leq 0$. In addition, we readily obtain

$$h_1^T Q + h_2^T S < 0, \quad h_1^T P + h_2^T R = -\varepsilon \mathbf{1}^{n_1 T} < 0.$$

Again, from (iii) of Lemma 1, this shows that the Metzler matrix Π in (i) is Hurwitz stable. ■

B. Proof of Theorem 1

Proof of Theorem 1: (ii)$\Rightarrow$(i) Suppose (ii) holds for some $h > 0$. Then $A \in \mathbb{M}^n$ is obviously Hurwitz from (iii) of Lemma 1. In addition, there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} h^T A + q_z^T C & h^T B + q_z^T D - (\gamma - \varepsilon) q_w^T \end{bmatrix} < 0.$$

It follows that, for any $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{n_w}$ satisfying $[\ x^T\ \ w^T\ ]^T \geq 0$, we have

$$\begin{bmatrix} h^T A + q_z^T C & h^T B + q_z^T D - (\gamma - \varepsilon) q_w^T \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0. \tag{41}$$

Since G is positive, we note that $x(t) \geq 0\ \forall t \in [0, \infty)$ holds for any input signal $w \in L_{1+}^{n_w}$ and $x(0) = 0$. From this fact and (41), we see that along the trajectory of the system G the following relation holds:

$$h^T \dot{x}(t) + q_z^T z(t) - (\gamma - \varepsilon) q_w^T w(t) \leq 0 \quad \forall t \in [0, \infty) \quad \forall w \in L_{1+}^{n_w}. \tag{42}$$

By integrating the above inequality over $[0, T]$, we have

$$h^T x(T) + \int_0^T q_z^T z(t) dt - (\gamma - \varepsilon) \int_0^T q_w^T w(t) dt \leq 0 \quad \forall w \in L_{1+}^{n_w}.$$

By noting $h^T x(T) \geq 0$, it is obvious that

$$\int_0^T q_z^T z(t) dt - (\gamma - \varepsilon) \int_0^T q_w^T w(t) dt \leq 0 \quad \forall w \in L_{1+}^{n_w}.$$

Moreover, by restricting w to ~~those~~ such that $\|q_w^T w\|_1 = 1$ and letting $T \to \infty$, we see that

$$\int_0^\infty q_z^T z(t) dt - (\gamma - \varepsilon) \leq 0$$

holds for all $w \in L_{1+}^{n_w}$ such that $\|q_w^T w\|_1 = 1$. It follows that (i) is satisfied.

(i)⇒(ii) To prove the assertion by contradiction, suppose (ii) does not hold for any $h > 0$. Then only the following two cases are possible:

(a) A is not Hurwitz stable.

(b) A is Hurwitz stable ~~but (5) does not hold for any $h > 0$~~.

Since (a) clearly contradicts (i), we only consider the case (b). Then, from the strong alternative for linear inequalities [3, Section 5.8], there exist $g_1 \in \mathbb{R}_+^n$ and $g_2 \in \mathbb{R}_+^{n_w}$, not simultaneously zero, such that

$$A g_1 + B g_2 \geq 0, \quad q_z^T C g_1 + (q_z^T D - \gamma q_w^T) g_2 \geq 0.$$

If $g_2 = 0$, we have $g_1 \neq 0$, $g_1 \geq 0$, and $A g_1 \geq 0$, which contradicts the Hurwitz stability of A (see (iv) of Lemma 1). Therefore it suffices to consider the case where A is Hurwitz stable and $g_2 \neq 0$. With this in mind, let us note that the first inequality above implies $g_1 \leq -A^{-1} B g_2$

since $A^{-1} \le 0$ from (ii) of Lemma 1. By substituting this into the second inequality, we obtain $(q_z^T G(0) - \gamma q_w^T) g_2 \ge 0$. Moreover, since $g_2 \ge 0$ and $g_2 \ne 0$ as noted above, the following inequality must hold for at least one index $j^\star$ $(1 \le j^\star \le n_w)$:

$$(q_z^T G(0))_{j^\star} - \gamma q_{w,j^\star} \ge 0. \tag{43}$$

In the following, we assume $q_{w,j^\star} = 1$ without loss of generality. For a given $T > 0$, we also define a linear operator $\mathbb{I}_T$ as follows:

$$\mathbb{I}_T \zeta := \begin{cases} \zeta(t) & (0 \le t \le T) \\ 0 & (T < t) \end{cases} .$$

