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Brief paper

On the structure of generalized plant convexifying static H_∞ control problems[☆]

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ABSTRACT

This paper shows that, under specific structures of generalized plants, the set of static controllers satisfying internal stability and a certain level of H_∞ performance becomes convex. More precisely, we characterize such static H_∞ controllers by an LMI with controller variables being kept directly as decision variables. The structural conditions on the generalized plant are not too strict, and we show that generalized plants corresponding to a sort of mixed sensitivity problems indeed satisfy these conditions. For the generalized plants of interest, we further prove that full-order dynamical H_∞ controllers can be characterized by an LMI with a simple change of variables. In stark contrast to the known LMI-based H_∞ controller synthesis, the change of variables is free from the coefficient matrices of the generalized plant and this property is promising when dealing with a variety of robust control problems. Related issues such as robust controller synthesis against real parametric uncertainties are also discussed.

1. Introduction

In modern control system analysis and synthesis, linear matrix inequality (LMI) and semidefinite programming (SDP) are widely accepted as useful tools with the help of freely available powerful softwares. In retrospect, one of the reasons why LMI attracted such intensive attention would be the fact that the H_∞ control problem, the central issue of the robust control theory, has been solved completely by means of LMIs. The prominent result is the elimination of controller variables approach independently conceived by Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994). Subsequently, Scherer, Gahinet, and Chilali (1997) and Masubuchi, Ohara, and Suda (1998) independently proposed the linearizing change of controller variables approach. These pioneering works are then extended to other problems such as gain-scheduled controller synthesis.

Even though these works opened up a new horizon for H_∞ control theory, one of the possible criticisms is that these approaches do not provide LMIs that keep controller variables directly as decision variables, and in particular, the controller variables are characterized as a function of the plant data. This surely restricts the scope of their application. For example, it is hard to design robust H_∞ controllers for the plants with parametric uncertainties.

In this paper, we show that, under specific structures of generalized plants, the set of static controllers satisfying internal stability and a certain level of H_∞ performance becomes convex. More precisely, we characterize such static H_∞ controllers by an LMI with controller variables being kept directly as decision variables. Even in the case of dynamical controller synthesis, we can conceive novel convexity results for the generalized plant of interest. In particular, we show that full-order H_∞ controllers can be characterized by an LMI with a simple change of variables. In stark contrast to Masubuchi et al. (1998) and Scherer et al. (1997), the change of variables in the present paper does not involve plant data and this property is promising when dealing with a variety of robust control problems. As an example, we show that we can design robust H_∞ controllers for the plant with parametric uncertainties, where we can employ parameter-dependent Lyapunov functions so that less conservative results can be achieved.

The synthesis of static output feedback controllers that meet desired performances and/or robustness specifications has been a challenging issue (Syrmos, Abdallah, Dorato, & Grigoriadis, 1997).

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In recent years, many attempts have been made to give efficient numerical procedures to solve related problems (de Oliveira, Geromel, & Bernussou, 2002; El Ghaoui, Oustry, & AitRami, 1997; Geromel, Peres, & Souza, 1996; Grigoriadis & Beran, 2000; Iwasaki, 1999; Suplin & Shaked, 2005). In de Oliveira and Geromel (1997), a numerical comparison was performed and classification into three categories (nonlinear programming, parametric optimization and convex programming approaches) was proposed. Among them, LMI-based methods are relevant to the current study, examples of which are efficient iterative algorithms (El Ghaoui et al., 1997; Grigoriadis & Beran, 2000; Iwasaki, 1999) and extended LMI approach with structural constraints on LMI variables to handle static output feedback synthesis (de Oliveira et al., 2002; Suplin & Shaked, 2005). We also note that, recently, the static and fixed-order controller synthesis problems under structural constraints have been studied in Henrion (2005), Henrion and Lasserre (2005, 2006) and Hol and Scherer (2005) by means of positive polynomials and real algebraic geometry. Differently from these studies, the basic spirit of the present study is overcoming nonconvexity by exploiting specific structures of the H_∞ control problem.

On the other hand, the change of variables for full-order H_∞ controllers presented in this paper is closely related to those for (robust) filter synthesis (de Souza & Trofino, 2000). We thus expect that H_∞ controller synthesis problems treated in this paper is somehow related to filter synthesis, even though we have not obtained definite results along this direction.

We use the following notations: for $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, we define $\mathrm{He}\{A\} := A+A^T$ and $\mathrm{Sq}\{B\} = BB^T$. The set of positive definite (Hurwitz stable) matrices of the size n is denoted by $\mathbb{S}^n_{++}$ ($\mathbb{H}^n$). In symmetric matrices, we denote by $*$ those blocks that are obvious by symmetry. Other notations are standard.

The preliminary version of this paper has been presented in Ebihara, Peaucelle, and Arzelier (2011). We refined the results in Ebihara et al. (2011) so that we can provide a useful way for constructing generalized plants to cope with practical control specifications. Moreover, we will illustrate the effectiveness of our design strategy by a practical mixed sensitivity problem for a radar antenna studied in Gahinet, Nemirovski, Laub, and Chilali (1995).

2. Static H_∞ controller synthesis

In this section, we show that a set of static controllers satisfying internal stability and a certain level of H_∞ performance becomes convex when the generalized plant satisfies specific structural conditions. Concrete examples of the generalized plant satisfying the required conditions are also given.

2.1. Specific structures of generalized plant ensuring convexity

Let us consider the closed-loop system depicted in Fig. 1 where G_γ denotes the generalized plant and K denotes the controller to be designed. Suppose the state space realization of G_γ is given by

$$G_\gamma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_{1,\gamma} x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w. \end{cases} \tag{1}$$

Here, $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^{n_w}$ the disturbance input, $u \in \mathbb{R}^{n_u}$ the control input, $z \in \mathbb{R}^{n_z}$ the performance output, and $y \in \mathbb{R}^{n_y}$ the measured output, respectively. We consider the case where C_1 is a continuous function of $\gamma > 0$ as in $C_{1,\gamma}$, where γ stands for the H_∞ performance level to be minimized. It might be natural to let D_{11} and D_{12} as functions of γ but we assume that they are independent of γ to derive the main result, Theorem 1.[2]

[2] In fact we further assume $D_{11} = 0$ in Theorem 1.

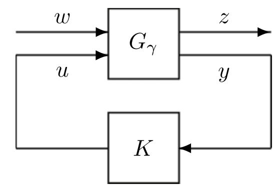

Fig. 1. Generalized plant for the H_∞ controller synthesis.

For a given controller K, static or dynamic, let us denote by $T(G_\gamma, K)$ the closed-loop system shown in Fig. 1. Moreover, for static controllers $K \in \mathbb{R}^{n_u\times n_y}$, we define the set

$$\mathcal{K}_\gamma := \Big\{K : K \in \mathbb{R}^{n_u\times n_y},\ A + B_2KC_2 \in \mathbb{H}^n \\ \text{and } \|T(G_\gamma, K)\|_\infty < 1\Big\}. \tag{2}$$

Our main concern in this paper is under what condition on G_γ the set $\mathcal{K}_\gamma$ defined by (2) becomes convex. It turns out that the following assumption will suffice:

Assumption 1. (i) $D_{11} = 0$ and $D_{21} = 0$;
(ii) the matrix B_1 can be partitioned as $B_1 = [B_2\ B_{12}]$ for some matrix B_{12}. The matrix B_{12} can be null;
(iii) $D_{12}^T D_{12} \succeq I_{n_u}$.

