Global Probability of Collision:

Problem modeling via occupation measures

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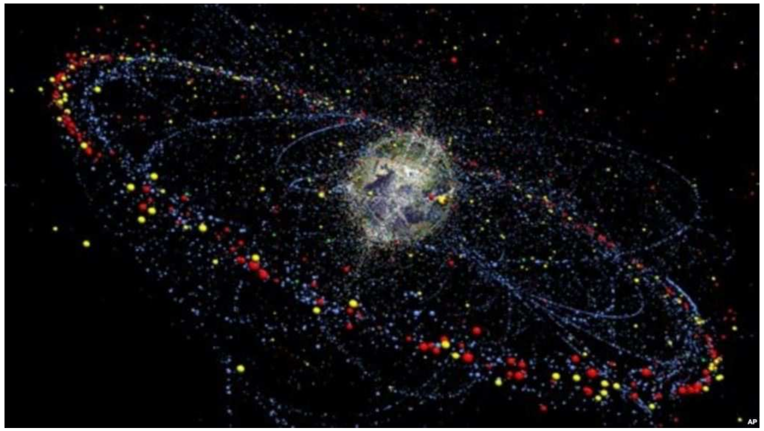
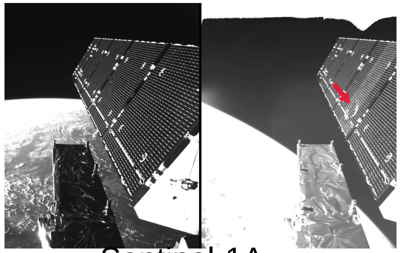
Joint Project with CNES



Credit: ESA

Context


Sentinel-1A

Prediction of the risk by space surveillance centers (JSpOC)

Goal: detect, track, identify and catalogue all in-orbit space objects

Procedure:

❶ Detection of debris ($> 5 - 10$ cm for LEO and $> 0.3 - 1.0$ m elsewhere)

❷ Propagation of trajectories

❸ Sending alert reports to operators or/and owners (if there is a risk of collision)

- Reference time t_{ca} (*Time of Closest Approach* e.g.)
- Information on the geometry of the 2 objects
- Positions and velocities of the 2 objects at t_{ca} + statistical uncertainty information

Risk management by operators or owners (ex: Airbus Defence & Space)

❶ Collision risk assessment

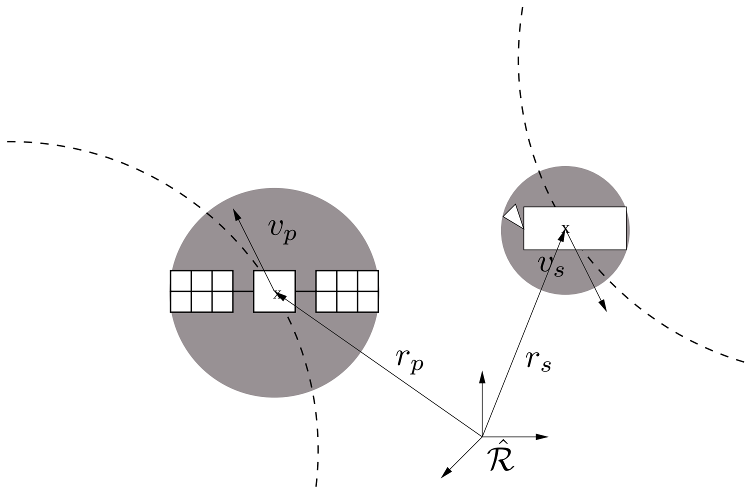
❷ Performing one or several evasive maneuvers if the predicted risk is too high

Single encounter modelling

✔ Geometry

Operational satellite p

$$x_p(t) = (r_p(t), v_p(t))$$



Orbital debris s

$$x_s(t) = (r_s(t), v_s(t))$$

Let:

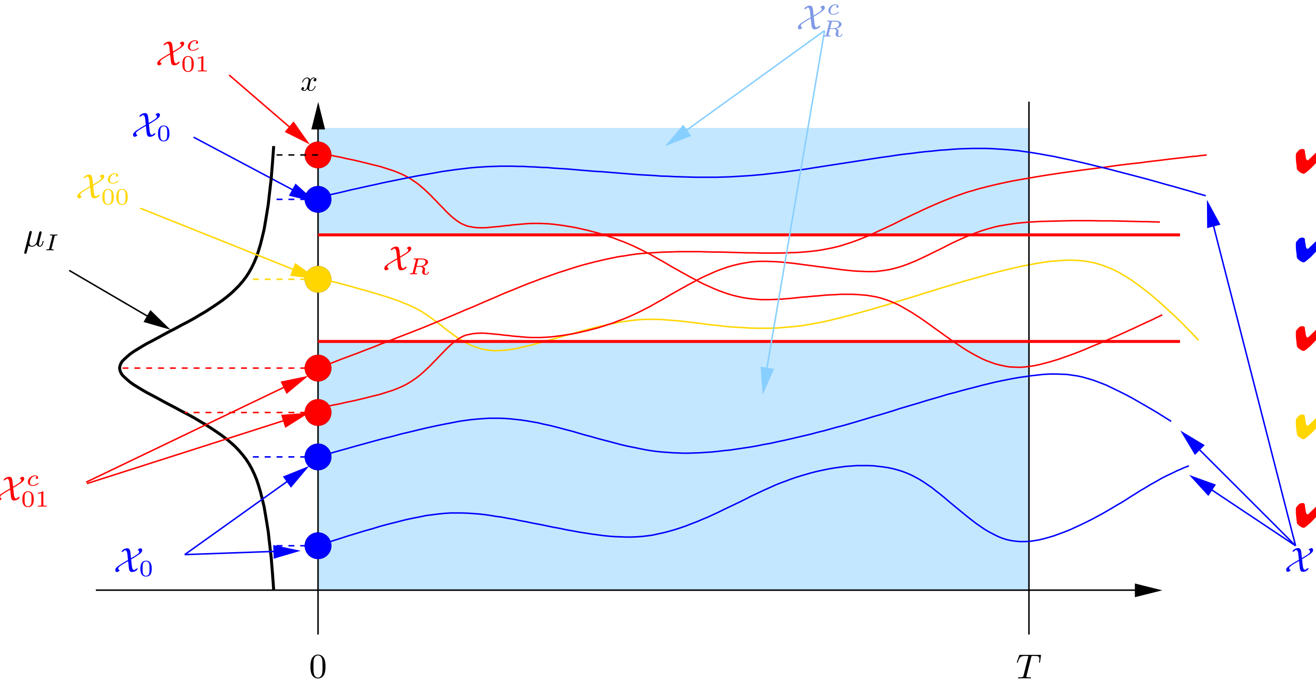
$$x(t) = (r_p(t), v_p(t), r_s(t), v_s(t)) \in \mathbb{R}^{n=2\times 6}, \qquad t \in [0, T]$$

✔ Deterministic dynamics

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \quad t \in [0, T] \\ x(0) = x_0 \end{cases} \quad \rightsquigarrow \quad x(\cdot|x_0) \text{ sample path from } x_0$$

✔ Uncertainties: initial condition $x_0 \in \mathbb{R}^{12}$ not exactly known

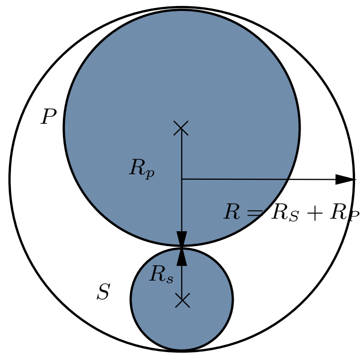
- ▶ Random vector $X_0 \sim \mathcal{N}(m_0, \Sigma_0)$ according to a given Gaussian probability measure μ_I
- ▶ (m_0, Σ_0) given by the CDM (Conjunction Data Message) or CSM (Conjunction Summary Message)



- ✔ A forbidden region $\mathcal{X}_R$
- ✔ Safe initial conditions $\mathcal{X}_0$
- ✔ Collision-prone intial states $\mathcal{X}_0^c = \mathbb{R}^n \setminus \mathcal{X}_0$
- ✔ Collision-prone initial states $\mathcal{X}_{00}^c$ at $t = 0$
- ✔ Collision-prone intial states $\mathcal{X}_{01}^c = \mathcal{X}_0^c \setminus \mathcal{X}_{00}^c$

Ex:

$$\mathcal{X}_R = \{x = (r_p, v_p, r_s, v_s) \in \mathbb{R}^{12} \mid \|r_p - r_s\|_2^2 \leq R^2\}$$



▼ **Definition 1** *Given* $x_0 \in \mathbb{R}^n$, $T > 0$, *and* $\mathcal{X}_R$, ***a collision*** *occurs if* $\exists\, t \in [0, T]$ *s.t.* $x(t|x_0) \in \mathcal{X}_R$