Now we move on to the final stage of the proof. To this end, let us define a constant input signal $w_{\rm st}(t) := e_{j^\star} \in \mathbb{R}_+^{n_w}$, where e_i is the i-th standard basis of $\mathbb{R}^{n_w}$. We also denote by $z_{\rm st}(t)$ the response of the system G ~~for~~ the input $w_{\rm st}(t)$. Then, in view of the steady-state output, we see that for any $\varepsilon > 0$ satisfying $\gamma - \varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$q_z^T z_{\rm st}(t) - q_z^T G(0) w_{\rm st}(t) > -\frac{\varepsilon}{2} \quad \forall t > T_\varepsilon.$$

From (43), this implies

$$q_z^T z_{\rm st}(t) - \gamma > -\frac{\varepsilon}{2} \quad \forall t > T_\varepsilon$$

or equivalently,

$$q_z^T z_{\rm st}(t) > \gamma - \frac{\varepsilon}{2} > 0 \quad \forall t > T_\varepsilon.$$

If we define another input signal $w_T^\star(t) := \mathbb{I}_T w_{\rm st}$ for a given $T(> T_\varepsilon)$ and denote by $z_T^\star(t)$ the corresponding output signal, then we have $||q_w^T w_T^\star||_1 = T$, $z_T^\star(t) = z_{\rm st}(t)$ $(0 \le t \le T)$ and hence

$$\begin{aligned} \frac{\|q_z^T z_T^\star\|_1}{\|q_w^T w_T^\star\|_1} &= \frac{1}{T}\left(\int_0^{T_\varepsilon} q_z^T z_T^\star(t)dt + \int_{T_\varepsilon}^{T} q_z^T z_T^\star(t)dt + \int_T^\infty q_z^T z_T^\star(t)dt\right) \\ &\ge \frac{1}{T}\int_{T_\varepsilon}^{T} q_z^T z_T^\star(t)dt \\ &\ge \frac{(\gamma - \frac{\varepsilon}{2})(T - T_\varepsilon)}{T} \\ &\ge \gamma - \frac{\varepsilon}{2} - (\gamma - \frac{\varepsilon}{2})\frac{T_\varepsilon}{T}. \end{aligned}$$

Therefore, for the particular choice of

$$T > \frac{\gamma - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} T_\varepsilon = \frac{2\gamma - \varepsilon}{\varepsilon} T_\varepsilon (> T_\varepsilon),$$

we have

$$\frac{\|q_z^T z_T^\star\|_1}{\|q_w^T w_T^\star\|_1} > \gamma - \varepsilon.$$

Since $\varepsilon > 0$ can be taken arbitrarily small, this implies $\|G_{q_z,q_w}\|_{1+} \geq \gamma$, which contradicts (i).

(ii)⇒(iii) The linear inequality (5) implies $A \in \mathbb{H}^n$ and

$$h^T > -q_z^T C A^{-1}, \quad h^T B + q_z^T D < \gamma q_w^T$$

since we have $A^{-1} \leq 0$ from (ii) of Lemma 1. By substituting the former into the latter, we obtain (6).