Indeed, under this assumption, we can establish the following theorem.

Theorem 1. *For given $\gamma > 0$ and the generalized plant G_γ satisfying Assumption 1, the set of static controllers $\mathcal{K}_\gamma$ defined by (2) is convex if it is not empty. In particular, the set $\mathcal{K}_\gamma$ can be characterized by an LMI as follows:*

$$\mathcal{K}_\gamma = \Big\{K : K \in \mathbb{R}^{n_u\times n_y},\ \exists P \succ 0 \text{ such that} \\ L(M_\gamma, P, K) \prec 0\Big\}, \tag{3}$$
$$M_\gamma := \{A, B_1, B_2, C_{1,\gamma}, D_{12}, C_2\}.$$

Here, $L(M_\gamma, P, K)$ is defined in (4) given in Box I.

Proof. Let us denote by $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ the state space matrices of the closed-loop system $T(G_\gamma, K)$. They can be represented by

$$A_{\rm cl} = A + B_2KC_2, \qquad B_{\rm cl} = B_1, \\ C_{\rm cl} = C_{1,\gamma} + D_{12}KC_2, \qquad D_{\rm cl} = 0 \tag{5}$$

where the assumption (i) is used tacitly. Then, from bounded real lemma (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Skelton, Iwasaki, & Grigoriadis, 1997), we see that $A_{\rm cl}$ is Hurwitz stable and $\|T(G_\gamma, K)\| < 1$ holds if and only if there exists $P \in \mathbb{S}^n_{++}$ such that

$$\mathrm{He}\{PA_{\rm cl}\} + \mathrm{Sq}\{PB_{\rm cl}\} + \mathrm{Sq}\{C_{\rm cl}^T\} \prec 0.$$

From (5) and the assumption (ii), this inequality can be rewritten equivalently as

$$\mathrm{He}\{PA + C_{1,\gamma}^T D_{12}KC_2\} + \mathrm{Sq}\{PB_2 + C_2^T K^T\} + \mathrm{Sq}\{PB_{12}\} \\ + \mathrm{Sq}\{C_{1,\gamma}^T\} + C_2^T K^T (D_{12}^T D_{12} - I)KC_2 \prec 0.$$

Since $D_{12}^T D_{12} - I \succeq 0$ from (iii), we can rewrite the above inequality as in $L(M_\gamma, P, K) \prec 0$ via Schur complement, where $L(M_\gamma, P, K)$ is given in (4). This completes the proof. ■

Remark 1. In our preliminary result in Ebihara et al. (2011), we have imposed additional conditions $C_{1,\gamma}^T D_{12} = 0$ and A is Hurwitz in Assumption 1. In the present paper we have shown that the convexity of the set $\mathcal{K}_\gamma$ is ensured even if we remove these two conditions.

Remark 2. For the case where B_{12} is null (i.e., $B_1 = B_2$), the corresponding LMI can be obtained by removing the third row and column from (4).

$$L(M_\gamma, P, K) := \begin{bmatrix} \mathrm{He}\{PA + C_{1,\gamma}^T D_{12} K C_2\} + \mathrm{Sq}\{C_{1,\gamma}^T\} & PB_2 + C_2^T K^T & PB_{12} & C_2^T K^T (D_{12}^T D_{12} - I)^{1/2} \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix}. \quad (4)$$

Box I.

Remark 3. We arrive at Assumption 1 by extending the simple fact that the set of internally stabilizing static output feedback controllers that render the H_∞ norm of unweighted input-side complementary sensitivity function less than unity is convex (if the set is not empty). This corresponds to the case where the generalized plant (1) satisfies Assumption 1 with null B_{12} and $D_{12} = I$. Under Assumption 1 that does not necessarily require that B_{12} is null and $D_{12} = I$, we can handle a variant of mixed sensitivity problems as shown in Section 2.2.

Remark 4. The LMI $L(M_\gamma, P, K) \prec 0$ is not convex in γ in general. Therefore we need a bisection search over γ for its minimization.

We can obtain a similar result to Theorem 1 by replacing Assumption 1 by Assumption 2 given below. This fact readily follows from the concept of system duality.

Assumption 2. (i) $D_{11} = 0$ and $D_{12} = 0$;
(ii) the matrix C_1 can be partitioned as $C_1 = [C_2^T \ C_{12}^T]^T$ for some matrix C_{12}. The matrix C_{12} can be null;
(iii) $D_{21} D_{21}^T \succeq I_{n_y}$.

When stating the above assumption, we implicitly assume that C_1 is a constant matrix whereas B_1 is a function of γ as in $B_{1,\gamma}$. Again, this is natural from of the system duality.

Remark 5. Assumption 1 never holds if $n_w < n_u$ and Assumption 2 never holds if $n_z < n_y$. Therefore at least one of the conditions $n_w \geq n_u$ and $n_w \geq n_u$ needs to be satisfied for applying the design method developed in this paper.

2.2. Modified generalized plant for mixed sensitivity problem

Let us consider the mixed sensitivity problem for the plant P described by $P(s) = C_P(sI - A_P)^{-1} B_P$. We assume that P is a SISO system just for simplicity. As usual, we assume that weighting functions W_S and W_T are appropriately designed for the shaping of the sensitivity and complementary sensitivity functions, respectively. Since W_S and W_T are typically chosen to be low-pass and high-pass, suppose that their state space realizations are given by

$$W_S(s) = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & 0 \end{array}\right], \qquad W_T(s) = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right]. \quad (6)$$

If we place these weighting functions on the input-side of P, the block-diagram for the mixed sensitivity problem can be represented as Fig. 2. Then, one of the standard settings for the mixed sensitivity problem will be

$$\inf_K \gamma \text{ subject to } \|T(G_\gamma, K)\|_\infty < 1 \quad (7)$$

where G_γ can be written explicitly as

$$G_\gamma(s) = \left[\begin{array}{ccc|c:c} A_P & 0 & 0 & B_P & B_P \\ 0 & A_S & 0 & B_S & B_S \\ 0 & 0 & A_T & 0 & B_T \\ \hline 0 & \frac{1}{\gamma} C_S & 0 & 0 & 0 \\ 0 & 0 & C_T & 0 & D_T \\ \hdashline C_P & 0 & 0 & 0 & 0 \end{array}\right]. \quad (8)$$

Unfortunately, as expected, this generalized plant does not satisfy Assumption 1 unless $B_T = 0$ and $D_T \geq 1$. It is obvious that Assumption 2 is never satisfied. In view of these facts, we try to modify Fig. 2 while preserving the basic spirit of the sensitivity and complementary sensitivity function shaping via W_S and W_T. By introducing a parameter α $(0 \leq \alpha < 1)$ and defining $\beta := \sqrt{1 - \alpha^2}$ $(0 < \beta \leq 1)$, we propose a possible modification in Fig. 3. The state space realization of the corresponding $G_{\gamma,\alpha}$ is given by

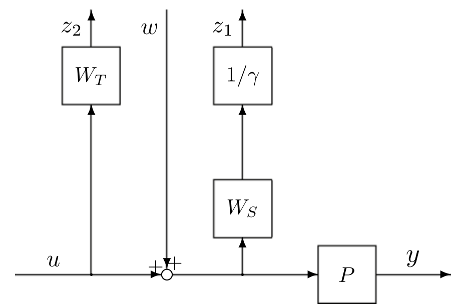

Fig. 2. Standard generalized plant with weightings on input-side.