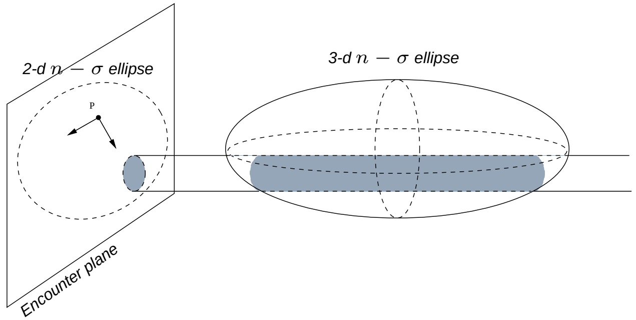
Probability of collision:

$$\mathcal{P}_c := \mathbb{P}\left(X_0 \in \mathcal{X}_0^c\right) = \mu_I(\mathcal{X}_0^c) = \int_{\mathcal{X}_0^c} \mathrm{d}\mu_I = 1 - \mathcal{P}_{nc} := 1 - \mathbb{P}(X_0 \in \mathcal{X}_0) = 1 - \mu_I(\mathcal{X}_0)$$

Collision risk assessment for specific encounter models

Short-term encounter model (Low Earth Orbits (LEO) - Relative velocity > 1 km/s)

$$\mathcal{P}_c = \left(2\pi \prod_{i=1}^{2} \sigma_i\right)^{-1} \int_{\mathcal{B}_2(0,R)} \exp\left(-\frac{1}{2}\sum_{i=1}^{2}\frac{(x_i-\mu_i)^2}{\sigma_i^2}\right) \mathrm{d}x_1 \mathrm{d}x_2$$



- ✔ Methods based on numerical integration schemes
- ✔ Analytical formula in the form of a series with positive terms:
  - ► [Chan 1997]: based on a simplifying approximation (isotropic vs anisotropic) of the initial model
  - ► [Serra 2015 - JGCD 2016]: exact analytical formula, analytical bounds, numerically efficient, validated on more than 220 000 test-cases by CNES
  - ► [Garcia-Pelayo 2015 - JGCD 2016]: exact analytical formula, analytical bounds, numerically efficient, expansion valid for arbitrary pdf and involving Hermite polynomials for Gaussian pdf

Long-term encounter model (Geostationary orbits - Relative velocities: < 10 m/s)

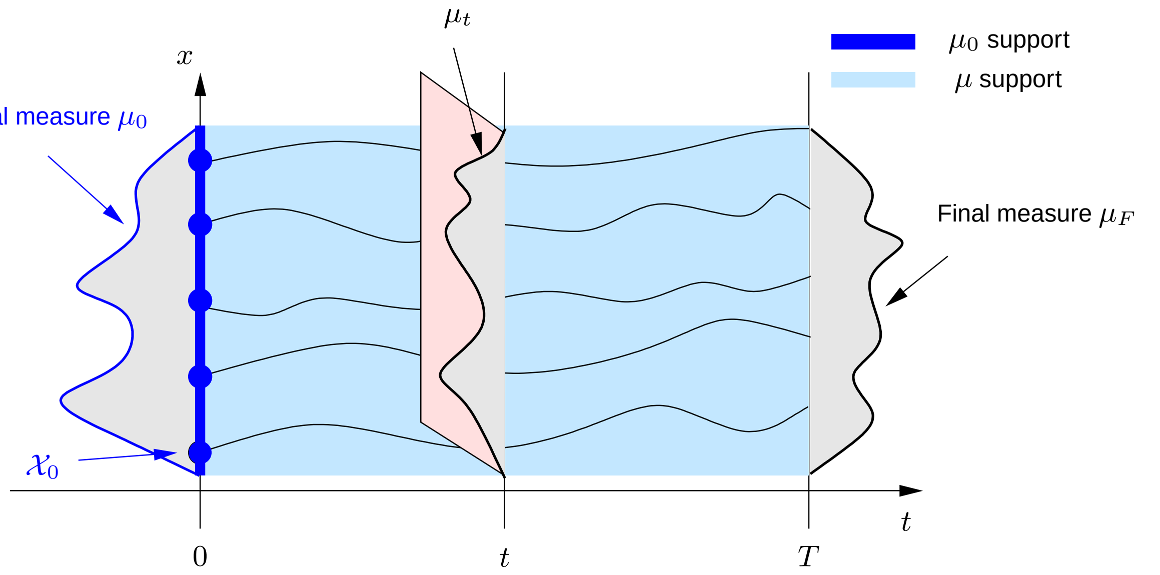
- ► Piece-wise linear approximation methods [Chan 2008], [Coppola 2012, Krier 2017]

Propose an exact, mostly general and rigorous modelling for the computation of the probability of collision

New formulation via occupation measures†

Given μ_I and $\mathcal{X}_R^c$, seek for the measure $\mu_0^* = 1_{\mathcal{X}_0}\mu_I \geq 0$ which measures safe initial states in $\mathcal{X}_0$

- ✔ $\mu_0^* = 1_{\mathcal{X}_0}\mu_I \leq \mu_I$
- ✔ $\mathcal{X}_0 \subseteq \mathcal{X}_R^c$ and $\mathrm{supp}(\mu_0^*) \subset \mathcal{X}_R^c$
- ✔ $\mu_0^* = \mathrm{Arg}[\sup_{\mu_0} \mu_0(\mathcal{X}_R^c)] \rightsquigarrow \mathrm{supp}(\mu_0^*) = \mathcal{X}_0$
- ✔ Transport of $x_0 \in \mathcal{X}_0$ to $x(T) \in \mathcal{X}_R^c$ via $x(t|x_0)$
- ✔ Family of trajectories from $\mathcal{X}_0$
  - ▶ Occupation measures $\mu(\cdot \times \cdot | x_0)$ and $\mu(\cdot \times \cdot)$
  - ▶ Lifting dynamics: Liouville's equation and operator



Any family of $x(t|x_0)$ with $x_0 \in \mathcal{X}_0$ distributed by the measure μ_0 generates:

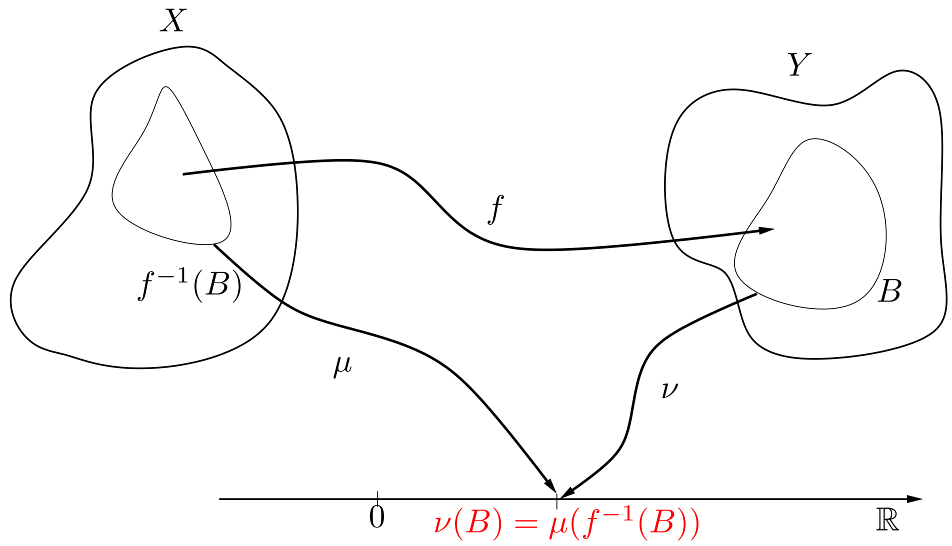
- an occupation measure μ
- a final measure μ_F

s.t. (μ_0, μ, μ_F) satisfies Liouville's equation

† based on works of D. Henrion and M. Korda

▼ **Definition 2** *(Pushforward measure)*

- $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *two measurable spaces*
- $f : X \to Y$ *a* $(\mathcal{A}, \mathcal{B})$*-measurable mapping*
- $\mu \in M(X)_+$