(iii)⇒(ii) Let us fix $v \in \mathbb{R}_{++}^n$ such that $v^T A < 0$. Then, the condition (6) implies that there exists $\varepsilon > 0$ such that

$$q_z^T D + (-q_z^T C A^{-1} + \varepsilon v^T) B < \gamma q_w^T.$$

If we define $h := (-q_z^T C A^{-1} + \varepsilon v^T)^T > 0$, we readily obtain

$$h^T A + q_z^T C = \varepsilon v^T A < 0, \quad h^T B + q_z^T D - \gamma q_w^T < 0.$$

This clearly shows that (5) holds. ■

C. Proof of Theorem 2

Proof of Theorem 2: For each subsystem, let us define

$$B_i := [\ B_{i,1}\ \cdots\ B_{i,i-1}\ B_{i,i+1}\ \cdots B_{i,N}\ ],$$

$$C_i := \begin{bmatrix} C_{1,i} \\ \vdots \\ C_{i-1,i} \\ C_{i+1,i} \\ \vdots \\ C_{N,i} \end{bmatrix}, \quad D_i := \begin{bmatrix} D_{1,i,1} & \cdots & D_{1,i,i-1} & D_{1,i,i+1} & \cdots & D_{1,i,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{i-1,i,1} & \cdots & D_{i-1,i,i-1} & D_{i-1,i,i+1} & \cdots & D_{i-1,i,N} \\ D_{i+1,i,1} & \cdots & D_{i+1,i,i-1} & D_{i+1,i,i+1} & \cdots & D_{i+i,i,N} \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{N,i,1} & \cdots & D_{N,i,i-1} & D_{N,i,i+1} & \cdots & D_{N,i,N} \end{bmatrix}$$

and z_i, w_i $(i = 1, \cdots, N)$ by (19). Then, the system $\mathcal{G}$ defined by (10) can be written in the form of (11) with (12) and (13). Therefore the interconnection of subsystems G_i with (17) can be seen as an interconnection with $\mathcal{G}$ and a matrix Ω precisely given in the following. From (17) and (19), we see that the interconnection matrix Ω of this case is nothing but a permutation matrix that permutes z_{ij} and z_{ji} in $\widehat{z}$, i.e.,

$$\Omega \begin{bmatrix} \vdots \\ z_{ij} \\ \vdots \\ z_{ji} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ z_{ji} \\ \vdots \\ z_{ij} \\ \vdots \end{bmatrix}. \tag{44}$$

A concrete example of Ω is given in (20). Since Ω is a permutation matrix, we see that $\Omega \geq 0$. It follows from Lemma 3 that the interconnected system is admissible and stable if and only if the Metzler matrix

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}\Omega \\ \mathcal{C} & \mathcal{D}\Omega - I \end{bmatrix}$$

is Hurwitz stable. This can be restated equivalently that there exists $h_i \in \mathbb{R}_{++}^{n_i}$ and $q_{ij} \in \mathbb{R}_{++}^{n_{w_{ij}}}$ $(i, j = 1, \cdots, N, i \neq j)$ such that

$$\begin{bmatrix} \widehat{h} \\ \widehat{q}_z \end{bmatrix}^T \begin{bmatrix} \mathcal{A} & \mathcal{B}\Omega \\ \mathcal{C} & \mathcal{D}\Omega - I \end{bmatrix} < 0,\ \widehat{h} := \begin{bmatrix} h_1^T & \cdots & h_N^T \end{bmatrix}^T,\ \widehat{q}_z := \begin{bmatrix} q_{z,1}^T & \cdots & q_{z,N}^T \end{bmatrix}^T. \tag{45}$$

Here, $q_{z,i}$ $(i = 1, \cdots, N)$ are given by (18). Since Ω is a permutation matrix, we see that (45) holds if and only if

$$\begin{bmatrix} \widehat{h} \\ \widehat{q}_z \end{bmatrix}^T \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} - \Omega^T \end{bmatrix} < 0 \tag{46}$$

Moreover, we see from the property of Ω represented by (44) that

$$\widehat{q}_z^T \Omega^T = (\Omega \widehat{q}_z)^T = \widehat{q}_w^T, \quad \widehat{q}_w := \begin{bmatrix} q_{w,1}^T & \cdots & q_{w,N}^T \end{bmatrix}^T.$$