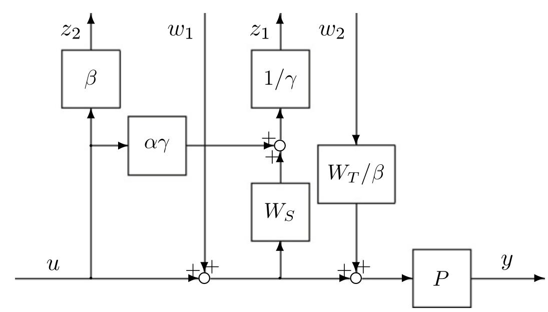

Fig. 3. Modified generalized plant for Fig. 2.

$$G_{\gamma,\alpha}(s) = \left[\begin{array}{ccc|cc:c} A_P & 0 & B_P C_T & B_P & \beta^{-1} B_P D_T & B_P \\ 0 & A_S & 0 & B_S & 0 & B_S \\ 0 & 0 & A_T & 0 & \beta^{-1} B_T & 0 \\ \hline 0 & \frac{1}{\gamma} C_S & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & \beta \\ \hdashline C_P & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$(0 \leq \alpha < 1, \ \beta = \sqrt{1 - \alpha^2}). \quad (9)$$

We can confirm that this generalized plant certainly satisfies Assumption 1 irrespective of $0 \leq \alpha < 1$ since $D_{12}^T D_{12} = \alpha^2 + \beta^2 = 1$.

It is of course important to compare the two strategies of H_∞ controller synthesis based on the generalized plants in Figs. 2 and 3. If we employ Fig. 2, the constraint $\|T(G_\gamma, K)\|_\infty < 1$ can be written explicitly as

$$\left\| \begin{bmatrix} \frac{1}{\gamma} W_S S \\ -W_T T \end{bmatrix} \right\|_\infty < 1, \quad (10)$$

$$S := (1 + KP)^{-1}, \qquad T := (1 + KP)^{-1} KP.$$

On the other hand, in the case of Fig. 3, the constraint $\|T(G_{\gamma,\alpha}, K)\|_\infty < 1$ becomes

$$\left\| \begin{bmatrix} \frac{1}{\gamma} W_S S - \alpha T & -\frac{1}{\beta}\left(\frac{1}{\gamma} W_S + \alpha\right) T W_T \\ -\beta T & -T W_T \end{bmatrix} \right\|_\infty < 1. \quad (11)$$

We have the following remarks regarding the comparison between (10) and (11).

(a) The constraint (11) implies $\left\|\frac{\beta}{\gamma}W_S S\right\|_\infty < 1$ and $\|TW_T\|_\infty < 1$. The former inequality follows from

$$\begin{aligned}\left\|\frac{\beta}{\gamma}W_S S\right\|_\infty &= \left\|\begin{bmatrix}\beta & -\alpha\end{bmatrix}\begin{bmatrix}\frac{1}{\gamma}W_S S - \alpha T \\ -\beta T\end{bmatrix}\right\|_\infty \\ &\le \left\|\begin{bmatrix}\beta & -\alpha\end{bmatrix}\right\| \left\|\begin{bmatrix}\frac{1}{\gamma}W_S S - \alpha T \\ -\beta T\end{bmatrix}\right\|_\infty \\ &\le \left\|\begin{bmatrix}\beta & -\alpha\end{bmatrix}\right\| = 1.\end{aligned}$$

Therefore the shaping of S and T via W_S and W_T is basically achieved, even though the former is slightly loosened to $\left\|\frac{1}{\gamma}W_S S\right\|_\infty < \frac{1}{\beta}$ $(0 < \beta \le 1)$. Note that the above evaluation $\left\|\frac{\beta}{\gamma}W_S S\right\|_\infty < 1$ is conservative due to the application of the triangular inequality.

(b) The constraint (11) implies $\|T\|_\infty < \beta^{-1}$, i.e., it restricts the H_∞ norm of the unweighted complementary sensitivity function under β^{-1}. In well-designed feedback control systems, the frequency response $T(j\omega)$ typically satisfies $\|T(j\omega)\| \simeq 1$ at low frequency range and $\|T(j\omega)\| \ll 1$ at high frequency range. Therefore, the restriction $\|T(j\omega)\| < \beta^{-1}$ would not be stringent in these frequency ranges. However, if we take β very close to 1 (i.e., α very close to 0), the restriction $\|T\|_\infty < \beta^{-1}$ can be a source of conservatism when we want to improve the frequency response of overall system at middle frequency range.

(c) The constraint (11) implies

$$\left\|\frac{1}{\beta}\left(\frac{1}{\gamma}W_S + \alpha\right)TW_T\right\|_\infty < 1$$

and again this could be a source of conservatism. However, the defect can be reduced if $\|W_S W_T\|_\infty \ll 1$ and $\alpha \ll \beta$. We can expect that the former condition is usually satisfied since W_S and W_T are typically chosen as low-pass and high-pass, respectively.

(d) Since $\frac{1}{\gamma}W_S S$ and TW_T are on the diagonal blocks in (11), we cannot draw any definite conclusion on the inclusion relationship among the two sets K_γ corresponding to the generalized plants in Figs. 2 and 3.

In our preliminary paper (Ebihara et al., 2011), we only dealt with the case $\alpha = 0$ (and hence $\beta = 1$). As we will illustrate in Section 2.3, it is often the case that we cannot shape the sensitivity function adequately if we let $\alpha = 0$, probably because of the conservatism stated in (b). This is the motivation why we introduce the parameter α. If we increase α from 0, then we can expect that the conservatism arising from (b) can be reduced gradually. However, if we take α too large, the conservatism arising from (c) becomes serious. In addition, the constraint $\left\|\frac{1}{\gamma}W_S S\right\|_\infty < \frac{1}{\beta}$ can be excessively loose. Consequently, by gradually increasing α from 0 and repeat trial and error procedure, we can expect that desirable α can be obtained.

Even though we have placed weighting functions on the input-side of P, it is possible to place them on the output-side of P and construct a generalized plant (with a parameter) that satisfies Assumption 2.

2.3. Numerical example

Let us consider a mixed sensitivity problem borrowed from Gahinet et al. (1995), where the plant is a radar antenna whose transfer function[3] is given by

$$\begin{aligned}P(s) &= \frac{30}{(s+0.002)(s^2+0.099s+300.3)} \\ &=: \frac{K\beta\omega_n^2}{(s+\beta)(s^2+2\zeta\omega_n s+\omega_n^2)}\end{aligned} \quad (12)$$

where $K \approx 49.95$, $\beta = 0.002$, $\zeta \approx 0.0029(=:\zeta_0)$ and $\omega_n \approx 17.3292$. The weightings used in Gahinet et al. (1995) for the shaping of S and T are follows:

$$\begin{aligned}W_{S0}(s) &= \frac{0.640s^3+1.045s^2+1.4928s+2.587\times10^{-2}}{s^3+0.120s^2+0.0083s+8.31\times10^{-7}}, \\ W_{T0}(s) &= 100\frac{3.41s+21.775}{s+3.54\times10^3}.\end{aligned}$$

Since we assume that W_S is strictly proper in (6), and since $\|W_S W_T\|_\infty \ll 1$ is desirable in the generalized plant in Fig. 3, we first slightly modify these weightings and redesign W_S and W_T as follows:

$$\begin{aligned}W_S(s) &= \frac{1.6556s^2+1.4472s+2.5192\times10^{-2}}{s^3+0.1165s^2+0.00808s+8.092\times10^{-7}}, \\ W_T(s) &= 100\frac{3.41s^2+0.2\omega_r+10^{-5}\omega_r^2}{s^2+0.06\omega_r s+10^{-2}\omega_r^2}\times\frac{s}{s+4\times10^3},\end{aligned} \quad (13)$$

$\omega_r = 173$.