The image measure under the mapping f *is:*

$$\nu(B) = f_\star\mu(B) = \mu(f^{-1}(B))$$

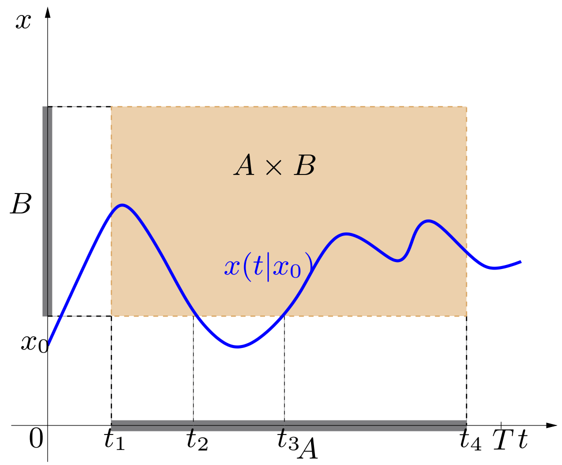
❒ **Theorem 1** *(Change of variables)*

Let $\mu \in M(X)_+$ *and* g *a measurable function on* Y

$$\int_Y g \, \mathrm{d}(f_\star\mu) = \int_X g \circ f \mathrm{d}\mu$$

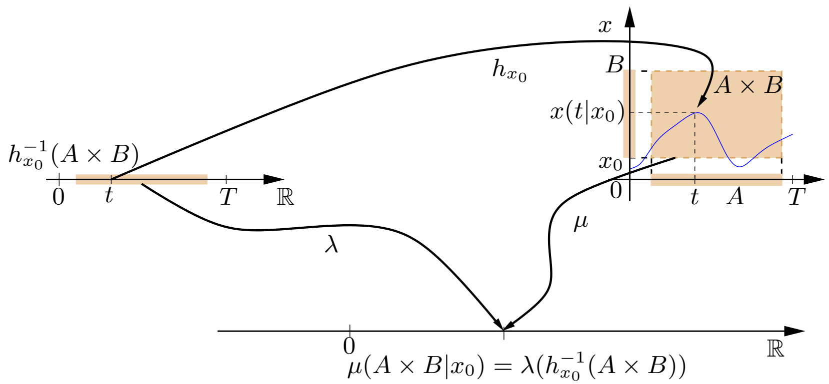
✔ Conditional occupation measure

$$\mu\left(A\times B|x_0\right) := \int_0^T 1_{A\times B}\left(t, x\left(t|x_0\right)\right)\mathrm{d}t$$



Ex :

$$\mu\left(A\times B|x_0\right) = t_4 - t_3 + t_2 - t_1$$



$$\mu\left(A\times B|x_0\right) = h_{x_0\,\star}\lambda(A\times B)$$

$$\begin{array}{rccc} h_{x_0}: & [0,T] & \rightarrow & [0,T]\times\mathbb{R}^n \\ & t & \mapsto & (t, x(t|x_0)) \end{array}$$

For any measurable function $v \in \mathcal{C}^1([0,T]\times\mathbb{R}^n)$

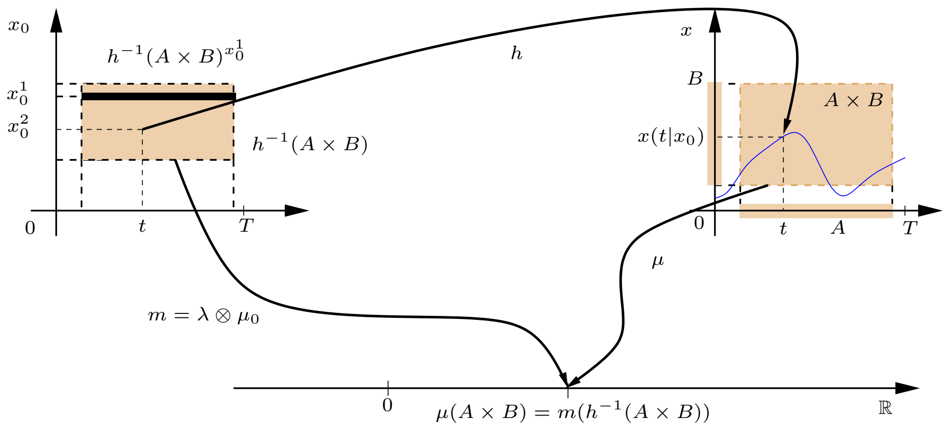
$$\int_{[0,T]\times\mathbb{R}^n} v(t,x)\mathrm{d}\mu\left(t,x|x_0\right) = \int_{[0,T]\times\mathbb{R}^n} v(t,x)\mathrm{d}(h_{x_0\star}\lambda) = \int_0^T v\circ h_{x_0}(t)\mathrm{d}t = \int_0^T v(t,x(t|x_0))\mathrm{d}t$$

Occupation measures as pushforward measures (II)

✔ Average occupation measure

$$\mu(A\times B) := \int_{\mathbb{R}^n} \mu(A\times B|x_0)\,\mathrm{d}\mu_0(x_0) = h_\star(\lambda\otimes\mu_0)(A\times B)$$

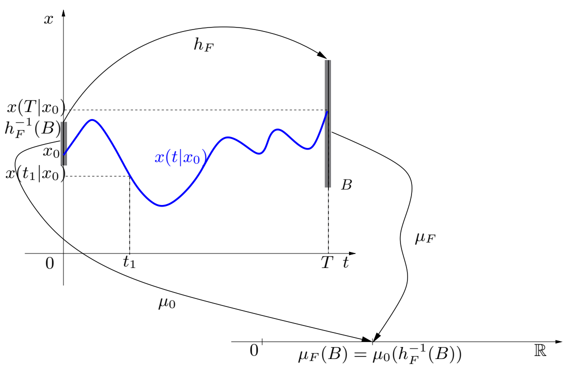
$$\begin{array}{rccc} h: & [0,T]\times\mathbb{R}^n & \to & [0,T]\times\mathbb{R}^n \\ & (t,x_0) & \mapsto & (t,x(t|x_0)) \end{array}$$



For any measurable function $v\in\mathcal{C}^1([0,T]\times\mathbb{R}^n)$

$$\int_{[0,T]\times\mathbb{R}^n} v(t,x)\mathrm{d}\mu(t,x) = \int_0^T\int_{\mathbb{R}^n} v\circ h(t,x_0)\mathrm{d}\mu_0(x_0)\mathrm{d}t = \int_0^T\int_{\mathbb{R}^n} v(t,x(t|x_0))\mathrm{d}\mu_0(x_0)\mathrm{d}t$$

✔ Final measure



$$\mu_F(B) := \int_{\mathbb{R}^n} 1_B(x(T|x_0))\mathrm{d}\mu_0(x_0) = h_{F\star}\mu_0(B)$$

$$\begin{array}{rccc} h_F: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x_0 & \mapsto & x(T|x_0) \end{array}$$

$$\int_{\mathbb{R}^n} v(T,x)\mathrm{d}\mu_F(x) = \int_{\mathbb{R}^n} v(T,x(T|x_0))\mathrm{d}\mu_0(x_0)$$

For any test function $v \in \mathcal{C}^1([0,T] \times \mathbb{R}^n)$

- Liouville's operator: $\mathcal{L} : \mathcal{C}^1([0,T] \times \mathbb{R}^n) \to \mathcal{C}([0,T] \times \mathbb{R}^n)$

$$v \mapsto \mathcal{L}v := \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i = \frac{\partial v}{\partial t} + \nabla v \cdot f$$

- Liouville's equation:

$$\boxed{\int_{\mathbb{R}^n} v(T,x)\mathrm{d}\mu_F(x) - \int_{\mathbb{R}^n} v(0,x_0)\mathrm{d}\mu_0(x_0) = \int_{[0,T]\times\mathbb{R}^n} (\mathcal{L}v)(t,x)\mathrm{d}\mu(t,x)}$$

$$\begin{aligned}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} v(t,x(t|x_0))\mathrm{d}t &= v(T,x(T|x_0)) - v(0,x_0) = \int_0^T \frac{\partial v}{\partial t}(t,x(t|x_0)) + \nabla v(t,x(t|x_0)) \cdot f(t,x(t|x_0))\mathrm{d}t \\ &= \int_{[0,T]\times\mathbb{R}^n} \frac{\partial v}{\partial t}(t,x) + \nabla v(t,x) \cdot f(t,x)\mathrm{d}\mu(t,x|x_0)\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^n} v(T,x(T|x_0))\mathrm{d}\mu_0(x_0) - \int_{\mathbb{R}^n} v(0,x_0)\mathrm{d}\mu_0(x_0) &= \int_{\mathbb{R}^n}\int_{[0,T]\times\mathbb{R}^n} (\mathcal{L}v)(t,x)\mathrm{d}\mu(t,x|x_0)\mathrm{d}\mu_0(x_0) \\ &= \int_{[0,T]\times\mathbb{R}^n} (\mathcal{L}v)(t,x)\mathrm{d}\mu(t,x)\end{aligned}$$

Problem 1 *[Direct problem]*

Solve for (μ_0, μ, μ_F):

$$p^* = \sup_{\mu_0,\mu,\mu_F} \mu_0(\mathcal{X}_R^c)$$

$$\int_{\mathcal{X}_R^c} v(T,\cdot)\mathrm{d}\mu_F = \int_{\mathcal{X}_R^c} v(0,\cdot)\mathrm{d}\mu_0 + \int_{[0,T]\times\mathcal{X}_R^c} (\mathcal{L}v)\mathrm{d}\mu, \ \forall\, v \in \mathcal{C}^1([0,T]\times\mathbb{R}^n)$$