Here, $q_{w,i}$ $(i = 1, \cdots, N)$ are given by (18). It follows that (46) can be divided into N inequalities as in

$$\begin{bmatrix} h_i^T A_i + q_{z,i}^T C_i & h_i^T B_i + q_{z,i}^T D_i - q_{w,i}^T \end{bmatrix} < 0 \quad (i = 1, \cdots, N). \tag{47}$$

From Theorem 1, the inequalities above hold if and only if

$$\|G_{i,q_{z,i},q_{w,i}}\|_{1+} < 1 \ (i = 1, \cdots, N).$$

This completes the proof. ■

D. Proof of Lemma 4

Proof of Lemma 4: We give the proof for the controllability only. The result for the observability readily follows from the system duality. For contradiction, suppose $v := A^{-1}B < 0$ does not hold. From the underlying assumptions $A \in \{\mathbb{M}^n \cap \mathbb{H}^n\}$ and $B \in \mathbb{R}_+^n$, we see that $v \leq 0$ definitely holds since $A^{-1} \leq 0$. Therefore there exists a nonempty index set $\mathcal{I} \subset \{1, \cdots, n\}$ such that

$$v_i = 0 \ (i \in \mathcal{I}), \quad v_i < 0 \ (i \in \mathcal{I}^{\mathrm{c}})$$

where $\mathcal{I}^{\mathrm{c}}$ is the complement of $\mathcal{I}$. Again from $B(= Av) \in \mathbb{R}_+^n$, it follows that

$$Av \geq 0, \quad v_i = 0 \ (i \in \mathcal{I}), \quad v_i < 0 \ (i \in \mathcal{I}^{\mathrm{c}}). \tag{48}$$

Since $A \in \mathbb{M}^n$, the above conditions imply $A_{ij} = 0 \ (i \in \mathcal{I}, \ j \in \mathcal{I}^{\mathrm{c}})$. Therefore we have $(Av)_i = 0 \ (i \in \mathcal{I})$. Repeating the same argument, we obtain $(A^k v)_i = 0 \ (i \in \mathcal{I}, \ k = 0, 1, \cdots, n-1)$. Then, if we denote by U_{c} the controllability matrix for the pair (A, B), we have

$$\begin{aligned} \mathrm{rank}(U_{\mathrm{c}}) &= \mathrm{rank}([B \ AB \ \cdots A^{n-1}B \ ]) \\ &= \mathrm{rank}([Av \ A^2 v \ \cdots A^n v \ ]) \\ &= \mathrm{rank}([v \ Av \ \cdots A^{n-1} v \ ]) \\ &\leq n - |\mathcal{I}| \end{aligned}$$

where $|\mathcal{I}|$ stands for the cardinality of the index set $\mathcal{I}$. This implies that (A, B) is not controllable and hence the proof is completed. ■

E. Proof of (IV) in Theorem 5

For the proof, we need the next lemma.

Lemma 5: For given $A \in \mathbb{M}^n$, $B \in \mathbb{R}_+^{n\times 1}$, and $C \in \mathbb{R}_+^{1\times n}$, suppose (A,B) is controllable and (A,C) is observable. Then, for a given $\alpha \in \mathbb{R}$ such that $\alpha I + A \in \mathbb{R}_+^{n\times n}$, we have $C(\alpha I + A)^i B > 0$ for at least one index $i \in \{0,\cdots,n-1\}$.