The weighting W_S is obtained by applying balanced truncation to $W_{S0}(s)\frac{1}{0.01s+1}$, while $W_T(s)$ is designed to actively suppress the resonance of the plant around ω_r (rad/s). The bode gain plot of these weightings are given in Figs. 4 and 5.

With these P, W_{S0}, and W_{T0}, we first constructed the standard generalized plant in Fig. 2 and designed a full-order H_∞ optimal controller by minimizing γ subject to $\|T(G_\gamma, K)\|_\infty < 1$ via bisection search. Here, we have used the LMI-based method in Scherer et al. (1997) for the H_∞ controller synthesis. Then, we next designed an H_∞ optimal controller for the generalized plant in Fig. 2 with the modified weightings W_S and W_T. The Bode gain plots of the resulting S and T are shown in Figs. 6 and 7. We can confirm that adequate (and almost identical) shaping of S and T is achieved in both cases. We therefore employ W_S and W_T in the sequel.

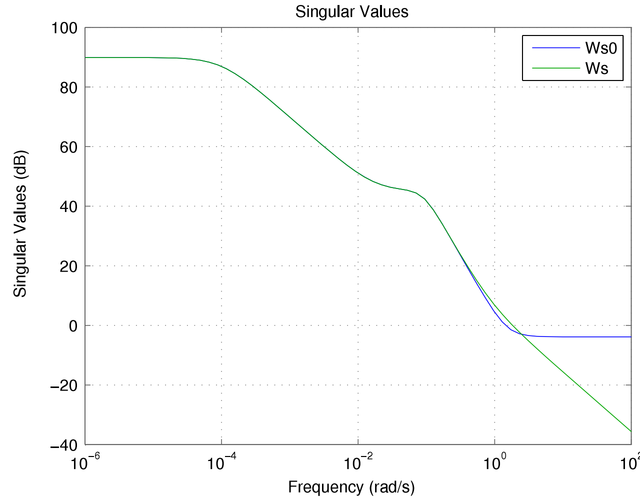

Fig. 4. Bode gain plots of W_{S0} and W_S.

[3] We have scaled the frequency variable s with the factor 0.1 to avoid possible numerical problems.

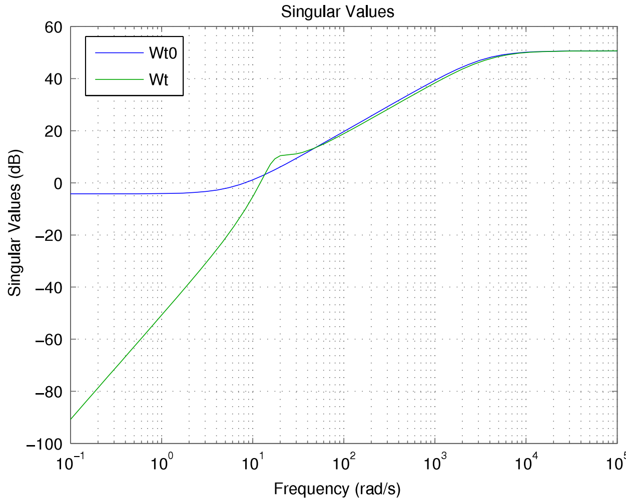

Fig. 5. Bode gain plots of W_{T0} and W_T.

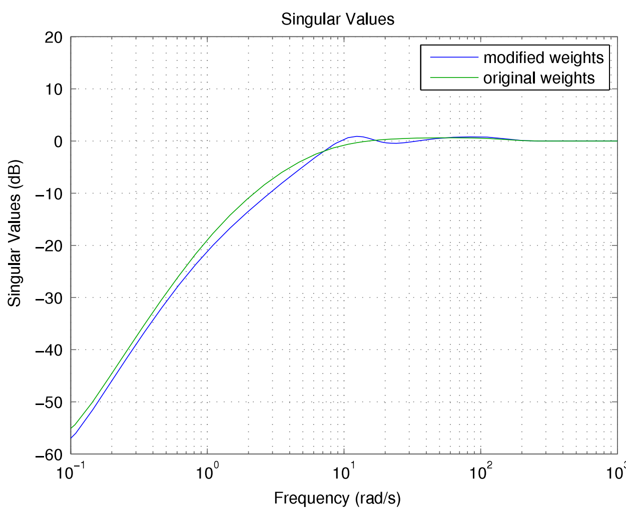

Fig. 6. Bode gain plots of S shaped under the generalized plant in Fig. 2.

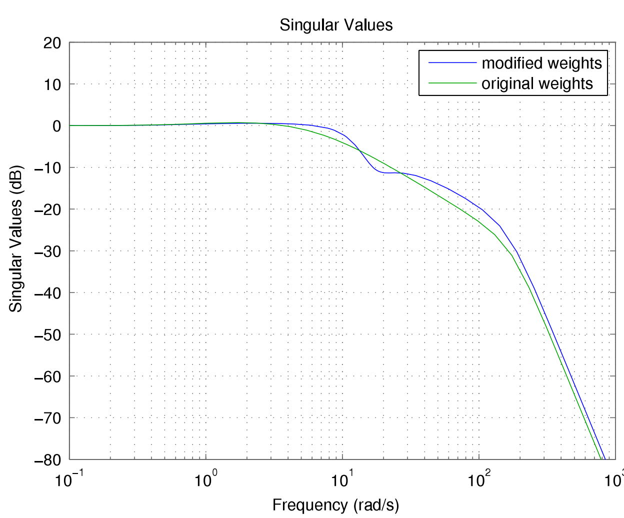

Fig. 7. Bode gain plots of T shaped under the generalized plant in Fig. 2.

We next constructed the generalized plant shown in Fig. 3 with P, W_S, and W_T. We then designed a full-order H_∞ optimal controller by minimizing γ subject to $\|T(G_{\gamma,\alpha}, K)\|_\infty < 1$ via bisection search. The bisection terminated at 85.0235 for $\alpha = 0$ and 0.1988

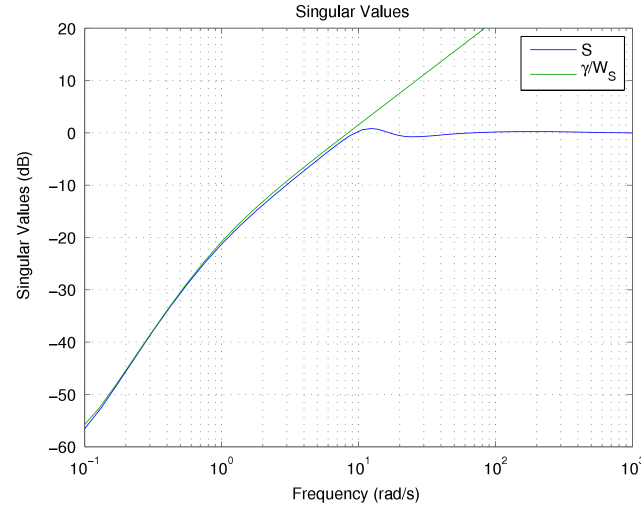

Fig. 8. Bode gain plot of S shaped under the proposed generalized plant in Fig. 3 with $\alpha = 0.5$.