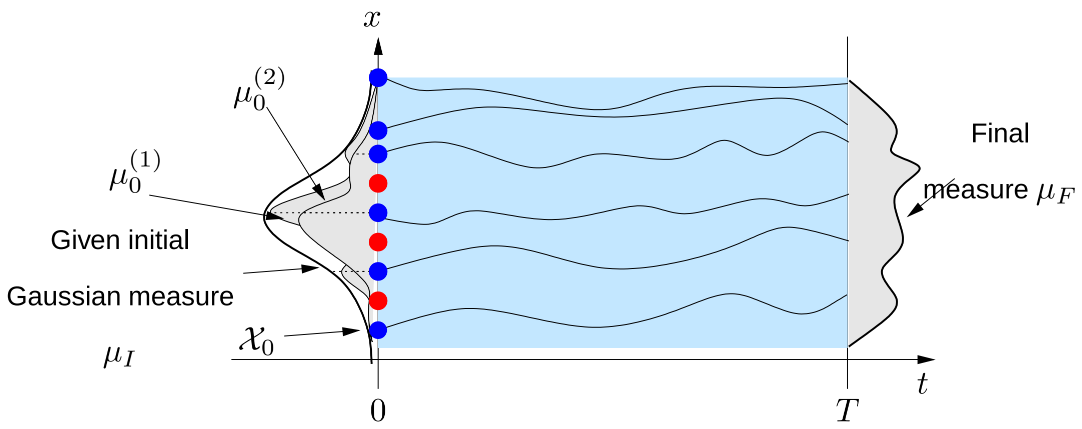
$$\text{s.t. } \mu_0 \leq \mu_I$$

$$\mu_0 \geq 0, \ \mu \geq 0, \ \mu_F \geq 0$$

$$\mathrm{supp}(\mu_0) \subseteq \mathcal{X}_R^c, \ \mathrm{supp}(\mu) \subseteq [0,T]\times\mathcal{X}_R^c, \ \mathrm{supp}(\mu_F) \subseteq \mathcal{X}_R^c$$

Theorem 2

$$p^* = \mu_I(\mathcal{X}_0) = \mathcal{P}_{nc} \quad \text{and} \quad \mu_0^* = 1_{\mathcal{X}_0}\mu_I$$

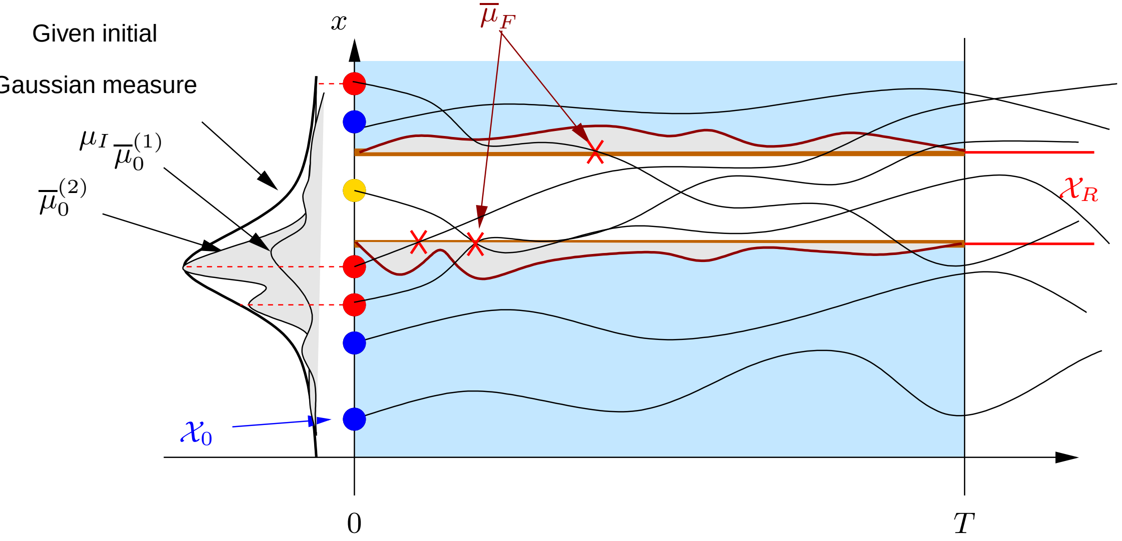


LP formulation based on collision-prone initial states $\mathcal{X}_0^c$

☛ **Problem 2** *[Indirect problem]*

Solve for $(\overline{\mu}_0, \overline{\mu}, \overline{\mu}_F)$:

$$q^* = \sup_{\overline{\mu}_0, \overline{\mu}, \overline{\mu}_F} \overline{\mu}_0 \left(\mathcal{X}_R^c\right)$$
$$\text{s.t.} \quad \int_{[0,T]\times\partial\mathcal{X}_R^c} v \mathrm{d}\overline{\mu}_F = \int_{\mathcal{X}_R^c} v(0,\cdot)\mathrm{d}\overline{\mu}_0 + \int_{[0,T]\times\mathcal{X}_R^c} (\mathcal{L}v)\mathrm{d}\overline{\mu}, \ \forall\, v \in \mathcal{C}^1([0,T]\times\mathbb{R}^n)$$
$$\overline{\mu}_0 \leq \mu_I, \ \overline{\mu}_0 \geq 0, \ \overline{\mu} \geq 0, \ \overline{\mu}_F \geq 0$$
$$\mathrm{supp}(\overline{\mu}_0) \subseteq \mathcal{X}_R^c, \ \mathrm{supp}(\overline{\mu}) \subseteq [0,T]\times\mathcal{X}_R^c, \ \mathrm{supp}(\overline{\mu}_F) \subseteq \partial\mathcal{X}_R^c$$



❐ **Theorem 3**

$$q^* = \mu_I(\mathcal{X}_{01}^c) = \mathcal{P}_c \quad \text{and} \quad \overline{\mu}_0^* = 1_{\mathcal{X}_{01}^c}\mu_I$$

☛ **Problem 3** *[Direct problem]*

$$p^* = \sup_{\mu_0 \ \mu \ \mu_F} \mu_0 \left(\mathcal{X}_R^c\right)$$

$$\int_{\mathcal{X}_R^c} v(T,\cdot)\mathrm{d}\mu_F = \int_{\mathcal{X}_R^c} v(0,\cdot)\mathrm{d}\mu_0 + \int_{[0,T]\times\mathcal{X}_R^c} (\mathcal{L}v)\mathrm{d}\mu, \ \forall\, v \in \mathcal{C}^1([0,T]\times\mathbb{R}^n)$$

$$\text{s.t.} \ \mu_0 \leq \mu_I$$

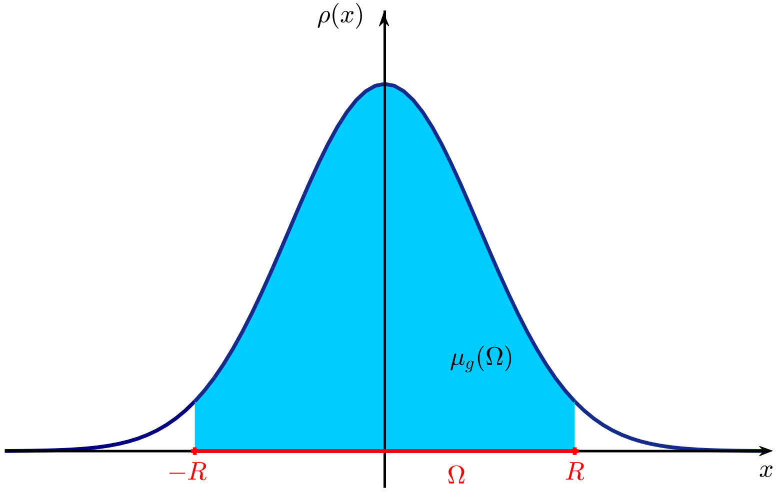
$$\mu_0 \geq 0, \ \mu \geq 0, \ \mu_F \geq 0$$

$$\operatorname{supp}(\mu_0) \subseteq \mathcal{X}_R^c \ \operatorname{supp}(\mu) \subseteq [0,T]\times\mathcal{X}_R^c, \ \operatorname{supp}(\mu_F) \subseteq \mathcal{X}_R^c$$

☛ **Problem 4** *[GM] Compute* $\mu_g(\Omega)$ *with*

$$\mu_g(B) = \frac{1}{\sqrt{(2\pi)^n}} \int_B \exp\left(-\frac{\|x\|^2}{2}\right) \mathrm{d}x = \int_B \rho(x) \mathrm{d}x$$

and $\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0,\ g_i \in \mathbb{R}^n[x],\ \forall\, i = 1, \cdots, m\}$



❒ **Theorem 4** *[Lasserre 2015]*

$$\mu_g(\Omega) = p^* = \sup_{\phi} \phi(\Omega)$$

$$\text{s.t.} \quad \begin{array}{rcll} \phi & \leq & \mu_g & \text{(a)} \\ \mathrm{supp}(\phi) & \subseteq & \Omega & \text{(b)} \\ \phi & \geq & 0 & \text{(c)} \end{array}$$