Proof of Lemma 5: Since (A,B) is controllable and (A,C) is observable, $(\alpha I + A, B)$ is controllable and $(\alpha I + A, C)$ is observable. Therefore we see that $U := U_{\mathrm{o},\alpha} U_{\mathrm{c},\alpha} \in \mathbb{R}_+^{n\times n}$ is nonsingular where $U_{\mathrm{c},\alpha}$ and $U_{\mathrm{o},\alpha}$ stand for the controllability and observability matrices for the pairs $(\alpha I + A, B)$ and $(\alpha I + A, C)$, respectively. The first row of U given by $[\ CB\ C(\alpha I + A)B\ \cdots C(\alpha I + A)^{n-1}B]$ is nonzero since U is nonsingular from the controllability and observability assumption. Therefore the assertion readily follows. ■

Proof of (IV) in Theorem 5: Let us define $\Omega_{\mathcal{D}} := \Omega(I - \mathcal{D}\Omega)^{-1}$ as in the proof of (III). Then, from the assertion (I) already validated, the matrix $\mathcal{D}\Omega - I$ is Hurwitz and hence $\rho(\mathcal{D}\Omega) = \lambda_{\mathrm{F}}(\mathcal{D}\Omega) < 1$. It follows that

$$\Omega_{\mathcal{D}} = \Omega(I - \mathcal{D}\Omega)^{-1} = \Omega \sum_{i=0}^{\infty} (\mathcal{D}\Omega)^i \geq \Omega \geq 0.$$

Since Ω is irreducible from (iii), the above inequality implies $\Omega_{\mathcal{D}}$ is also irreducible and $\Omega_{\mathcal{D}} \in \mathbb{R}_+^{N\times N}$.

With this in mind, suppose $\mathcal{A}_{\mathrm{cl}}$ is reducible for contradiction. Then, for $\alpha > 0$ such that $\alpha I + \mathcal{A} \geq 0$, there exists a permutation matrix P such that

$$\begin{aligned} & P^T(\alpha I + \mathcal{A} + \mathcal{B}\Omega_{\mathcal{D}}\mathcal{C})P \in \mathbb{W}_+, \\ & \mathbb{W} := \left\{ W = \begin{bmatrix} Q & R \\ 0_{n_{\widehat{x}}-r,r} & S \end{bmatrix} :\ Q \in \mathbb{R}^{r\times r},\ S \in \mathbb{R}^{(n_{\widehat{x}}-r)\times(n_{\widehat{x}}-r)},\ 1 \leq r \leq n_{\widehat{x}} - 1. \right\}, \\ & \mathbb{W}_+ := \mathbb{W} \cap \mathbb{R}_+^{n_{\widehat{x}}\times n_{\widehat{x}}}. \end{aligned}$$

Since $\alpha I + \mathcal{A}$ and $\mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}$ are both nonnegative, the above condition implies

$$\mathcal{Y}_{\mathcal{A}} := P^T(\alpha I + \mathcal{A})P \in \mathbb{W}_+, \tag{49}$$

$$\mathcal{Y}_{\mathcal{BC}} := P^T\mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}P \in \mathbb{W}_+. \tag{50}$$

To proceed, let us define

$$n_{\max} := \max_{i=1,\cdots,N} n_{x_i}, \quad U := \sum_{i=0}^{n_{\max}-1} (P^T(\alpha I + \mathcal{A})P)^i \in \mathbb{W}_+.$$

Then, from Lemma 5, we have

$$\mathcal{X} := \mathcal{C}PUP^T\mathcal{B} \in \mathbb{D}_{++}^N. \tag{51}$$

With the matrix $U \in \mathbb{W}_+$ defined above, we also have

$$\begin{aligned}
&\mathcal{Y}_{\mathcal{BC}} = P^T\mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}P \in \mathbb{W}_+,\\
&\mathcal{Y}_{\mathcal{BC}}U\mathcal{Y}_{\mathcal{BC}} = P^T\mathcal{B}\Omega_{\mathcal{D}}\mathcal{X}\Omega_{\mathcal{D}}\mathcal{C}P \in \mathbb{W}_+,\\
&\mathcal{Y}_{\mathcal{BC}}(U\mathcal{Y}_{\mathcal{BC}})^2 = P^T\mathcal{B}\Omega_{\mathcal{D}}(\mathcal{X}\Omega_{\mathcal{D}})^2\mathcal{C}P \in \mathbb{W}_+,\\
&\qquad\vdots\\
&\mathcal{Y}_{\mathcal{BC}}(U\mathcal{Y}_{\mathcal{BC}})^{N-1} = P^T\mathcal{B}\Omega_{\mathcal{D}}(\mathcal{X}\Omega_{\mathcal{D}})^{N-1}\mathcal{C}P \in \mathbb{W}_+.
\end{aligned}$$