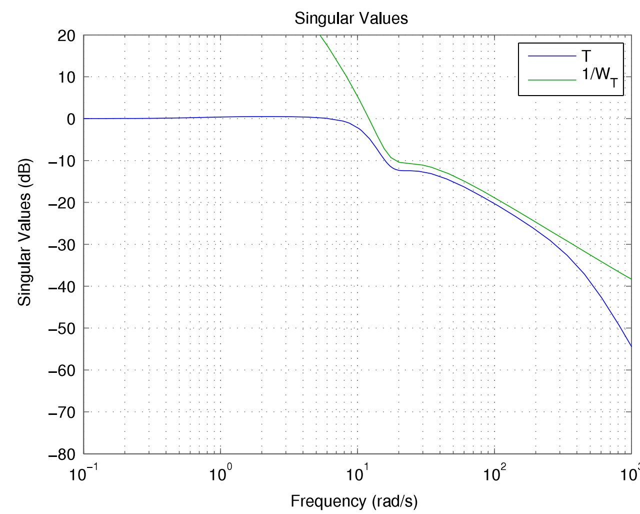

Fig. 9. Bode gain plot of T shaped under the proposed generalized plant in Fig. 3 with $\alpha = 0.5$.

for $\alpha = 0.5$. The Bode gain plots of the resulting shaped S and T for the case $\alpha = 0.5$ are shown in Figs. 8 and 9. It can be seen that we can shape S and T as desired by letting $\alpha = 0.5$. In particular, the results in Figs. 8 and 9 are almost identical to those in Figs. 6 and 7, respectively. We thus confirmed the usefulness of the parameter α.

Since $G_{\gamma,\alpha}$ satisfies Assumption 1, we can compute the H_∞-optimal static gain by carrying out a bisection search over γ subject to $L(M_\gamma, P, K) \prec 0$. The bisection search terminated at $\gamma = 2.4079 \times 10^3$ yielding an optimal static controller $K_{\mathrm{opt,st}} = -0.2228$. The Bode gain plot of the corresponding S and T are shown in Figs. 10 and 11. Even though $K_{\mathrm{opt,st}}$ is surely optimal static controller, we can see that the shaping of S and T is inadequate. This clarifies the limitation of the performance achievable by static controllers in this case.

3. Dynamical H_∞ controller synthesis

Let us move on to the synthesis of dynamical controllers of the form

$$K : \begin{cases} \dot{x}_c = A_c x_c + B_c y, \\ u = C_c x_c + D_c y \end{cases} \tag{14}$$

where $x_c \in \mathbb{R}^{n_c}$. As in the preceding section, we are interested in whether the set of controllers K satisfying internal stability and

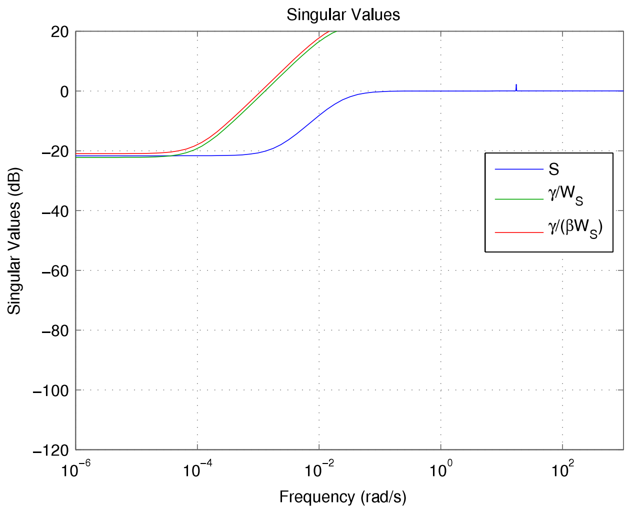

Fig. 10. Bode gain plot of S shaped with the static H_∞-optimal controller.

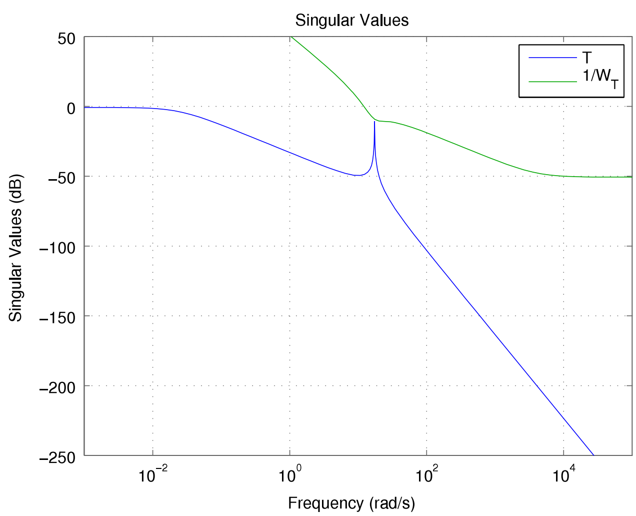

Fig. 11. Bode gain plot of T shaped with the static H_∞-optimal controller.

$\|T(G_\gamma, K)\| < 1$ becomes convex in the variables (A_c, B_c, C_c, D_c) under Assumption 1 (or 2). Unfortunately, this seems too demanding and beyond reach as explicated later on. However, it turns out that Assumption 1 (or 2) still brings novel convexity results for dynamical controller synthesis. In the sequel, we only concentrate on Assumption 1 to avoid duplicated arguments.

3.1. Convexity of (C_c, D_c) for fixed (A_c, B_c)

By following the standard procedure for dynamical controller synthesis, let us first write the state space matrices $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ of the closed-loop system $T(G_\gamma, K)$ as

$$\begin{aligned} A_{\rm cl} &= \widetilde{\mathsf{A}} + \widetilde{\mathsf{B}}_2\widetilde{\mathsf{K}}\widetilde{\mathsf{C}}_2, \qquad B_{\rm cl} = \widetilde{\mathsf{B}}_1, \\ C_{\rm cl} &= \widetilde{\mathsf{C}}_{1,\gamma} + \widetilde{\mathsf{D}}_{12}\widetilde{\mathsf{K}}\widetilde{\mathsf{C}}_2, \qquad D_{\rm cl} = 0. \end{aligned} \tag{15}$$

Here, we defined

$$\begin{aligned} &\widetilde{\mathsf{A}} := \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \quad \widetilde{\mathsf{B}}_1 := \begin{bmatrix} B_1 \\ 0_{n_c,n_w} \end{bmatrix}, \quad \widetilde{\mathsf{B}}_2 := \begin{bmatrix} B_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}, \\ &\widetilde{\mathsf{C}}_{1,\gamma} := \begin{bmatrix} C_{1,\gamma} & 0_{n_z,n_c} \end{bmatrix}, \quad \widetilde{\mathsf{D}}_{12} := \begin{bmatrix} D_{12} & 0_{n_z,n_c} \end{bmatrix}, \\ &\widetilde{\mathsf{C}}_2 := \begin{bmatrix} C_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}, \quad \widetilde{\mathsf{K}} := \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \end{aligned} \tag{16}$$

Furthermore, let us define

$$\widetilde{G}_\gamma(s) := \left[\begin{array}{c|cc} \widetilde{\mathsf{A}} & \widetilde{\mathsf{B}}_1 & \widetilde{\mathsf{B}}_2 \\ \hline \widetilde{\mathsf{C}}_{1,\gamma} & 0 & \widetilde{\mathsf{D}}_{12} \\ \widetilde{\mathsf{C}}_2 & 0 & 0 \end{array}\right]. \tag{17}$$