Definition 3 *[Moment of order $\alpha \in \mathbb{N}^n$ of a positive measure μ]*

$$y_\alpha = \int_X x^\alpha \, \mathrm{d}\mu$$

Notation: $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$ is the real infinite sequence of the moments

Example 1

Let $X = \mathbb{R}$ ($n = 1$) and let μ_g be the Gaussian measure with mean $\mu = 0$ and variance σ^2, $\forall \alpha \in \mathbb{N}$:

$$y_\alpha = \int_X x^\alpha \, \mathrm{d}\mu_g = \int_{\mathbb{R}} x^\alpha \rho(x) \, \mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} x^\alpha e^{-\frac{x^2}{2\sigma^2}} \, \mathrm{d}x = \begin{cases} 0 & \text{if } \alpha \text{ odd} \\ \frac{\sigma^\alpha 2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha+1}{2}\right) & \text{if } \alpha \text{ even} \end{cases}$$

Definition 4 *[Riesz Linear functional associated to $(y_\alpha)_{\alpha \in \mathbb{N}^n}$]*

$$\begin{array}{rrcl} L_y : & \mathbb{R}^n[x] & \rightarrow & \mathbb{R} \\ & f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha & \mapsto & L_y(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha \end{array}$$

Example 2

For $p : x \in \mathbb{R}^2 \mapsto p(x) = 1 + 3x_1 + 2x_1^2 - x_1 x_2$, $L_y(p) = y_{00} + 3y_{10} + 2y_{20} - y_{11}$

▼ **Definition 5** *[Moment Matrix] Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ and a given sequence $(y_\alpha)_{\alpha \in \mathbb{N}^n}$, the moment matrix $M_k(y) \in \mathcal{S}^{\binom{n+k}{n}}$ of order k with rows and columns labelled by $\alpha \in \mathbb{N}^n_k$:*

$$M_k(y)_{\alpha,\beta} = L_y(x^\alpha x^\beta) = y_{\alpha+\beta},\ \forall\, \alpha, \beta \in \mathbb{N}^n_k$$

Example 3

Let $n = 2$ and $k = 2$, then $M_2(y)$ is:

$$M_2(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix}$$

$M_k(y)$ defines a bilinear form $\langle ., . \rangle_y$ on $\mathbb{R}^n[x]_k$:

$$\langle p, q \rangle_y = L_y(pq) = \langle p, M_k(y)q \rangle_y = p^T M_k(y) q,\ \forall\, p, q \in \mathbb{R}^{\binom{n+k}{n}}$$

For a given sequence of real numbers $(y_\alpha)_{\alpha \in \mathbb{N}^n}$

- $\exists$ a representing finite Borel positive measure μ s.t. $\mathrm{supp}(\mu) = X$ and $y_\alpha = \int_X x^\alpha \, \mathrm{d}\mu$?
- Is μ determinate (uniquely determined by $(y_\alpha)_{\alpha \in \mathbb{N}^n}$)?

X is a basic semi-algebraic set, $X = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0 \, \forall \, j = 1, \cdots, m\}$

Solutions for the classical one-dimensional real moment problems [Simon 1998]:

- Stieltjes (1894), when $n = 1$, $X = \mathbb{R}_{\geq 0}$, $(y_\alpha)_{\alpha \in \mathbb{N}^n} \subset \mathbb{R}_{\geq 0}$
- Hamburger (1921), when $n = 1$, $X = \mathbb{R}$, $(y_\alpha)_{\alpha \in \mathbb{N}^n} \subset \mathbb{R}$
- Hausdorff (1923), when $n = 1$, $X = [0, 1]$, $(y_\alpha)_{\alpha \in \mathbb{N}^n} \subset \mathbb{R}$

No general solution for the multi-dimensional moment problem ($n > 1$) for general sets X

❒ **Theorem 5** *[Riesz-Haviland]*

Let $(y_\alpha)_{\alpha \in \mathbb{N}^n}$ and the closed set $X \subseteq \mathbb{R}^n$.

$\exists$ a finite representing Borel positive measure μ on X if and only if $L_y(f) \geq 0$, $\forall \, f \in \mathbb{R}[x]$ non negative on X

Nota: conditions based on representation of nonnegative polynomials on basic semi-algebraic sets and SemiDefinite Programming (SDP)

From moments to measures: the moment problem (II)

- ✔ if μ is a representing measure for $(y_\alpha)_{\alpha\in\mathbb{N}^n}$, then $\forall q \in \mathbb{R}^n[x]$, $\langle q, M_k(y)q\rangle_y = L_y(q^2) = \int q^2 \,\mathrm{d}\mu \geq 0$, and $M_k(y) \succeq 0$, $\forall\, k \in \mathbb{N}$. if $n=1$ then NSC
- ✔ Existence sufficient condition of Carleman (multivariate case, $X = \mathbb{R}^n$, $X = [-a,a]^n$)
- ✔ LMI conditions for the X-moment problem when X is a basic semi-algebraic set

Definition 6 *[Localizing Matrix]*

Let $n \in \mathbb{N}$, $(y_\alpha)_{\alpha\in\mathbb{N}^n}$ *and* $u \in \mathbb{R}^n[x]$. $M_k(uy) \in \mathcal{S}^{\binom{n+k}{n}}$ *is the localizing matrix of order* k *w.r.t* $(y_\alpha)_{\alpha\in\mathbb{N}^n}$ *and* u*:*

$$M_k(uy)_{\alpha,\beta} = L_y\left(u(x)\,x^\alpha x^\beta\right) = \sum_{\gamma\in\mathbb{N}^n} u_\gamma y_{\gamma+\alpha+\beta},\ \forall\,\alpha,\beta \in \mathbb{N}_k^n$$

Example 4

Let $n=2$, $k=1$, $(y_\alpha)_{\alpha\in\mathbb{N}^2}$ *and* $u \in \mathbb{R}^2[x]$ *with* $u : x \mapsto a + bx_1 + cx_2^2$*:*

$$M_1(uy) = \begin{pmatrix} ay_{00}+by_{10}+cy_{02} & ay_{10}+by_{20}+cy_{12} & ay_{01}+by_{11}+cy_{03} \\ ay_{10}+by_{20}+cy_{12} & ay_{20}+by_{30}+cy_{22} & ay_{11}+by_{21}+cy_{13} \\ ay_{01}+by_{11}+cy_{13} & ay_{11}+by_{21}+cy_{13} & ay_{02}+by_{12}+cy_{04} \end{pmatrix}$$

- ✔ μ is a representing measure for $(y_\alpha)_{\alpha\in\mathbb{N}^n}$ with $\mathrm{supp}(\mu) \subset X$ iff $M_k(g_J y) \succeq 0$, $\forall\, J \subset \{1,\cdots,m\}$, $\forall\, k \in \mathbb{N}$ and $g_J = \prod_{j\in J} g_j$, $g_\emptyset = 1$

Finite SDP relaxation of order k of the problem GM (Upper bounds): $\overline{p}^{(k)*} \xrightarrow[k\to+\infty]{\downarrow} \mu_g(\Omega)$

$$\overline{p}^{(k)*} = \sup_{u\in\mathbb{R}^{l(2k)}} u_0$$
$$\text{s.t.}\quad \begin{aligned} M_k(y-u) &\succeq 0 \\ M_k(u) &\succeq 0 \\ M_{k-d_j}(g_j u) &\succeq 0, \quad \forall j=1,\cdots,m,\ d_j = \lceil \deg(g_j)/2 \rceil \text{ and } k \geq \max_{j=1,\cdots,m} d_j \end{aligned}$$

Example 5

Given a standard Gaussian distribution $\mu_g \sim \mathcal{N}(m,\sigma)$*, compute* $\mathcal{P}_c = \mathbb{P}(x_0 \in [-R,R]) = \varphi([-R,R])$*. The theoretical value of* $\mathcal{P}_c$ *is:*

$$\mathcal{P}_c = \frac{1}{\sigma\sqrt{2\pi}}\int_{-R}^{R} e^{-\frac{(x-m)^2}{2\sigma^2}}\mathrm{d}x = \frac{1}{2}\left(\mathrm{erf}\left(\frac{R-m}{\sigma\sqrt{2}}\right) + \mathrm{erf}\left(\frac{R+m}{\sigma\sqrt{2}}\right)\right) = p^* = \sup_{\varphi} \varphi([-R,R])$$

$$\varphi^* = 1_{[-R,R]}\mu_g \qquad \text{s.t.}\quad \varphi \leq \mu_g,\ \varphi \geq 0,\ \mathrm{supp}(\varphi) \subset [-R,R]$$

- *Objective function: if* $\mathrm{supp}(\varphi) \subset [-R,R]$*, then* $\varphi([-R,R]) = \int_{[-R,R]} \mathrm{d}\varphi = \int_{\mathbb{R}} 1_{[-R,R]}\mathrm{d}\varphi = u_0$
- *Domination constraint:* $\forall k \in \mathbb{N}^*,\ M_k(y-u) \succeq 0$