It follows that

$$P^T\mathcal{B}\mathcal{Z}\mathcal{C}P \in \mathbb{W}_+, \quad \mathcal{Z} := \Omega_{\mathcal{D}}\sum_{i=0}^{N-1}(\mathcal{X}\Omega_{\mathcal{D}})^i. \tag{52}$$

Since $\Omega_{\mathcal{D}}$ is irreducible and $\mathcal{X} \in \mathbb{D}_{++}^N$, it is obvious that $\mathcal{X}\Omega_{\mathcal{D}}$ is irreducible. Moreover, since $\mathcal{X}\Omega_{\mathcal{D}} \in \mathbb{R}_+^{N\times N}$, we see from (b) of Proposition 2 that $\sum_{i=0}^{N-1}(\mathcal{X}\Omega_{\mathcal{D}})^i \in \mathbb{R}_{++}^{N\times N}$. Since $\Omega_{\mathcal{D}}$ is irreducible and $\Omega_{\mathcal{D}} \in \mathbb{R}_+^{N\times N}$, this further indicates that $\mathcal{Z} \in \mathbb{R}_{++}^{N\times N}$.

Now we move onto the final stage of the proof. To this end, let us define

$$\mathcal{A}_P := P^T\mathcal{A}P, \quad \mathcal{B}_P := P^T\mathcal{B}, \quad \mathcal{C}_P := \mathcal{C}P$$

and partition $\mathcal{B}_P$ and $\mathcal{C}_P$ as follows:

$$\begin{aligned}
&\mathcal{B}_P =: \begin{bmatrix} \mathcal{B}_{P,1} \\ \mathcal{B}_{P,2} \end{bmatrix}, \quad \mathcal{B}_{P,1} \in \mathbb{R}_+^{r\times N}, \quad \mathcal{B}_{P,2} \in \mathbb{R}_+^{(n_{\hat{x}}-r)\times N},\\
&\mathcal{C}_P =: \begin{bmatrix} \mathcal{C}_{P,1} & \mathcal{C}_{P,2} \end{bmatrix}, \quad \mathcal{C}_{P,1} \in \mathbb{R}_+^{N\times r}, \quad \mathcal{C}_{P,2} \in \mathbb{R}_+^{N\times(n_{\hat{x}}-r)}.
\end{aligned}$$

Then, from (49) and (52), we have

$$\mathcal{A}_P \in \mathbb{W}, \tag{53}$$

$$\mathcal{B}_P\mathcal{Z}\mathcal{C}_P \in \mathbb{W}_+, \quad \mathcal{Z} \in \mathbb{R}_{++}^{N\times N}. \tag{54}$$

Here, in relation to (54), suppose $\mathcal{B}_{P,2} \neq 0$. Then, from (54), we have $\mathcal{C}_{P,1} = 0$. On the other hand, suppose $\mathcal{C}_{P,1} \neq 0$. Then, again from (54), we have $\mathcal{B}_{P,2} = 0$. It follows that $\mathcal{B}_{P,2} = 0$ or $\mathcal{C}_{P,1} = 0$ holds. From the form of $\mathcal{A}_P$ given by (53), the former case implies $(\mathcal{A}_P, \mathcal{B}_P)$ is not controllable, and the latter case implies $(\mathcal{A}_P, \mathcal{C}_P)$ is not observable. This contradicts to the assumption that (A_i, B_i) is controllable and (A_i, C_i) is observable for $i = 1, \cdots, N$ (and hence $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$ are controllable and observable, respectively). This completes the proof. ■