As is well-known, this procedure enables us to deal with the dynamical controller synthesis problem as a static controller synthesis problem since $T(G_\gamma, K) = T(\widetilde{G}_\gamma, \widetilde{\mathsf{K}})$ holds. In particular, since the structure of (15) conforms to (5), we can conclude that the set $\mathcal{K}_\gamma^{\rm dy}$, defined by

$$\begin{aligned} \mathcal{K}_\gamma^{\rm dy} := \Big\{ \widetilde{\mathsf{K}} :\ & \widetilde{\mathsf{K}} \in \mathbb{R}^{(n_c+n_u)\times(n_c+n_y)}, \\ & \widetilde{A} + \widetilde{B}_2\widetilde{K}\widetilde{C}_2 \in \mathbb{H}^{n+n_c} \text{ and } \|T(\widetilde{G}_\gamma, \widetilde{\mathsf{K}})\|_\infty < 1 \Big\}, \end{aligned}$$

becomes convex if the state space matrices in (17) satisfy the conditions in Assumption 1. Unfortunately, it is obvious from (16) that the condition (ii) is never satisfied unless we let $n_c = 0$.

To examine which convexity results can be obtained under Assumption 1, let us focus on the alternative representation of $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ given by

$$\begin{aligned} A_{\rm cl} &= \mathsf{A}(A_c, B_c) + \mathsf{B}_2\mathsf{K}\mathsf{C}_2, \qquad B_{\rm cl} = \mathsf{B}_1, \\ C_{\rm cl} &= \mathsf{C}_{1,\gamma} + \mathsf{D}_{12}\mathsf{K}\mathsf{C}_2, \qquad D_{\rm cl} = 0. \end{aligned} \tag{18}$$

Here, we defined

$$\begin{aligned} &\mathsf{A}(A_c, B_c) := \begin{bmatrix} A & 0 \\ B_cC_2 & A_c \end{bmatrix}, \qquad \mathsf{B}_1 := \widetilde{\mathsf{B}}_1, \\ &\mathsf{B}_2 := \begin{bmatrix} B_2 \\ 0_{n_c,n_u} \end{bmatrix}, \\ &\mathsf{C}_{1,\gamma} := \widetilde{\mathsf{C}}_{1,\gamma}, \qquad \mathsf{D}_{12} := D_{12}, \qquad \mathsf{C}_2 := \widetilde{\mathsf{C}}_2, \\ &\mathsf{K} := \begin{bmatrix} D_c & C_c \end{bmatrix}. \end{aligned} \tag{19}$$

If we fix the matrices A_c and B_c and incorporate them into the plant side, we can confirm that $T(G_\gamma, K) = T(G_{\gamma,A_c,B_c}, \mathsf{K})$ holds where

$$G_{\gamma,A_c,B_c}(s) := \left[\begin{array}{c|cc} \mathsf{A}(A_c, B_c) & \mathsf{B}_1 & \mathsf{B}_2 \\ \hline \mathsf{C}_{1,\gamma} & 0 & \mathsf{D}_{12} \\ \mathsf{C}_2 & 0 & 0 \end{array}\right]. \tag{20}$$

Moreover, it is straightforward to see that the above state space matrices satisfy Assumption 1. It follows that the set of admissible $\mathsf{K} = [D_c\ C_c]$ becomes convex for fixed A_c and B_c. This result can be stated formally as in the next theorem.

Theorem 2. *For given $\gamma > 0$ and the generalized plant G_γ satisfying Assumption 1, let us consider the synthesis of dynamical controller K of the form (14). Then, for each fixed (A_c, B_c), the set $\mathcal{K}_{\gamma,A_c,B_c}$ defined by*

$$\begin{aligned} \mathcal{K}_{\gamma,A_c,B_c} := \Big\{ \mathsf{K} :\ & \mathsf{K} = [D_c\ C_c] \in \mathbb{R}^{n_u\times(n_y+n_c)}, \\ & \mathsf{A}(A_c, B_c) + \mathsf{B}_2\mathsf{K}\mathsf{C}_2 \in \mathbb{H}^{n+n_c} \\ & \text{and } \|T(G_{\gamma,A_c,B_c}, \mathsf{K})\|_\infty < 1 \Big\} \end{aligned} \tag{21}$$

is convex if it is not empty. In particular, the set $\mathcal{K}_{\gamma,A_c,B_c}$ can be characterized by an LMI as follows:

$$\begin{aligned} \mathcal{K}_{\gamma,A_c,B_c} = \Big\{ \mathsf{K} :\ & \mathsf{K} = [D_c\ C_c] \in \mathbb{R}^{n_u\times(n_y+n_c)}, \\ & \exists P \in \mathbb{S}_{++}^{n+n_c} \text{ such that } L(\mathsf{M}_{\gamma,A_c,B_c}, P, \mathsf{K}) \prec 0 \Big\}. \end{aligned} \tag{22}$$

Here, $\mathsf{M}_{\gamma,A_c,B_c}$ is defined through (19) as $\mathsf{M}_{\gamma,A_c,B_c} := \{\mathsf{A}(A_c, B_c), \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}_{1,\gamma}, \mathsf{D}_{12}, \mathsf{C}_2\}$.

In view of the results in Theorems 1 and 2, we could say that Assumption 1 has the effect that it convexifies (C_c, D_c) of the controller to be designed. The usefulness of Theorem 2 is briefly sketched in Ebihara et al. (2011).

3.2. Full-order controller synthesis via new change of variables

In the preceding subsection, we have shown an LMI-based strategy for dynamical H_∞ controller synthesis of any order. The variables (C_c, D_c) are kept directly as LMI variables at the expense of freezing (A_c, B_c). It is nonetheless useful if we can directly optimize (A_c, B_c) as well in most problem instances.

In this subsection, we consider the case of full-order controller synthesis (i.e., $n_c = n$ in (14)) and show that such direct optimization of (A_c, B_c) is indeed possible, provided that we allow them being involved in a linearizing change of variables. It should be noted that, if we give up the objective of deriving LMI conditions that keep the controller variables directly as decision variables, the problem of LMI-based H_∞ controller characterization has been solved completely in the literature. In fact, the elimination of controller variables approach has been proposed independently in Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994), while the linearizing change of controller variables approach has been proposed independently in Masubuchi et al. (1998) and Scherer et al. (1997). In comparison with these known approaches, the present approach still has its own benefits mainly because of the following reasons:

(i) contrary to Gahinet and Apkarian (1994), Iwasaki and Skelton (1994), the controller variables are kept as decision variables without being eliminated;
(ii) contrary to Masubuchi et al. (1998), Scherer et al. (1997), the linearizing change of variables in the present paper does not involve the state space matrices of the generalized plant in the controller parametrization.

We emphasize that these benefits are obtained under the restriction that the generalized plant satisfies Assumptions 1 or 2.

On the other hand, our approach has the same drawback as Gahinet and Apkarian (1994), Iwasaki and Skelton (1994), Masubuchi et al. (1998), Scherer et al. (1997) and cannot deal with structural constraints on the controller due to change of variables. Nevertheless, we emphasize that the properties stated above are particularly useful when dealing with robust controller synthesis problems for plants affected by parametric uncertainties (Barmish, 1994; Boyd et al., 1994). Roughly speaking, in the present approach, we can deal with dynamical output-feedback controller synthesis as for static state-feedback controller synthesis by a well-known simple change of variables (Scherer et al., 1997). The rest of this section is devoted to the technical details to verify these assertions.