- *Support constraint:* $\forall k \in \mathbb{N}^*,\ M_{k-1}(gu) \succeq 0,\ \text{ where } g(x) = R^2 - x^2$
- *Existence and uniqueness of positive representative measures:* $\forall k \in \mathbb{N}^*,\ M_k(u) \succeq 0$

The relaxation of order k reads:

$$\overline{p}^{(k)*} = \sup_{(u_\alpha)_{0\le\alpha\le l(2k)}} u_0 \quad \text{s.t.} \quad M_k(u) \succeq 0,\ M_k(y-u) \succeq 0,\ M_{k-1}(gu) \succeq 0, \forall\, k \ge 1$$

For example, when $m = 0$ and $\sigma = 1$:

$$\overline{p}^{(1)*} = \sup_{u_0, u_1, u_2} u_0 \quad \text{s.t.} \quad \begin{bmatrix} u_0 & u_1 \\ u_1 & u_2 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} y_0 - u_0 & y_1 - u_1 \\ y_1 - u_1 & y_2 - u_2 \end{bmatrix} \succeq 0$$

$$\overline{p}^{(2)*} = \sup_{u_{0\le i\le 4}} u_0 \quad \text{s.t.} \quad \begin{bmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} y_0 - u_0 & y_1 - u_1 & y_2 - u_2 \\ y_1 - u_1 & y_2 - u_2 & y_3 - u_3 \\ y_2 - u_2 & y_3 - u_3 & y_4 - u_4 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} R^2 u_0 - u_2 & R^2 u_1 - u_3 \\ R^2 u_1 - u_3 & R^2 u_2 - u_4 \end{bmatrix} \succeq 0$$

Primal :

$$\overline{p}^{(k)*} = \inf_{u} \; -u_0 \quad \text{s.t.} \quad M_k(u) \succeq 0, \quad M_k(y-u) \succeq 0, \quad M_{k-d_j}(g_j u) \succeq 0$$

Dual :

$$\overline{p}_D{}^{(k)*} = \inf_{X,S,(Z_j)} \; \langle M_k(y), S\rangle = \mathrm{trace}(M_k(y)S)$$

$$\text{s.t.} \quad \langle A_0, S - X\rangle - \sum_{1\le j\le m} \langle B_{j,0}, Z_j\rangle = 1$$

$$\langle A_\alpha, S - X\rangle - \sum_{1\le j\le m} \langle B_{j,\alpha}, Z_j\rangle = 0$$

$$S \succeq 0, \; X \succeq 0, \; Z_j \succeq 0$$

Nota : Lagrangian duality and $M_k(u) = \sum_{|\alpha|\le 2k} A_\alpha u_\alpha$ and $M_k(g_j u) = \sum_{|\alpha|\le 2k} B_{j,\alpha} u_\alpha$

$$\mathcal{L}(u, X, S, (Z_j)_j) = -c^T u - \langle M_k(u), X\rangle - \langle M_k(y-u), S\rangle - \sum_{1\le j\le m} \langle M_{k-d_j}(g_j u), Z_j\rangle$$

✔ SDP dual as a polynomial optimization problem:

$$\overline{p}_D{}^{(k)*} = \inf_{\sigma_0, h, \sigma_{1\le j\le m}} \int_\Omega h \, d\mu_g$$

$$\text{s.t.} \quad h - \sigma_0 - \sum g_j \sigma_j = 1, \quad h \in \Sigma^2[x]_{2k}, \; \sigma_0 \in \Sigma^2[x]_{2k}, \quad \sigma_j \in \Sigma^2[x]_{2(k-d_j)}$$

$$\inf_{\substack{h \in \mathbb{R}^n[x]_{2k} \\ h \ge 1_\Omega}} \int_{\mathbb{R}^n} |h(x) - 1_\Omega(x)| \, d\mu_g(x)$$

Example 6

Following Example 5, we get the SDP dual of SDP relaxation of order 1 as:

$$\overline{p}_D{}^{(1)*} = \sup_{x_0,x_1,x_2} \quad -(x_0+1)y_0 - 2x_1y_1 - x_2y_2$$
$$\begin{bmatrix} x_0 & x_1 \\ x_1 & x_2 \end{bmatrix} \succeq 0$$

and:

$$\overline{p}_D{}^{(2)*} = \sup_{\substack{x_{0\le i\le 5}\\ s_{0\le i\le 5}\\ z_{0\le i\le 5}}} \quad -s_0y_0 - 2s_1y_1 - 2s_2y_2 - 2s_4y_4 - y_3s_3 - s_5y_5$$
$$s_0 - x_0 - z_0 = 1$$
$$2s_1 - 2x_1 - 2R^2z_1 = 0$$
$$2s_2 + s_3 - 2x_2 - x_3 + z_0 - R^2z_2 = 0$$
$$2s_4 - 2x_4 + z_1 = 0$$
$$s_5 - x_5 - 2R^2z_1 = 0$$
$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix} \succeq 0, \begin{bmatrix} s_0 & s_1 & s_2 \\ s_1 & s_3 & s_4 \\ s_2 & s_4 & s_5 \end{bmatrix} \succeq 0, \begin{bmatrix} z_0 & z_1 \\ z_1 & z_2 \end{bmatrix} \succeq 0$$

Lower bounds for problem GM

Finite SDP relaxation of order k of the problem GM (Lower bounds):

Let $\Omega^c = \Omega \backslash \mathrm{supp}(\varphi)$ and $\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0,\ g_i \in \mathbb{R}^n[x],\ \forall\, i = 1, \cdots, m\}$

❒ **Corollary 1** *Assume that:*

$$\Omega^c = \bigcup_{l=1}^{s} \Omega_l^c \ \text{ with } \ \varphi(\Omega^c) = \sum_{l=1}^{s} \varphi(\Omega_l^c)$$

where:

$$\Omega_l^c = \left\{x \in \mathbb{R}^n :\ g_{l_j}(x) \geq 0,\ g_{l_j} \in \mathbb{R}[x],\ j = 1, \cdots, m_l\right\},\ \ l = 1, \cdots, s$$

Let $d_j = \lceil (\deg g_{l_j})/2 \rceil$ *and* $d_0 = \max_j d_j$

Let $\overline{p}_l^{(k)^*}$ *for all* $l = 1, \cdots, s$ *be the optimal value of the SDP relaxation of order* k *with* g_{l_j} *instead of* g_j *(and* m_l *instead of* m*). If*

$$\underline{p}^{(k)^*} = \varphi(\mathbb{R}^n) - \left(\sum_{l=1}^{s} \overline{p}_l^{(k)^*}\right)$$

Then, $(\underline{p}^{(k)^*})_{k \in \mathbb{N}}$ *is monotone non decreasing with:*

$$\varphi(\Omega) \geq \underline{p}^{(k)^*}\ \forall\, k \geq d_0 \quad \text{and} \quad \underline{p}^{(k)^*} \xrightarrow[k \to +\infty]{\uparrow} \varphi(\Omega)$$

✔ Liouville's equation as an infinite system of linear equations on moments

➥ **Assumption 1** *The real vector field f defining the dynamics of the objects involved in the encounter is:*

$$f_i : (t, x) \mapsto \sum_{\gamma_1 + |\gamma_2| \leq d_{f_i}} p_{i,\gamma_1,\gamma_2} t^{\gamma_1} x^{\gamma_2}$$

❍ **Proposition 1** *For test functions $v_{\alpha,\beta} = v_\gamma : (t, x) \mapsto t^\alpha x^\beta$, the weak form of the Liouville equation is an infinite-dimensional linear system of equation on $(u^0_\gamma)_{\gamma\in\mathbb{N}^n}$, $(u_\gamma)_{\gamma\in\mathbb{N}^{n+1}}$ and $(u^F_\gamma)_{\gamma\in\mathbb{N}^n}$ of μ_F:*

$$AU = A \begin{bmatrix} u^{0\,T}_\gamma & u^T_\gamma & u^{F\,T}_\gamma \end{bmatrix}^T = 0$$

where A is a linear operator defined by the structure of U and given as:

$$\begin{aligned}
u^F_\beta - u^0_\beta - \sum_{i=1}^n \beta_i \sum_{\gamma_1 + |\gamma_2| \leq d_{f_i}} p_{i,\gamma_1,\gamma_2} u_{\gamma_1, \beta^{(i)}+\gamma_2} &= 0 \quad \text{if} \, \alpha = 0 \\
T^\alpha u^F_\beta - \alpha u_{\alpha-1,\beta} - \sum_{i=1}^n \beta_i \sum_{\gamma_1 + |\gamma_2| \leq d_{f_i}} p_{i,\gamma_1,\gamma_2} u_{\alpha+\gamma_1, \beta^{(i)}+\gamma_2} &= 0 \quad \text{if } \alpha \geq 1
\end{aligned}$$

where $\beta^{(i)} = \beta^{(i)}_1 \cdots \beta^{(i)}_l \cdots \beta^{(i)}_n,\ \beta^{(i)}_l \in \mathbb{N}$