To derive the desired LMI condition, let us revisit the matrix inequality condition

$$L(\mathsf{M}_{\gamma,A_c,B_c}, P, \mathsf{K}) \prec 0 \tag{23}$$

which is presented in (22). If we consider (A_c, B_c) as decision variables as well, this matrix inequality condition is a BMI since a bilinear term among (A_c, B_c) and $P \in \mathbb{S}^{2n}_{++}$ appears. To get around this difficulty, let us first consider partitioning P. Due to the freedom of the similarity transformation of the controller, it is shown in Masubuchi et al. (1998) that we can select P as follows without introducing any conservatism:

$$P = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}, \quad X, Z \in \mathbb{S}^n_{++}. \tag{24}$$

Then, the sole bilinear term in (23) given by $P\mathsf{A}(A_c, B_c)$ can be linearized as

$$\begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B_cC_2 & A_c \end{bmatrix} \leftrightarrow \begin{bmatrix} XA + Y_{B_c}C_2 & Y_{A_c} \\ ZA + Y_{B_c}C_2 & Y_{A_c} \end{bmatrix}. \tag{25}$$

Here, we introduced the linearizing change of variables $Y_{A_c} := ZA_c$ and $Y_{B_c} := ZB_c$. Thus the BMI (23) can be reduced into the LMI of the form

$$\widehat{L}(M_\gamma, X, Z, Y_{A_c}, Y_{B_c}, C_c, D_c) \prec 0. \tag{26}$$

Here, $\widehat{L}(\cdot)$ is an affine function with respect to the decision variables $X \in \mathbb{S}^n_{++}$, $Z \in \mathbb{S}^n_{++}$, $Y_{A_c} \in \mathbb{R}^{n\times n}$, $Y_{B_c} \in \mathbb{R}^{n\times n_y}$, $C_c \in \mathbb{R}^{n_u\times n}$, and $D_c \in \mathbb{R}^{n_u\times n_y}$. If this LMI is feasible, then the desired full-order H_∞ controller K can be reconstructed by

$$K(s) = \left[\begin{array}{c|c} Z^{-1}Y_{A_c} & Z^{-1}Y_{B_c} \\ \hline C_c & D_c \end{array}\right]. \tag{27}$$

As noted, the BMI (23) can be linearized also by the known approaches in Gahinet and Apkarian (1994), Iwasaki and Skelton (1994), Masubuchi et al. (1998) and Scherer et al. (1997). However, in stark contrast with (27), the controller parametrization there involves state space matrices of the generalized plant and this is undesirable in several applications. One of the typical examples is the robust controller synthesis against parametric uncertainties. We briefly discuss this issue in the sequel and show the usefulness of the present approach.

Consider the case where the state space matrices of G_γ are affected by polytopic-type uncertainty as follows:

$$\begin{bmatrix} A & B_1 & B_2 \\ C_{1,\gamma} & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix} \in \left\{ \sum_{l=1}^{L} \theta_l \begin{bmatrix} A^{[l]} & B_1^{[l]} & B_2^{[l]} \\ C_{1,\gamma} & 0 & D_{12} \\ C_2^{[l]} & 0 & 0 \end{bmatrix} : \theta \in \boldsymbol{\theta} \right\}, \tag{28}$$
$$\boldsymbol{\theta} := \left\{ \theta : \theta \in \mathbb{R}^L, \ \sum_{l=1}^{L} \theta_l = 1, \ \theta_l \geq 0 \right\}.$$

Here, $A^{[l]}, B_1^{[l]}, B_2^{[l]}, C_2^{[l]}$ $(l = 1, \ldots, L)$, $C_{1,\gamma}$ and D_{12} are known matrices. On the other hand, θ is a time-invariant uncertain parameter whose only available information is $\theta \in \boldsymbol{\theta}$. In (28), we have assumed that $C_{1,\gamma}$ and D_{12} are parameter-independent to derive concise convex formulation.

For ease of description, we denote by G_γ^θ the generalized plant for the parameter $\theta \in \boldsymbol{\theta}$. Then, our goal here is to design a full-order robust controller K satisfying

$$\|T(G_\gamma^\theta, K)\|_\infty < 1 \quad \forall \theta \in \boldsymbol{\theta}. \tag{29}$$

Since the LMI (26) is affine with respect to the plant data M_γ for frozen $C_{1,\gamma}$ and D_{12}, and since the parametrization (27) does not depend on the plant data, such robust controller can be sought by solving the following LMI problem:

$$\widehat{L}(M_\gamma^{[l]}, X, Z, Y_{A_c}, Y_{B_c}, C_c, D_c) \prec 0 \quad (l = 1, \ldots, L). \tag{30}$$

Here, we defined

$$M_\gamma^{[l]} := \{A^{[l]}, B_1^{[l]}, B_2^{[l]}, C_{1,\gamma}, D_{12}, C_2^{[l]}\} \quad (l = 1, \ldots, L).$$

If this LMI is feasible, then the desired robust controller can be reconstructed via (27).

This approach is based on the well-known concept of quadratic stabilization (Bernussou, Geromel, & Peres, 1989), since we seek for a single Lyapunov matrix P of the form (24) that ensures the H_∞ performance over the whole uncertainty domain. Due to this restriction, the LMI approach (30) is surely conservative, but there is no other source of conservatism. To this date, such effective and efficient quadratic-stability-based approach is only available for the static state-feedback controller synthesis. In fact, for the present robust output-feedback H_∞ controller synthesis problem, we cannot apply the approaches in Gahinet and Apkarian (1994), Iwasaki and Skelton (1994) from the outset since the elimination of the controller variables does not preserve the constraint that we have to generate a single (parameter-independent) robust controller. Similarly, the direct application of the approaches in Masubuchi et al. (1998), Scherer et al. (1997) results in a controller that depends on θ (and hence cannot be implemented). In the present approach, we have successfully circumvented these difficulties by exploiting the underlying assumptions on the generalized plant.

$$\begin{bmatrix} \mathrm{He}\{Y_{B_c}C_2^{[l]} + C_{1,\gamma}^T D_{12} D_c C_2^{[l]}\} + \mathrm{Sq}\{C_{1,\gamma}^T\} & Y_{A_c} + (ZA^{[l]} + Y_{B_c}C_2^{[l]})^T + C_{1,\gamma}^T D_{12} C_c & C_2^{[l]T} D_c^T & 0 & X^{[l]} \\ * & \mathrm{He}\{Y_{A_c}\} & ZB_2^{[l]} + C_c^{[l]T} & ZB_{12}^{[l]} & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & 0 \end{bmatrix}$$
$$+ \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} \begin{bmatrix} A^{[l]} & 0 & B_2^{[l]} & B_{12}^{[l]} & -I \end{bmatrix} \right\} \prec 0, \qquad \begin{bmatrix} X^{[l]} & Z \\ Z & Z \end{bmatrix} \succ 0 \quad (l = 1, \ldots, L). \tag{32}$$

Box II.