Direct :

$$\overline{p}^{(k)^*} = \sup_{u^0, u, u^F} u_0^0$$

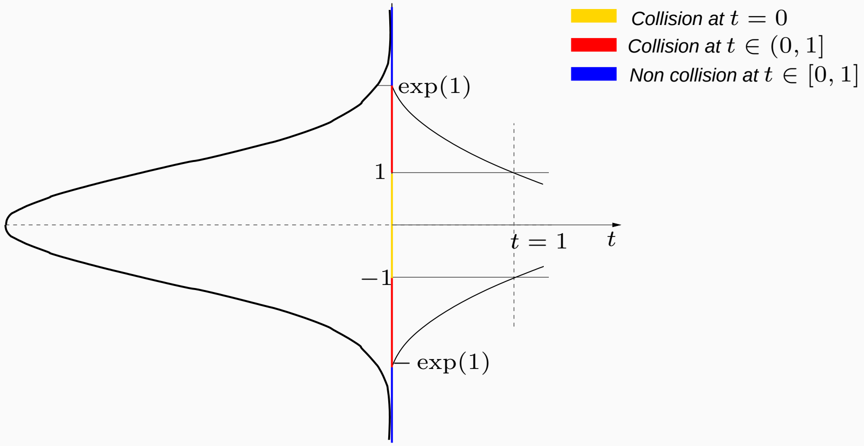
$$\text{s.t.} \quad \begin{aligned} & A_k\left(u^0, u, u^F\right) = 0 \\ & M_k\left(u^0\right) \succeq 0 \\ & M_k\left(y^0 - u^0\right) \succeq 0 \\ & M_k\left(u\right) \succeq 0 \\ & M_k\left(u^F\right) \succeq 0 \\ & M_{k-d_{d_i}}\left(g_i^d u^0\right) \succeq 0 \\ & M_{k-d_{d_i}}\left(g_i^d u\right) \succeq 0 \\ & M_{k-2}\left(t\left(T-t\right) u\right) \succeq 0 \\ & M_{k-d_{d_i}}\left(g_i^d u^F\right) \succeq 0 \end{aligned}$$

Indirect :

$$\overline{q}^{(k)^*} = \sup_{\overline{u}^0, \overline{u}, \overline{u}^F} \overline{u}_0^0$$

$$\text{s.t.} \quad \begin{aligned} & \overline{A}_k\left(\overline{u}^0, \overline{u}, \overline{u}^F\right) = 0 \\ & M_k\left(\overline{u}^0\right) \succeq 0 \\ & M_k\left(y - \overline{u}^0\right) \succeq 0 \\ & M_k\left(\overline{u}\right) \succeq 0 \\ & M_k\left(\overline{u}^F\right) \succeq 0 \\ & M_{k-d_{d_i}}\left(g_i^d \overline{u}^0\right) \succeq 0 \\ & M_{k-d_{d_i}}\left(g_i^d \overline{u}\right) \succeq 0 \\ & M_{k-1}\left(t\left(T-t\right) \overline{u}\right) \succeq 0 \\ & \sum_{\gamma \leq d_{d_i}} g_\gamma \overline{u}_{\alpha,\beta+\gamma}^F = 0 \\ & M_{k-2}\left(t\left(T-t\right) \overline{u}^F\right) \succeq 0 \end{aligned}$$

Example 7



- ✔ $X_0 \sim \mathcal{N}(\mu, \sigma)$
- ✔ $\dot{x}(t) = -x(t)$, x_0 and $t \in [0, 1]$
- ✔ $x(t) = x_0 \exp(-t)$
- ✔ $\mathcal{X}_R^c = \{x | x^2 - 1 \geq 0\}$

$$\mathcal{P}_{nc} = \mathbb{P}(x_0 \in (-\infty, -\exp(1)] \cup [\exp(1), \infty)) = \mu_g((-\infty, -\exp(1)] \cup [\exp(1), \infty)) = 1 - \text{erf}\left(\frac{\sqrt{2}\exp(1)}{2}\right)$$

The relaxation of order k reads:

$$\begin{aligned} \overline{p}^{(k)*} = \sup_{(u^0_\alpha)_{\alpha \leq 2k}, (u_\alpha)_{\alpha \leq 2k}, (u^F_\alpha)_{\alpha \leq 2k}} \quad & u^0_0 \\ \text{s.t.} \quad & M_k(u^0) \succeq 0,\ M_k(y - u^0) \succeq 0 \\ & M_{k-2}(g u^0) \succeq 0,\ M_k(u) \succeq 0 \\ & M_{k-2}(g u) \succeq 0,\ M_{k-2}(t(1-t)u) \succeq 0 \\ & M_k(u^F) \succeq 0,\ M_{k-2}(g u^F) \succeq 0 \\ & A_k(u^0, u, u^F) = 0 \end{aligned}$$

Relaxation of order $k = 1$*:*

$$\overline{p}^{(1)*} = \sup_{\substack{u_0^0, u_1^0, u_2^0 \\ (u_{ij})_{0\le i,j\le 1},\ u_0^F, u_1^F, u_2^F}} u_0^0$$

$$\text{s.t.} \quad \begin{bmatrix} u_0^0 & u_1^0 \\ u_1^0 & u_2^0 \end{bmatrix} \succeq 0, \begin{bmatrix} y_0^0 - u_0^0 & y_1^0 - u_1^0 \\ y_1^0 - u_1^0 & y_2^0 - u_2^0 \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} u_0^F & u_1^F \\ u_1^F & u_2^F \end{bmatrix} \succeq 0 \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \succeq 0$$

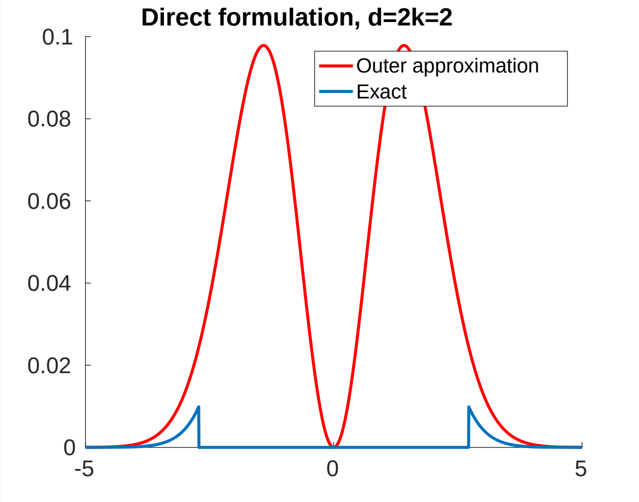
$$u_0^0 - u_0^F = 0,\ u_1^0 - u_{01} - u_1^F = 0$$

$$u_{00} - u_0^F = 0$$

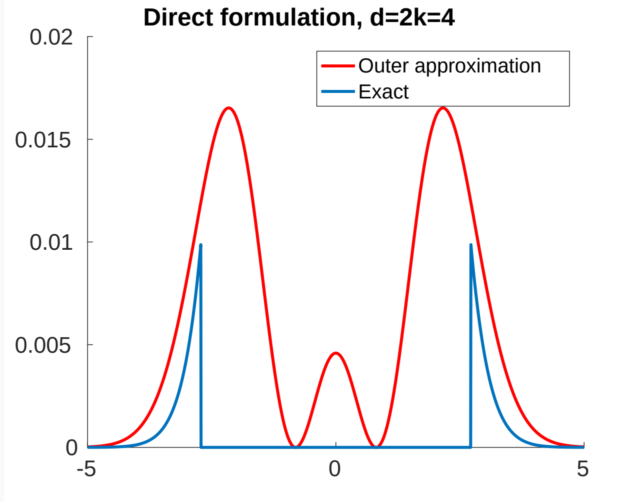
$$p^* = \int_{-\infty}^{\infty} 1_{(-\infty,-\exp(1)]\cup[\exp(1),\infty)}(x) \frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi}} dx = 1 - \mathrm{erf}(\exp(1)\sqrt{2}/2) \simeq 0.0065$$

- ✔ $1_{(-\infty,-\exp(1)]\cup[\exp(1),\infty)}(x) \dfrac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi}}$
- ✔ *Approximation* $p_d(x) \dfrac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi}}$

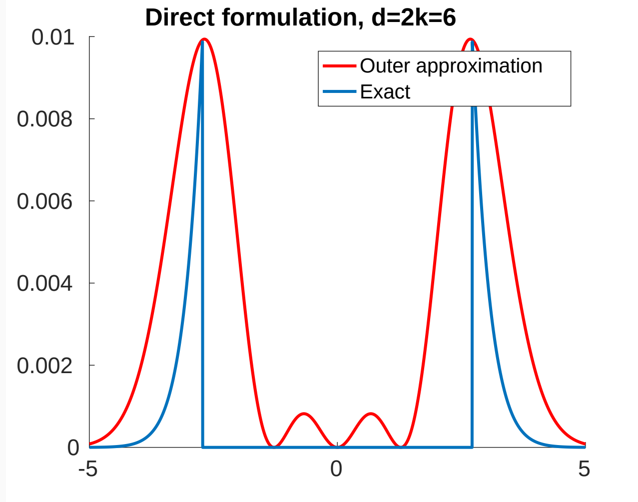
Relaxations for the direct problem: numerical example



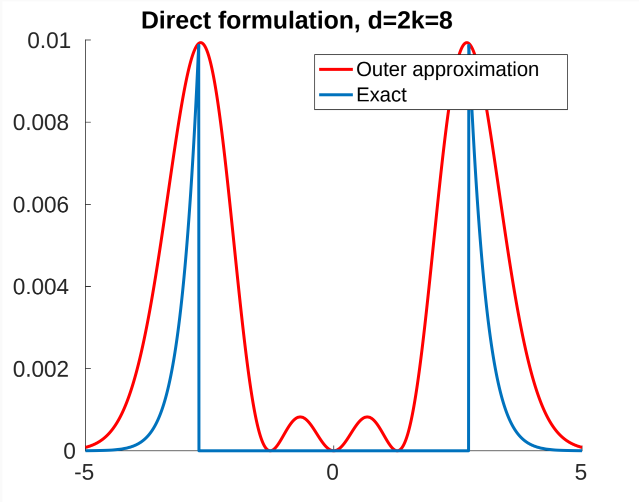
(a) $d = 2$, $\overline{p}^{(1)^*} = 0.333$



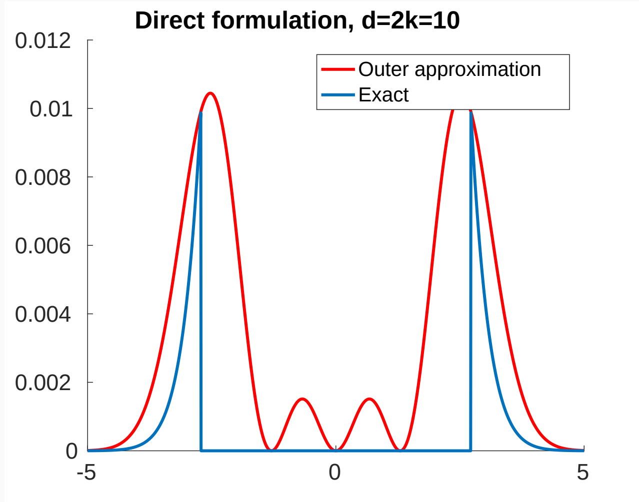
(b) $d = 4$, $\overline{p}^{(2)^*} = 0.056$



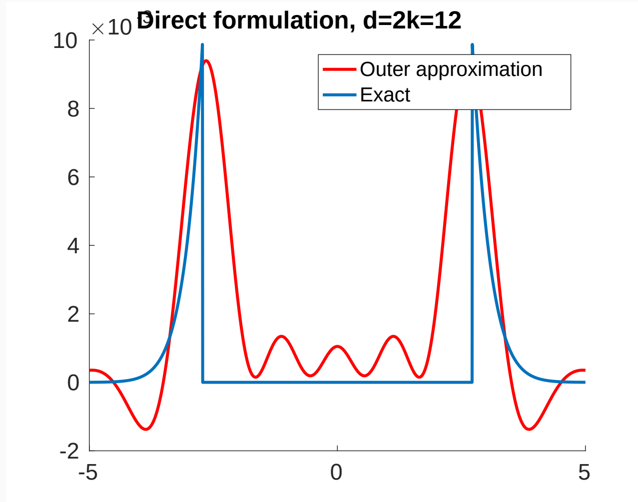
(c) $d = 6$, $\overline{p}^{(3)^*} = 0.032$



(d) $d = 8$, $\overline{p}^{(4)^*} = 0.032$

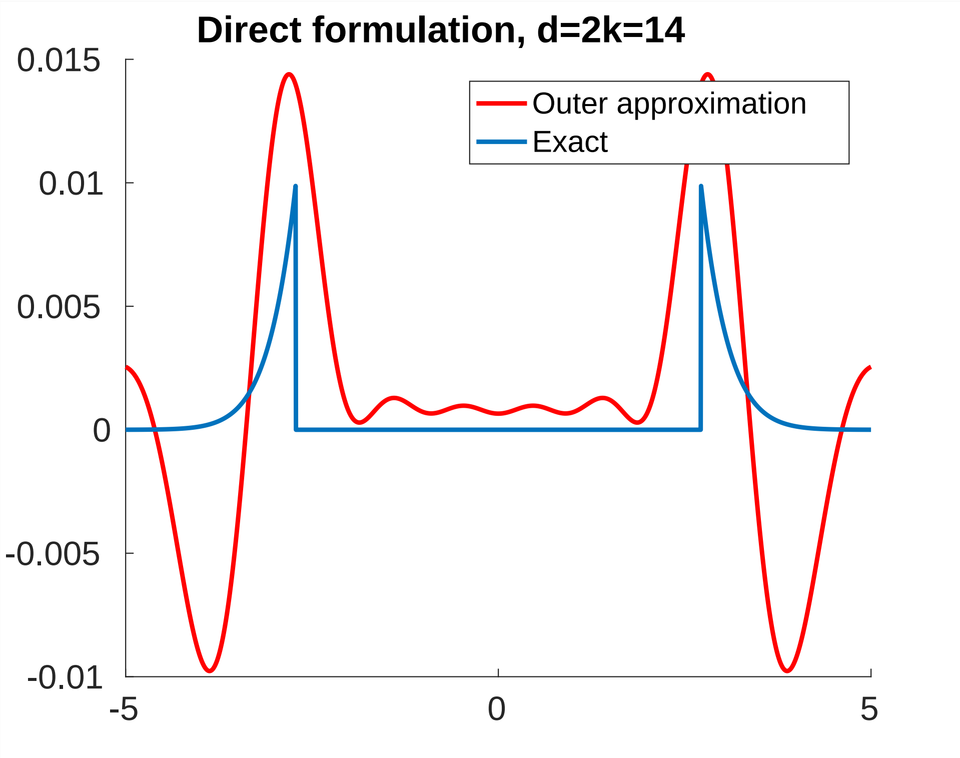


(e) $d = 12$, $\overline{p}^{(6)^*} = 0.031$

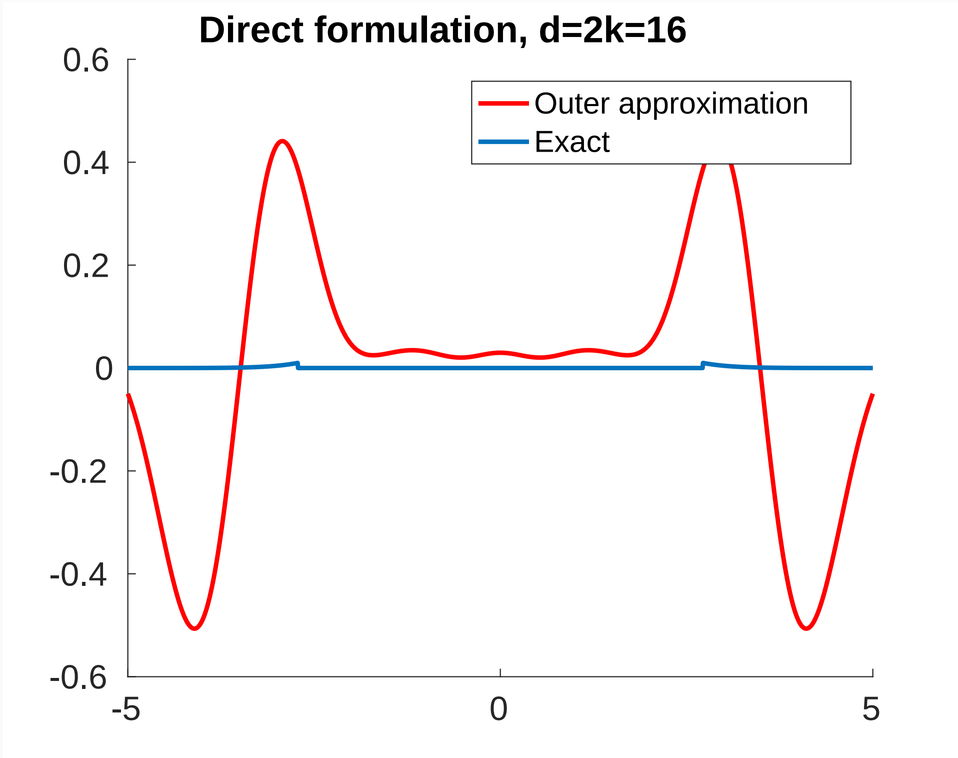


(f) $d = 14$, $\overline{p}^{(7)^*} = 0.019$

Relaxations for the direct problem: numerical example



(g) $d = 16, \overline{p}^{(8)^*} = 0.017$



(h) $d = 18, \overline{p}^{(9)^*} = 0.018$