To reduce the conservatism of the quadratic-stability-based approach, we further note that the parameter dependent Lyapunov matrix of the form

$$P(\theta) = \begin{bmatrix} X(\theta) & Z \\ Z & Z \end{bmatrix} \tag{31}$$

can be employed. The variable Z needs to remain constant since it is involved in the linearizing change of variables. If we resort to the parameter-dependent Lyapunov matrix (31), the resulting controller synthesis problem becomes a robust SDP for which powerful approaches are available nowadays. In particular, if we let $X(\theta)$ to be linear on θ and apply the idea in Ebihara and Hagiwara (2005), Peaucelle, Arzelier, Bachelier, and Bernussou (2000) known as slack-variable LMI or LMI dilation, we can derive a tractable LMI problem ensuring that we can yield better (no worse) upper bound than the quadratic-stability-based approach. For reference, the dilated LMI condition in the case where $D_{12}^T D_{12} - I = 0$ is shown in (32) given in Box II. In (32), the variables are $X^{[l]}$ $(l = 1, \ldots, L)$, Z, Y_{A_c}, Y_{B_c}, C_c, D_c and F_i $(i = 1, \ldots, 5)$. It can be easily seen that if the LMI (30) is feasible with $\gamma = \gamma_0$, $X = X_0$, $Z = Z_0$, $Y_{A_c} = Y_{A_c,0}$, $Y_{B_c} = Y_{B_c,0}$, $C_c = C_{c,0}$, and $D_c = D_{c,0}$, then there exists sufficiently small $\varepsilon > 0$ such that the dilated LMI (32) is feasible with $\gamma = \gamma_0$, $X^{[l]} = X_0$ $(l = 1, \ldots, L)$, $Z = Z_0$, $Y_{A_c} = Y_{A_c,0}$, $Y_{B_c} = Y_{B_c,0}$, $C_c = C_{c,0}$, $D_c = D_{c,0}$, and $[F_1\ F_2\ F_3\ F_4\ F_5] = [X_0\ 0\ 0\ 0\ \varepsilon I]$. The dilated LMI (32) enables us to design a robust controller via a parameter-dependent Lyapunov matrix of the form

$$P(\theta) = \sum_{i=1}^{L} \theta_l \begin{bmatrix} X^{[l]} & Z \\ Z & Z \end{bmatrix}. \tag{33}$$

In the next section, we will design a robust controller by means of (32).

3.3. Numerical examples

Let us consider again the mixed sensitivity problem discussed in Section 2.3. For the plant (12), we selected weighting functions as in (13) and constructed the generalized plant as in Fig. 3 with $\alpha = 0.5$. For this generalized plant, we consider the problem (7) by concentrating on the full-order controllers. Obviously, this problem can be cast as

$$\inf_{X,Z,Y_{A_c},Y_{B_c},C_c,D_c} \gamma \quad \text{subject to (26)}.$$

By a bisection search over γ, we successfully designed an H_∞ optimal controller that achieves the optimal H_∞ performance $\gamma_{\mathrm{opt}} = 0.1962$. As expected, the resulting value is very close to $\gamma_{\mathrm{opt}} = 0.1988$ obtained by applying the method (Scherer et al., 1997) to exactly the same problem.[4] We confirmed that the designed optimal controller achieves the desirable shaping of S and T that is almost the same as in Figs. 8, 9.

[4] Theoretically these two values should coincide with each other. However, due to unavoidable numerical error, we cannot expect the coincidence in practice.

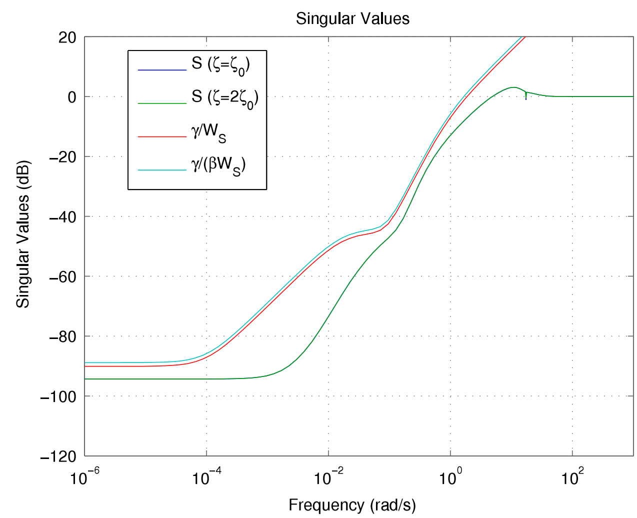

Fig. 12. Bode gain plots of S shaped with robust controller.

We finally deal with the case where the parameter ζ of the plant (12) is uncertain but bounded as $\zeta_0 \le \zeta \le 2\zeta_0$. The corresponding generalized plant can be modeled as (28) by appropriately defining $M_\gamma^{[l]}$ $(l = 1, 2)$. Under this setting, we aim at designing a robust H_∞ controller. More precisely, we want to solve

$$\gamma_{\mathrm{rob}} := \inf_K \gamma \quad \text{subject to (29)}.$$

To this end, we first solve the following problem:

$$\inf_{X,Z,Y_{A_c},Y_{B_c},C_c,D_c} \gamma \quad \text{subject to (30)}.$$

The best achievable performance by this quadratic-stability-based approach turned out to be $\gamma = 728.6299$. In view of the fact that we have achieved $\gamma \approx 0.20$ in the case where $\zeta = \zeta_0$, this result is very conservative, possibly due to the use of common (parameter-independent) Lyapunov matrix. To reduce the conservatism of the design, we next apply the dilated LMI approach that enables us to employ a parameter-dependent Lyapunov matrix of the form (33). Indeed, by minimizing γ subject to (32) via bisection search, we were able to design a suboptimal robust controller that achieves an upper bound $\gamma_{\mathrm{pd}} = 0.9704$. The performance of this controller is illustrated by Figs. 12 and 13. In these figures, the gain plots of S for $\zeta = \zeta_0$ and $\zeta = 2\zeta_0$ turn out to be almost the same. Similarly for the gain plots of T. We see that, by means of the parameter-dependent Lyapunov matrix, satisfactory shaping of S and T has been achieved robustly against the variation of the parameter ζ over $\zeta_0 \le \zeta \le 2\zeta_0$.

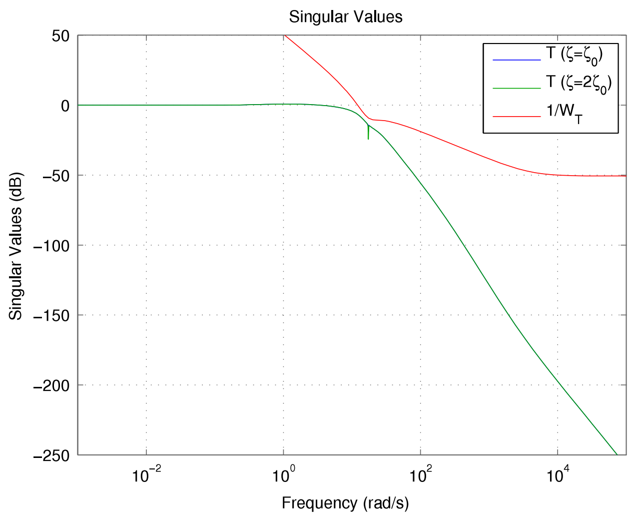

Fig. 13. Bode gain plots of T shaped with robust controller.

4. Conclusion

In this paper, we clarified several sufficient conditions on the generalized plant under which the set of static H_∞ controllers becomes convex. For the generalized plant satisfying these conditions, we further clarified that novel convexity results can be obtained even in the case of dynamical controller synthesis.

We finally note that Assumptions 1 and 2 are of course demanding for the convexity of the set of admissible static H_∞ controllers. It is undoubtfully an important issue to investigate how to loosen these assumptions further.

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